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New Symmetries for Ordinary Differential Equations

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In this paper we present a theory for calculating new symmetries for ordinary differential equations. These new symmetries lead to a systematic reduction of the order of a differential equation. Our approach depends on computing the Lie symmetries of a differential equation related to a given differential equation by a Bäcklund transformation. Consequently we induce new symmetries of the given differential equation which are not in general of Lie, contact, or Lie–Bäcklund type.

We obtain new symmetries and corresponding new analytic results for a class of ordinary differential equations arising from nonlinear diffusion.

1. Introduction

IN THIS paper we present an algorithm to find new symmetry groups for ordinary differential equations (ODE). These new symmetries reduce the order of a given ODE in cases where a direct application of Lie's method fails. With respect to the given independent and dependent variables of an ODE, our new symmetries are in general not of point, contact or Lie–Bäcklund type.

We define a *symmetry group of a differential equation* to be a group which maps solutions to other solutions of the differential equation.

In the latter part of the nineteenth century Sophus Lie introduced the notion of continuous groups, now known as Lie groups, in order to unify and extend various ad hoc methods for solving ODEs. Such groups are characterized completely by their infinitesimal generators. Lie showed that the order of an ODE can be reduced by one, constructively, if it is invariant under a one-parameter Lie group of point transformations acting on the space of its independent and dependent variables (cf. (Cohen, 1911; Dickson, 1924; Bluman & Cole, 1974; Olver, 1986)). Moreover Lie gave an algorithm to determine the infinitesimal generators admitted by a given ODE. If no such generator exists, Lie's method is not applicable. Lie extended his work to groups of contact transformations which are point groups acting on the space of the independent and dependent variables and the derivative of the dependent variable of a given ODE. (Such transformations are rare.)

Lie's work has been extended to include Lie–Bäcklund symmetries for which the infinitesimals are allowed to depend on derivatives of the dependent variable up to any finite order (cf. (Anderson & Ibragimov, 1979). We are unaware of examples of such symmetries for nonlinear ODEs).

For an ODE with independent variable x and dependent variable u we show how to obtain new symmetries by computing the Lie group of point transformations admitted by a related ODE with the same independent variable x and different dependent variable ϕ. Although the connection between the given ODE and its related ODE is not one-to-one, any solution of the related ODE maps into a solution of the given ODE. (Often the connection is such that the general solution of the related ODE maps into the general solution of the given ODE.) Hence a Lie symmetry group of the related ODE induces a *symmetry group* of the given ODE. We shall show, theoretically and by example, that such a symmetry group can be a new symmetry group of the given ODE.

We find the related ODE by expressing the given ODE in a conserved form through the introduction of an auxiliary variable ϕ. This new variable ϕ is associated with the original variables (x, u) by a Bäcklund transformation.

Bluman & Kumei (1987) found classes of new symmetries for the wave equation. Bluman, Kumei, & Reid (1987) give a general procedure for finding new symmetries for partial differential equations (PDE). New symmetries for PDEs are discovered by a different procedure than that presented here for ODEs.

Sayegh & Jones (1986) give a theoretical framework which in principle can lead to new symmetries for ODEs. They provide no examples of new symmetries. In their approach a new symmetry is a solution of an underdetermined system of PDEs. In our approach a new symmetry is a solution of an overdetermined system of PDEs. It appears that our new symmetries do not correspond to solutions of the underdetermined system of Sayegh & Jones.

In Section 2 we present our method for finding new symmetries for ODEs. Non-trivial examples, originating from the nonlinear diffusion equation, yield new symmetries which are discussed in Section 3. In Section 4 we develop features of the calculus to prepare ODEs in a form suitable for applying our method.

2. Method for finding new symmetries

Consider a second-order ODE

$$F\left(x, u, \frac{du}{dx}, \frac{d^2u}{dx^2}\right) = 0, \tag{2.1}$$

with independent variable x and dependent variable u. We assume that there exists a Bäcklund transformation, defining an auxiliary variable ϕ,

$$u = f\left(x, \phi, \frac{d\phi}{dx}\right), \tag{2.2}$$

such that (2.1) can be expressed in the form

$$\frac{d}{dx} G\left(x, u, \frac{du}{dx}, \phi, \frac{d\phi}{dx}\right) = 0 \tag{2.3}$$

for some function G; the function f must depend explicitly on $d\phi/dx$. Our second-order ODE, related to (2.1), is

$$G\left(x, f\left(x, \phi, \frac{d\phi}{dx}\right), \frac{d}{dx} f\left(x, \phi, \frac{d\phi}{dx}\right), \phi, \frac{d\phi}{dx}\right) = 0. \tag{2.4}$$

If $\phi(x)$ solves ODE (2.4) then u, given by (2.2), solves ODE (2.1).

Next we assume that (2.4) admits a one-parameter (ε) Lie group of point transformations

$$\left.\begin{aligned} x^* &= X(x, \phi; \varepsilon) = x + \varepsilon\xi(x, \phi) + O(\varepsilon^2), \\ \phi^* &= \Phi(x, \phi; \varepsilon) = \phi + \varepsilon\zeta(x, \phi) + O(\varepsilon^2), \end{aligned}\right\} \tag{2.5}$$

where ξ and ζ are the infinitesimals of the group. This group maps a solution of ODE (2.4) into another solution of (2.4) and hence maps a solution of ODE (2.1) into another solution of (2.1). Thus the group (2.5) induces a symmetry group of ODE (2.1). This symmetry group is a new symmetry group of (2.1) if ξ depends explicitly on ϕ. Such a new symmetry is not a point, contact or Lie–Bäcklund symmetry of ODE (2.1) since ϕ, as defined by (2.2), cannot be expressed in terms of x, u and derivatives of u to some finite order.

A generalization of this procedure to higher-order ODEs is straightforward.

3. Examples of new symmetries

Consider the nonlinear diffusion equation

$$\frac{\partial}{\partial y}\left(K(v)\frac{\partial v}{\partial y}\right) - \frac{\partial v}{\partial t} = 0, \tag{3.1}$$

where $K(v)$ is a concentration-dependent conductivity. Such equations arise in plasma and solid-state physics (Berryman & Holland, 1978; Baeri, Campisano, Foti, & Rimini, 1979) and polymer science (Fujita, 1952a,b, 1954). For (3.1) an important class of solutions arises from its invariance under scalings $y^* = \alpha y$, $t^* = \alpha^2 t$, namely

$$v = u(x), \qquad x = yt^{-\frac{1}{2}}, \tag{3.2}$$

where $u(x)$ satisfies the ODE

$$2\frac{\mathrm{d}}{\mathrm{d}x}\left(K(u)\frac{\mathrm{d}u}{\mathrm{d}x}\right) + x\frac{\mathrm{d}u}{\mathrm{d}x} = 0. \tag{3.3}$$

Boundary-value problems leading to (3.2), (3.3) are considered in detail in Fujita's papers. For applications it is important to be able to reduce the order of ODE (3.3). After a lengthy calculation one can show that ODE (3.3) admits a one-parameter Lie group of point transformations

$$\left.\begin{aligned} x^* &= X(x, u; \varepsilon) = x + \varepsilon\xi(x, u) + O(\varepsilon^2), \\ u^* &= U(x, u; \varepsilon) = u + \varepsilon\eta(x, u) + O(\varepsilon^2), \end{aligned}\right\} \tag{3.4}$$

if and only if

$$K(u) = \lambda(u + \kappa)^{\nu}, \tag{3.5}$$

where $\{\lambda, \kappa, \nu\}$ are arbitrary constants, or the limiting case

$$K(u) = \lambda \mathrm{e}^{\nu u}. \tag{3.6}$$

Correspondingly one can reduce the order of ODE (3.3) by Lie's standard procedure if $K(u)$ is given by (3.5) or (3.6). Moreover these are the only forms of $K(u)$ for which the order of ODE (3.3) can be reduced by Lie's method.

We can apply the procedure introduced in Section 2 to ODE (3.3) by setting

$$u = f\left(x, \phi, \frac{d\phi}{dx}\right) = \frac{d\phi}{dx}. \tag{3.7}$$

Consequently

$$x\frac{du}{dx} = \frac{d}{dx}\left[x\frac{d\phi}{dx} - \phi\right] \tag{3.8}$$

and hence ODE (3.3) becomes

$$\frac{d}{dx}\left[2K(u)\frac{du}{dx} + \left(x\frac{d\phi}{dx} - \phi\right)\right] = 0. \tag{3.9}$$

Thus our related ODE is

$$2K\left(\frac{d\phi}{dx}\right)\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} - \phi = 0. \tag{3.10}$$

Let $\phi' = d\phi/dx$. Then ODE (3.10) admits a one-parameter Lie group of the form (2.5) if and only if there exists a non-trivial solution $\{\xi(x, \phi), \zeta(x, \phi)\}$ of the determining equation

$$\begin{aligned}
&\frac{1}{K(\phi')}\frac{dK(\phi')}{d\phi'}\left[\phi\frac{\partial\zeta}{\partial x} + \left(\phi\frac{\partial\zeta}{\partial\phi} - \phi\frac{\partial\xi}{\partial x} - x\frac{\partial\zeta}{\partial x}\right)\phi'\right.\\
&\quad\left.+\left(x\frac{\partial\xi}{\partial x} - x\frac{\partial\zeta}{\partial\phi} - \phi\frac{\partial\xi}{\partial\phi}\right)(\phi')^2 + x\frac{\partial\xi}{\partial\phi}(\phi')^3\right]\\
&\quad+2K(\phi')\left[\frac{\partial^2\zeta}{\partial x^2} + \left(2\frac{\partial^2\zeta}{\partial x\,\partial\phi} - \frac{\partial^2\xi}{\partial x^2}\right)\phi' + \left(\frac{\partial^2\zeta}{\partial\phi^2} - 2\frac{\partial^2\xi}{\partial x\,\partial\phi}\right)(\phi')^2\right.\\
&\quad\left.-\frac{\partial^2\xi}{\partial\phi^2}(\phi')^3\right] + \left[\left(x\frac{\partial\zeta}{\partial x} - \zeta + \phi\frac{\partial\zeta}{\partial\phi} - 2\phi\frac{\partial\xi}{\partial x}\right)\right.\\
&\qquad\qquad\left.+\left(\xi + x\frac{\partial\xi}{\partial x} - 3\phi\frac{\partial\xi}{\partial\phi}\right)\phi' + 2x\frac{\partial\xi}{\partial\phi}(\phi')^2\right] = 0.
\end{aligned} \tag{3.11}$$

One can show that such a non-trivial solution exists if

$$K(u) = \frac{1}{au^2 + bu + c}\exp\left[\lambda\int\frac{1}{au^2 + bu + c}\,du\right], \tag{3.12}$$

where $\{a, b, c, \lambda\}$ are arbitrary constants.

In (3.12) we set $b = 0$, $a = 1$, $c = \pm 1$. Then for the bounded conductivity

$$K(u) = \frac{1}{u^2 + 1}\exp[\lambda \arctan u], \tag{3.13}$$

ODE (3.3) has a new symmetry group with infinitesimals

$$\xi(x, \phi) = \tfrac{1}{2}\lambda x - \phi, \qquad \zeta(x, \phi) = \tfrac{1}{2}\lambda\phi + x. \tag{3.14}$$

For the unbounded conductivity

$$K(u) = \frac{1}{u^2-1}\left(\frac{u-1}{u+1}\right)^{\frac{1}{2}\lambda}, \qquad (3.15)$$

ODE (3.3) has a new symmetry group with infinitesimals

$$\xi(x, \phi) = \tfrac{1}{2}\lambda x - \phi, \qquad \zeta(x, \phi) = \tfrac{1}{2}\lambda\phi - x. \qquad (3.16)$$

For the rest of this section we only consider the bounded conductivity (3.13).

If $K(u)$ is given by (3.13) then the corresponding ODE (3.10) is invariant under the group

$$R^* = R, \qquad \theta^* = \theta + \varepsilon, \qquad (3.17)$$

for the canonical coordinates

$$R = (x^2 + \phi^2)^{\frac{1}{2}} \exp[-\tfrac{1}{2}\lambda \arctan(\phi/x)], \qquad \theta = \arctan(\phi/x). \qquad (3.18)$$

(In terms of the canonical coordinates $(R, \theta): x = e^{\frac{1}{2}\lambda\theta}R\cos\theta$, $\phi = e^{\frac{1}{2}\lambda\theta}R\sin\theta$.) Consequently ODE (3.10), expressed in terms of the canonical coordinates (3.18), reduces to a first order ODE with respect to independent variable R and dependent variable $W = d\theta/dR$:

$$2e^{\lambda(\beta+\gamma)}\left[R\frac{dW}{dR} + 2W + \lambda RW^2 + \left(1 + \frac{\lambda^2}{4}\right)R^2W^3\right] + R^2W\left[1 + \lambda RW + \left(1 + \frac{\lambda^2}{4}\right)R^2W^2\right] = 0, \qquad (3.19)$$

where

$$\left.\begin{aligned} \beta &= \arctan\left(\frac{RW}{2}\left[1 + \left(1 + \frac{2\lambda}{RW}\right)^{\frac{1}{2}}\right]\right), \\ \gamma &= \arctan\left(\frac{RW}{2}\left[1 - \left(1 + \frac{2\lambda}{RW}\right)^{\frac{1}{2}}\right]\right). \end{aligned}\right\} \qquad (3.20)$$

If $\lambda = 0$ in (3.13), that is,

$$K(u) = \frac{1}{u^2+1}, \qquad (3.21)$$

the canonical coordinates corresponding to (3.14):

$$r = (x^2 + \phi^2)^{\frac{1}{2}}, \qquad \theta = \arctan(\phi/x), \qquad (3.22)$$

lead to expressing ODE (3.10) in the form

$$2\frac{d^2\theta}{dr^2} + \left(\frac{4}{r} + r\right)\frac{d\theta}{dr} + (2r + r^3)\left(\frac{d\theta}{dr}\right)^3 = 0. \qquad (3.23)$$

The substitution $P = d\theta/dr$ reduces ODE (3.23) to a Bernoulli equation with general solution

$$P(r, A) = \frac{A}{r(r^2 \exp\frac{1}{2}r^2 - A)^{\frac{1}{2}}}, \qquad (3.24)$$

where A is an arbitrary constant. Then the general solution of ODE (3.10) is

$$\phi = \phi(x, A, B) = (x^2 + \phi^2)^{\frac{1}{2}} \sin\left[\int^{(x^2+\phi^2)^{\frac{1}{2}}} P(\rho, A)\, \mathrm{d}\rho + B\right], \tag{3.25}$$

where A and B are arbitrary constants. The general solution of the corresponding ODE (3.3) is

$$u = u(x, A, B) = \tan\left[\int^{r} P(\rho, A)\, \mathrm{d}\rho + B + \arctan\,[rP(r, A)]\right], \tag{3.26}$$

where

$$r \cos\left[\int^{r} P(\rho, A)\, \mathrm{d}\rho + B\right] = x. \tag{3.27}$$

For a given problem the constants A and B are determined from boundary data.

Fujita (1954) obtained solution (3.26) of ODE (3.3) with $K(u)$ given by (3.21). He found this solution after making a number of ingenious substitutions and transformations.

Our reduction of ODE (3.3), for $K(u)$ given by (3.13) with $\lambda \neq 0$, appears to be a new result.

4. Features of the calculus to find a related ODE

Our method depends on finding a Bäcklund transformation (2.2) such that ODE (2.1) can be expressed in the form (2.3). We indicate a class of ODEs for which this is possible. Say that

$$F\left(x, u, \frac{\mathrm{d}u}{\mathrm{d}x}, \frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right) = 0$$

can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[A\left(x, u, \frac{\mathrm{d}u}{\mathrm{d}x}\right)\right] + b(x)\frac{\mathrm{d}u}{\mathrm{d}x} + c(x)u = 0, \qquad b'(x) \not\equiv c(x). \tag{4.1}$$

Then for ODE (4.1) the Bäcklund transformation

$$u = \frac{1}{b' - c}\frac{\mathrm{d}\phi}{\mathrm{d}x} \tag{4.2}$$

leads to the related second-order ODE

$$B\left(x, \frac{\mathrm{d}\phi}{\mathrm{d}x}, \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right) + \frac{b}{b' - c}\frac{\mathrm{d}\phi}{\mathrm{d}x} - \phi = 0. \tag{4.3}$$

In (4.2), (4.3) a prime denotes differentiation with respect to x;

$$B\left(x, \frac{\mathrm{d}\phi}{\mathrm{d}x}, \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right) = A\left(x, u, \frac{\mathrm{d}u}{\mathrm{d}x}\right). \tag{4.4}$$

The example of Section 3 is of the form (4.1) with

$$\left.\begin{aligned} A\left(x, u, \frac{du}{dx}\right) &= 2K(u)\frac{du}{dx}, \\ b(x) = x, \qquad c(x) &= 0, \end{aligned}\right\} \tag{4.5}$$

yielding the Bäcklund transformation $u = d\phi/dx$.

The van der Pol equation

$$\frac{d^2u}{dx^2} + (u^2 - 1)\frac{du}{dx} + u = 0 \tag{4.6}$$

is also of the form (4.1) with

$$\left.\begin{aligned} A\left(x, u, \frac{du}{dx}\right) &= -\frac{du}{dx} + u - \frac{1}{3}u^3, \\ b(x) = 0, \qquad c(x) &= -1. \end{aligned}\right\} \tag{4.7}$$

Here the Bäcklund transformation

$$u = d\phi/dx \tag{4.8}$$

leads to the related ODE

$$\frac{d^2\phi}{dx^2} - \frac{d\phi}{dx} + \frac{1}{3}\left(\frac{d\phi}{dx}\right)^3 + \phi = 0, \tag{4.9}$$

which is the Rayleigh equation. The transformation (4.8) is the well-known mapping of any solution of the Rayleigh equation to a solution of the van der Pol equation (Kevorkian & Cole, 1981).

5. Discussion

In this paper we have presented a theory for obtaining new symmetries for ordinary differential equations. Such new symmetries lead to a reduction in order of a given differential equation. We have shown how to prepare an ordinary differential equation in order to discover these new symmetries. For a prepared differential equation we have given an explicit algorithm to compute new symmetries.

Our work should prove fruitful in extending symmetry methods to wider classes of ordinary differential equations than those amenable to Lie's classical approach.

References

Anderson, R., & Ibragimov, N. 1979 *Lie–Bäcklund Transformations in Applications.* Studies in Applied Mathematics. Philadelphia: SIAM.

Baeri, P., Campisano, S., Foti, G., & Rimini, E. 1979 A melting model for pulsing-laser annealing of implanted semiconductors. *J. appl. Phys.* **50**, 788–797.

BERRYMAN, J., & HOLLAND, C. 1978 Nonlinear diffusion problem arising in plasma physics. *Phys. Rev. Lett.* **40**, 1720–1722.

BLUMAN, G., & COLE, J. 1974 *Similarity Methods for Differential Equations*. New York: Springer.

BLUMAN, G., & KUMEI, S. 1987 On invariance properties of the wave equation. *J. math. Phys.* **28**, 307–318.

BLUMAN, G., KUMEI, S., & REID, G. 1987 New classes of symmetries for partial differential equations. *J. math. Phys.* to appear.

COHEN, A. 1911 *An Introduction to the Lie Theory of One-Parameter Groups with Applications to the Solution of Differential Equations*. New York: D. C. Heath.

DICKSON, L. 1924 Differential equations from the group standpoint. *Ann. Math.* **25**, 287–378.

FUJITA, H. 1952a The exact pattern of a concentration-dependent diffusion in a semi-infinite medium, Part I. *Text. Res. J.* **22**, 757–760.

FUJITA, H. 1952b The exact pattern of a concentration-dependent diffusion in a semi-infinite medium, Part II. *Text. Res. J.* **22**, 823–827.

FUJITA, H. 1954 The exact pattern of a concentration-dependent diffusion in a semi-infinite medium, Part III. *Text. Res. J.* **24**, 234–240.

KEVORKIAN, J., & COLE, J. 1981 *Perturbation Methods in Applied Mathematics*. New York: Springer.

OLVER, P. 1986 *Applications of Lie Groups to Differential Equations*. New York: Springer.

SAYEGH, S., & JONES, G. 1986 Symmetries of differential equations. *J. Phys. A. Math. Gen.* **19**, 1793–1800.