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# Use and Construction of Potential Symmetries

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**Abstract**—Group-theoretic methods based on local symmetries are useful to construct invariant solutions of PDEs and to linearize nonlinear PDEs by invertible mappings. Local symmetries include point symmetries, contact symmetries and, more generally, Lie-Bäcklund symmetries. An obvious limitation in their utility for particular PDEs is the non-existence of local symmetries.

A given system of PDEs with a conserved form can be embedded in a related auxiliary system of PDEs. A local symmetry of the auxiliary system can yield a nonlocal symmetry (potential symmetry) of the given system. The existence of potential symmetries leads to the construction of corresponding invariant solutions as well as to the linearization of nonlinear PDEs by non-invertible mappings.

Recent work considers the problem of finding all potential symmetries of given systems of PDEs. Examples include linear wave equations with variable wave speeds as well as nonlinear diffusion, reaction-diffusion, and gas dynamics equations.

## 1. INTRODUCTION

Group-theoretic methods based on local symmetries are useful to construct invariant solutions (similarity solutions) of PDEs (see [1–11]), and to linearize nonlinear PDEs by invertible mappings [12,13]. Local symmetries include point symmetries, contact symmetries and, more generally, Lie-Bäcklund symmetries. An obvious limitation in their utility for particular PDEs is the non-existence of local symmetries.

A given system of PDEs with a conserved form can be embeddded in a related auxiliary system of PDEs. A local symmetry of the auxiliary system can yield a nonlocal symmetry (potential symmetry) of the given system [3,14–16]. The existence of potential symmetries leads to the construction of corresponding invariant solutions as well as to the linearization of nonlinear PDEs by non-invertible mappings [3,17].

In this article, we review the uses of symmetries for finding invariant solutions and linearizations of nonlinear systems, as well as literature on the use of symbolic manipulation to construct symmetries. Then we consider the problem of algorithmically finding nonlocal symmetries of given systems of PDEs. Examples will include nonlinear diffusion, reaction-diffusion, and gas dynamics equations.

## 2. POINT SYMMETRIES AND THEIR USES

Consider a system of $m$ PDEs $R\{x,u\}$ given by the relations

$$G^\sigma\left(x, u, \underset{1}{u}, \underset{2}{u}, \dots, \underset{k}{u}\right) = 0, \qquad \sigma = 1, 2, \dots, m, \tag{2.1}$$

with independent variables $x = (x_1, x_2, \dots, x_n)$, dependent variables $u = (u^1, u^2, \dots, u^m)$; $\underset{j}{u}$ denotes the set of coordinates corresponding to all $j^{\text{th}}$ order partial derivatives of $u$ with respect

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Corresponding subsystems are

$$R_1\{x,v\} : v_t = \left(\frac{1}{v_x} + bx^2\right)_x; \tag{6.10}$$

$$\begin{aligned} S_1\{x,v,w\} : w_x &= v, \\ w_t &= -\left(\frac{1}{v_x} + bx^2\right); \end{aligned} \tag{6.11}$$

$$\begin{aligned} S_2\{x,u,w\} : w_t &= -\left(\frac{1}{u} + bx^2\right), \\ w_{xx} &= u; \end{aligned} \tag{6.12}$$

$$S_3\{x,w\} : w_t = -\left(\frac{1}{w_{xx}} + bx^2\right). \tag{6.13}$$

One can show that $T\{x,u,v,w\}$ admits

$$\begin{aligned} \mathbf{X}_\infty^T = e^{b(w-xv)} \Big\{ &(F^1 - bxF^3)\frac{\partial}{\partial x} + (2bxu^2F^1 - u^2F^2 + bu(1-bx^2u)F^3)\frac{\partial}{\partial u} \\ &+ (vF^1 - (1+bxv)F^3)\frac{\partial}{\partial w} \Big\} \end{aligned} \tag{6.14}$$

where $F^1 = F^1(v,t), F^2 = F^2(v,t), F^3 = F^3(v,t)$ satisfy linear system (6.7). It is easy to check that the criteria of Theorems 2.1, 2.2 are satisfied and hence $T\{x,u,v,w\}$ is linearizable by an invertible mapping. From the form of $\mathbf{X}_\infty^T$, we see that it projects to a point symmetry of $S_1\{x,v,w\}$; induces a contact symmetry of $S_3\{x,w\}$, a Lie-Bäcklund symmetry of $S_2\{x,u,w\}$ and a nonlocal symmetry of $R_1\{x,v\}, R\{x,u\}, S\{x,u,v\}$.

Hence $S_1\{x,v,w\}$ and $S_3\{x,w\}$ are linearizable by invertible mappings, whereas all other systems are linearizable by non-invertible mappings. (Consequently, although the point symmetries of $S\{x,u,v\}$ yield the linearization of $R\{x,u\}$ when $b=0$, this is not the case when $b \neq 0$ where the discovery of linearization requires consideration of the point symmetries of $T(x,u,v,w\}$ or $S_1\{x,v,w\}$.)

Use of Theorem 2.2 yields the mapping

$$\begin{aligned} z_1 &= t, \\ z_2 &= v, \\ w^1 &= xe^{b(xv-w)} \\ w^2 &= \left(bx^2 + \frac{1}{u}\right)e^{b(xv-w)}, \\ w^3 &= \frac{1}{b}(e^{b(xv-w)} - 1), \end{aligned}$$

which invertibly transforms $T\{x,u,v,w\}$ to the linear system

$$\begin{aligned} \frac{\partial w^3}{\partial v} &= w^1, \\ \frac{\partial w^3}{\partial t} &= w^2, \\ \frac{\partial w^1}{\partial v} &= w^2. \end{aligned}$$

Conjectures (5.1), (5.2) are easily checked.

to $x$ (a coordinate in $\underset{j}{u}$ is denoted by $u^{\gamma}_{i_1 i_2 \cdots i_j} = \frac{\partial^j u^{\gamma}}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_j}}$ with $\gamma = 1, 2, \dots, m; i_j = 1, 2, \dots, n;$ $j = 1, 2, \dots, k$).

A *symmetry* admitted by $R\{x, u\}$ is a transformation which maps any solution of (2.1) into another solution of (2.1). Note that a symmetry transformation has no restriction to action on a particular set of coordinates.

A *point symmetry* admitted by $R\{x, u\}$ is characterized by infinitesimal generators of the form

$$\mathbf{X} = \xi_i(x, u) \frac{\partial}{\partial x_i} + \eta^{\mu}(x, u) \frac{\partial}{\partial u^{\mu}}, \tag{2.2}$$

(summation over a repeated index is assumed throughout this article) corresponding to one-parameter ($\epsilon$) Lie groups of point transformations

$$\begin{aligned} x_i^* &= x_i + \epsilon \xi_i(x, u) + O(\epsilon^2), & i &= 1, 2, \dots, n, \\ (u^{\mu})^* &= u^{\mu} + \epsilon \eta^{\mu}(x, u) + O(\epsilon^2), & \mu &= 1, 2, \dots, m. \end{aligned}$$

Corresponding to point symmetry (2.2), admitted by $R\{x, u\}$, one can construct *invariant (similarity) solutions* $u = \theta(x)$ of (2.1) satisfying

$$\xi_i(x, u) \frac{\partial u^{\nu}}{\partial x_i} - \eta^{\nu}(x, u) = 0, \qquad \nu = 1, 2, \dots, m. \tag{2.3}$$

Substituting (2.3) into $R\{x, u\}$, one obtains a reduced system of $m$ PDEs in $n - 1$ independent variables. It is not necessary to solve explicitly (2.3) in order to accomplish this reduction (see [3, p. 198]). In the case of a specific boundary value problem (BVP), this reduced system is useful for obtaining its solution, provided the boundary conditions are also invariant. In the case of a BVP for a nonlinear system of PDEs, invariance of boundary conditions means that both the boundary of the domain as well as all boundary conditions specified on the boundary must be separately invariant. (The restriction is much less severe for BVPs posed for linear systems. See [3, Section 4.4] for details on applications to BVPs.)

For each point symmetry admitted by $R\{x, u\}$, one can map any solution of (2.1) (provided it is not an invariant solution corresponding to (2.2)) into a one-parameter family of solutions of (2.1) (see [3, Section 4.2.2]).

If one knows all infinitesimal generators of point symmetries admitted by $R\{x, u\}$ when $m \geq 2$, (in the scalar case, $m = 1$, one must know all infinitesimal generators of admitted contact symmetries) one can determine *whether or not* $R\{x, u\}$ can be *linearized* by an *invertible mapping* and construct such a mapping when it exists. This result follows from the following two theorems.

THEOREM 2.1. (NECESSARY CONDITIONS). *If there is an invertible mapping of $R\{x, u\}$, $m \geq 2$, into a linear system with independent variables $z = (z_1, z_2, \dots, z_n)$ and dependent variables $w = (w^1, w^2, \dots, w^m)$, then*

(1) *the mapping is a point transformation*

$$\begin{aligned} z_j &= \phi_j(x, u), & j &= 1, 2, \dots, n, \\ w^{\gamma} &= \psi^{\gamma}(x, u), & \gamma &= 1, 2, \dots, m. \end{aligned}$$

(2) *$R\{x, u\}$ must admit infinitesimal generators of the form (2.2) with*

$$\begin{aligned} \xi_i(x, u) &= \alpha_i^{\sigma}(x, u) F^{\sigma}(x, u), & i &= 1, 2, \dots, n, \\ \eta^{\mu}(x, u) &= \beta_{\mu}^{\sigma}(x, u) F^{\sigma}(x, u), & \mu &= 1, 2, \dots, m, \end{aligned}$$

*where* $\alpha_i^\sigma, \beta_\mu^\sigma$ *are specific functions of* $(x,u)$, *and* $F = (F^1, F^2, \dots, F^m)$ *is an arbitrary solution of some linear system*

$$L[X]F = 0,$$

*with* $L[X]$ *a linear operator depending on independent variables*

$$X = (X_1(x,u), X_2(x,u), \dots, X_n(x,u)).$$

THEOREM 2.2. (SUFFICIENT CONDITIONS). *If the linear system of* $m$ *first order PDEs*

$$\alpha_i^\sigma \frac{\partial \Phi}{\partial x_i} + \beta_\mu^\sigma \frac{\partial \Phi}{\partial u^\mu} = 0, \qquad \sigma = 1, 2, \dots, m,$$

*has* $X_1(x,u), X_2(x,u), \dots, X_n(x,u)$, *as* $n$ *functionally independent solutions, and the linear system of* $m^2$ *first order PDEs*

$$\alpha_i^\sigma \frac{\partial \psi^\gamma}{\partial x_i} + \beta_\mu^\sigma \frac{\partial \psi^\gamma}{\partial u^\mu} = \delta^{\gamma\sigma}, \qquad \gamma, \sigma = 1, 2, \dots, m,$$

($\delta^{\gamma\sigma}$ *is the Kronecker symbol:* $\delta^{\gamma\gamma} = 1, \delta^{\gamma\sigma} = 0$ *if* $\gamma \neq \sigma$) *has a solution*

$$\psi = (\psi^1(x,u), \psi^2(x,u), \dots, \psi^m(x,u)),$$

*then the invertible mapping*

$$\begin{aligned} z_j &= \phi_j(x,u) = X_j(x,u), \qquad && j = 1, 2, \dots, n, \\ w^\gamma &= \psi^\gamma(x,u), && \gamma = 1, 2, \dots, m, \end{aligned}$$

*transforms* $R\{x,u\}$ *to a linear system* $\hat{R}\{z,w\}$ *given by*

$$L[z]w = g(z),$$

*for some nonhomogeneous term* $g(z)$.

The proofs of Theorems 2.1, 2.2 and extensions to the scalar case $m = 1$ are found in [3,13]. An earlier version of these theorems is discussed in [12].

## 3. DETERMINATION OF POINT SYMMETRIES

Point symmetries acting on the space of independent and dependent variables naturally extend to symmetry transformations acting on the space of independent and dependent variables and their derivatives to some fixed order by requiring the preservation of contact conditions (see [3,5–8,10]). In terms of total derivative operators

$$D_i = \frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + u_i^\gamma \frac{\partial}{\partial u^\gamma} + u_{ij}^\gamma \frac{\partial}{\partial u_j^\gamma} + u_{ii_1i_2\cdots i_\ell}^\gamma \frac{\partial}{\partial u_{i_1i_2\cdots i_\ell}^\gamma} + \cdots, \qquad i = 1, 2, \dots, n,$$

the infinitesimal generator (2.2) extends to

$$\mathbf{X}^{(j)} = \mathbf{X} + \eta_i^{(1)\mu}\left(x, u, \underset{1}{u}\right) \frac{\partial}{\partial u_i^\mu} + \cdots + \eta_{i_1i_2\cdots i_j}^{(j)\mu}\left(x, u, \underset{1}{u}, \underset{2}{u}, \cdots, \underset{j}{u}\right) \frac{\partial}{\partial u_{i_1i_2\cdots i_j}^\mu}, \quad j = 1, 2, \dots, \tag{3.1}$$

where

$$\eta_i^{(1)\mu} = D_i \eta^\mu - (D_i \xi_\ell) u_\ell^\mu,$$

and

$$\eta^{(j)\mu}_{i_1 i_2 \cdots i_j} = D_{i_j} \eta^{(j-1)\mu}_{i_1 i_2 \cdots i_{j-1}} - (D_{i_j}\xi_\ell) u^\mu_{i_1 i_2 \cdots i_{j-1}\ell},$$

$$i_q = 1, 2, \ldots, n \quad \text{for} \quad q = 1, 2, \ldots, j \quad \text{with} \quad j = 2, 3, \ldots.$$

The $j^{\text{th}}$ extended infinitesimal generator (3.1) defines a one-parameter ($\epsilon$) Lie group of transformations acting on $(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{j}{u})$-space:

$$\begin{aligned} x_i^* &= x_i + \epsilon \xi_i(x, u) + O(\epsilon^2), \\ (u^\mu)^* &= u^\mu + \epsilon \eta^\mu(x, u) + O(\epsilon^2), \\ (u_i^\mu)^* &= u_i^\mu + \epsilon \eta_i^{(1)\mu}(x, u, \underset{1}{u}) + O(\epsilon^2), \\ (u^\mu_{i_1 i_2 \cdots i_j})^* &= u^\mu_{i_1 i_2 \cdots i_j} + \epsilon \eta^{(j)\mu}_{i_1 i_2 \cdots i_j} \left( x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{j}{u} \right) + O(\epsilon^2). \end{aligned}$$

Consequently, the algorithm to find the point symmetries admitted by $R\{x, u\}$ reduces to finding the components $\{\xi_i, \eta^\mu\}$ of the infinitesimal generators (2.2) admitted by $R\{x, u\}$. In particular,

$$\mathbf{X}^{(k)} G^\sigma \left( x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{k}{u} \right) = 0, \tag{3.2}$$

must hold for *any* solution of $R\{x, u\}$. The conditions (3.2) impose severe restrictions on $\{\xi_i, \eta^\mu\}$ and result in a set of over-determined linear PDEs (*determining equations*) satisfied by $\{\xi_i, \eta^\mu\}$. The general solution of the determining equations yields the group of point symmetries admitted by $R\{x, u\}$.

In recent years, the development of symbolic manipulation programs has made the use of group methods more accessible to non-specialists. Programs have been developed which set up the determining equations automatically and sometimes yield all point symmetries of a given $R\{x, u\}$ automatically (see [18–21]). Kersten [22] has developed an interactive symbolic manipulation program which can significantly simplify and often solve the determining equations. Unfortunately these programs may not succeed when symmetry groups are nontrivial and usually will not handle the group classification program.

Reid [23,24] has developed a symbolic manipulation algorithm which automatically determines the dimension of the Lie algebra of infinitesimal generators admitted by $R\{x, u\}$, i.e., the number of linearly independent infinitesimal generators admitted by $R\{x, u\}$, without having to explicitly compute the generators themselves, when the dimension is finite. Moreover, Reid's algorithm is able to handle the group classification problem where one is interested in finding the point symmetries of a system $R\{x, u\}$ containing an arbitrary (model) function. Here different forms of the model function yield different groups of point symmetries. Reid is able to find all splittings of the model function (each splitting corresponds to a specific DE satisfied by the model function) and dimensions of the resulting symmetry groups. Related work appears in Topunov [11,25].

Often the calculations for the group classification problem, even using Reid's algorithm, are too lengthy to go to completion. Recently, Lisle [26] has developed an algorithm to determine equivalent model functions, including how to find explicit equivalence transformations. Using ideas from differential geometry, Lisle is able to incorporate his group equivalence algorithm to Reid's algorithm and handle previously intractable group classification problems. For example, Lisle obtains the (highly nontrivial) group classification of the nonlinear diffusion-convection equation with arbitrary model functions for both diffusion and convection.

We conclude this section by giving all infinitesimal generators of admitted point symmetries for three examples ($x_1 = x, x_2 = t$).

EXAMPLE 1. GROUP CLASSIFICATION FOR THE NONLINEAR DIFFUSION EQUATION.

Here $R\{x,u\}$ is given by

$$u_t = (L(u))_{xx} = (K(u)u_x)_x, \tag{3.3}$$

with $K(u) = L'(u) \neq \text{const.}$

Modulo scalings and translations in $u$, the following splittings arise:

(1) $K(u)$ arbitrary:

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \qquad \mathbf{X}_2 = \frac{\partial}{\partial t}, \qquad \mathbf{X}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}.$$

(2) $K(u) = u^\lambda$:

$$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 = \frac{1}{2}\lambda x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

(3) $K(u) = u^{-4/3}$:

$$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 \quad \text{with} \quad \lambda = -\frac{4}{3}, \qquad \mathbf{X}_5 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}.$$

EXAMPLE 2. A REACTION-DIFFUSION EQUATION.

Consider the reaction-diffusion equation

$$U_t = U^2 U_{xx} + 2bU^2, \qquad b = \text{ const. } \neq 0. \tag{3.4}$$

Multiply (3.4) by $-\frac{1}{U^2}$ and let $u = \frac{1}{U}$. Then, one can show that $R\{x,u\}$, given by

$$u_t = -\left(\frac{1}{u} + bx^2\right)_{xx}, \tag{3.5}$$

admits

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \qquad \mathbf{X}_2 = \frac{\partial}{\partial t}, \qquad \mathbf{X}_3 = x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.$$

(When $b = 0$, $R\{x,u\}$ also admits $\mathbf{X}_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$.)

EXAMPLE 3. ONE-DIMENSIONAL PLANAR GAS DYNAMICS EQUATIONS.

Let $u = (v,p,\rho)$ where $v, p$, and $\rho$, respectively, are velocity, pressure, and density functions for a fluid. Then the equations of one-dimensional planar gas dynamics are

$$\rho_t + v\rho_x + \rho v_x = 0, \tag{3.6a}$$

$$\rho(v_t + vv_x) + p_x = 0, \tag{3.6b}$$

$$\rho(p_t + vp_x) + B(p,\rho)v_x = 0, \tag{3.6c}$$

where $B(p,\rho)$ satisfies some constitutive equation. If one multiplies (3.6a) by $v$ and adds the resulting PDE to (3.6b), then (3.6a–c) is equivalent to

$$\rho_t + (\rho v)_x = 0, \tag{3.7a}$$

$$(\rho v)_t + (p + \rho v^2)_x = 0, \tag{3.7b}$$

$$\rho(p_t + vp_x) + B(p,\rho)v_x = 0. \tag{3.7c}$$

If $B \equiv 1$, the corresponding system $R\{x,u\}$ admits [27]

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \qquad \mathbf{X}_2 = \frac{\partial}{\partial t}, \qquad \mathbf{X}_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t},$$
$$\mathbf{X}_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}, \qquad \mathbf{X}_5 = t\frac{\partial}{\partial t} - v\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho},$$
$$\mathbf{X}_6 = \frac{\partial}{\partial p}.$$

Except for the symmetry $\mathbf{X}_5$ for the nonlinear diffusion equation with $K(u) = u^{-4/3}$, the point symmetries for these three examples are "obvious" since they can be seen by inspection.

# 4. SYMMETRY EXTENSIONS; NONLOCAL SYMMETRIES; POTENTIAL SYMMETRIES

In principle, from the definition of a symmetry transformation, every system of PDEs with topologically continuous solution sets admits symmetries. The problem is to find explicit symmetries and to be able to exploit them to determine something concrete about the system such as particular solutions, linearization, conservation laws, variational principles, equivalences, etc. A first explicit generalization of point symmetry was due to Lie who showed that scalar PDEs could admit contact symmetries. Noether [28], in her celebrated paper on conservation laws, mentioned the possibility of using symmetries whose infinitesimal generators allowed $(\xi, \eta)$ to depend on derivatives of $u$ to some finite order. (Generators of Lie's contact symmetries restricted dependence to first order derivatives of $u$.) Such symmetries are now commonly called *Lie-Bäcklund symmetries* (higher symmetries) which are *local symmetries*, defined by infinitesimal generators of the form

$$\mathbf{X} = \xi_i \left( x, u, \underset{1}{u}, \underset{2}{u}, \dots, \underset{p}{u} \right) \frac{\partial}{\partial x_i} + \eta^\nu \left( x, u, \underset{1}{u}, \underset{2}{u}, \dots, \underset{p}{u} \right) \frac{\partial}{\partial u^\nu}. \tag{4.1}$$

They appear to have been first discovered for explicit PDEs by Anderson, Kumei, and Wulfman [29]. Except in the special case of contact symmetries, some important properties of point symmetries are not inherited by Lie-Bäcklund symmetries. In particular, Lie-Bäcklund symmetries cannot be integrated to global transformations by solving characteristic equations and in general cannot be used for linearizations. However, the existence of Lie-Bäcklund symmetries appears to be a characteristic property of the various evolutionary nonlinear scalar PDEs in two independent variables which exhibit soliton behaviour and are linearizable by an inverse scattering transform for particular initial data. Vinogradov [11,30] gives informal explanations of "why differential equations with more than two independent variables generally have no higher symmetries." For details on Lie-Bäcklund symmetries, see [3,5,6,11].

It turns out that PDEs can admit *nonlocal symmetries* whose infinitesimal generators are not of the form (4.1). A formal *ad-hoc* way to obtain such symmetries for some PDEs is to allow $(\xi, \eta)$ to depend on integrals of $u$ in some specific manner.

Krasil'shchik and Vinogradov [11,31,32] give criteria which must be satisfied by a class of nonlocal symmetries of $R\{x,u\}$ when realized as local symmetries of a system of PDEs which "covers" $R\{x,u\}$. Their papers appear to exhibit no new examples.

Akhatov, Gazizov, and Ibragimov [27] give nontrivial examples of nonlocal symmetries generated by heuristic procedures. Their paper is rich in examples. Exhibited calculations are sources of useful data for developing the following generalizations of our previous algorithms [3,14–16] to obtain nonlocal symmetries.

Suppose one of the PDEs of system $R\{x,u\}$, given by (2.1), in particular $G^m = 0$, is a conserved form

$$\frac{D}{Dx_i} f^i \left( x, u, \underset{1}{u}, \underset{2}{u}, \dots, \underset{k-1}{u} \right) = 0.$$

Then $R\{x,u\}$ is the system

$$G^\sigma \left( x, u, \underset{1}{u}, \underset{2}{u}, \dots, \underset{k}{u} \right) = 0, \qquad \sigma = 1, 2, \dots, m-1, \tag{4.2a}$$

$$\frac{D}{Dx_i} f^i \left( x, u, \underset{1}{u}, \underset{2}{u}, \dots, \underset{k-1}{u} \right) = 0. \tag{4.2b}$$

Through (4.2b), we can introduce $n-1$ auxiliary variables (*potentials*) $v^1, v^2, \dots, v^{n-1}$ and form an auxiliary system $S\{x,u,v\}$ of $m+n-1$ PDEs with $m+n-1$ dependent variables. In particular,

$S\{x,u,v\}$ is the system

$$\begin{aligned} f^1\left(x,u,\underset{1}{u},\underset{2}{u},\dots,\underset{k-1}{u}\right) &= \frac{\partial v^1}{\partial x_2}, \\ f^\ell\left(x,u,\underset{1}{u},\underset{2}{u},\dots,\underset{k-1}{u}\right) &= (-1)^{\ell-1}\left[\frac{\partial v^\ell}{\partial x_{\ell+1}}+\frac{\partial v^{\ell-1}}{\partial x_{\ell-1}}\right], \quad 1<\ell<n, \\ f^n\left(x,u,\underset{1}{u},\underset{2}{u},\dots,\underset{k-1}{u}\right) &= (-1)^{n-1}\frac{\partial v^{n-1}}{\partial x_{n-1}}, \\ G^\sigma\left(x,u,\underset{1}{u},\underset{2}{u},\dots,\underset{k}{u}\right) &= 0, \ \sigma=1,2,\dots,m-1. \end{aligned} \tag{4.3}$$

Now, consider the relationship between the solutions of the systems $R\{x,u\}, S\{x,u,v\}$:

If $(u(x),v(x))$ solves $S\{x,u,v\}$, then $u(x)$ solves $R\{x,u\}$, since (4.2a,b) is an integrability condition for (4.3). If $u(x)$ solves $R\{x,u\}$, then there is some $v(x)$ such that $(u(x),v(x))$ solves $S\{x,u,v\}$. Clearly $v(x)$ is not unique. Hence it follows that even though all solutions of $R\{x,u\}$ can be found from knowledge of all solutions of $S\{x,u,v\}$ and, conversely, all solutions of $S\{x,u,v\}$ can be determined from knowledge of all solutions of $R\{x,u\}$, the relationship between $R\{x,u\}$ and $S\{x,u,v\}$ is non-invertible.

A symmetry of $S\{x,u,v\}$ defines (induces) a symmetry of $R\{x,u\}$; conversely, any symmetry of $R\{x,u\}$ determines a symmetry of $S\{x,u,v\}$. But, since the solutions of $R\{x,u\}$ and $S\{x,u,v\}$ are not in one-to-one correspondence, it follows that a point symmetry of $S\{x,u,v\}$ could correspond to a nonlocal symmetry (*potential symmetry*) of $R\{x,u\}$ and also that a point symmetry (or, more generally, a Lie-Bäcklund symmetry) of $R\{x,u\}$ could yield a nonlocal symmetry of $S\{x,u,v\}$.

More generally, if $(u^{i_1},u^{i_2},\dots,u^{i_\alpha},v^{j_1},v^{j_2},\dots,v^{j_\beta}), i_1<i_2<\dots<i_\alpha\le m, j_1<j_2<\dots<j_\beta\le n-1, \alpha+\beta<m+n-1$, solves a subsystem of PDEs $R\{x,u^{i_1},u^{i_2},\dots,u^{i_\alpha},v^{j_1},v^{j_2},\dots,v^{j_\beta}\}$, arising from integrability conditions of $S\{x,u,v\}$, it follows that all solutions of any such subsystem yields all solutions of any other subsystem, as well as all solutions of $R\{x,u\}=R\{x,u^1,u^2,\dots,u^m\}$ and $S\{x,u,v\}=R\{x,u^1,u^2,\dots,u^m,v^1,v^2,\dots,v^{n-1}\}$. (Note that a subsystem itself could play the role of a given system of PDEs which has no conserved forms!)

*Consequently all such subsystems of PDEs, $R\{x,u\}$, and $S\{x,u,v\}$ have the same symmetries.* But a local symmetry of one subsystem could induce a nonlocal symmetry of another subsystem. Since local symmetries and, in particular, point symmetries yield invariant solutions, it follows that invariant solutions constructed for one subsystem can yield solutions of another subsystem which are not invariant solutions for any local symmetries admitted by the second subsystem. Nonlocal symmetries arising through this process as point symmetries of a related subsystem can be useful for solving BVPs, since any BVP posed for one subsystem can be posed as a BVP for all other related subsystems.

Suppose $R_1$ and $R_2$ are distinct systems. Let $G_1$ and $G_2$ be their respective point symmetry groups. Then, the symmetries $G_1\cup G_2$ (which do not necessarily form a group) represent a "symmetry covering" for both of the systems $R_1$ and $R_2$.

To simplify matters, we specialize to the case when $m=1$, although later on in this article we will consider examples when $m>1$. Let $R=R\{x,u\}$, $S=S\{x,u,v\}$. Then, $R_1=R_1\{x,v\}$ is a related subsystem provided integrability conditions of $S\{x,u,v\}$ lead to a PDE satisfied by $v(x)$.

Suppose

$$\mathbf{X}_S=\xi_i^S(x,u,v)\frac{\partial}{\partial x_i}+\eta^S(x,u,v)\frac{\partial}{\partial u}+\zeta^S(x,u,v)\frac{\partial}{\partial v} \tag{4.4}$$

defines a point symmetry of $S\{x,u,v\}$. $\mathbf{X}_S$ induces a *nonlocal symmetry* (*potential symmetry*) of $R\{x,u\}$ if and only if $(\xi^S,\eta^S)$ depends essentially on $v$; otherwise, $\mathbf{X}_S$ projects onto a point symmetry of $R\{x,u\}$.

Suppose

$$\mathbf{X}_R = \xi_i^R(x,u)\frac{\partial}{\partial x_i} + \eta^R(x,u)\frac{\partial}{\partial u} \tag{4.5}$$

defines a point symmetry of $R\{x,u\}$. Then, $\mathbf{X}_R$ yields a *nonlocal symmetry* of $S\{x,u,v\}$ if and only if $\mathbf{X}_S = \mathbf{X}_R + \zeta(x,u,v)\frac{\partial}{\partial v}$ defines no point symmetry admitted by $S\{x,u,v\}$ for any choice of $\zeta(x,u,v)$; otherwise $\mathbf{X}_R$ induces a point symmetry of $S\{x,u,v\}$.

For the rest of this section, we assume that $R_1 = R_1\{x,v\}$ is a related subsystem.

Suppose

$$\mathbf{X}_{R_1} = \xi_i^{R_1}(x,v)\frac{\partial}{\partial x_i} + \zeta^{R_1}(x,v)\frac{\partial}{\partial v}$$

defines a point symmetry of $R_1\{x,v\}$. Then $\mathbf{X}_{R_1}$ yields a *nonlocal (potential) symmetry* of $R\{x,u\}$ if and only if either (1) $\xi^{R_1}(x,v)$ depends essentially on $v$, or (2) $\xi^{R_1} \equiv \xi^{R_1}(x)$ and $\mathbf{X}_R = \xi_i^{R_1}(x)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u}$ defines no point symmetry of $R\{x,u\}$ for any choice of $\eta(x,u)$; otherwise $\mathbf{X}_{R_1}$ induces a point symmetry of $R\{x,u\}$. $\mathbf{X}_{R_1}$ yields a *nonlocal symmetry* of $S\{x,u,v\}$ if and only if $\mathbf{X}_S = \mathbf{X}_{R_1} + \eta(x,u,v)\frac{\partial}{\partial u}$ defines no point symmetry admitted by $S\{x,u,v\}$ for any choice of $\eta(x,u,v)$; otherwise $\mathbf{X}_{R_1}$ induces a point symmetry admitted by $S\{x,u,v\}$.

In turn, a point symmetry $\mathbf{X}_S$ of $S\{x,u,v\}$, given by (4.4), induces a symmetry of $R_1\{x,v\}$, which is not an admitted point symmetry of $R_1\{x,v\}$, if and only if $(\xi^S,\zeta^S)$ depends essentially on $u$; otherwise $\mathbf{X}_S$ projects onto a point symmetry of $R_1\{x,v\}$. A point symmetry $\mathbf{X}_R$ admitted by $R\{x,u\}$, given by (4.5), yields a symmetry of $R_1\{x,v\}$ which is not a point symmetry admitted by $R_1\{x,v\}$ if and only if either

(1) $\xi^R(x,u)$ depends essentially on $u$,
  or
(2) $\xi^R \equiv \xi^R(x)$, and $\mathbf{X}_{R_1} = \xi_i^R(x)\frac{\partial}{\partial x_i} + \zeta(x,v)\frac{\partial}{\partial v}$ defines no point symmetry admitted by $R_1\{x,v\}$ for any choice of $\zeta(x,v)$; otherwise $\mathbf{X}_R$ induces a point symmetry of $R_1\{x,v\}$.

## 5. FURTHER EXTENSIONS; TWO CONJECTURES

One can extend the process described in the previous section to determine "potentially" more nonlocal symmetries admitted by given systems of PDEs. Let $v^{(1)} = v, S^{(1)} = S^{(1)}\{x,u,v^{(1)}\} = S\{x,u,v\}$. Suppose one of the PDEs of $S^{(1)}$ can be replaced by an equivalent conserved form, leading to the introduction of $n-1$ more potential variables $v^{(2)}$ and another auxiliary system $S^{(2)} = S^{(2)}\{x,u,v^{(1)},v^{(2)}\}$ of $m+2(n-1)$ PDEs with $m+2(n-1)$ dependent variables. Correspondingly, there may be more related subsystems of PDEs and further nonlocal symmetries, contact symmetries, and Lie-Bäcklund symmetries could arise for $R\{x,u\}$, $S\{x,u,v\}$ and previous subsystems.

Suppose we can continue this process to some auxiliary system $S^{(N)} = S^{(N)}\{x,u,v^{(1)},v^{(2)},\dots, v^{(N)}\}$. At any given step, the union of point symmetry groups for all subsystems will not necessarily yield a group—in particular commutation relations may not exist connecting infinitesimal generators of point symmetries for different subsystems. However, all known calculations to date lead to the following two conjectures which, if correct, result in a "complete" algorithm for finding nonlocal symmetries through conserved forms.

CONJECTURE 5.1.

The process of obtaining auxiliary systems $S^{(1)}\{x,u,v^{(1)}\}$, $S^{(2)}\{x,u,v^{(1)},v^{(2)}\},\dots,S^{(N)}\{x,u,v^{(1)}, v^{(2)},\dots,v^{(N)}\}$ terminates at some finite $N$ where either

(1) $S^{(N)}$ has no conserved forms equivalent to one of its $m+N(n-1)$ PDEs (An *equivalent conserved form* has the property that its replacement of one of the PDEs of a system leads to no change in the solution set of the system with such a replacement) or
(2) $S^{(N)}$ can be linearized by some invertible point transformation.

In particular, $S^{(N)}$ has point symmetries which satisfy the criteria of Theorems 2.1, 2.2.

CONJECTURE 5.2.

The group of all point symmetries of $S^{(N)}$ yields, through projections, the groups of all point symmetries of *any* subsystem of $S^{(N)}$ including $R\{x,u\}$, $S^{(1)}, S^{(2)}, \ldots, S^{(N-1)}$. (In other words, one only needs to find the point symmetries of $S^{(N)}$ in order to determine all nonlocal symmetries of the various related subsystems obtained through the use of conserved forms.)

## 6. EXAMPLES

We now seek nonlocal symmetries for the three examples considered in Section 3.

EXAMPLE 1. NONLINEAR DIFFUSION EQUATION.

$R\{x,u\}$ is given by the conserved form (3.3). Then, system $S^{(1)} = S\{x,u,v\}$ is given by

$$v_x = u, \tag{6.1a}$$

$$v_t = (L(u))_x. \tag{6.1b}$$

The conserved form (6.1b), with the introduction of potential $w$, yields system $S^{(2)} = T\{x,u,v,w\}$ given by

$$\begin{aligned} v_x &= u, \\ w_t &= L(u), \\ w_x &= v. \end{aligned} \tag{6.2}$$

The subsystem emanating from (6.1a,b) is $R_1\{x,v\}$ given by

$$v_t = (L(v_x))_x. \tag{6.3}$$

Subsystems emanating from (6.2) are

$$\begin{aligned} S_1\{x,v,w\}:\ w_x &= v, \\ w_t &= L(v_x); \end{aligned} \tag{6.4}$$

$$\begin{aligned} S_2\{x,u,w\}:\ w_t &= L(u), \\ w_{xx} &= u; \end{aligned} \tag{6.5}$$

$$S_3\{x,w\}:\ w_t = L(w_{xx}). \tag{6.6}$$

It turns out, that for any choice of diffusion function $K(u) = L'(u) \neq$ const., the group of point symmetries of $T\{x,u,v,w\}$ yields, through projections, the point symmetry groups of all subsystems: $R\{x,u\}$, $S\{x,u,v\}$, $R_1\{x,v\}$, $S_1\{x,v,w\}$, $S_2\{x,u,w\}$, $S_3\{x,w\}$. In particular, modulo scalings and translation in $u$, the group classification of $T\{x,u,v,w\}$ is as follows:

(1) $K(u)$ arbitrary:

$$\mathbf{X}_1^T = \frac{\partial}{\partial x}, \qquad \mathbf{X}_2^T = \frac{\partial}{\partial t}, \qquad \mathbf{X}_3^T = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v} + 2w\frac{\partial}{\partial w}.$$

(All project to point symmetries for each subsystem.)

(2) $K(u) = u^\lambda$:

$$\mathbf{X}_1^T, \mathbf{X}_2^T, \mathbf{X}_3^T, \mathbf{X}_4^T = \frac{1}{2}\lambda x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}.$$

(All project to point symmetries for each subsystem.)

(3) $K(u) = u^{-4/3}$:

$$\mathbf{X}_1^T, \mathbf{X}_2^T, \mathbf{X}_3^T, \mathbf{X}_4^T \quad \text{with} \quad \lambda = -\frac{4}{3}, \qquad \mathbf{X}_5^T = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u} + (w - xv)\frac{\partial}{\partial v} + xw\frac{\partial}{\partial w}.$$

($\mathbf{X}_5^T$ projects to a point symmetry of $R\{x,u\}, S_1\{x,v,w\}, S_2\{x,u,w\}, S_3\{x,w\}$; $\mathbf{X}_5$ induces a nonlocal (potential) symmetry admitted by $S\{x,u,v\}$ and a nonlocal symmetry of $R_1\{x,v\}$.)

(4) $K(u) = u^{-2/3}$:

$$\mathbf{X}_1^T, \mathbf{X}_2^T, \mathbf{X}_3^T, \mathbf{X}_4^T \quad \text{with} \quad \lambda = -\frac{2}{3}, \mathbf{X}_6^T = w\frac{\partial}{\partial x} - 3uv\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v}.$$

($\mathbf{X}_6^T$ projects to a point symmetry admitted by $S_1\{x,v,w\}, S_3\{x,w\}$; $\mathbf{X}_6^T$ induces a nonlocal (potential) symmetry of $R\{x,u\}, S\{x,u,v\}$, a nonlocal symmetry of $R_1\{x,v\}$, and a Lie-Bäcklund symmetry of $S_2\{x,u,w\}$ since $v = w_x$.)

(5) $K(u) = \frac{1}{1+u^2}$:

$$\mathbf{X}_1^T, \mathbf{X}_2^T, \mathbf{X}_3^T, \mathbf{X}_7^T = v\frac{\partial}{\partial x} + (1+u^2)\frac{\partial}{\partial u} + x\frac{\partial}{\partial v} + \left(t + \frac{1}{2}(x^2 - v^2)\right)\frac{\partial}{\partial w}.$$

($\mathbf{X}_7^T$ projects to a point symmetry of $S\{x,u,v\}, R_1\{x,v\}, S_1\{x,v,w\}$; $\mathbf{X}_7^T$ induces a nonlocal (potential) symmetry of $R\{x,u\}$, a Lie-Bäcklund symmetry of $S_2\{x,u,w\}$, and a Lie-Bäcklund symmetry equivalent to a contact symmetry of $S_3\{x,w\}$.)

(6) $K(u) = \frac{1}{1+u^2}e^{\lambda \arctan u}, \lambda \neq 0$: This case admits four infinitesimal generators exhibiting the same symmetry properties for the various subsystems as when $\lambda = 0$.

(7) $K(u) = u^{-2}$: Here $T\{x,u,v,w\}$ admits

$$\mathbf{X}_\infty^T = F^1(v,t)\frac{\partial}{\partial x} - u^2F^2(v,t)\frac{\partial}{\partial u} + [vF^1(v,t) - F^3(v,t)]\frac{\partial}{\partial w}$$

where

$$\begin{aligned}\frac{\partial F^3}{\partial v} &= F^1,\\ \frac{\partial F^3}{\partial t} &= F^2,\\ \frac{\partial F^1}{\partial v} &= F^2.\end{aligned} \tag{6.7}$$

From the form of symmetry (6.7), it follows that it has the same symmetry properties for the various subsystems as $\mathbf{X}_7^T$. In addition, (6.7) satisfies the criteria of Theorems 2.1 and 2.2 for systems $T\{x,u,v,w\}$, $S\{x,u,v\}$, $R_1\{x,v\}$, $S_1\{x,v,w\}$ and the extension of these theorems to the contact symmetries admitted by $S_3\{x,w\}$. Hence, using Theorem 2.2, and its extension to scalar PDEs, one can construct invertible mappings which linearize these five systems and non-invertible mappings which linearize $R\{x,u\}, S_2\{x,u,w\}$. One can check that Conjecture 5.1 holds for $R\{x,u\}$ with $N = 2\,(N = 1$ when $K(u) = u^{-2})$ and that Conjecture 5.2 holds for all six subsystems related to $S^{(2)} = T\{x,u,v,w\}$. (Note that if the original system had been $R_1\{x,v\}$, then the only conserved form (6.3) would lead to the terminating potential system $S_1\{x,v,w\}$. Again, one can check that Conjectures 5.1, 5.2 hold for all forms of $K(u)$ with $N = 1, S^{(1)} = S_1\{x,v,w\}$. Here subsystems are $R_1\{x,v\}, S_3\{x,w\}$.)

EXAMPLE 2. REACTION-DIFFUSION EQUATION.

Here $R\{x,u\}$ is given by (3.5). Correspondingly, we obtain systems $S^{(1)} = S\{x,u,v\}$:

$$\begin{aligned}v_x &= u,\\ v_t &= -\left(\frac{1}{u} + bx^2\right)_x,\end{aligned} \tag{6.8}$$

and $S^{(2)} = T\{x,u,v,w\}$:

$$\begin{aligned}v_x &= u,\\ w_t &= -\left(\frac{1}{u} + bx^2\right),\\ w_x &= v.\end{aligned} \tag{6.9}$$

EXAMPLE 3. GAS DYNAMICS EQUATIONS.

Now suppose $R\{x,u\} = R\{x,v,p,\rho\}$ is represented by system (3.7a–c). Then conserved forms (3.7a,b) yield systems $S^{(1)} = S\{x,v,p,\rho,V\}$:

$$\begin{aligned} V_x &= \rho, \\ V_t &= -\rho v, \\ (\rho v)_t + (p+\rho v^2)_x &= 0, \\ \rho(p_t + vp_x) + B(p,\rho)v_x &= 0, \end{aligned} \tag{6.15}$$

and $S^{(2)} = T\{x,v,p,\rho,V,W\}$:

$$\begin{aligned} V_x &= \rho, \\ V_t &= -\rho v, \\ W_x &= \rho v, \\ W_t &= -(p+\rho v^2), \\ \rho(p_t + vp_x) + B(p,\rho)v_x &= 0. \end{aligned} \tag{6.16}$$

In this case, if $B(p,p)$ is arbitrary, not all possible subsystems yield systems of PDEs. The ones that do are:

$R_1\{x,p,\rho,V\}$, $R_2\{x,v,p,V\}$, $R_3\{x,p,V\}$, $S_1\{x,v,p,V,W\}$, $S_2\{x,v,p,\rho,W\}$, $S_3\{x,p,\rho,V,W\}$, $S_4\{x,v,\rho,V,W\}$, $S_5\{x,p,\rho,W\}$, $S_6\{x,v,p,W\}$, $S_7\{x,p,V,W\}$, $S_8\{x,v,\rho,W\}$, $S_9\{x,\rho,V,W\}$, $S_{10}\{x,v,V,W\}$, $S_{11}\{x,V,W\}$, $S_{12}\{x,p,W\}$, $S_{13}\{x,\rho,W\}$, $S_{14}\{x,v,W\}$.

If $B \equiv 1, T\{x,v,p,\rho,V,W\}$ admits

$$\mathbf{X}_\infty^T = F^1\frac{\partial}{\partial x} + F^2\frac{\partial}{\partial v} - F^3\frac{\partial}{\partial p} - \rho^2F^4\frac{\partial}{\partial \rho} + F^5\frac{\partial}{\partial W},$$

where $\{F^i = F^i(t,V)\}$ satisfies the linear system

$$\begin{aligned} \frac{\partial F^1}{\partial t} &= F^2, \\ \frac{\partial F^1}{\partial V} &= F^3, \\ F^4 &= F^3, \\ \frac{\partial F^5}{\partial V} &= F^2, \\ \frac{\partial F^5}{\partial t} &= F^3. \end{aligned} \tag{6.17}$$

Consequently, from the form of (6.17), and then using Theorems 2.1, 2.2, one can invertibly linearize systems $T, S, R_1, R_2, R_3, S_1, S_3, S_4, S_7, S_9, S_{10}, S_{11}$ whereas $R, S_2, S_5, S_6, S_8, S_{12}, S_{13}, S_{14}$ are non-invertibly linearized.

For example, from the form of (6.17) and then application of Theorem 2.2, one can show that $R_2\{x,p,v,V\}$, given by the system of PDEs

$$\begin{aligned} V_t + vV_x &= 0, \\ V_x(v_t + vv_x) + p_x &= 0, \\ V_x(p_t + vp_x) + v_x &= 0, \end{aligned}$$

with $\rho = V_x$, is invertibly transformed by the mapping

$$\begin{aligned} z_1 &= t, \\ z_2 &= V, \\ w^1 &= x, \\ w^2 &= v, \\ w^3 &= -p, \end{aligned}$$

to the linear system

$$\frac{\partial w^1}{\partial t} = w^2,$$
$$\frac{\partial w^1}{\partial v} = w^3,$$
$$\frac{\partial w^2}{\partial t} = \frac{\partial w^3}{\partial v}.$$

## 7. DISCUSSION

The problem of finding useful conserved forms to determine potential symmetries for given systems of PDEs will be discussed in a future paper. Attempts to find useful conserved forms through a change of coordinates are useless [33]. Moreover, one can show that most conservation laws (conserved forms) arising from invariance (Noether's theorem) are of no value to obtain useful potential systems.

## REFERENCES


1. G.I. Barenblatt, *Similarity, Self-Similarity, and Intermediate Asymptotics*, Consultants Bureau, New York, (1979).
2. G.W. Bluman and J.D. Cole, *Similarity Methods for Differential Equations,* (Appl. Math. Sci. No. 13), Springer, New York, (1974).
3. G.W. Bluman and S. Kumei, *Symmetries and Differential Equations,* (Appl. Math. Sci. No. 81), Springer, New York, (1989).
4. J.M. Hill, *Solution of Differential Equations by Means of One-Parameter Groups*, Pitman, Boston, (1982).
5. N.H. Ibragimov, *Transformation Groups Applied to Mathematical Physics*, Reidel, Boston, (1985).
6. P.J. Olver, *Applications of Lie Groups to Differential Equations,* (GTM, No. 107), Springer, New York, (1986).
7. L.V. Ovsiannikov, *Group Properties of Differential Equations* (in Russian), Novosibirsk, (1962).
8. L.V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, (1982).
9. C. Rogers and W.F. Ames, *Nonlinear Boundary Value Problems in Science and Engineering*, Academic Press, Boston, (1989).
10. H. Stephani, *Differential Equations—Their Solution Using Symmetries*, Cambridge U.P., Cambridge, U.K., (1989).
11. A.M. Vinogradov, (Ed.), *Symmetries of Partial Differential Equations*, Kluwer, Dordrecht, (1989).
12. S. Kumei and G.W. Bluman, When nonlinear differential equations are equivalent to linear differential equations, *SIAM J. Appl. Math.* **42**, 1157–1173 (1982).
13. G.W. Bluman and S. Kumei, Symmetry-based algorithms to relate partial differential equations: I. Local symmetries, *Euro. J. Appl. Math.* **1**, 189–216 (1990).
14. G.W. Bluman and S. Kumei, On invariance properties of the wave equation, *J. Math. Phys.* **28**, 307–318 (1987).
15. G.W. Bluman, S. Kumei and G.J. Reid, New classes of symmetries for partial differential equations, *J. Math. Phys.* **29**, 806–811, 2320 (1988).
16. G.W. Bluman, Potential symmetries, In *Proceedings of the Annual Seminar of the Canadian Mathematical Society on Lie Theory, Differential Equations and Representation Theory,* (Edited by V. Hussin), Centre de Recherches Mathématiques, Montreal, 85–100, (1990).
17. G.W. Bluman and S. Kumei, Symmetry-based algorithms to relate partial differential equations: II. Linearization by nonlocal symmetries, *Euro. J. Appl. Math* **1**, 217–223 (1990).
18. A.K. Head, *LIE: A muMATH Program for the Calculation of the LIE Algebra of Differential Equations*, CSIRO Division of Material Sciences, Clayton, Australia, (1990, 1992).
19. F. Schwarz, Automatically determining symmetries of partial differential equations, *Computing* **34**, 91–106 (1985).
20. T. Wolf and A. Brand, The computer algebra package CRACK for investigating PDEs, In *ERCIM Advanced Course on Partial Differential Equations and Group Theory*, Bonn, (1992).
21. B. Champagne, W. Hereman and P. Winternitz, The computer calculation of Lie point symmetries of large systems of differential equations, *Comp. Phys. Comm.* **66**, 319–340 (1991).
22. P.H.M. Kersten, *Infinitesimal Symmetries: A Computational Approach* (CWI Tract No. 34), Centrum voor Wiskunde en Informatica, Amsterdam, (1987).
23. G.J. Reid, Algorithms for reducing a system of PDEs to standard form, determining the dimension of its solution space and calculating its Taylor series solution, *Euro. J. Appl. Math.* **2**, 293–318 (1991).
24. G.J. Reid, Finding abstract Lie symmetry algebras of differential equations without integrating determining equations, *Euro. J. Appl. Math.* **2**, 319–340 (1991).

25. V.L. Topunov, Reducing systems of linear differential equations to passive form, *Acta Applic. Math.* **16**, 191–206 (1989).
26. I. Lisle, Equivalence transformations for classes of differential equations, Ph.D. Thesis, University of British Columbia, (1992).
27. I.S. Akhatov, R.K. Gazizov and N.K. Ibragimov, Nonlocal symmetries. Heuristic approach, *J. Sov. Math,* 1401–1450 (1991).
28. E. Noether, Invariante Variationsprobleme. Nachr. König. Gesell. Wissen. Göttingen, *Math-Phys.* **Kl**, 235–257 (1918).
29. R.L. Anderson, S. Kumei and C.E. Wulfman, Generalization of the concept of invariance of differential equations, *Phys. Rev. Lett.* **28**, 988–991 (1972).
30. A.M. Vinogradov, Symmetries and conservation laws of partial differential equations: Basic notions and results, *Acta Applic. Math.* **15**, 3–21 (1989).
31. I.S. Krasil'shchik and A.M. Vinogradov, Nonlocal symmetries and the theory of coverings: An addendum to A.M. Vinogradov's local symmetries and conservation laws, *Acta Applic. Math.* **2**, 79–96. (1984).
32. I.S. Krasil'shchik and A.M. Vinogradov, Nonlocal trends in the geometry of differential equations: Symmetries, conservation laws, and Bäcklund transformations, *Acta Applic. Math.* **15**, 161–209 (1989).
33. G.W. Bluman, Invariance of conserved forms under contact transformations, (Preprint) (1992).