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Use of Group Analysis in Solving Overdetermined Systems of Ordinary Differential Equations

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Group analysis is applied to overdetermined systems of ODEs. If each ODE of the system admits the same r-parameter solvable Lie group, then the use of the corresponding differential invariants greatly simplifies the analysis of the system. Moreover this can even lead to its explicit solution. Examples are given.

I. INTRODUCTION

We show how group analysis can be used to solve an overdetermined system of two ordinary differential equations (ODEs) of orders m and n, respectively, $m \leqslant n$:

$$f\left(x, y, \frac{dy}{dx}, ..., \frac{d^m y}{dx^m}\right) = 0, \tag{1.1a}$$

$$g\left(x, y, \frac{dy}{dx}, ..., \frac{d^n y}{dx^n}\right) = 0. \tag{1.1b}$$

Let D be a set of solutions common to (1.1a), (1.1b). Each member of D lies on the surface S defined by the intersection of the surfaces

$$f(z_1, z_2, z_3, ..., z_{m+2}) = 0, \tag{1.2a}$$

$$g(z_1, z_2, z_3, ..., z_{n+2}) = 0, \tag{1.2b}$$

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with the correspondence $z_1 = x$, $z_2 = y$, $z_3 = dy/dx, ..., z_{n+2} = d^n y/dx^n$. Obviously the dimensionality of the surface S is at most $m+1$.

We assume that each of the ODEs (1.1a) and (1.1b) is invariant under the same r-parameter $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_r)$ solvable Lie group of point transformations $G^{(r)}$ [1, p. 154]:

$$x^* = X(x, y; \varepsilon_1, \varepsilon_2, ..., \varepsilon_r), \tag{1.3a}$$

$$y^* = Y(x, y; \varepsilon_1, \varepsilon_2, ..., \varepsilon_r), \tag{1.3b}$$

$1 \leqslant r \leqslant m$. Consequently there exist differential invariants [1, 2],

$$u\left(x, y, \frac{dy}{dx}, ..., \frac{d^{r-1}y}{dx^{r-1}}\right), \tag{1.4a}$$

$$v\left(x, y, \frac{dy}{dx}, ..., \frac{d^r y}{dx^r}\right), \tag{1.4b}$$

such that Eqs. (1.1a) and (1.1b), respectively, reduce to the equivalent overdetermined system of equations

$$F\left(u, v, \frac{dv}{du}, ..., \frac{d^{m-r}v}{du^{m-r}}\right) = 0, \tag{1.5a}$$

$$G\left(u, v, \frac{dv}{du}, ..., \frac{d^{n-r}v}{du^{n-r}}\right) = 0, \tag{1.5b}$$

for some functions F and G.

Thus by group analysis we see that the surface S containing a set of solutions D common to (1.1a), (1.1b) is a surface S of dimensionality at most $m+1-r$ in $(z_1, z_2, z_3, ..., z_{m+2})$-space since it is now constrained by (1.4a), (1.4b), (1.5a), (1.5b). In particular the surface S lies in $(Z_1, Z_2, Z_3, ..., Z_{m+2-r})$-space with the correspondence $Z_1 = u$, $Z_2 = v$, $Z_3 = dv/du, ..., Z_{m+2-r} = d^{m-r}v/du^{m-r}$.

Suppose a curve

$$v = \Phi(u) \tag{1.6}$$

solves (1.5a), (1.5b). Then any solution of the ODE

$$v\left(x, y, \frac{dy}{dx}, ..., \frac{d^r y}{dx^r}\right) = \Phi\left(u\left(x, y, \frac{dy}{dx}, ..., \frac{d^{r-1}y}{dx^{r-1}}\right)\right) \tag{1.7}$$

is a common solution of (1.1a), (1.1b). Since u and v are invariants of $G^{(r)}$ it follows that Eq. (1.7) is invariant under the r-parameter solvable group $G^{(r)}$ defined by (1.3a), (1.3b). Hence Eq. (1.7) can be reduced constructively to r quadratures which introduce r constants labelled $c_1, c_2, ..., c_r$. Thus we obtain explicitly a function $\psi(x, y; c_1, c_2, ..., c_r)$ for which the equation

$$\psi(x, y; c_1, c_2, ..., c_r) = 0 \tag{1.8}$$

defines an implicit common solution of (1.1a), (1.1b) with $c_1, c_2, ..., c_r$ as essential constants.

Equations (1.5a), (1.5b) also can have point solutions

$$u = A, \tag{1.9a}$$

$$v = B, \tag{1.9b}$$

for some constants A and B. In this case a common solution of (1.1a), (1.1b), if any exists, satisfies each of the ODEs

$$u\left(x, y, \frac{dy}{dx}, ..., \frac{d^{r-1}y}{dx^{r-1}}\right) = A, \tag{1.10a}$$

$$v\left(x, y, \frac{dy}{dx}, ..., \frac{d^{r}y}{dx^{r}}\right) = B. \tag{1.10b}$$

Since both of the ODEs (1.10a), (1.10b) are invariant under the r-parameter group $G^{(r)}$ defined by (1.3a), (1.3b), it follows that the solution of each of these ODEs is constructively reduced to quadratures. Hence one can determine explicitly all common solutions of (1.10a), (1.10b).

Note that with our procedure one may find the set of all common solutions of (1.1a), (1.1b) without determining the general solution of either (1.1a) or (1.1b).

II. A Simple Example

Consider the system of ODEs ($m = 2, n = 2$):

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = 0, \tag{2.1a}$$

$$y \frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 = 0. \tag{2.1b}$$

Equations (2.1a) and (2.1b) are both invariant under three Lie groups of point transformations ($r = 1$ or 2):

(i) The one-parameter (ε_1) group $G^{(1)}(\varepsilon_1)$

$$x^* = e^{\varepsilon_1}x, \tag{2.2a}$$

$$y^* = y. \tag{2.2b}$$

(ii) The one-parameter (ε_2) group $G^{(1)}(\varepsilon_2)$

$$x^* = x, \tag{2.3a}$$

$$y^* = e^{\varepsilon_2}y. \tag{2.3b}$$

(iii) The two-parameter ($\varepsilon_1, \varepsilon_2$) group $G^{(2)}(\varepsilon_1, \varepsilon_2)$

$$x^* = e^{\varepsilon_1}x, \tag{2.4a}$$

$$y^* = e^{\varepsilon_2}y. \tag{2.4b}$$

These three groups are all solvable since each has at most two parameters. We demonstrate our procedure for each of these groups.

(i) $G^{(1)}(\varepsilon_1)$. Differential invariants for $G^{(1)}(\varepsilon_1)$ are

$$u = y, \tag{2.5a}$$

$$v = x\frac{dy}{dx}. \tag{2.5b}$$

Correspondingly, Eqs. (2.1a), (2.1b) reduce to

$$v\frac{dv}{du} - u = 0, \tag{2.6a}$$

$$v\left[u\frac{dv}{du} - u - 2v\right] = 0. \tag{2.6b}$$

Substituting (2.6a) into (2.6b) one obtains

$$(v+u)(2v-u) = 0$$

so that either

$$v = \frac{u}{2} \tag{2.7}$$

or

$$v = -u. \tag{2.8}$$

Then substituting (2.7) into (2.6a) yields $v = u = 0$ and hence the common trivial solution $y = 0$. Corresponding to (2.8) we find that the set D of all common solutions of (2.1a), (2.1b) is given by the ODE $v = \Phi(u) = -u$, namely,

$$x\frac{dy}{dx} = -y. \tag{2.9}$$

Hence

$$y = \frac{c_1}{x} \tag{2.10}$$

for arbitrary constant c_1 represents the set of all common solutions of (2.1a), (2.1b).

(ii) $G^{(1)}(\varepsilon_2)$. Differential invariants for $G^{(1)}(\varepsilon_2)$ are

$$u = x, \tag{2.11a}$$

$$v = \frac{1}{y}\frac{dy}{dx}. \tag{2.11b}$$

In this case Eqs. (2.1a), (2.1b) become

$$u^2\frac{dv}{du} + u^2v^2 + uv - 1 = 0, \tag{2.12a}$$

$$\frac{dv}{du} - v^2 = 0. \tag{2.12b}$$

Substitution of (2.12b) into (2.12a) leads to

$$(2uv - 1)(uv + 1) = 0$$

so that either

$$v = \frac{1}{2u} \tag{2.13}$$

or

$$v = -\frac{1}{u}. \tag{2.14}$$

Then again using (2.12b) we see that all common solutions of (2.1a), (2.1b) result from $v=\Phi(u)=-1/u$, i.e.,

$$\frac{1}{y}\frac{dy}{dx}=-\frac{1}{x}, \tag{2.15}$$

the same ODE as defined by Eq. (2.9).

(iii) $G^{(2)}(\varepsilon_1, \varepsilon_2)$. Differential invariants are

$$u=\frac{x}{y}\frac{dy}{dx}, \tag{2.16a}$$

$$v=\frac{x^2}{y}\frac{d^2y}{dx^2}. \tag{2.16b}$$

Thus Eqs. (2.1a), (2.1b) become

$$v+u-1=0, \tag{2.17a}$$

$$v-2u^2=0. \tag{2.17b}$$

The solutions of (2.17a), (2.17b) are the points

$$(u, v)=(\tfrac{1}{2}, \tfrac{1}{2}) \tag{2.18}$$

and

$$(u, v)=(-1, 2). \tag{2.19}$$

The point (2.18) leads to

$$\frac{x}{y}\frac{dy}{dx}=\frac{1}{2}, \tag{2.20a}$$

$$\frac{x^2}{y}\frac{d^2y}{dx^2}=\frac{1}{2}. \tag{2.20b}$$

The solution of (2.20a) is $y=\beta x^{1/2}$ for any constant β. Substitution of this expression into (2.20b) leads to $\beta=0$. The point (2.19) leads to

$$\frac{x}{y}\frac{dy}{dx}=-1, \tag{2.21a}$$

$$\frac{x^2}{y}\frac{d^2y}{dx^2}=2. \tag{2.21b}$$

Equation (2.21a) is the same as Eq. (2.9) with solution given by Eq. (2.10). This solution satisfies Eq. (2.21b).

III. A Problem Arising From Group Analysis of the Wave Equation

We showed [3] that the wave equation

$$y^2(x)\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}=0 \tag{3.1}$$

is invariant under a four-parameter Lie group of point transformations if and only if the wave speed $y(x)$ satisfies the fifth order ODE

$$\frac{d}{dx}\left\{y^2\left[\frac{\dfrac{d^3H}{dx^3}}{2\dfrac{dH}{dx}+H^2}+\frac{3\left[2\left(\dfrac{dH}{dx}\right)^3-2H\dfrac{dH}{dx}\dfrac{d^2H}{dx^2}-\left(\dfrac{d^2H}{dx^2}\right)^2\right]}{\left[2\dfrac{dH}{dx}+H^2\right]^2}\right]\right\}=0, \tag{3.2}$$

where

$$H=\frac{1}{y}\frac{dy}{dx}. \tag{3.3}$$

We also showed that the related system of partial differential equations

$$\frac{\partial v}{\partial t}-\frac{\partial u}{\partial x}=0, \tag{3.4a}$$

$$\frac{\partial u}{\partial t}-y^2(x)\frac{\partial v}{\partial x}=0, \tag{3.4b}$$

is invariant under a four-parameter Lie group of point transformations if and only if $y(x)$ solves the fourth order ODE

$$\frac{d}{dx}\left[y\frac{dy}{dx}\frac{d^2}{dx^2}\left(\frac{y}{dy/dx}\right)\right]=0. \tag{3.5}$$

Equation (3.1) and system (3.4a), (3.4b) are related as follows: If (u, v) solves (3.4a), (3.4b), then u solves (3.1); if $u=\phi(x, t)$ solves (3.1) then $(u, v)=(\partial\phi/\partial t, \partial\phi/\partial x)$ solves (3.4a), (3.4b). A natural question arises: What is the set D of common solutions of the ODEs (3.2) and (3.5)? Without some simplifying procedure it is unclear how to find D. The method outlined in Section I yields D simply and elegantly.

Each of the ODEs (3.2) and (3.5) admits the same three-parameter $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ solvable Lie group of point transformations $G^{(3)}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$:

$$x^*=e^{\varepsilon_2}(x+\varepsilon_1), \tag{3.6a}$$

$$y^*=e^{\varepsilon_3}y. \tag{3.6b}$$

Differential invariants corresponding to (1.4a), (1.4b) are

$$u = y\frac{d^2y}{dx^2}\bigg/\left(\frac{dy}{dx}\right)^2, \tag{3.7a}$$

$$v = y^2\frac{d^3y}{dx^3}\bigg/\left(\frac{dy}{dx}\right)^3. \tag{3.7b}$$

Consequently Eqs. (3.2) and (3.5), respectively, reduce to the equation

$$2u\Gamma + \left(\frac{\partial \Gamma}{\partial u} + \dot{v}\frac{\partial \Gamma}{\partial v} + \ddot{v}\frac{\partial \Gamma}{\partial \dot{v}}\right)(u + v - 2u^2) = 0, \tag{3.8a}$$

where $\dot{v} = dv/du$, $\ddot{v} = d^2v/du^2$, and

$$\Gamma(u, v, \dot{v}) = (1-u) + \frac{(2u^2 - 3uv - 9u + 6v + 4) + \dot{v}(2u^2 - u - v)}{2u-1} + \frac{3(v - 2u + 1)^2}{(2u-1)^2}, \tag{3.8b}$$

and to the equation

$$(2u^2 - u - v)(\dot{v} - 2u + 1) = 0. \tag{3.9}$$

From Eq. (3.9) two cases arise:

$$v = 2u^2 - u; \tag{3.10}$$

$$v = u^2 - u + \delta, \tag{3.11}$$

where δ is an arbitrary constant. We determine separately the compatibility of Eqs. (3.10) and (3.11) with Eqs. (3.8a), (3.8b).

Case 1. $v = 2u^2 - u$. It is easy to check that Eqs. (3.8a), (3.8b) are satisfied identically so that we have common solutions defined by $v = \Phi_1(u) = 2u^2 - u$. The ODE corresponding to (1.7), namely

$$y^2\frac{d^3y}{dx^3}\bigg/\left(\frac{dy}{dx}\right)^3 = 2y^2\left(\frac{d^2y}{dx^2}\right)^2\bigg/\left(\frac{dy}{dx}\right)^4 - y\frac{d^2y}{dx^2}\bigg/\left(\frac{dy}{dx}\right)^2, \tag{3.12}$$

is invariant under the solvable group $G^{(3)}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ given by Eqs. (3.6a), (3.6b). An integration of (3.12) leads to

$$u = y\frac{d^2y}{dx^2}\bigg/\left(\frac{dy}{dx}\right)^2 = \text{constant} = \lambda. \tag{3.13}$$

From this equation it is easy to show that the general solution of ODE (3.12) is

$$y = (c_1 + c_2 x)^{c_3}, \tag{3.14}$$

where c_1, c_2, c_3 are arbitrary constants, with $c_3 = 1/(1-\lambda)$. Let D_1 denote the common solution set (3.14) of ODEs (3.2) and (3.5).

Case 2. $v = u^2 - u + \delta$. The substitution of Eq. (3.11) into Eqs. (3.8a), (3.8b) leads to the compatibility equation

$$(u^2 - \delta)(1 - 4\delta) = 0. \tag{3.15}$$

If in Eq. (3.15) the first factor is set to zero, we obtain Eq. (3.13) which leads to the common solution set D_1 defined by Eq. (3.14). Setting the second factor to zero, i.e., $\delta = \frac{1}{4}$, we see that $y(x)$ is a common solution of Eqs. (3.2) and (3.5) if it satisfies the ODE corresponding to (1.7), $v = \Phi_2(u) = u^2 - u + \frac{1}{4}$, namely,

$$y^2 \frac{d^3y}{dx^3} \Big/ \left(\frac{dy}{dx}\right)^3 = y^2 \left(\frac{d^2y}{dx^2}\right)^2 \Big/ \left(\frac{dy}{dx}\right)^4 - y \frac{d^2y}{dx^2} \Big/ \left(\frac{dy}{dx}\right)^2 + \frac{1}{4}. \tag{3.16}$$

From our remarks in Section I, Eq. (3.16) admits the solvable group $G^{(3)}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and using a chain of Lie subgroups corresponding to the solvability of $G^{(3)}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, one can reduce this ODE to three explicit quadratures. The calculations follow.

Let

$$U = \frac{dy}{dx}, \tag{3.17a}$$

$$V = y \frac{d^2y}{dx^2}, \tag{3.17b}$$

be differential invariants of $G^{(3)}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. Then ODE (3.16) reduces to

$$\frac{dV}{dU} = \frac{V}{U} + \frac{U^3}{4V} \tag{3.18}$$

with general solution

$$V^2 = \frac{U^4}{4} + \alpha U^2, \tag{3.19}$$

where α is an arbitrary constant. From (3.17a), (3.17b) we see that

$$V = yU \frac{dU}{dy}. \tag{3.20}$$

Thus Eq. (3.19) becomes

$$y^2\left(\frac{dU}{dy}\right)^2=\frac{U^2}{4}+\alpha. \qquad (3.21)$$

Consequently $U=dy/dx$ can be determined explicitly as a function of y and two arbitrary constants $\{\gamma>0, \beta\}$:

$$\frac{dy}{dx}=\beta[(\gamma y)^{1/2}\pm(\gamma y)^{-1/2}]. \qquad (3.22)$$

Finally, corresponding to the two signs in (3.22) we obtain the common solution set D_2 of ODEs (3.2) and (3.5) consisting of the two families of solutions

$$\sqrt{y}-\arctan c_1\sqrt{y}=c_2(x+c_3), \qquad (3.23a)$$

$$\sqrt{y}+\log\left|\frac{\sqrt{y}-c_1}{\sqrt{y}+c_1}\right|^{1/2}=c_2(x+c_3), \qquad (3.23b)$$

where in each family c_1, c_2, c_3 are arbitrary constants.

Let us determine the set of all solutions $D_1\cap D_2$ common to ODEs (3.12) and (3.16).

A solution belonging to $D_1\cap D_2$ must lie on both parabolas

$$v=\Phi_1(u)=2u^2-u \qquad (3.24a)$$

and

$$v=\Phi_2(u)=u^2-u+\tfrac{1}{4}. \qquad (3.24b)$$

An intersection point of these parabolas defines a solution in $D_1\cap D_2$ since $u=$ constant solves (3.24a) for any constant. The intersection points of these parabolas are located at $u=\pm\frac{1}{2}$. The point $u=\frac{1}{2}$ corresponds to the family of solutions

$$y^2=(ax+b)^4, \qquad (3.25)$$

where a, b are arbitrary constants, and the point $u=-\frac{1}{2}$ corresponds to the family of solutions

$$y^2=(cx+d)^{4/3}, \qquad (3.26)$$

where c, d are arbitrary constants. The set $D_1\cap D_2$ consists of the two families of solutions (3.25), (3.26).

It is interesting to note that the wave speeds $y(x)$, defined by Eqs. (3.25), (3.26), have special significance for the wave equation (3.1): One can obtain explicitly [4, 5] the general solution of (3.1) when the wave speed $y(x)$ satisfies either Eq. (3.25) or (3.26). If $y(x)$ is given by (3.25), the general solution of (3.1) is obtained through a point transformation mapping this wave equation into one with a constant wave speed $y(x)=1$. If $y(x)$ is given by (3.26), the general solution of (3.1) is obtained through a Bäcklund transformation which relates the wave equation with wave speed $y(x)=(cx+d)^{2/3}$ to the wave equation with wave speed $y(x)=(ax+b)^2$.

REFERENCES

1. P. J. OLVER, "Applications of Lie Groups to Differential Equations," Springer-Verlag, New York/Berlin/Heidelberg, 1986.
2. G. BLUMAN AND S. KUMEI, "Infinitesimal Group Transformation Methods for Differential Equations," Springer-Verlag, New York/Berlin/Heidelberg, 1989.
3. G. BLUMAN AND S. KUMEI, On invariance properties of the wave equation, *J. Math. Phys.* **28** (1987), 307–318.
4. G. BLUMAN AND G. REID, "A Bäcklund Transformation Theory Generating Chains of Second-Order Linear PDE's and Related Solution Sequences," IAM Technical Report No. 87-4, University of British Columbia, Vancouver, 1987.
5. G. BLUMAN AND G. REID, Sequences of related linear PDES, *J. Math. Anal. Appl.*, to appear.