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# A REDUCTION ALGORITHM FOR AN ORDINARY DIFFERENTIAL EQUATION ADMITTING A SOLVABLE LIE GROUP*

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**Abstract.** An iterative algorithm is presented for reducing an $n$th-order ordinary differential equation to an $(n-r)$th-order ordinary differential equation (ODE) plus $r$ quadratures when it admits an $r$-parameter solvable Lie group of transformations. The procedure is automatic. The reduced $(n-r)$th-order ODE is obtained without determining intermediate ODEs of orders $n-r+1, \cdots n-1$. This reduced ODE and the $r$ quadratures are deduced directly after iteratively computing $2r$ invariant coordinates $\{x_{(i)}, y_{(i)}\}$ and $3(r-1)$ coefficients $\{\alpha_j, \beta_j, \gamma_j\}$ of infinitesimal generators associated with an admitted $r$-parameter solvable Lie group.

The reduction algorithm is illustrated by several examples including the third-order Blasius equation which admits a two-parameter group and a fourth-order ODE admitting a three-parameter solvable group which arises in studying the group properties of the wave equation in an inhomogeneous medium.

**Key words.** reduction algorithm, Lie group, Lie algebra, solvable group, Blasius equation, wave equation, differential invariant, canonical coordinates, quadrature

**AMS(MOS) subject classifications.** primary 34A05, 58F35; secondary 22E25

**1. Introduction.** In this paper we construct an iterative algorithm for reducing an $n$th-order ordinary differential equation which admits an $r$-parameter Lie group of transformations, $2 \leqq r \leqq n$. If the Lie group is *solvable* then we will show that the given $n$th-order ODE can be reduced iteratively to an $(n-r)$th-order ODE plus $r$ quadratures. The reduced $(n-r)$th-order ODE will arise directly from the given $n$th-order ODE without the need to determine any intermediate ODEs of orders $n-r+1$ to $n-1$. The results presented in this paper appear in a slightly less general form in Bluman and Kumei (1989, § 3.4).

Bianchi (1918, § 167) (cf. Eisenhart (1933, § 36)) used solvable Lie groups (called integrable groups in earlier literature!) to reduce the order of a system of first-order ODEs. Olver (1986, pp. 154–157) gives an existence theorem which shows that if an $n$th-order ODE admits an $r$-parameter solvable Lie group of transformations, then its general solution can be found by quadratures from the general solution of an $(n-r)$th-order ODE. However Olver's proof of his existence theorem does not yield an iterative reduction algorithm. In particular his proof, as illustrated by an example, requires us to determine all intermediate ODEs.

We briefly summarize some important results concerning symmetries and differential equations necessary for the construction of the reduction algorithm. For details see Olver (1986) or Bluman and Kumei (1989).

Consider an $n$th-order ODE

$$F(x, y, y_1, \cdots, y_n) = 0, \tag{1.1}$$

where

$$y_k = \frac{d^k y}{dx^k}, \qquad k = 1, 2, \cdots, n,$$

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and a one-parameter Lie group of transformations

$$x^* = X(x, y; \varepsilon) = x + \varepsilon\xi(x, y) + O(\varepsilon^2),$$
$$y^* = Y(x, y; \varepsilon) = y + \varepsilon\eta(x, y) + O(\varepsilon^2), \tag{1.2}$$

with infinitesimal generator given by

$$\mathbf{X} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{1.3}$$

The $k$th extended infinitesimal generator of (1.3) is given by

$$\mathbf{X}^{(k)} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \eta^{(1)}(x, y, y_1)\frac{\partial}{\partial y_1} + \cdots + \eta^{(k)}(x, y, y_1, \cdots, y_k)\frac{\partial}{\partial y_k},$$

where, in terms of the total derivative operator

$$\frac{D}{Dx} = \frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + y_2\frac{\partial}{\partial y_1} + \cdots + y_{n+1}\frac{\partial}{\partial y_n},$$

$$\eta^{(k)}(x, y, y_1, \cdots, y_k) = \frac{D\eta^{(k-1)}}{Dx} - y_k\frac{D\xi}{Dx}, \qquad k = 1, 2, \cdots, n$$

with

$$\eta^{(0)} = \eta(x, y).$$

Then the group (1.2) is admitted by the ODE (1.1) (ODE (1.1) is invariant under the group (1.2)) if and only if

$$\mathbf{X}^{(n)}F = 0$$

when $F = 0$.

If (1.1) admits (1.2) then ODE (1.1) can be reduced constructively to an $(n-1)$th-order ODE plus a quadrature. This reduction can be accomplished in terms of either differential invariants or canonical coordinates as follows.

For the first extension of (1.2) there exist invariants $u(x, y)$, $v(x, y, y_1)$ which satisfy

$$\mathbf{X}u(x, y) = 0, \qquad \mathbf{X}^{(1)}v(x, y, y_1) = 0$$

with $\partial v/\partial y_1 \neq 0$. Then

$$v_k(x, y, y_1, \cdots, y_k) = \frac{d^{k-1}v}{du^{k-1}}$$

is an invariant (*differential invariant*) of the $k$th extension of (1.2) which satisfies

$$\mathbf{X}^{(k)}v_k(x, y, y_1, \cdots, y_k) = 0$$

with $\partial v_k/\partial y_k \neq 0$, $k = 2, \cdots, n$. In terms of these differential invariants the ODE (1.1) reduces to an $(n-1)$th-order ODE plus a quadrature: In particular, (1.1) reduces to

$$G\left(u, v, \frac{dv}{du}, \cdots, \frac{d^{n-1}v}{du^{n-1}}\right) = 0 \tag{1.4}$$

for some function $G(u, v, dv/du, \cdots, d^{n-1}v/du^{n-1})$. If $\phi(u, v, C_1, C_2, \cdots, C_{n-1})=0$ is a general solution of (1.4), then a general solution of (1.1) is found by solving the first-order ODE

$$\phi(u(x,y), v(x,y,y_1); C_1, C_2, \cdots, C_{n-1})=0. \tag{1.5}$$

The ODE (1.5) reduces to quadrature since it admits (1.2).

Alternatively, let $r(x,y)$, $s(x,y)$ be *canonical coordinates* of (1.2) which satisfy $\mathbf{X}r=0$, $\mathbf{X}s=1$. Let

$$z=\frac{ds}{dr}.$$

Then (1.1) reduces to an $(n-1)$th-order ODE

$$H\left(r, z, \frac{dz}{dr}, \cdots, \frac{d^{n-1}z}{dr^{n-1}}\right)=0 \tag{1.6}$$

for some function $H(r, z, dz/dr, \cdots, d^{n-1}z/dr^{n-1})$. If

$$\psi(r, z; C_1, C_2, \cdots, C_{n-1})=0$$

is a general solution of (1.6), then a general solution of ODE (1.1) is found by solving the first-order ODE

$$\psi\left(r(x,y), \frac{s_x+s_y y_1}{r_x+r_y y_1}; C_1, C_2, \cdots, C_{n-1}\right)=0,$$

which reduces to quadrature since it admits (1.2).

An $r$-parameter Lie group of transformations is generated by $r$ infinitesimal generators

$$\mathbf{X}_\alpha=\xi_\alpha(x,y)\frac{\partial}{\partial x}+\eta_\alpha(x,y)\frac{\partial}{\partial y}, \qquad \alpha=1,2,\cdots,r,$$

of an $r$-dimensional Lie algebra $L^r$. The commutator of $\mathbf{X}_\alpha$ and $\mathbf{X}_\beta$, given by the operator

$$[\mathbf{X}_\alpha, \mathbf{X}_\beta]=\mathbf{X}_\alpha\mathbf{X}_\beta-\mathbf{X}_\beta\mathbf{X}_\alpha,$$

satisfies a commutation relation

$$[\mathbf{X}_\alpha, \mathbf{X}_\beta]=\sum_{\gamma=1}^{r} C^\gamma_{\alpha\beta}\mathbf{X}_\gamma, \tag{1.7}$$

where the coefficients $C^\gamma_{\alpha\beta}$, $\alpha, \beta, \gamma=1,2,\cdots,r$, are real constants called the structure constants of $L^r$. The $k$th extended infinitesimal generators satisfy

$$[\mathbf{X}^{(k)}_\alpha, \mathbf{X}^{(k)}_\beta]=\sum_{\gamma=1}^{r} C^\gamma_{\alpha\beta}\mathbf{X}^{(k)}_\gamma \tag{1.8}$$

with the same structure constants as in (1.7) for $k=1,2,\cdots$.

A subalgebra $J\subset L^r$ is called an ideal (normal subalgebra) of $L^r$ if for any $\mathbf{X}\in J$, $\mathbf{Y}\in L^r$, $[\mathbf{X},\mathbf{Y}]\in J$. $L^r$ is an $r$-dimensional *solvable Lie algebra* (the corresponding $r$-parameter Lie group is an $r$-parameter solvable Lie group) if there exists a chain of subalgebras

$$L^{(1)}\subset L^{(2)}\subset\cdots\subset L^{(r-1)}\subset L^{(r)}=L^r$$

such that $L^{(k)}$ is a $k$-dimensional Lie algebra and $L^{(k-1)}$ is an ideal of $L^{(k)}$, $k=1,2,\cdots,,r$. Most importantly we can show that if $L^r$ is solvable then it has a basis set $\{\mathbf{X}_1,\mathbf{X}_2,\cdots,\mathbf{X}_r\}$ satisfying commutation relations of the form

$$[\mathbf{X}_\alpha,\mathbf{X}_\beta]=\sum_{\gamma=1}^{\beta-1} C^{\gamma}_{\alpha\beta}\mathbf{X}_j,\qquad 1\leqq\alpha<\beta,\quad \beta=2,\cdots,r \tag{1.9}$$

for some real structure constants $C^{\gamma}_{\alpha\beta}$.

It is easy to show that any two-dimensional Lie algebra is solvable (cf. Bluman and Kumei (1989, p. 85)). Moreover, every even-dimensional ($r=2m$ for some integer $m$) Lie algebra contains a two-dimensional subalgebra (cf. Cohen (1911, p. 150), Dickson (1924)). It turns out that there is precisely one Lie algebra acting on $\mathbf{R}^2$ which does not contain a two-dimensional subalgebra (Olver (1989)).

**2. Invariance of a second-order ODE under a two-parameter Lie group.** We show that if a second-order ODE

$$F(x,y,y_1,y_2)=0 \tag{2.1}$$

admits a two-parameter Lie group of transformations, then we can construct the general solution of (2.1) through a reduction to two quadratures.

Let $\mathbf{X}_1$, $\mathbf{X}_2$ be basis generators of the Lie algebra of the given two-parameter Lie group of transformations and let $\mathbf{X}_i^{(k)}$ denote the $k$th extended infinitesimal generator of $\mathbf{X}_i$, $i=1, 2$. Without loss of generality we can assume that

$$[\mathbf{X}_1,\mathbf{X}_2]=\lambda\mathbf{X}_1 \tag{2.2}$$

for some constant $\lambda$.

Let $u(x,y)$, $v(x,y,y_1)$ be invariants of $\mathbf{X}_1^{(2)}$ such that

$$\mathbf{X}_1 u=0,\qquad \mathbf{X}_1^{(1)}v=0. \tag{2.3}$$

Then the differential invariant $dv/du$ satisfies the equation

$$\mathbf{X}_1^{(2)}\frac{dv}{du}=0,$$

and hence (2.1) reduces to

$$G\left(u,v,\frac{dv}{du}\right)=0 \tag{2.4}$$

for some function $G(u,v,dv/du)$. (Note that $\partial v/\partial y_1\neq 0$.) From the commutation relation (2.2) it follows that

$$\mathbf{X}_1\mathbf{X}_2 u=\mathbf{X}_2\mathbf{X}_1 u+\lambda\mathbf{X}_1 u=0.$$

Hence

$$\mathbf{X}_2 u=\alpha(u) \tag{2.5a}$$

for some function $\alpha(u)$.

Then from (2.3), (2.2), and (1.8) it follows that

$$\mathbf{X}_1^{(1)}\mathbf{X}_2^{(1)}v=0,\qquad \mathbf{X}_1^{(2)}\mathbf{X}_2^{(2)}\frac{dv}{du}=0.$$

Hence

$$\mathbf{X}_2^{(1)}v=\beta(u,v) \tag{2.5b}$$

for some function $\beta(u, v)$. Since (2.1) admits $\mathbf{X}_2$ it follows that

$$\mathbf{X}_2^{(2)} G\left(u, v, \frac{dv}{du}\right) = 0 \quad \text{when } G\left(u, v, \frac{dv}{du}\right) = 0.$$

From (2.5a), (2.5b) it follows that in terms of $(u, v)$ coordinates $\mathbf{X}_2^{(1)}$ becomes

$$\mathbf{X}_2^{(1)} = \alpha(u)\frac{\partial}{\partial u} + \beta(u, v)\frac{\partial}{\partial v},$$

and that this infinitesimal generator is admitted by (2.4). Let canonical coordinates $(R(u, v), S(u, v))$ be such that

$$\mathbf{X}_2^{(1)} R = 0, \qquad \mathbf{X}_2^{(1)} S = 1.$$

Then $(R(u, v), S(u, v))$ satisfy

$$\alpha(u)\frac{\partial R}{\partial u} + \beta(u, v)\frac{\partial R}{\partial v} = 0,$$

$$\alpha(u)\frac{\partial S}{\partial u} + \beta(u, v)\frac{\partial S}{\partial v} = 1.$$

Thus the one-parameter Lie group of transformations

$$R^* = R, \qquad S^* = S + \varepsilon,$$

is admitted by (2.4). Hence (2.4) reduces to

$$H\left(R, \frac{dS}{dR}\right) = 0 \tag{2.6}$$

for some function $H(R, dS/dR)$. In terms of a solved form

$$\frac{dS}{dR} = I(R),$$

the first-order ODE (2.6) integrates out to

$$S(u, v) = \int^{R(u,v)} I(R) \; dR + C_1,$$

where $C_1$ is an arbitrary constant. The first-order ODE

$$S(u(x, y), v(x, y, y_1)) = \int^{R(u(x,y), v(x,y,y_1))} I(R)\, dR + C_1$$

admits $\mathbf{X}_1$ and hence reduces to quadrature by the method of canonical coordinates after we determine $(r(x, y), s(x, y))$ such that

$$\mathbf{X}_1 r = 0, \qquad \mathbf{X}_1 s = 1.$$

*Consequently, any second-order* ODE *which admits a two-parameter Lie group of transformations reduces completely to quadratures.*

As an example consider the second-order linear nonhomogeneous ODE

$$y_2 + p(x)y_1 + q(x)y = g(x). \tag{2.7}$$

Let $z=\phi_1(x)$, $z=\phi_2(x)$ be linearly independent solutions of the corresponding homogeneous equation

$$z''+p(x)z'+q(x)z=0.$$

Then (2.7) admits the two-parameter $(\varepsilon_1, \varepsilon_2)$ Lie group of transformations

$$x^*=x, \qquad y^*=y+\varepsilon_1\phi_1(x)+\varepsilon_2\phi_2(x).$$

The corresponding infinitesimal generators are

$$\mathbf{X}_1=\phi_1(x)\frac{\partial}{\partial y}, \qquad \mathbf{X}_2=\phi_2(x)\frac{\partial}{\partial y}$$

with $[\mathbf{X}_1, \mathbf{X}_2]=0$. Then

$$\mathbf{X}_i^{(1)}=\phi_i(x)\frac{\partial}{\partial y}+\phi_i'(x)\frac{\partial}{\partial y_1}, \qquad i=1,2,$$

$$u=x, \qquad v=\frac{y_1}{\phi_1'(x)}-\frac{y}{\phi_1(x)},$$

$$\mathbf{X}_2u=\mathbf{X}_2x=0, \qquad \mathbf{X}_2^{(1)}v=\frac{\phi_2'(x)}{\phi_1'(x)}-\frac{\phi_2(x)}{\phi_1(x)}=\frac{W(x)}{\phi_1(x)\phi_1'(x)},$$

where $W(x)$ is the Wronskian $W(x)=\phi_1\phi_2'-\phi_2\phi_1'$. Now in terms of $x$ and $v$, $\mathbf{X}_2^{(1)}=(W(x)/\phi_1(x)\phi_1'(x))(\partial/\partial v)$. Canonical coordinates $(R(x, v), S(x, v))$ satisfy

$$\mathbf{X}_2^{(1)}R=\frac{W}{\phi_1\phi_1'}\frac{\partial R}{\partial v}=0, \qquad \mathbf{X}_2^{(1)}S=\frac{W}{\phi_1\phi_1'}\frac{\partial S}{\partial v}=1,$$

and hence

$$R=x, \qquad S=\frac{v\phi_1\phi_1'}{W}.$$

Consequently, by a simple calculation,

$$\frac{dS}{dx}=\frac{g(x)\phi_1(x)}{W(x)},$$

so that

$$S=\frac{y'\phi_1-y\phi_1'}{W}=\int\frac{g\phi_1}{W}\,dx+C_1, \tag{2.8}$$

where $C_1$ is an arbitrary constant.

By construction the first-order ODE (2.8) admits $\mathbf{X}_1=\phi_1(x)\,\partial/\partial y$. In terms of canonical coordinates $r=x$, $s=y/\phi_1(x)$, (2.8) reduces to

$$\frac{ds}{dx}=\frac{W}{(\phi_1)^2}\left[\int\frac{g\phi_1}{W}\,dx+C_1\right].$$

But $W/(\phi_1)^2=(\phi_2/\phi_1)'$. Hence

$$\frac{W}{(\phi_1)^2}\int\frac{g\phi_1}{W}\,dx=\frac{d}{dx}\left[\frac{\phi_2}{\phi_1}\int\frac{g\phi_1}{W}\,dx\right]-\frac{g\phi_2}{W}.$$

Thus

$$s=C_1\frac{\phi_2}{\phi_1}+\frac{\phi_2}{\phi_1}\int\frac{g\phi_1}{W}\,dx-\int\frac{g\phi_2}{W}\,dx+C_2,$$

which leads to the familiar general solution

$$y=C_1\phi_2+C_2\phi_1+\phi_2\int\frac{g\phi_1}{W}\,dx-\phi_1\int\frac{g\phi_2}{W}\,dx$$

of (2.7).

**3. Invariance of an nth-order ODE under a two-parameter Lie group.** Now consider the $n$th-order ODE

$$F(x, y, y_1, \cdots, y_n)=0, \tag{3.1}$$

$n\geqq 3$, assumed to be invariant under a two-parameter Lie group of transformations. Without loss of generality there exist infinitesimal generators $\mathbf{X}_1, \mathbf{X}_2$ such that $[\mathbf{X}_1, \mathbf{X}_2]=\lambda\mathbf{X}_1$ for some constant $\lambda$.

As in § 2 let $u(x, y)$, $v(x, y, y_1)$ be invariants of $\mathbf{X}_1^{(2)}$. Then $\mathbf{X}_1^{(2)}\dot{v}=0$ where $\dot{v}=dv/du$, and (3.1) reduces to

$$G\left(u, v, \frac{dv}{du}, \cdots, \frac{d^{n-1}v}{du^{n-1}}\right)=0 \tag{3.2}$$

for some function $G(u, v, dv/du, \cdots, d^{n-1}v/du^{n-1})$.

Since $[\mathbf{X}_1^{(k)}, \mathbf{X}_2^{(k)}]=\lambda\mathbf{X}_1^{(k)}$, $k=1, 2, \cdots$, it follows that

$$\mathbf{X}_2u=\alpha(u),\quad \mathbf{X}_2^{(1)}v=\beta(u, v),\quad \mathbf{X}_2^{(2)}\dot{v}=\gamma(u, v, \dot{v}),$$

for some functions $\alpha(u)$, $\beta(u, v)$, $\gamma(u, v, \dot{v})$. Then

$$\mathbf{X}_2^{(1)}=\alpha(u)\frac{\partial}{\partial u}+\beta(u, v)\frac{\partial}{\partial v},$$

with first extension given by

$$\mathbf{X}_2^{(2)}=\alpha(u)\frac{\partial}{\partial u}+\beta(u, v)\frac{\partial}{\partial v}+\gamma(u, v, \dot{v})\frac{\partial}{\partial \dot{v}},$$

is admitted by (3.2). (Note that $\partial\gamma/\partial\dot{v}\neq 0$.) Let $U(u, v)$, $V(u, v, \dot{v})$ be such that

$$\mathbf{X}_2^{(1)}U=0,\qquad \mathbf{X}_2^{(2)}V=0.$$

Then

$$\mathbf{X}_2^{(3)}\frac{dV}{dU}=0.$$

Consequently, (3.2) and hence (3.1) reduces to

$$H\left(U, V, \frac{dV}{dU}, \cdots, \frac{d^{n-2}V}{dU^{n-2}}\right)=0 \tag{3.3}$$

for some function $H(U, V, dV/dU, \cdots, d^{n-2}V/dU^{n-2})$. If

$$\phi(U, V; C_1, C_2, \cdots, C_{n-2})=0$$

is the general solution of (3.3), then the first-order ODE

$$\phi\left(U(u, v), V\left(u, v, \frac{dv}{du}\right); C_1, C_2, \cdots, C_{n-2}\right)=0 \tag{3.4}$$

admits $\mathbf{X}_2^{(1)}=\alpha(u)\,\partial/\partial u+\beta(u,v)\,\partial/\partial v$. Thus (3.4) reduces to quadrature

$$\psi(u, v; C_1, C_2, \cdots, C_{n-2}, C_{n-1})=0$$

for some function $\psi(u, v; C_1, C_2, \cdots, C_{n-2}, C_{n-1})$. But the first-order ODE

$$\psi(u(x, y), v(x, y, y_1); C_1, C_2, \cdots, C_{n-2}, C_{n-1})=0 \tag{3.5}$$

admits $\mathbf{X}_1$. Thus (3.5) reduces to quadrature which leads to a general solution of (3.1).

Hence we have shown that *if an nth-order* ODE ($n \geqq 3$) *admits a two-parameter Lie group of transformations, then it can be reduced constructively to an* $(n-2)$*th-order* ODE *plus two quadratures.* Note that the order of using the operators $\mathbf{X}_1$ and $\mathbf{X}_2$ is *crucial* if $\lambda \neq 0$.

As an example consider the Blasius equation

$$y_3+\tfrac{1}{2}yy_2=0, \tag{3.6}$$

which admits the two-parameter $(\varepsilon_1, \varepsilon_2)$ Lie group of transformations

$$x^*=e^{\varepsilon_2}(x+\varepsilon_1), \qquad y^*=e^{-\varepsilon_2}y,$$

with infinitesimal generators given by

$$\mathbf{X}_1=\frac{\partial}{\partial x}, \qquad \mathbf{X}_2=x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}.$$

Then

$$[\mathbf{X}_1, \mathbf{X}_2]=\mathbf{X}_1.$$

Invariants of $\mathbf{X}_1^{(2)}$ are

$$u=y, \quad v=y_1, \quad \dot{v}=\frac{dv}{du}=\frac{y_2}{y_1}.$$

It follows that

$$\mathbf{X}_2^{(2)}=x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}-2y_1\frac{\partial}{\partial y_1}-3y_2\frac{\partial}{\partial y_2},$$

$$\mathbf{X}_2 u=-y=-u, \qquad \mathbf{X}_2^{(1)}v=-2y_1=-2v,$$

$$\mathbf{X}_2^{(2)}\dot{v}=-\frac{y_2}{y_1}=-\dot{v}.$$

Without loss of generality we set

$$\mathbf{X}_2^{(2)}=u\frac{\partial}{\partial u}+2v\frac{\partial}{\partial v}+\dot{v}\frac{\partial}{\partial \dot{v}}.$$

Then

$$\mathbf{X}_2^{(1)}U(u, v)=0 \quad \text{leads to } U=\frac{v}{u^2},$$

and

$$\mathbf{X}_2^{(2)}V(u, v, \dot{v})=0 \quad \text{leads to } V=\frac{\dot{v}}{u}.$$

Then the third-order Blasius equation (3.6) reduces to the first-order ODE

$$\frac{dV}{dU}=\frac{V}{U}\left[\frac{\frac{1}{2}+V+U}{2U-V}\right]. \tag{3.7}$$

If in solved form $V=\phi(U;\,C_1)$ is a general solution of (3.7), then the first-order ODE

$$\dot{v}=\frac{dv}{du}=u\phi\left(\frac{v}{u^2};\,C_1\right) \tag{3.8}$$

admits $\mathbf{X}_2^{(1)}=u(\partial/\partial u)+2v(\partial/\partial v)$. In terms of corresponding canonical coordinates $s=\log v$, $r=v/u^2$, (3.8) becomes

$$\frac{ds}{dr}=\frac{\phi(r;\,C_1)}{r[\phi(r;\,C_1)-2r]}.$$

This leads to the quadrature

$$v=C_2\exp\left[\int^r\frac{\phi(\rho;\,C_1)}{\rho[\phi(\rho;\,C_1)-2\rho]}\,d\rho\right], \tag{3.9}$$

where $v=y_1$, $r=y_1/y^2$. In principle (3.9) can be expressed in a solved form

$$y_1=\psi(y;\,C_1,\,C_2),$$

which admits $\mathbf{X}_1=\partial/\partial x$, and hence reduces to quadrature

$$\int\frac{dy}{\psi(y;\,C_1,\,C_2)}=x+C_3. \tag{3.10}$$

Equation (3.10) represents a general solution of the Blasius equation.

**4. Invariance of an *n*th-order ODE under an *r*-parameter Lie group with a solvable Lie algebra.** If an $r$-parameter Lie group ($r\geqq 3$) is admitted by an $n$th-order ODE ($n\geqq r$) it does not always follow that we can have a reduction to an $(n-r)$th-order ODE plus $r$ quadratures. We show that such a reduction is always possible if the Lie algebra $L^r$, formed by the infinitesimal generators of the group, is a solvable Lie algebra.

Consider an $n$th-order ODE

$$F_n(x,y,y_1,\cdots,y_n)=0. \tag{4.1}$$

Assume that (4.1) admits an $r$-parameter Lie group of transformations ($3\leqq r\leqq n$) whose infinitesimal generators form a solvable Lie algebra. Without loss of generality we can assume that the infinitesimal generators $\{\mathbf{X}_i\}$, $i=1,2,\cdots,r$, satisfy commutation relations of the form (1.9).

Let $x_{(1)}(x,y)$, $y_{(1)}(x,y,y_1)$ be such that

$$\mathbf{X}_1x_{(1)}=0,\qquad \mathbf{X}_1^{(1)}y_{(1)}=0.$$

Then

$$\mathbf{X}_1^{(k+1)}\frac{d^ky_{(1)}}{dx_{(1)}^k}=0,\qquad k=1,2,\cdots,n-1.$$

Let

$$y_{(1)k}=\frac{d^ky_{(1)}}{dx_{(1)}^k},\qquad k=1,2,\cdots,n+1.$$

In terms of the invariants $x_{(1)}$, $y_{(1)}$, and the differential invariants $\{y_{(1)k}\}$, $k=1,2,\cdots,n-1$ of $\mathbf{X}_1^{(n)}$, ODE (4.1) reduces to an $(n-1)$th-order ODE

$$F_{n-1}(x_{(1)},y_{(1)},y_{(1)1},\cdots,y_{(1)n-1})=0 \tag{4.2}$$

for some function $F_{n-1}$ of the indicated invariants of $\mathbf{X}_1^{(n)}$.

From (1.7), (1.8) it follows that

$$\mathbf{X}_2 x_{(1)} = \alpha_1(x_{(1)}),$$
$$\mathbf{X}_2^{(1)} y_{(1)} = \beta_1(x_{(1)} y_{(1)}),$$
$$\mathbf{X}_2^{(2)} y_{(1)1} = \gamma_1(x_{(1)}, y_{(1)}, y_{(1)1}),$$

for some functions $\alpha_1$, $\beta_1$, $\gamma_1$ of the indicated arguments. Hence

$$\mathbf{X}_2^{(1)} = \alpha_1(x_{(1)}) \frac{\partial}{\partial x_{(1)}} + \beta_1(x_{(1)}, y_{(1)}) \frac{\partial}{\partial y_{(1)}},$$

with first extension given by

$$\mathbf{X}_2^{(2)} = \mathbf{X}_2^{(1)} + \gamma_1(x_{(1)}, y_{(1)}, y_{(1)1}) \frac{\partial}{\partial y_{(1)1}},$$

is admitted by (4.2).

Let $x_{(2)}(x_{(1)}, y_{(1)})$, $y_{(2)}(x_{(1)}, y_{(1)}, y_{(1)1})$ be such that

$$\mathbf{X}_2^{(1)} x_{(2)} = 0, \qquad \mathbf{X}_2^{(2)} y_{(2)} = 0. \tag{4.3}$$

Then

$$\mathbf{X}_2^{(2+k)} \frac{d^k y_{(2)}}{dx_{(2)}^k} = 0, \qquad k = 1, 2, \cdots, n-2.$$

Let

$$y_{(2)k} = \frac{d^k y_{(2)}}{dx_{(2)}^k}, \qquad k = 1, 2, \cdots, n-2.$$

In terms of the invariants $x_{(2)}$, $y_{(2)}$, $\{y_{(2)k}\}$, $k = 1, 2, \cdots, n-2$ of $\mathbf{X}_2^{(n)}$ (which are also invariants of $\mathbf{X}_1^{(n)}$), ODE (4.2), and hence ODE (4.1), reduces to the $(n-2)$th-order ODE

$$F_{n-2}(x_{(2)}, y_{(2)}, y_{(2)1}, \cdots, y_{(2)n-2}) = 0 \tag{4.4}$$

for some function $F_{n-2}$ of invariants of $\mathbf{X}_2^{(n)}$, $\mathbf{X}_1^{(n)}$.

from (1.7), (1.8) it follows that

$$\mathbf{X}_1^{(1)} \mathbf{X}_3^{(1)} x_{(2)} = 0, \tag{4.5a}$$

$$\mathbf{X}_2^{(1)} \mathbf{X}_3^{(1)} x_{(2)} = 0. \tag{4.5b}$$

Then (4.5a) leads to

$$\mathbf{X}_3^{(1)} x_{(2)} = A(x_{(1)}, y_{(1)}),$$

for some function $A(x_{(1)}, y_{(1)})$. From (4.5b) we have

$$\mathbf{X}_2^{(1)} A(x_{(1)}, y_{(1)}) = 0. \tag{4.6}$$

Then (4.3) leads to

$$\mathbf{X}_3^{(1)} x_{(2)} = A(x_{(1)}, y_{(1)}) = \alpha_2(x_{(2)}),$$

for some function $\alpha_2(x_{(2)})$. Similarly,

$$\mathbf{X}_3^{(2)} y_{(2)} = \beta_2(x_{(2)}, y_{(2)}),$$
$$\mathbf{X}_3^{(3)} y_{(2)1} = \gamma_2(x_{(2)}, y_{(2)}, y_{(2)1}),$$

for some functions $\beta_2$, $\gamma_2$ of the indicated arguments. Hence

$$\mathbf{X}_3^{(2)}=\alpha_2(x_{(2)})\frac{\partial}{\partial x_{(2)}}+\beta_2(x_{(2)},y_{(2)})\frac{\partial}{\partial y_{(2)}},$$

with first extension given by

$$\mathbf{X}_3^{(3)}=\mathbf{X}_3^{(2)}+\gamma_2(x_{(2)},y_{(2)},y_{(2)1})\frac{\partial}{\partial y_{(2)1}},$$

is admitted by (4.4).

Then let $x_{(3)}(x_{(2)},y_{(2)})$, $y_{(3)}(x_{(2)},y_{(2)},y_{(2)1})$ be such that

$$\mathbf{X}_3^{(2)}x_{(3)}=0,\qquad \mathbf{X}_3^{(3)}y_{(3)}=0.$$

Consequently,

$$\mathbf{X}_3^{(3+k)}\frac{d^k y_{(3)}}{dx_{(3)}^k}=0,\qquad k=1,2,\cdots,n-3.$$

Let

$$y_{(3)k}=\frac{d^k y_{(3)}}{dx_{(3)}^k},\qquad k=1,2,\cdots,n-3.$$

In terms of the invariants $x_{(3)}$, $y_{(3)}$, $\{y_{(3)k}\}$, $k=1,2,\cdots,n-3$, of $\mathbf{X}_3^{(n)}$ (which are also invariants of $\mathbf{X}_2^{(n)}$, $\mathbf{X}_1^{(n)}$), ODE (4.4), and hence ODE (4.1), reduces to the $(n-3)$th-order ODE

$$F_{n-3}(x_{(3)},y_{(3)},y_{(3)1},\cdots,y_{(3)n-3})=0$$

for some function $F_{n-3}$ of the indicated invariants.

Continue inductively and suppose that for $q=3,\cdots,m$, $m<r$,

$$x_{(q)}(x_{(q-1)},y_{(q-1)}),\qquad y_{(q)}(x_{(q-1)},y_{(q-1)},y_{(q-1)1})$$

are such that

$$\mathbf{X}_p^{(q-1)}x_{(q)}=0,\quad \mathbf{X}_p^{(q)}y_{(q)}=0,\quad p=1,2,\cdots,q,$$

$$\mathbf{X}_p^{(q+k)}\frac{d^k y_{(q)}}{dx_{(q)}^k}=0,\quad k=1,2,\cdots,n-q\quad \text{for } 1\leqq p\leqq q,$$

with $y_{(q)k}=d^k y_{(q)}/dx_{(q)}^k$, $k=1,2,\cdots,n-q$, so that the $n$th-order ODE (4.1) reduces to the $(n-m)$th-order ODE

$$F_{n-m}(x_{(m)},y_{(m)},y_{(m)1},\cdots,y_{(m)n-m})=0 \tag{4.7}$$

for some function $F_{n-m}$ of invariants of $\mathbf{X}_m^{(n)}$, $\mathbf{X}_{m-1}^{(n)}$, $\cdots$, $\mathbf{X}_2^{(n)}$, $\mathbf{X}_1^{(n)}$.

To go from step $m$ to step $m+1$ we proceed as follows.

From (1.7), (1.8) it follows that

$$\mathbf{X}_j^{(m-1)}\mathbf{X}_{m+1}^{(m-1)}x_{(m)}=0,\qquad j=1,2,\cdots,m.$$

The equation $\mathbf{X}_1^{(m-1)}\mathbf{X}_{m+1}^{(m-1)}x_{(m)}=0$ leads to

$$\mathbf{X}_{m+1}^{(m-1)}x_{(m)}=A_1(x_{(1)},y_{(1)},y_{(1)1},\cdots,y_{(1)m-2})$$

for some function $A_1$ of the invariants of $\mathbf{X}_1^{(m-1)}$; $\mathbf{X}_2^{(m-1)}\mathbf{X}_{m+1}^{(m-1)}x_{(m)}=0$ leads to

$$A_1=A_2(x_{(2)},y_{(2)},y_{(2)1},\cdots,y_{(2)m-3})$$

for some function $A_2$ of the invariants of $\mathbf{X}_2^{(m-1)}$, $\mathbf{X}_1^{(m-1)}$; $\mathbf{X}_l^{(m-1)}\mathbf{X}_{m+1}^{(m-1)}x_{(m)}=0$ leads to

$$A_1=A_l(x_{(l)},y_{(l)},y_{(l)1},\cdots,y_{(l)m-l-1})$$

for some function $A_l$ of the invariants of $\mathbf{X}_l^{(m-1)}$, $\mathbf{X}_{l-1}^{(m-1)}, \cdots, \mathbf{X}_1^{(m-1)}$, $1 \leqq l \leqq m-1$. Then the equation $\mathbf{X}_{m-1}^{(m-1)}\mathbf{X}_{m+1}^{(m-1)}x_{(m)}=0$ leads to

$$A_1 = A_{m-1}(x_{(m-1)}, y_{(m-1)})$$

for some function $A_{m-1}(x_{(m-1)}, y_{(m-1)})$ of the invariants of $\mathbf{X}_{m-1}^{(m-1)}, \mathbf{X}_{m-2}^{(m-1)}, \cdots, \mathbf{X}_1^{(m-1)}$; finally, $\mathbf{X}_m^{(m-1)}\mathbf{X}_{m+1}^{(m-1)}x_{(m)}=0$ leads to

$$\mathbf{X}_{m+1}^{(m-1)}x_{(m)} = A_1 = \alpha_m(x_{(m)})$$

for some function $\alpha_m(x_{(m)})$.

Similarly, we can show that

$$\mathbf{X}_{m+1}^{(m)}y_{(m)} = \beta_m(x_{(m)}, y_{(m)}),$$

$$\mathbf{X}_{m+1}^{(m+1)}y_{(m)1} = \gamma_m(x_{(m)}, y_{(m)}, y_{(m)1}),$$

for some functions $\beta_m$, $\gamma_m$ of the indicated arguments. Hence

$$\mathbf{X}_{m+1}^{(m)} = \alpha_m(x_{(m)})\frac{\partial}{\partial x_{(m)}} + \beta_m(x_{(m)}, y_{(m)})\frac{\partial}{\partial y_{(m)}},$$

with first extension given by

$$\mathbf{X}_{m+1}^{(m+1)} = \mathbf{X}_{m+1}^{(m)} + \gamma_m(x_{(m)}, y_{(m)}, y_{(m)1})\frac{\partial}{\partial y_{(m)1}},$$

is admitted by (4.7) since ODE (4.1) admits $\mathbf{X}_{m+1}$. Now let $x_{(m+1)}(x_{(m)}, y_{(m)})$, $y_{(m+1)}(x_{(m)}, y_{(m)}, y_{(m)1})$ be such that

$$\mathbf{X}_{m+1}^{(m)}x_{(m+1)} = 0, \qquad \mathbf{X}_{m+1}^{(m+1)}y_{(m+1)} = 0.$$

Then

$$\mathbf{X}_{m+1}^{(m+1+k)}\frac{d^k y_{(m+1)}}{dx_{(m+1)}^k} = 0, \qquad k = 1, 2, \cdots, n-m-1.$$

Let

$$y_{(m+1)k} = \frac{d^k y_{(m+1)}}{dx_{(m+1)}^k}, \qquad k = 1, 2, \cdots, n-m-1.$$

In terms of the invariants $x_{(m+1)}$, $y_{(m+1)}$, $\{y_{(m+1)k}\}$, $k=1, 2, \cdots, n-m-1$, of $\mathbf{X}_{m+1}^{(n)}$ (which are also invariants of $\mathbf{X}_1^{(n)}, \mathbf{X}_2^{(n)}, \cdots, \mathbf{X}_m^{(n)}$), ODE (4.7) and hence ODE (4.1) reduces to an $(n-m-1)$th-order ODE

$$F_{n-m-1}(x_{(m+1)}, y_{(m+1)}, y_{(m+1)1}, \cdots, y_{(m+1)n-m-1}) = 0,$$

for some function $F_{n-m-1}$ of invariants of $\mathbf{X}_{m+1}^{(n)}$.

Finally, two cases are distinguished.

*Case* I ($3 \leqq r < n$). Here ODE (4.1) reduces to an $(n-r)$th-order ODE

$$F_{n-r}(x_{(r)}, y_{(r)}, y_{(r)1}, \cdots, y_{(r)n-r}) = 0 \tag{4.8}$$

for some function $F_{n-r}$ of invariants of $\mathbf{X}_r^{(n)}$ plus $r$ quadratures. The quadratures arise as follows.

Suppose

$$\phi_r(x_{(r)}, y_{(r)}; C_1, C_2, \cdots, C_{n-r}) = 0$$

is a general solution of ODE (4.8). Then the first-order ODE

$$\phi_r(x_{(r)}(x_{(r-1)}, y_{(r-1)}), y_{(r)}(x_{(r-1)}, y_{(r-1)}, y_{(r-1)1}); C_1, C_2, \cdots, C_{n-r})=0$$

admits

$$\mathbf{X}_r^{(r-1)}=\alpha_{r-1}(x_{(r-1)})\frac{\partial}{\partial x_{(r-1)}}+\beta_{r-1}(x_{(r-1)}, y_{(r-1)})\frac{\partial}{\partial y_{(r-1)}},$$

which leads to a quadrature

$$\phi_{r-1}(x_{(r-1)}, y_{(r-1)}; C_1, C_2, \cdots, C_{n-r+1})=0$$

for some function $\phi_{r-1}$ of the indicated arguments. Continuing inductively, assume that we have obtained

$$\phi_k(x_{(k)}, y_{(k)}; C_1, C_2, \cdots, C_{n-k})=0.$$

Then the first-order ODE

$$\phi_k(x_{(k)}(x_{(k-1)}, y_{(k-1)}), y_{(k)}(x_{(k-1)}, y_{(k-1)}, y_{(k-1)1}); C_1, C_2, \cdots, C_{n-k})=0$$

admits

$$\mathbf{X}_k^{(k-1)}=\alpha_{k-1}(x_{(k-1)})\frac{\partial}{\partial x_{(k-1)}}+\beta_{k-1}(x_{(k-1)}, y_{(k-1)})\frac{\partial}{\partial y_{(k-1)}},$$

which leads to quadrature

$$\phi_{k-1}(x_{(k-1)}, y_{(k-1)}; C_1, C_2, \cdots, C_{n-k+1})$$

for some function $\phi_{k-1}$ of the indicated arguments, $k=r, r-1, \cdots, 1$ $(y_{(0)}=y)$.

*Case* II ($3\leqq r=n$). Here ODE (4.1) reduces to a first-order ODE

$$F_1(x_{(n-1)}, y_{(n-1)}, y_{(n-1)1})=0 \tag{4.9}$$

for some function $F_1$ of the invariants $x_{(n-1)}$, $y_{(n-1)}$ of $\mathbf{X}_{n-1}^{(n)}$ plus $n-1$ quadratures which are obtained as demonstrated for Case I. The first-order ODE (4.9) reduces to quadrature since (4.9) admits

$$\mathbf{X}_n^{(n-1)}=\alpha_{n-1}(x_{(n-1)})\frac{\partial}{\partial x_{(n-1)}}+\beta_{n-1}(x_{(n-1)}, y_{(n-1)})\frac{\partial}{\partial y_{(n-1)}}.$$

Thus the solution of ODE (4.1) is reduced to $n$ quadratures.

Consequently, we have proved that if an $n$th-order ODE is invariant under an $r$-parameter solvable Lie group of transformations, then it can be reduced algorithmically to an $(n-r)$th-order ODE plus $r$ quadratures. Note that in applying this reduction algorithm *we do not need to determine the intermediate* ODEs *of orders* $n-1$, $n-2, \cdots, n-r+2$; *in Case* I *we do not need to determine the intermediate* ODE *of order* $n-r+1$.

As an example consider the fourth-order ODE

$$\left[yy'\left(\frac{y}{y'}\right)''\right]'=0, \tag{4.10}$$

which arises in studying the group properties of the linear wave equation in an inhomogeneous medium (Bluman and Kumei (1987)). The ODE (4.10) obviously admits the three-parameter $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ Lie group of transformations

$$x^*=e^{\varepsilon_2}(x+\varepsilon_1), \qquad y^*=e^{\varepsilon_3}y.$$

Corresponding infinitesimal generators $\mathbf{X}_1=\partial/\partial x$, $\mathbf{X}_2=x(\partial/\partial x)$, and $\mathbf{X}_3=y(\partial/\partial y)$ satisfy the commutation relations $[\mathbf{X}_1,\mathbf{X}_2]=\mathbf{X}_1$, $[\mathbf{X}_2,\mathbf{X}_3]=0$, $[\mathbf{X}_1,\mathbf{X}_3]=0$, and thus commutation relations of the form (1.9). To carry out the reduction algorithm, we first need the following extended infinitesimal generators:

$$\mathbf{X}_1^{(1)}=\frac{\partial}{\partial x},\quad \mathbf{X}_2^{(1)}=x\frac{\partial}{\partial x}-y_1\frac{\partial}{\partial y_1},\quad \mathbf{X}_2^{(2)}=x\frac{\partial}{\partial x}-y_1\frac{\partial}{\partial y_1}-2y_2\frac{\partial}{\partial y2},$$

$$\mathbf{X}_3^{(1)}=y\frac{\partial}{\partial y}+y_1\frac{\partial}{\partial y_1},\quad \mathbf{X}_3^{(2)}=y\frac{\partial}{\partial y}+y_1\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial y_2},$$

$$\mathbf{X}_3^{(3)}=y\frac{\partial}{\partial y}+y_1\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial y_2}+y_3\frac{\partial}{\partial y_3}.$$

From $\mathbf{X}_1x_{(1)}=0$, $\mathbf{X}_1^{(1)}y_{(1)}=0$, $y_{(1)1}=dy_{(1)}/dx_{(1)}$, we get

$$x_{(1)}=y,\quad y_{(1)}=y_1,\quad y_{(1)1}=\frac{y_2}{y_1}.$$

Then

$$\alpha_1(x_{(1)})=\mathbf{X}_2x_{(1)}=0,\qquad \beta_1(x_{(1)},y_{(1)})=\mathbf{X}_2^{(1)}y_{(1)}=-y_{(1)},$$

$$\gamma_1(x_{(1)},y_{(1)},y_{(1)1})=\mathbf{X}_2^{(2)}y_{(1)1}=-\frac{y_2}{y_1}=-y_{(1)1}.$$

Thus in terms of $x_{(1)}$, $y_{(1)}$, $y_{(1)1}$, we have

$$\mathbf{X}_2^{(1)}=-y_{(1)}\frac{\partial}{\partial y_{(1)}},\qquad \mathbf{X}_2^{(2)}=-y_{(1)}\frac{\partial}{\partial y_{(1)}}-y_{(1)1}\frac{\partial}{\partial y_{(1)1}}.$$

Now from $\mathbf{X}_2^{(1)}x_{(2)}=0$, $\mathbf{X}_2^{(2)}y_{(2)}=0$, $y_{(2)1}=dy_{(2)}/dx_{(2)}$, we find

$$x_{(2)}=x_{(1)}=y,\quad y_{(2)}=\frac{y_{(1)1}}{y_{(1)}}=\frac{y_2}{(y_1)^2},\quad y_{(2)1}=\frac{y_1y_3-2(y_2)^2}{(y_1)^4}.$$

Then

$$\alpha_2=\mathbf{X}_3^{(1)}x_{(2)}=y=x_{(2)},\qquad \beta_2=\mathbf{X}_3^{(2)}y_{(2)}=-\frac{y_2}{(y_1)^2}=-y_{(2)},$$

$$\gamma_2=\mathbf{X}_3^{(3)}y_{(2)1}=\frac{4(y_2)^2-2y_1y_3}{(y_1)^4}=-2y_{(2)1}.$$

Thus in terms of $x_{(2)}$, $y_{(2)}$, $y_{(2)1}$, we have

$$\mathbf{X}_3^{(2)}=x_{(2)}\frac{\partial}{\partial x_{(2)}}-y_{(2)}\frac{\partial}{\partial y_{(2)}},$$

$$\mathbf{X}_3^{(3)}=x_{(2)}\frac{\partial}{\partial x_{(2)}}-y_{(2)}\frac{\partial}{\partial y_{(2)}}-2y_{(2)1}\frac{\partial}{\partial y_{(2)1}}.$$

Now from $\mathbf{X}_3^{(2)}x_{(3)}=0$, $\mathbf{X}_3^{(3)}y_{(3)}=0$, we get

$$x_{(3)}=x_{(2)}y_{(2)}=\frac{yy_2}{(y_1)^2},\qquad y_{(3)}=(x_{(2)})^2y_{(2)1}=\frac{y^2[y_1y_3-2(y_2)^2]}{(y_1)^4}.$$

It now must follow that ODE (4.10) reduces to a solved form

$$\frac{dy_{(3)}}{dx_{(3)}}=J(x_{(3)},y_{(3)}) \tag{4.11}$$

for some function $J(x_{(3)}, y_{(3)})$ since (4.10) can be written in solved form in terms of $y_4$. It turns out that for ODE (4.10)

$$J(x_{(3)}, y_{(3)}) = -(1+2x_{(3)}),$$

and hence (4.11) *fortunately* reduces to quadrature. Then

$$(x_{(2)})^2 y_{(2)1} = -x_{(2)} y_{(2)} - [x_{(2)} y_{(2)}]^2 - c_1 \tag{4.12}$$

admits $\mathbf{X}_3^{(2)} = x_{(2)}(\partial/\partial x_{(2)}) - y_{(2)}(\partial/\partial y_{(2)})$, with corresponding canonical variables

$$R = x_{(2)} y_{(2)}, \qquad S = \log y_{(2)}.$$

In terms of these variables (4.12) becomes

$$\frac{dS}{dR} = \frac{1}{R} + \frac{1}{R^2 - c_1}. \tag{4.13}$$

Consider the case $c_1 > 0$, and let $c_1 = (C_1)^2$. Then

$$S = \log R + \log\left(\frac{R - C_1}{R + C_1}\right)^{1/2C_1} + c_2,$$

and consequently

$$\begin{aligned} y_{(2)} &= \Phi(x_{(2)}; C_1, C_2) \\ &= \frac{C_1}{x_{(2)}}\left(\frac{1 + B(x_{(2)})}{1 - B(x_{(2)})}\right), \end{aligned} \tag{4.14}$$

where $B(x_{(2)}) = (C_2/x_{(2)})^{2C_1}$, with arbitrary constants $C_1$, $C_2$. Then the first-order ODE resulting from (4.14), i.e.,

$$\frac{y_{(1)1}}{y_{(1)}} = \Phi(x_{(1)}; C_1, C_2), \tag{4.15}$$

admits $\mathbf{X}_2^{(1)} = -y_{(1)}(\partial/\partial y_{(1)})$. Hence (4.15) reduces to

$$\frac{dy_{(1)}}{y_{(1)}} = \Phi(x_{(1)}; C_1, C_2)\, dx_{(1)}$$

which integrates out to

$$\begin{aligned} y_{(1)} &= \Psi(y; C_1, C_2, C_3) \\ &= C_3 \exp\left[\int^y \Phi(x_{(1)}; C_1, C_2)\, dx_{(1)}\right]. \end{aligned}$$

Finally the first-order ODE

$$y_1 = \frac{dy}{dx} = \Psi(y; C_1, C_2, C_3)$$

admits $\mathbf{X}_1 = \partial/\partial x$ and reduces to quadrature

$$\int^y \frac{dy}{\Psi(y; C_1, C_2, C_3)} = x + C_4,$$

yielding a general solution of (4.10). The case $c_1 = -(C_1)^2$ substituted into (4.13) would yield another general solution of (4.10).

An alternative way of using the group properties of the fourth-order ODE (4.10) to obtain general solutions was considered in Bluman and Kumei (1987).

**5. Summary.** In using the reduction algorithm developed in § 4, from the infinitesimal generators of the admitted solvable Lie group we determine iteratively coordinates $\{x_{(i)}, y_{(i)}, y_{(i)1}\}$ and coefficients

$$\{\alpha_i(x_{(i)}), \beta_i(x_{(i)}, y_{(i)}), \gamma_i(x_{(i)}, y_{(i)}, y_{(i)1})\}:$$

$$\begin{aligned}(x_{(1)}, y_{(1)}, y_{(1)1}) &\to (\alpha_1, \beta_1, \gamma_1) \to (x_{(2)}, y_{(2)}, y_{(2)1}) \\ &\to (\alpha_2, \beta_2, \gamma_2) \to \cdots \to (x_{(r-1)}, y_{(r-1)}, y_{(r-1)1}) \\ &\to (\alpha_{r-1}, \beta_{r-1}, \gamma_{r-1}) \to (x_{(r)}, y_{(r)}).\end{aligned}$$

The $n$th-order ODE reduces directly to an $(n-r)$th-order ODE in coordinates $(x_{(r)}, y_{(r)})$. The quadratures follow from reversing the arrows of the iterative procedure.

In the case of an $n$th-order ODE (4.1) admitting a three-parameter Lie group with infinitesimal generators $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ satisfying commutation relations of the form (1.9), the procedure simplifies to:

(1) Determine coordinates $x_{(1)}(x, y)$, $y_{(1)}(x, y, y_1)$ and hence $y_{(1)1}$, invariants of $\mathbf{X}_1^{(2)}$.

(2) Apply $\mathbf{X}_2^{(2)}$ to $x_{(1)}$, $y_{(1)}$, $y_{(1)1}$, respectively, and find $\alpha_1(x_{(1)})$, $\beta_1(x_{(1)}, y_{(1)})$, $\gamma_1(x_{(1)}, y_{(1)}, y_{(1)1})$. Then

$$\mathbf{X}_2^{(2)} = \alpha_1 \frac{\partial}{\partial x_{(1)}} + \beta_1 \frac{\partial}{\partial y_{(1)}} + \gamma_1 \frac{\partial}{\partial y_{(1)1}}.$$

(3) Determine the invariants $x_{(2)}(x_{(1)}, y_{(1)})$, $y_{(2)}(x_{(1)}, y_{(1)}, y_{(1)1})$ of $\mathbf{X}_2^{(2)}$ and hence the differential invariant $y_{(2)1}$ of $\mathbf{X}_2^{(3)}$.

(4) Apply $\mathbf{X}_3^{(3)}$ to $x_{(2)}$, $y_{(2)}$, $y_{(2)1}$, respectively, and find $\alpha_2(x_{(2)})$, $\beta_2(x_{(2)}, y_{(2)})$, $\gamma_2(x_{(2)}, y_{(2)}, y_{(2)1})$. Then

$$\mathbf{X}_3^{(3)} = \alpha_2 \frac{\partial}{\partial x_{(2)}} + \beta_2 \frac{\partial}{\partial y_{(2)}} + \gamma_2 \frac{\partial}{\partial y_{(2)1}}.$$

(5) Determine the invariants $x_{(3)}(x_{(2)}, y_{(2)})$, $y_{(3)}(x_{(2)}, y_{(2)}, y_{(2)1})$ of $\mathbf{X}_3^{(3)}$ and hence the corresponding differential invariants

$$y_{(3)1}, y_{(3)2}, \cdots, y_{(3)n-3} \quad \text{of } \mathbf{X}_3^{(n)}.$$

(6) Find the reduced ODE

$$F_{n-3}(x_{(3)}, y_{(3)}, y_{(3)1}, \cdots, y_{(3)n-3}) = 0$$

of order $n-3$ with independent variable $x_{(3)}$ and dependent variable $y_{(3)}$.

(7) Let $\phi_3(x_{(3)}, y_{(3)}; C_1, C_2, \cdots, C_{n-3}) = 0$ be the general solution of $F_{n-3} = 0$. Then in terms of coordinates $(x_{(2)}, y_{(2)})$ the first-order ODE

$$\phi_3(x_{(3)}(x_{(2)}, y_{(2)}), y_{(3)}(x_{(2)}, y_{(2)}, y_{(2)1}); C_1, C_2, \cdots, C_{n-3}) = 0$$

admits $\mathbf{X}_3^{(2)} = \alpha_2(\partial/\partial x_{(2)}) + \beta_2(\partial/\partial y_{(2)})$.

(8) The invariance of $\phi_3 = 0$ under $\mathbf{X}_3^{(2)}$ leads to quadrature

$$\phi_2(x_{(2)}(x_{(1)}, y_{(1)}), y_{(2)}(x_{(1)}, y_{(1)}, y_{(1)1}); C_1, C_2, \cdots, C_{n-2}) = 0.$$

In terms of coordinates $(x_{(1)}, y_{(1)})$ the first-order ODE $\phi_2 = 0$ admits

$$\mathbf{X}_2^{(1)} = \alpha_1 \frac{\partial}{\partial x_{(1)}} + \beta_1 \frac{\partial}{\partial y_{(1)}}.$$

(9) Reduce $\phi_2 = 0$ to quadrature

$$\phi_1(x_{(1)}(x, y), y_{(1)}(x, y, y_1); C_1, C_2. \cdots, C_{n-1}) = 0.$$

In terms of coordinates $(x, y)$ the first-order ODE $\phi_1 = 0$ admits

$$\mathbf{X}_1 = \xi_1(x, y) \frac{\partial}{\partial x} + \eta_1(x, y) \frac{\partial}{\partial y}.$$

(10) Reduce $\phi_1 = 0$ to quadrature

$$\phi_0(x, y; C_1, C_2, \cdots, C_n) = 0,$$

a general solution of (4.1).

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