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Direct construction method for conservation laws of partial differential equations Part I: Examples of conservation law classifications

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An effective algorithmic method is presented for finding the local conservation laws for partial differential equations with any number of independent and dependent variables. The method does not require the use or existence of a variational principle and reduces the calculation of conservation laws to solving a system of linear determining equations similar to that for finding symmetries. An explicit construction formula is derived which yields a conservation law for each solution of the determining system. In the first of two papers (Part I), examples of nonlinear wave equations are used to exhibit the method. Classification results for conservation laws of these equations are obtained. In a second paper (Part II), a general treatment of the method is given.

1 Introduction

In the study of differential equations, conservation laws have many significant uses, particularly with regard to integrability and linearization, constants of motion, analysis of solutions, and numerical solution methods. Consequently, an important problem is how to calculate all of the conservation laws for given differential equations.

For a differential equation with a variational principle, Noether's theorem [12, 4, 6, 5, 14] gives a formula for obtaining the local conservation laws by use of symmetries of the action. One usually attempts to find these symmetries by noting that any symmetry of the action leaves invariant the extremals of the action and hence gives rise to a symmetry of the differential equation. However, all symmetries of a differential equation do not necessarily arise from symmetries of the action when there is a variational principle. For example, if a differential equation is scaling invariant, then the action is often not invariant. Indeed it is often computationally awkward to determine the symmetries of the action and carry out the calculation with the formula to obtain a conservation law. Moreover, in general a differential equation need not have a variational principle even allowing for a change of variables. Therefore, it is more effective to seek a direct, algorithmic method without involving an action principle to find the conservation laws of a given differential equation.

In earlier work [2], we presented an algorithmic approach replacing Noether's theorem so as to allow one to obtain all local conservation laws for any differential equation whether or not it has a variational principle. Details of this approach for the situation of Ordinary Differential Equations (ODEs) are given in Anco & Bluman Ref. [3]. Here we concentrate on the situation of Partial Differential Equations (PDEs).

In the case of a PDE with a variational principle, the approach shows how to use the symmetries of the PDE to directly construct the conservation laws. The symmetries of a PDE satisfy a linear determining equation, for which there is a standard algorithmic method [5, 14] to seek all solutions. There is also an invariance condition, involving just the PDE and its symmetries, which is necessary and sufficient for a symmetry of a PDE with a variational principle to correspond to a symmetry of the action. The invariance condition can be checked by an algorithmic calculation and, in addition, leads to a direct construction formula for a conservation law in terms of the symmetry and the PDE. This approach makes no use of the variational principle for the PDE.

In the case of a PDE without a variational principle, the approach involves replacing symmetries by adjoint symmetries of the PDE. The adjoint symmetries satisfy a linear determining equation that is the adjoint of the determining equation for symmetries. Geometrically, symmetries of a PDE describe motions on the solution space of the PDE. Adjoint symmetries in general do not have such an interpretation. The invariance condition on symmetries is replaced by an adjoint invariance condition on adjoint symmetries and there is a corresponding direct construction formula for obtaining the conservation laws in terms of the adjoint symmetries and the PDE. The adjoint invariance condition is a necessary and sufficient determining condition for an adjoint symmetry to yield a conservation law.

In general for any PDE, with or without a variational principle, the approach of Anco & Bluman Ref. [2] for finding all local conservation laws gave the following step-by-step method:

(1) Find the adjoint symmetries of the given PDE.
(2) Check the adjoint invariance condition on the adjoint symmetries.
(3) For each adjoint symmetry satisfying the adjoint invariance condition use the direct construction formula to obtain a conservation law.

If the adjoint symmetry determining equation is the same as the symmetry determining equation then adjoint symmetries are symmetries. In this case the given PDE can be shown to have a variational principle. Conversely, if a variational principle exists, then the symmetry determining equation of the PDE can be shown to be self-adjoint, so that symmetries are adjoint symmetries.

To solve the determining equation for adjoint symmetries, one works on the solution space of the given PDE. On the other hand, in order to check the adjoint invariance condition, one must move off the solution space by replacing the dependent variable(s) of the given PDE by functions with arbitrary dependence on the independent variables of the given PDE. The same situation arises when checking for invariance of the action in Noether's theorem.

These steps of the method are an algorithmic version of the standard treatment presented in Olver Ref. [14] for finding PDE conservation laws in terms of multipliers. In particular, multipliers can be characterized as adjoint symmetries that satisfy the adjoint invariance condition and thus can be calculated by the step-by-step algorithm (1), (2), (3).

In this paper (Part I) and a sequel (Part II), we significantly improve the effectiveness of this method by replacing the adjoint invariance condition by extra determining equations which allow one to work entirely on the solution space of the given PDE. Consequently, by augmenting the adjoint symmetry determining equation by these extra determining equations we obtain a linear determining system for finding only those adjoint symmetries that are multipliers yielding conservation laws. At first sight it is natural to proceed as in Anco & Bluman Ref. [2], by solving the adjoint symmetry determining equation and then checking which of the solutions satisfy the extra determining equations. On the other hand, all of the determining equations are on an equal footing, and hence there is no requirement to solve the adjoint symmetry determining equation first. Indeed, as illustrated later in the examples in this paper, it is much more effective to start with the extra determining equations before considering the adjoint symmetry determining equation. This is true even in the case when the given PDE has a variational principle.

In solving the determining system one works completely on the solution space of the given PDE. Hence one can use the same algorithmic procedures as for solving symmetry determining equations in order to solve the conservation law determining system. In particular, existing symbolic manipulation programs [8] that calculate symmetries can be readily adapted to calculate solutions of the conservation law determining system. Moreover, for each solution one can directly obtain the resulting conservation law by evaluating the construction formula working entirely on the solution space of the given PDE.

The conservation law determining system together with the conservation law construction formula give a general, direct, computational method for finding the local conservation laws of given PDEs. We refer to this as the direct conservation law method. As emphasized above, its effectiveness stems from allowing the calculation of conservation laws to be carried out algorithmically up to any given order by solving a linear determining system without moving off the solution space of the PDEs. Compared to the standard treatment of PDE conservation laws, the determining system solves a long-standing question of how one can delineate necessary and sufficient determining equations to find multipliers by working entirely on the solution space of the given PDE. Most important, by mingling the adjoint symmetry equations with the extra equations in the determining system, one can gain a significant computational advantage over the standard methods for finding multipliers.

In the next section we illustrate the direct conservation law method through classifying conservation laws for three PDE examples: a generalized Korteweg-de Vries equation, a nonlinear wave-speed equation, and a class of nonlinear Klein-Gordon equations. The classification results obtained are new in that they establish the completeness of certain families of conservation laws which are of interest for these PDEs. These examples show how to calculate all conservation laws up to a given order and also how to determine which PDEs in a specified class admit conservation laws of a given type.

In the second paper (Part II), we present a general derivation of the conservation law determining system and construction formula, and we also give a summary of the general method.

2 Examples of conservation law classifications

Here we illustrate the use of our direct conservation law method on three PDE examples. For each example we derive the conservation law determining system and use it to obtain a classification result for conservation laws.

The first example is a generalized Korteweg-de Vries equation in physical form, for which there is no direct variational principle. The ordinary Korteweg-de Vries (KdV) equation is well-known to have local conservation laws of every even order [11], which can be understood to arise from a recursion operator [13]. Using our conservation law determining system, we derive a direct, complete classification for all conservation laws up to second order for the generalized KdV equation. This example illustrates a general approach for finding and classifying conservation laws for non-variational evolution equations.

The second example is a scalar wave equation with non-constant wave speed depending on the wave amplitude. This wave equation has a variational principle and admits local conservation laws for energy and momentum arising by Noether's theorem from time- and space-invariance of its corresponding action. By applying our conservation law determining system, we classify all wave speeds for which there are extra conservation laws of first-order and obtain the resulting conserved quantities. This example illustrates a general approach for classifying nonlinear evolution PDEs that admit extra conservation laws.

The third example is a general class of nonlinear Klein-Gordon equations. The class includes the sine-Gordon equation, Liouville equation, and Tzetzeica equation, which are known to be integrable equations [1] with local conservation laws up to arbitrarily high orders, starting at first order. Through our conservation law method we give a classification of nonlinear Klein-Gordon equations admitting at least one second-order conservation law. As a by-product we obtain an integrability characterization for some of the equations in this class. This example shows a general approach to classifying integrable PDEs by means of conservation laws.

2.1 Generalized Korteweg-de Vries equations

Consider the generalized Korteweg-de Vries equation

$$G = u_t + u^n u_x + u_{xxx} = 0 \tag{2.1}$$

with parameter $n > 0$. This is a first order evolution PDE which has no variational principle directly in terms of u and which reduces for $n = 1, 2$ to the ordinary KdV equation and modified KdV equation, respectively. Its symmetries with infinitesimal generator $\mathrm{X}u = \eta$ [5, 14] satisfy the determining equation

$$0 = D_t\eta + u^n D_x\eta + nu^{n-1}u_x\eta + D_x^3\eta \quad \text{when } G = 0 \tag{2.2}$$

where $D_t = \partial_t + u_t\partial_u + u_{tx}\partial_{u_x} + u_{tt}\partial_{u_t} + \cdots$ and $D_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{tx}\partial_{u_t} + \cdots$ are total derivative operators with respect to t and x. The adjoint of Eq. (2.2) is given by

$$0 = -D_t\omega - u^n D_x\omega - D_x^3\omega \quad \text{when } G = 0 \tag{2.3}$$

which is the determining equation for the adjoint symmetries ω of the generalized KdV equation. (Note, since these determining equations are not self-adjoint, the only solution common to Eqs. (2.2) and (2.3) is $\omega = \eta = 0$.)

The generalized KdV equation (2.1) itself has the form of a local conservation law

$$D_t(u) + D_x\left(\frac{1}{n+1}u^{n+1} + u_{xx}\right) = 0. \tag{2.4}$$

We now consider, more generally, local conservation laws

$$D_t\Phi^t + D_x\Phi^x = 0 \tag{2.5}$$

on all solutions $u(t,x)$ of Eq. (2.1). Clearly, we are free without loss of generality to eliminate any dependence on u_t (and differential consequences) in the conserved densities Φ^t, Φ^x. All nontrivial conserved densities in this form can be constructed from multipliers Λ on the generalized KdV equation, analogous to integrating factors, where Λ depends only on t, x, u, and x derivatives of u. In particular, by moving off the generalized KdV solution space, we have

$$D_t\Phi^t + D_x\Phi^x = (u_t + u^n u_x + u_{xxx})\Lambda_0 + D_x(u_t + u^n u_x + u_{xxx})\Lambda_1 + \cdots \tag{2.6}$$

for some expressions $\Lambda_0, \Lambda_1, \ldots$ with no dependence on u_t and differential consequences. This yields (after integration by parts) the multiplier

$$D_t\Phi^t + D_x(\Phi^x - \Gamma) = (u_t + u^n u_x + u_{xxx})\Lambda, \quad \Lambda = \Lambda_0 - D_x\Lambda_1 + \cdots \tag{2.7}$$

where $\Gamma = 0$ when u is restricted to be a generalized KdV solution. We now derive an augmented adjoint symmetry determining system which completely characterizes all multipliers Λ.

The definition for multipliers $\Lambda(t, x, u, \partial_x u, \ldots, \partial_x^p u)$ is that $(u_t + u^n u_x + u_{xxx})\Lambda$ must be a divergence expression for all functions $u(t,x)$ (not just generalized KdV solutions). This determining condition is expressed by

$$\begin{aligned} 0 = {} & E_u((u_t + u^n u_x + u_{xxx})\Lambda) \\ = {} & -D_t\Lambda - u^n D_x\Lambda - D_x^3\Lambda + (u_t + u^n u_x + u_{xxx})\Lambda_u - D_x((u_t + u^n u_x + u_{xxx})\Lambda_{\partial_x u}) \\ & + \cdots + (-1)^p D_x^p((u_t + u^n u_x + u_{xxx})\Lambda_{\partial_x^p u}) \end{aligned} \tag{2.8}$$

where $E_u = \partial_u - D_t\partial_{u_t} - D_x\partial_{u_x} + D_tD_x\partial_{u_{tx}} + D_x^2\partial_{u_{xx}} + \cdots$ is the standard Euler operator which annihilates divergence expressions. Equation (2.8) is linear in $u_t, u_{tx}, u_{txx}, \ldots$, and thus the coefficients of u_t and x derivatives of u_t up to order p give rise to a split system of determining equations for Λ. The system is found to consist of the adjoint symmetry determining equation on Λ,

$$0 = -\mathscr{D}_t\Lambda - u^n D_x\Lambda - D_x^3\Lambda \tag{2.9}$$

and extra determining equations on Λ,

$$0 = \sum_{k=1}^{p}(-D_x)^k\Lambda_{\partial_x^k u} \tag{2.10 a}$$

$$0 = (1 - (-1)^q)\Lambda_{\partial_x^q u} + \sum_{k=q+1}^{p}\frac{k!}{q!(k-q)!}(-D_x)^{k-q}\Lambda_{\partial_x^k u}, \quad q = 1, \ldots, p-1 \tag{2.10 b}$$

$$0 = (1 - (-1)^p)\Lambda_{\partial_x^p u}. \tag{2.10 c}$$

Here $\mathscr{D}_t = \partial_t - (uu_x + u_{xxx})\partial_u - (uu_x + u_{xxx})_x\partial_{u_x} + \cdots$ is the total derivative operator which expresses t derivatives of u through the generalized KdV equation (2.1). Consequently, one is able to work on the space of generalized KdV solutions $u(t,x)$ to solve the determining system (2.9) and (2.10*a–c*) to find $\Lambda(t,x,u,\partial_x u,\ldots,\partial_x^p u)$. The determining system solutions are the multipliers that yield all nontrivial generalized KdV conservation laws.

The explicit relation between multipliers Λ and conserved densities Φ^t, Φ^x for generalized KdV conservation laws is summarized as follows. Given a conserved density Φ^t, we find that a direct calculation of the u_t terms in $D_t\Phi^t + D_x\Phi^x$ yields the multiplier equation

$$D_t\Phi^t + D_x\Phi^x = (u_t + u^n u_x + u_{xxx})\hat{E}_u(\Phi^t) + D_x\Gamma \tag{2.11}$$

where $\hat{E}_u = \partial_u - D_x\partial_{u_x} + D_x^2\partial_{u_{xx}} + \cdots$ is a restricted Euler operator and Γ is proportional to $u_t + u^n u_x + u_{xxx}$ and differential consequences. Thus we obtain the multiplier

$$\Lambda = \hat{E}_u(\Phi^t). \tag{2.12}$$

Conversely, given a multiplier Λ, we can invert the relation (2.12) by a standard method [14] using Eq. (2.11) to obtain the conserved density

$$\Phi^t = \int_0^1 d\lambda\ u\Lambda(t,x,\lambda u,\lambda\partial_x u,\lambda\partial_x^2 u,\ldots). \tag{2.13}$$

From Eqs. (2.12) and (2.13) it is natural to define the order of a generalized KdV conservation law as the order of the highest x derivative of u in its multiplier. Thus, we see that all nontrivial generalized KdV conservation laws up to order p are determined by multipliers of order p which are obtained as solutions of the augmented system of adjoint symmetry determining equations (2.9) and (2.10*a–c*).

Through the determining system (2.9) and (2.10*a–c*) we now derive a *complete* classification of all conservation laws (2.5) up to second order, corresponding to multipliers of the form

$$\Lambda(t,x,u,u_x,u_{xx}). \tag{2.14}$$

The classification results are summarized by the following theorem.

Theorem 2.1.1 *The generalized KdV equation* (2.1) *for all* $n > 0$ *admits the multipliers*

$$\Lambda = 1,\ \Lambda = u,\ \Lambda = u_{xx} + \frac{1}{n+1}u^{n+1}. \tag{2.15}$$

The only additional admitted multipliers of the form (2.14) *are given by*

$$\Lambda = tu - x,\ \text{if } n = 1, \tag{2.16}$$

$$\Lambda = t\left(u_{xx} + \frac{1}{3}u^3\right) - \frac{1}{3}xu,\ \text{if } n = 2. \tag{2.17}$$

This classifies all nontrivial conservation laws up to second order for the generalized KdV equation for any $n > 0$.

The conserved densities for these conservation laws are easily obtained using the construction formula (2.13) as follows. For the multipliers (2.15) we find, respectively,

$$\Phi^t = u, \tag{2.18}$$

$$\Phi^t = \frac{1}{2}u^2, \tag{2.19}$$

$$\Phi^t = -\frac{1}{2}{u_x}^2 + \frac{1}{(n+1)(n+2)}u^{n+2} + D_x\theta, \tag{2.20}$$

where $D_x\theta$ is a trivial conserved density. In physical terms, if we regard u as a wave amplitude as in the ordinary ($n = 1$) KdV equation, then these conserved densities represent mass, momentum and energy [15].

For the additional multipliers (2.16) and (2.17) we find

$$\Phi^t = \frac{1}{2}tu^2 - xu, \text{ if } n = 1, \tag{2.21}$$

$$\Phi^t = -\frac{1}{2}t{u_x}^2 + \frac{1}{12}tu^4 - \frac{1}{6}xu^2 + D_x\theta, \text{ if } n = 2. \tag{2.22}$$

Proof of Theorem 2.1.1 For multipliers of the form (2.14) the adjoint symmetry equation (2.9) becomes

$$\begin{aligned} 0 = \ & -D_x^3\Lambda - u^n D_x\Lambda - \Lambda_t + \Lambda_u(u^n u_x + u_{xxx}) + \Lambda_{u_x}(u^n u_{xx} + nu^{n-1}{u_x}^2 + u_{xxxx}) \\ & + \Lambda_{u_{xx}}(u^n u_{xxx} + 3nu^{n-1}u_x u_{xx} + n(n-1)u^{n-2}{u_x}^3 + u_{xxxxx}) \end{aligned} \tag{2.23}$$

and the extra equations (2.10*a*–*c*) reduce to

$$0 = -D_x\Lambda_{u_x} + D_x^2\Lambda_{u_{xx}}, \tag{2.24}$$

$$0 = \Lambda_{u_x} - D_x\Lambda_{u_{xx}}. \tag{2.25}$$

Note that Eq. (2.24) is a differential consequence of Eq. (2.25).

We start from equation (2.25). Its term with highest order derivatives is $u_{xxx}\Lambda_{u_{xx}u_{xx}}$, and hence $\Lambda_{u_{xx}u_{xx}} = 0$. This yields that Λ is linear in u_{xx},

$$\Lambda = a(t,x,u,u_x)u_{xx} + b(t,x,u,u_x). \tag{2.26}$$

Then the remaining terms in Eq. (2.25), after some cancellations, are of first order

$$0 = b_{u_x} - a_u u_x - a_x. \tag{2.27}$$

We now turn to the adjoint symmetry equation (2.23) and separate it into highest derivative terms in descending order. The highest order terms cancel. The second highest order terms involve u_{xxxx}, and these yield

$$0 = D_x a = a_x + a_u u_x + a_{u_x}u_{xx}. \tag{2.28}$$

Clearly, Eq. (2.28) separates into

$$a_{u_x} = a_u = a_x = 0. \tag{2.29}$$

Hence $b_{u_x} = 0$ from Eq. (2.27).

The next highest order terms remaining in the adjoint symmetry equation (2.23) involve u_{xx}. These terms yield

$$0 = -a_t - 3b_{xu} + 3(nu^{n-1}a - b_{uu})u_x \tag{2.30}$$

which separates into $b_{uu} = nu^{n-1}a$ and $a_t = -3b_{xu}$. Hence we have

$$b = \frac{1}{n+1}au^{n+1} + \left(b_1(t) - \frac{1}{3}xa_t\right)u + b_2(t,x). \tag{2.31}$$

This simplifies the remaining terms in Eq. (2.23),

$$0 = -b_{2xxx} - b_{2t} - \left(b_{1t} - \frac{1}{3}xa_{tt}\right)u + b_{2x}u^n + \left(\frac{1}{3} - \frac{1}{n+1}\right)a_t u^{n+1}. \tag{2.32}$$

Clearly, the terms here can be separated, yielding

$$(n-2)a_t = 0, \tag{2.33}$$

$$b_{2t} = -b_{2xxx}, \tag{2.34}$$

$$b_{1t} - \frac{1}{3}xa_{tt} = -b_{2x}u^{n-1}. \tag{2.35}$$

From Eq. (2.35) there are two cases to consider.

Case (i): $n = 1$

Here Eq. (2.33) yields

$$a_t = 0 \tag{2.36}$$

and hence Eq. (2.35) becomes $b_{1t} = -b_{2x}$. Since b_1 depends only on t, using Eq. (2.34) we find $b_{2xx} = b_{2t} = 0$, and thus

$$b_1 = b_3 t + b_4,\ b_2 = -b_3 x + b_5,\ b_3 = \text{const},\ b_4 = \text{const},\ b_5 = \text{const}. \tag{2.37}$$

Finally, from Eqs. (2.29) and (2.36) it follows that

$$a = \text{const}. \tag{2.38}$$

Therefore, through Eq. (2.26), Eq. (2.31), Eqs. (2.37) and (2.38), we have

$$\Lambda = \frac{1}{2}au^2 + au_{xx} + b_3 - b_5 x + (b_4 + b_5 t)u, \text{ if } n = 1, \tag{2.39}$$

which is a linear combination of the multipliers (2.15) and (2.16) shown in Theorem 2.1.1.

Case (ii): $n > 1$

In this case Eq. (2.35) clearly splits into

$$b_{1t} = a_{tt} = 0, \tag{2.40}$$

$$b_{2x} = 0. \tag{2.41}$$

Hence from Eq. (2.41) and Eq. (2.34) we have

$$b_2 = \text{const}, \tag{2.42}$$

and from Eq. (2.40) and Eq. (2.29) we have

$$a = a_1 t + a_2, a_1 = \text{const}, a_2 = \text{const}, \tag{2.43}$$

$$b_1 = \text{const}. \tag{2.44}$$

Therefore, Eq. (2.31) becomes

$$b = \frac{1}{n+1}(a_1 t + a_2)u^{n+1} + \left(b_1 - \frac{1}{3}xa_1\right)u + b_2. \tag{2.45}$$

Finally, consider Eq. (2.33). If $n \neq 2$ then we have $a_t = 0$ and hence $a_1 = 0$. Otherwise, if $n = 2$ then we have no restriction on a_1. Consequently, through Eq. (2.26), Eqs. (2.43) and (2.45), we find

$$\Lambda = (a_1 t + a_2)u_{xx} + \frac{1}{n+1}u^{n+1}(a_1 t + a_2) + \left(b_1 - \frac{1}{3}xa_1\right)u + b_2, \text{ if } n > 1, \tag{2.46}$$

where $a_1 = 0$ for all $n > 2$. This yields a linear combination of the multipliers (2.15) and (2.17) shown in Theorem 2.1.1. □

2.2 Nonlinear wave-speed equation

Consider the wave equation

$$G = u_{tt} - c(u)(c(u)u_x)_x = 0 \tag{2.47}$$

with a non-constant wave speed $c(u)$. This is a scalar second-order evolution PDE, which has a variational principle given by the physically-motivated action

$$S = \int \frac{1}{2}(-{u_t}^2 + c(u)^2{u_x}^2)dtdx. \tag{2.48}$$

Symmetries of the wave equation (2.47) with infinitesimal generator $\mathrm{X}u = \eta$ [5, 14] satisfy the determining equation

$$0 = D_t^2\eta - c(u)^2 D_x^2\eta - 2c(u)c'(u)u_x D_x\eta - 2c(u)c'(u)u_{xx}\eta - (c(u)c''(u) + c'(u)^2){u_x}^2\eta \quad \text{when } G = 0. \tag{2.49}$$

Since Eq. (2.47) is variational, the determining equation (2.49) is self-adjoint and hence the adjoint symmetries of the wave equation are the solutions of Eq. (2.49).

For any wave speed $c(u)$, the wave equation (2.47) clearly admits time and space translation symmetries

$$\eta = u_t, \eta = u_x, \tag{2.50}$$

which are symmetries of the action. In particular, the action is invariant up to a boundary-term

$$\mathrm{X}S = \int \frac{1}{2}D_t(-{u_t}^2 + c(u)^2{u_x}^2)dtdx \tag{2.51}$$

under time-translation $\mathrm{X}u = u_t$, and

$$\mathrm{X}S = \int \frac{1}{2}D_x(-{u_t}^2 + c(u)^2{u_x}^2)dtdx \tag{2.52}$$

under space-translation $\mathrm{X}u = u_x$. By Noether's theorem, combining the invariance of the action with the general variational identity

$$\mathrm{X}S = \int \Big((u_{tt} - c^2 u_{xx} - cc'{u_x}^2)\eta + D_t(-u_t\eta) + D_x(c^2 u_x \eta)\Big)dtdx, \quad \mathrm{X}u = \eta, \tag{2.53}$$

we obtain corresponding local conservation laws $D_t\Phi^t + D_x\Phi^x = 0$ on all solutions $u(t, x)$

of the wave equation (2.47). The conserved densities Φ^t, Φ^x are given by

$$\Phi^t = \frac{1}{2}{u_t}^2 + \frac{1}{2}c(u)^2{u_x}^2, \Phi^x = -c(u)^2 u_x u_t \tag{2.54}$$

for the time-translation, and

$$\Phi^t = u_x u_t, \Phi^x = -\frac{1}{2}{u_t}^2 - \frac{1}{2}c(u)^2{u_x}^2 \tag{2.55}$$

for the space-translation. These yield conservation of energy and momentum, respectively.

We now consider classifying wave speeds $c(u)$ that lead to additional local conservation laws of first-order for the wave equation (2.47). (Through Noether's theorem all first-order conservation laws correspond to invariance of the action S under contact symmetries.)

For any local conservation laws

$$D_t\Phi^t + D_x\Phi^x = 0 \tag{2.56}$$

on all solutions $u(t,x)$ of the wave equation (2.47), we are clearly free without loss of generality to eliminate any dependence on u_{tt} (and differential consequences) in the conserved densities Φ^t, Φ^x. All nontrivial conserved densities in this form can be constructed from multipliers Λ on the wave equation, analogous to integrating factors, where Λ has no dependence on u_{tt} and its differential consequences. In particular, by moving off the wave equation solution space, we have

$$D_t\Phi^t + D_x\Phi^x = (u_{tt} - c(u)(c(u)u_x)_x)\Lambda_0 + D_x(u_{tt} - c(u)(c(u)u_x)_x)\Lambda_1 + \cdots \tag{2.57}$$

for some expressions $\Lambda_0, \Lambda_1, \ldots$ with no dependence on u_{tt} and differential consequences. This yields (after integration by parts) the multiplier

$$D_t\Phi^t + D_x(\Phi^x - \Gamma) = (u_{tt} - c(u)(c(u)u_x)_x)\Lambda, \quad \Lambda = \Lambda_0 - D_x\Lambda_1 + \cdots \tag{2.58}$$

where $\Gamma = 0$ when u is restricted to be a wave equation solution. We now derive an augmented symmetry determining system which completely characterizes all multipliers Λ.

Multipliers Λ are defined by the condition that $(u_{tt} - c(u)(c(u)u_x)_x)\Lambda$ is a divergence expression for all functions $u(t,x)$ (not just wave equation solutions). We restrict attention to Λ of first-order, depending on t, x, u, u_t, u_x. This leads to the necessary and sufficient determining condition

$$\begin{aligned} 0 = \ & E_u((u_{tt} - c(u)(c(u)u_x)_x)\Lambda) \\ = \ & D_t^2\Lambda - c^2D_x^2\Lambda - 2cc'u_xD_x\Lambda - 2cc'u_{xx}\Lambda - (cc'' + c'^2){u_x}^2\Lambda \\ & +(u_{tt} - c^2u_{xx} - cc'{u_x}^2)\Lambda_u - D_t((u_{tt} - c^2u_{xx} - cc'{u_x}^2)\Lambda_{u_t}) \\ & -D_x((u_{tt} - c^2u_{xx} - cc'{u_x}^2)\Lambda_{u_x}) \end{aligned} \tag{2.59}$$

where $E_u = \partial_u - D_t\partial_{u_t} - D_x\partial_{u_x} + D_tD_x\partial_{u_{tx}} + D_x^2\partial_{u_{xx}} + \cdots$ is the standard Euler operator which annihilates divergence expressions. Equation (2.59) is quadratic in u_{tt} and linear in u_{ttt} and u_{ttx}, and thus splits into separate equations. We organize the splitting by considering terms in G^2, G, D_xG, D_tG, and remaining terms with no dependence on u_{tt} and differential consequences. The coefficients of D_xG, D_tG, and G^2 are found to vanish as a result of $\Lambda(t,x,u,u_t,u_x)$ being first-order. The other coefficients in the splitting do not vanish.

This leads to a split system of two determining equations for $\Lambda(t,x,u,u_t,u_x)$, consisting of

$$0=\mathscr{D}_t^2\Lambda-c(u)^2D_x^2\Lambda-2c(u)c'(u)u_xD_x\Lambda-2c(u)c'(u)u_{xx}\Lambda-(c(u)c''(u)+c'(u)^2){u_x}^2\Lambda \quad (2.60)$$

which is the symmetry determining equation (2.49) on Λ, and

$$0=2\Lambda_u+\mathscr{D}_t\Lambda_{u_t}-D_x\Lambda_{u_x} \quad (2.61)$$

which is an extra determining equation on Λ. Here $\mathscr{D}_t=\partial_t+u_t\partial_u+u_{tx}\partial_{u_x}+(c(u)^2u_{xx}+c(u)c'(u){u_x}^2)\partial_{u_t}+\cdots$ is the total derivative operator which expresses t derivatives through the wave equation (2.47). Consequently, one is able to work on the space of wave equation solutions $u(t,x)$ in order to solve the determining system (2.60) and (2.61) to find $\Lambda(t,x,u,u_t,u_x)$. The determining system solutions are the multipliers that yield all nontrivial first-order conservation laws of the wave equation (2.47).

The first determining equation (2.60) shows that all multipliers are symmetries of the wave equation, while the second determining equation (2.61) provides the necessary and sufficient condition for a symmetry to leave the action (2.48) invariant up to a boundary term. This is a consequence of the one-to-one correspondence between symmetries of the action and multipliers for nontrivial conservation laws of the wave equation (2.47) as shown by Noether's theorem [14].

The explicit relation between multipliers Λ and conserved densities Φ^t,Φ^x for first-order conservation laws (2.56) is summarized as follows. Given a conserved density Φ^t, we find that a direct calculation of the u_{tt} terms in $D_t\Phi^t+D_x\Phi^x$ yields the multiplier equation

$$D_t\Phi^t+D_x\Phi^x=(u_{tt}-c(u)(c(u)u_x)_x)\hat{E}_{u_t}(\Phi^t)+D_x\Gamma \quad (2.62)$$

where $\hat{E}_{u_t}=\partial_{u_t}$ is the truncation of a restricted Euler operator, and Γ is proportional to $u_{tt}-c(u)(c(u)u_x)_x$ and differential consequences. Thus we obtain the multiplier

$$\Lambda=\hat{E}_{u_t}(\Phi^t). \quad (2.63)$$

Conversely, given a multiplier Λ, we can invert the relation (2.63) using Eq. (2.62) to obtain the conserved density

$$\Phi^t=\int_0^1 d\lambda\Big((u_t-\tilde{u}_t)\Lambda[\lambda u+(1-\lambda)\tilde{u}]+(\tilde{u}-u)\mathscr{D}_t\Lambda[\lambda u+(1-\lambda)\tilde{u}]\Big)+t\int_0^1 d\lambda K(\lambda t,\lambda x) \quad (2.64)$$

where

$$\Lambda[u]=\Lambda(t,x,u,u_t,u_x), \quad (2.65)$$

$$\mathscr{D}_t\Lambda[u]=\Lambda_t(t,x,u,u_t,u_x)+u_t\Lambda_u(t,x,u,u_t,u_x)+u_{tx}\Lambda_{u_x}(t,x,u,u_t,u_x)+c(u)(c(u)u_x)_x\Lambda_{u_t}(t,x,u,u_t,u_x), \quad (2.66)$$

$$K(t,x)=(\tilde{u}_{tt}-c(\tilde{u})(c(\tilde{u})\tilde{u}_x)_x)\Lambda[\tilde{u}], \quad (2.67)$$

with $\tilde{u}=\tilde{u}(t,x)$ being any function chosen such that the expressions $\Lambda[\tilde{u}]$ and $\tilde{u}_{tt}-c(\tilde{u})(c(\tilde{u})\tilde{u}_x)_x$ are non-singular.

Thus, we see that all nontrivial first-order conservation laws of the wave equation (2.47) are determined by multipliers of first-order which are obtained as solutions of the augmented system of symmetry determining equations (2.60) and (2.61).

We now use the determining system (2.60) and (2.61) to *completely* classify all first-order conservation laws in terms of corresponding multipliers

$$\Lambda(t, x, u, u_t, u_x). \tag{2.68}$$

The classification results are summarized by the following two theorems.

Theorem 2.2.1 *For arbitrary wave speeds* $c(u)$*, the only multipliers of form* (2.68) *admitted by the wave equation* (2.47) *are*

$$\Lambda = u_t, \Lambda = u_x, \Lambda = tu_t + xu_x. \tag{2.69}$$

These multipliers define symmetries given by time-translation, space-translation, and time-space dilation, respectively, which lead to conservation laws for energy (2.54), momentum (2.55), and a dilational quantity.

Theorem 2.2.2 *The wave equation* (2.47) *for non-constant wave speed* $c(u)$ *admits additional multipliers of form* (2.68) *iff* $c(u) = c_0(u - u_0)^{-2}$ *in terms of constants* c_0, u_0*. For these wave speeds the additional admitted multipliers are given by*

$$\Lambda = t^2 u_t - t(u - u_0), \tag{2.70}$$

$$\Lambda = x^2 u_x + x(u - u_0), \tag{2.71}$$

$$\Lambda = tu_t - xu_x - (u - u_0). \tag{2.72}$$

The symmetries defined by these additional multipliers correspond to two conformal (Möbius) transformations and one scaling transformation on independent variables t, x, accompanied by a scaling and shift of u. The conserved densities Φ^t for the three corresponding conservation laws are obtained by the construction formula (2.64). We choose $\tilde{u} = \text{const} \neq u_0$ to avoid the singularity at $u = u_0$. This leads to $K = 0$. Using the linear dependence of $\Lambda(t, x, u, u_x, u_t)$ together with the property $\Lambda_{xu_t} = 0$ given by Eqs. (2.70) to (2.72), we find that after integration by parts the formula reduces to

$$\begin{aligned}\Phi^t &= \frac{1}{2}u_t\Big(\Lambda[u] + \Lambda[\tilde{u}] + ((u - \tilde{u})\Lambda_{u_x}[\tilde{u}])_x\Big) + \frac{1}{2}(\tilde{u} - u)\Big(\Lambda_t[u] + \Lambda_t[\tilde{u}] + u_t\Lambda_u[\tilde{u}]\Big)\\ &\quad + \frac{1}{2}u_x{}^2 c(u)^2 \Lambda_{u_t}[\tilde{u}]\end{aligned} \tag{2.73}$$

up to a trivial conserved density $D_x\theta$. Since the terms in Eq. (2.73) are non-singular for any constant $\tilde{u}$, we can now set $\tilde{u} = u_0$ without loss of generality. Then, substituting Eqs. (2.70) to (2.72) for Λ, we obtain, respectively,

$$\Phi^t = \frac{1}{2}t^2 u_t{}^2 - t(u - u_0)u_t + \frac{1}{2}(u - u_0)^2 + \frac{1}{2}t^2 u_x{}^2 c_0{}^2 (u - u_0)^{-4}, \tag{2.74}$$

$$\Phi^t = x^2 u_x u_t + x(u - u_0)u_t, \tag{2.75}$$

$$\Phi^t = \frac{1}{2}tu_t{}^2 - (xu_x + u - u_0)u_t + \frac{1}{2}tu_x{}^2 c_0{}^2 (u - u_0)^{-4}, \tag{2.76}$$

which represent two conformal quantities and a scaling quantity.

Proof of Theorems 2.2.1 and 2.2.2 We start from the determining equation (2.61) and expand it in explicit form using

$$\mathscr{D}_t\Lambda = \Lambda_t + \Lambda_u u_t + \Lambda_{u_x} u_{tx} + \Lambda_{u_t}(c^2 u_{xx} + cc' {u_x}^2), \tag{2.77}$$

$$D_x\Lambda = \Lambda_x + \Lambda_u u_x + \Lambda_{u_x} u_{xx} + \Lambda_{u_t} u_{tx}. \tag{2.78}$$

This yields

$$0 = 2\Lambda_u + \Lambda_{tu_t} + \Lambda_{uu_t} u_t + cc'{u_x}^2 \Lambda_{u_t u_t} - \Lambda_{xu_x} - \Lambda_{uu_x} u_x + (c^2 \Lambda_{u_t u_t} - \Lambda_{u_x u_x}) u_{xx}. \tag{2.79}$$

Since Λ does not depend on u_{xx}, Eq. (2.79) separates into

$$0 = c^2 \Lambda_{u_t u_t} - \Lambda_{u_x u_x}, \tag{2.80}$$

$$0 = 2\Lambda_u + \Lambda_{tu_t} + \Lambda_{uu_t} u_t - \Lambda_{xu_x} - \Lambda_{uu_x} u_x. \tag{2.81}$$

Next we bring in the symmetry determining equation (2.60). Expanding it similarly, we find after use of Eq. (2.80) that its highest derivative terms involve u_{xx} and u_{tx}. Hence, since Λ does not depend on second-derivatives of u, these terms can be separated, leading to

$$0 = (\Lambda_{tu_t} + \Lambda_{uu_t} u_t - \Lambda_{xu_x} - \Lambda_{uu_x} u_x)c + (\Lambda_{u_t} u_t + \Lambda_{u_x} u_x - \Lambda)c' + \Lambda_{u_t u_t} {u_x}^2 c^2 c', \tag{2.82}$$

$$0 = \Lambda_{tu_x} + \Lambda_{uu_x} u_t - (\Lambda_{xu_t} + \Lambda_{uu_t} u_x)c^2 + \Lambda_{u_x u_t} {u_x}^2 cc'. \tag{2.83}$$

To proceed, we combine Eqs. (2.81) and (2.82) to yield

$$0 = (\Lambda_{u_t} u_t + \Lambda_{u_x} u_x - \Lambda)c' - 2\Lambda_u c \tag{2.84}$$

which is a first order linear PDE for Λ in u, u_t, u_x. The solution of Eq. (2.84) is given by

$$\Lambda = c^{-1/2} f(t, x, \alpha, \beta), \quad \alpha = c^{1/2} u_t, \beta = c^{1/2} u_x. \tag{2.85}$$

Substituting Eq. (2.85) into Eq. (2.80), we obtain $c^2 f_{\alpha\alpha} = f_{\beta\beta}$ which separates into

$$f_{\alpha\alpha} = f_{\beta\beta} = 0 \tag{2.86}$$

since $c(u) \neq$ const. Hence we have

$$f = a(t,x)\alpha + b(t,x)\beta + g(t,x)\alpha\beta + h(t,x). \tag{2.87}$$

Next, we substitute Eqs. (2.87) and (2.85) into Eq. (2.83) to obtain

$$0 = b_t - c^2 a_x + \frac{1}{2} c^{-1/2} c' ({u_t}^2 + c^2 {u_x}^2) g. \tag{2.88}$$

This immediately yields

$$g = 0, b_t = a_x = 0. \tag{2.89}$$

Thus, so far we have

$$\Lambda = a(t)u_t + b(x)u_x + h(t,x)c(u)^{-1/2}. \tag{2.90}$$

Now, using Eq. (2.90) we see that Eqs. (2.81) and (2.82) both reduce to

$$0 = -c^{-3/2} c' h + a_t - b_x. \tag{2.91}$$

This leads to two cases to consider. If the wave speed $c(u)$ is arbitrary, then we must have the following Case (i):

$$a_t = b_x, h = 0. \tag{2.92}$$

Alternatively, the only other possibility is for the wave speed $c(u)$ to satisfy a first-order ODE, so we then have Case (ii):

$$c^{-3/2}c' = \tilde{c} = \text{const}, \tilde{c}h = a_t - b_x. \tag{2.93}$$

Note in this case we require $\tilde{c} \neq 0$ for $c(u)$ to be non-constant.

Finally, we return to the symmetry determining equation (2.60) and consider the terms that remain after we substitute Eq. (2.90). The analysis proceeds according to the two cases.

In Case (i), the remaining terms in Eq. (2.60) reduce to

$$0 = -c^2 b_{xx} u_x + a_{tt} u_t \tag{2.94}$$

which separates into

$$b_{xx} = a_{tt} = 0. \tag{2.95}$$

Solving this equation and using Eqs. (2.92) and (2.89), we obtain

$$a = a_0 + a_1 t, b = b_0 + a_1 x \tag{2.96}$$

where a_0, b_0, a_1 are constants. Consequently, from Eq. (2.90), we have

$$\Lambda = a_0 u_t + b_0 u_x + a_1(t u_t + x u_x). \tag{2.97}$$

Hence, these are the only multipliers of first-order admitted by the wave equation (2.47) for arbitrary wave speeds $c(u)$. This establishes Theorem 2.2.1.

Finally, in Case (ii), using Eq. (2.93) for h and Eq. (2.89) for a and b, then eliminating c' by Eq. (2.93), we find that the remaining terms in Eq. (2.60) simplify considerably to

$$0 = c^{-1/2} h_{tt} - c^{-3/2} h_{xx}. \tag{2.98}$$

Hence, since $c(u)$ is non-constant, we obtain

$$h_{tt} = h_{xx} = 0. \tag{2.99}$$

Now we solve for a, b by combining Eq. (2.99) with Eqs. (2.93) and (2.89), which gives

$$a_{ttt} = b_{xxx} = 0. \tag{2.100}$$

Hence we have

$$a = a_0 + a_1 t + a_2 t^2, b = b_0 + b_1 x + b_2 x^2, \tag{2.101}$$

where $a_0, a_1, a_2, b_0, b_1, b_2$ are constants. Then Eq. (2.93) yields

$$\tilde{c}h = a_1 + 2a_2 t - b_1 - 2b_2 x. \tag{2.102}$$

Consequently, from Eq. (2.90), we obtain

$$\begin{aligned} \Lambda = \; & a_0 u_t + b_0 u_x + a_1(t u_t + c^{-1/2}/\tilde{c}) + b_1(x u_x - c^{-1/2}/\tilde{c}) \\ & + a_2(t^2 u_t + 2t c^{-1/2}/\tilde{c}) + b_2(x^2 u_x - 2x c^{-1/2}/\tilde{c}) \end{aligned} \tag{2.103}$$

where the wave speed is given from Eq. (2.93) by

$$c(u) = c_0 (u - u_0)^{-2} \tag{2.104}$$

with $\tilde{c} = -2{c_0}^{-1/2}$, $c_0 > 0$.

In Eq. (2.103) we have a six parameter family of admitted multipliers. The parameters $a_0, b_0, a_1 = b_1$ yield the three multipliers of Case (i). Hence the present case has three additional multipliers arising from the parameters $a_1 = -b_1, a_2, b_2$. This establishes Theorem 2.2.2. □

2.3 Klein-Gordon wave equations

Consider the class of Klein-Gordon wave equations

$$G = u_{tx} - g(u) = 0 \tag{2.105}$$

with a nonlinear interaction $g(u)$. This class has a variational principle given by the action

$$S = -\int \left(\frac{1}{2}u_t u_x + h(u)\right) dtdx, \quad h'(u) = g(u). \tag{2.106}$$

Since the general Klein-Gordon equation (2.105) is variational, its symmetries with infinitesimal generator $\mathrm{X}u = \eta$ [5, 14] satisfy the determining equation

$$0 = D_t D_x \eta - g'(u)\eta \tag{2.107}$$

which is self-adjoint. Hence the adjoint symmetries of Eq. (2.105) are the solutions of Eq. (2.107).

The symmetries $\mathrm{X}u = \eta$ admitted by the general Klein-Gordon equation (2.105) clearly consist of t and x translations

$$\eta = u_t, \eta = u_x, \tag{2.108}$$

as well as a t, x boost

$$\eta = tu_t - xu_x. \tag{2.109}$$

These are easily checked to be symmetries of the action, leaving S invariant up to a boundary-term, with

$$\mathrm{X}S = -\int D_t\left(\frac{1}{2}u_t u_x + h(u)\right) dtdx, \quad \mathrm{X}S = -\int D_x\left(\frac{1}{2}u_t u_x + h(u)\right) dtdx \tag{2.110}$$

under the respective translations $\mathrm{X}u = u_t$ and $\mathrm{X}u = u_x$, and with

$$\mathrm{X}S = \int \left(-D_t\left(\frac{1}{2}tu_t u_x + th(u)\right) + D_x\left(\frac{1}{2}xu_t u_x + xh(u)\right)\right) dtdx \tag{2.111}$$

under the boost $\mathrm{X}u = tu_t - xu_x$. By Noether's theorem, combining the invariance of the action and the general variational identity

$$\mathrm{X}S = \int \left((u_{tx} - g(u))\eta - D_t\left(\frac{1}{2}\eta u_x\right) - D_x\left(\frac{1}{2}\eta u_t\right)\right) dtdx, \quad \mathrm{X}u = \eta, \tag{2.112}$$

we obtain corresponding local conservation laws $D_t\Phi^t + D_x\Phi^x = 0$ on all solutions $u(t,x)$ of the Klein-Gordon equation (2.105). The conserved densities are given by

$$\Phi^t = h(u), \Phi^x = -\frac{1}{2}{u_t}^2, \tag{2.113}$$

$$\Phi^t = -\frac{1}{2}{u_x}^2, \Phi^x = h(u), \tag{2.114}$$

from the translations, and

$$\Phi^t = \frac{1}{2}x{u_x}^2 + th(u),\ \ \Phi^x = -\frac{1}{2}t{u_t}^2 - xh(u), \tag{2.115}$$

from the boost. These comprise all of the first-order local conservation laws for the general Klein-Gordon equation (2.105).

The Klein-Gordon equations in the class (2.105) include

$$u_{tx} = \sin u, \quad \text{sine-Gordon equation} \tag{2.116}$$

$$u_{tx} = e^u \pm e^{-u}, \quad \text{sinh-Gordon equation} \tag{2.117}$$

$$u_{tx} = e^u \pm e^{-2u}, \quad \text{Tzetzeica equation} \tag{2.118}$$

$$u_{tx} = e^u. \quad \text{Liouville equation} \tag{2.119}$$

The first three are soliton equations while the last is a linearizable equation. These are singled out [1] as nonlinear wave equations that are known to be integrable in the sense of admitting an infinite number of higher-order local conservation laws [7, 10]. For each equation the conservation laws fall into two sequences where Φ^t and Φ^x depend purely on u and t derivatives of u in one sequence, and purely on u and x derivatives of u in the other sequence (corresponding to the reflection symmetry $t \leftrightarrow x$).

Here, for the class of nonlinear Klein-Gordon equations (2.105), we consider local conservation laws of higher-order

$$D_t\Phi^t + D_x\Phi^x = 0 \tag{2.120}$$

with Φ^t, Φ^x depending either purely on t, u, and t derivatives of u, or purely on x, u, and x derivatives of u, on all solutions $u(t,x)$ of Eq. (2.105).

All nontrivial conservation laws (2.120) with conserved densities of the particular form

$$\Phi^t(x, u, \partial_x u, \ldots, \partial_x^q u),\ \ \Phi^x(x, u, \partial_x u, \ldots, \partial_x^q u) \tag{2.121}$$

can be constructed from multipliers Λ on the Klein-Gordon equation (2.105), analogous to integrating factors, with the dependence

$$\Lambda(x, u, \partial_x u, \ldots, \partial_x^p u) \tag{2.122}$$

where $p = 2q - 1$. In particular, moving off the Klein-Gordon solution space, we have

$$D_t\Phi^t + D_x\Phi^x = (u_{tx} - g(u))\Lambda_0 + D_x(u_{tx} - g(u))\Lambda_1 + \cdots \tag{2.123}$$

for some expressions $\Lambda_0, \Lambda_1, \ldots$ with no dependence on u_t and differential consequences. This yields (after integration by parts) the multiplier

$$D_t\Phi^t + D_x(\Phi^x - \Gamma) = (u_{tx} - g(u))\Lambda, \quad \Lambda = \Lambda_0 - D_x\Lambda_1 + \cdots \tag{2.124}$$

where $\Gamma = 0$ when u is restricted to be a Klein-Gordon solution. There are corresponding conservation laws of mirror form produced by the transformation $x \leftrightarrow t$, $\partial_x^k u \leftrightarrow \partial_t^k u$, with $\Phi^t \leftrightarrow \Phi^x$. We now derive an augmented symmetry determining system which completely characterizes the conservation law multipliers (2.122).

The determining condition for multipliers Λ is that $(u_{tx} - g(u))\Lambda$ must be a divergence expression for all functions $u(t,x)$ (not just Klein-Gordon solutions). For multipliers of the form (2.122), this condition is expressed by

$$\begin{aligned} 0 = {} & E_u((u_{tx} - g)\Lambda) \\ = {} & D_t D_x \Lambda - g'\Lambda + \Lambda_u(u_{tx} - g) - D_x(\Lambda_{\partial_x u}(u_{tx} - g)) + \cdots + (-D_x)^p(\Lambda_{\partial_x^p u}(u_{tx} - g)) \end{aligned} \tag{2.125}$$

where $E_u = \partial_u - D_t\partial_{u_t} - D_x\partial_{u_x} + D_t D_x \partial_{u_{tx}} + D_x^2\partial_{u_{xx}} + D_t^2\partial_{u_{tt}} + \cdots$ is the standard Euler operator which annihilates divergence expressions. Equation (2.125) is linear in $u_{tx}, u_{txx}, \ldots,$ and hence splits into separate equations. We organize the splitting in terms of $G, D_x G, \ldots, D_x^p G$. This yields a split system of determining equations for $\Lambda(x, u, \partial_x u, \ldots, \partial_x^p u)$, which is found to consist of the Klein-Gordon symmetry determining equation on Λ

$$0 = \mathscr{D}_t D_x \Lambda - g'(u)\Lambda \tag{2.126}$$

and extra determining equations on Λ

$$0 = 2\Lambda_u + \sum_{k=2}^{p} (-D_x)^k \Lambda_{\partial_x^k u}, \tag{2.127}$$

$$0 = (1 + (-1)^j)\Lambda_{\partial_x^j u} + ((-1)^j - j - 1) D_x \Lambda_{\partial_x^{j+1} u} + \sum_{k=j+2}^{p} \frac{k!}{j!(k-j)!} (-D_x)^{k-j} \Lambda_{\partial_x^k u}, \quad j = 1, \ldots, p-1 \tag{2.128}$$

$$0 = (1 + (-1)^p)\Lambda_{\partial_x^p u}. \tag{2.129}$$

Here $\mathscr{D}_t = \partial_t + u_t\partial_u + g(u)\partial_{u_x} + g'(u)u_x\partial_{u_{xx}} + \cdots$ is the total derivative operator which expresses t derivatives of u through the Klein-Gordon equation (2.105). Consequently, one is able to work on the space of Klein-Gordon solutions $u(t, x)$ in order to solve the determining system (2.126) to (2.129) to find $\Lambda(x, u, \partial_x u, \ldots, \partial_x^p u)$.

The solutions of the determining system (2.126) to (2.129) are the multipliers (2.122) that yield all nontrivial conservation laws of the form (2.121). Noether's theorem establishes that there is a one-to-one correspondence between symmetries of the action and multipliers for nontrivial conservation laws of the Klein-Gordon equation (2.105). Consequently, it follows that the extra determining equations (2.127)–(2.129) represent the necessary and sufficient condition for a symmetry to leave the action (2.106) invariant up to a boundary term.

The explicit relation between multipliers Λ and conserved densities Φ^t, Φ^x for conservation laws (2.121) is summarized as follows. Given a conserved density Φ^t, we find that a direct calculation of the u_{tx} terms in $D_t\Phi^t + D_x\Phi^x$ yields the multiplier equation

$$D_t\Phi^t + D_x\Phi^x = (u_{tx} - g(u))\hat{E}_{u_x}(\Phi^t) + D_x\Gamma \tag{2.130}$$

where $\hat{E}_{u_x} = \partial_{u_x} - D_x\partial_{u_{xx}} + \cdots$ is a restricted Euler operator, and Γ is proportional to $u_{tx} - g(u)$ and differential consequences. Thus we obtain the multiplier

$$\Lambda = \hat{E}_{u_x}(\Phi^t). \tag{2.131}$$

Conversely, given a multiplier Λ, we can invert the relation (2.131) using Eq. (2.130) to obtain the conserved density

$$\Phi^t = \int_0^1 d\lambda (u_x - \tilde{u}_x) \Lambda[\lambda u + (1-\lambda)\tilde{u}] \tag{2.132}$$

where $\Lambda[u] = \Lambda(x, u, \partial_x u, \ldots, \partial_x^p u)$, and $\tilde{u} = \tilde{u}(x)$ is any function chosen so that the expressions $\Lambda[\tilde{u}]$ and $g(\tilde{u})$ are non-singular. In particular, if $\Lambda[0]$ and $g(0)$ are non-singular, then we can choose $\tilde{u} = 0$, which simplifies the integral (2.132).

Thus, from Eqs. (2.131) and (2.132), we see that all nontrivial Klein-Gordon conservation laws (2.121) of order q are determined by multipliers (2.122) of order $p = 2q - 1$ which are obtained as solutions of the augmented system of symmetry determining equations (2.126) to (2.129).

Through the determining system (2.126) to (2.129), we now derive a classification of nonlinear Klein-Gordon interactions $g(u)$ that admit higher-order conservation laws (2.121) starting at order $q = 2$. The classification results are summarized by three theorems.

Theorem 2.3.1 *The nonlinear Klein-Gordon equation* (2.105) *admits a conservation law* (2.121) *of order* $q = 2$ *iff the nonlinear interaction is given by one of*

$$g(u) = e^u, g(u) = \sin u, g(u) = e^u \pm e^{-u} \tag{2.133}$$

to within scaling of t, x, u*, and translation of* u*.*

(If we allow u to undergo complex-valued scalings and translations then the Klein-Gordon equations for $g(u) = \sin u$ and $g(u) = e^u \pm e^{-u}$ are equivalent.)

The Klein-Gordon equations arising from Theorem 2.3.1 are the Liouville equation (2.119), sine-Gordon equation (2.116) and related sinh-Gordon equation (2.117). (The Tzetzeica equation (2.118) is absent in this classification because its first admitted higher-order conservation law (2.121) is of order $q = 3$.) Since each of these Klein-Gordon equations is known to admit an infinite sequence of higher order conservation laws (2.121) for $q \geqslant 2$, this leads to an integrability classification.

Corollary 2.3.2 *A nonlinear Klein-Gordon equation* (2.105) *is integrable in the sense of having an infinite number of conservation laws* (2.121) *up to arbitrarily high orders* $q > 1$ *if it admits one of order* $q = 2$*.*

Theorem 2.3.3 *The multipliers for the second-order conservation laws* (2.121) *admitted by the Klein-Gordon equation* (2.105) *with nonlinear interactions* (2.133) *are given by*

$$\Lambda = u_{xxx} + \frac{1}{2} {u_x}^3, \tag{2.134}$$

$$\Lambda = u_{xxx} - \frac{1}{2} {u_x}^3, \tag{2.135}$$

for the sine-Gordon equation (2.116) *and sinh-Gordon equation* (2.117)*, respectively, and*

$$\Lambda = (u_{xxx} - u_x u_{xx}) f_\xi + f_x + u_x f = D_x f + u_x f, \tag{2.136}$$

depending on an arbitrary function

$$f = f(x, \xi), \quad \xi = u_{xx} - \frac{1}{2}{u_x}^2, \tag{2.137}$$

for the Liouville equation (2.119).

From the construction formula (2.132) the corresponding conserved densities for the multipliers (2.134) and (2.135), respectively, are given by

$$\Phi^t = \frac{1}{2}u_{xxx}u_x \pm \frac{1}{8}{u_x}^4 = -\frac{1}{2}{u_{xx}}^2 \pm \frac{1}{8}{u_x}^4 + D_x\theta, \tag{2.138}$$

where we have used $\tilde{u} = 0$, since $\Lambda[0] = 0$ and $g(0) = \text{const}$. Similarly, the conserved density for the multiplier (2.136) is given by

$$\begin{aligned}\Phi^t &= (u_x - \tilde{u}_x)D_x \int_0^1 d\lambda f(x, \lambda(u_{xx} - \tilde{u}_{xx}) - \frac{1}{2}\lambda^2(u_x - \tilde{u}_x)^2) \\ &+(u_x - \tilde{u}_x)^2 \int_0^1 d\lambda\ \lambda f(x, \lambda(u_{xx} - \tilde{u}_{xx}) - \frac{1}{2}\lambda^2(u_x - \tilde{u}_x)^2) \\ &= -\int_{\tilde{u}_{xx}-\frac{1}{2}{\tilde{u}_x}^2}^{u_{xx}-\frac{1}{2}{u_x}^2} f(x,\xi)d\xi + D_x\theta,\end{aligned} \tag{2.139}$$

where now $\tilde{u} = \tilde{u}(x)$ is any function chosen so that $\Lambda[\tilde{u}] = D_x f(x, \tilde{\xi}) + \tilde{u}_x f(x, \tilde{\xi})$ and $g(\tilde{u}) = \exp(\tilde{u})$ are non-singular expressions.

We remark that the existence of the conservation law (2.139) involving an arbitrary function reflects the well-known classical integrability [9] (in the sense of explicit integration) of the Liouville equation.

Proof of Theorem 2.3.1 From Eqs. (2.121) and (2.122), conservation laws with $\Phi^t(x, u, u_x, u_{xx})$ and $\Phi^x(x, u, u_x, u_{xx})$ of order $q = 2$ correspond to multipliers

$$\Lambda(x, u, u_x, u_{xx}, u_{xxx}). \tag{2.140}$$

We start with the extra determining equations (2.127) to (2.129). These become

$$0 = 2\Lambda_u + D_x^2\Lambda_{u_{xx}} - D_x^3\Lambda_{u_{xxx}}, \tag{2.141}$$

$$0 = D_x\Lambda_{u_{xx}} - D_x^2\Lambda_{u_{xxx}}, \tag{2.142}$$

$$0 = \Lambda_{u_{xx}} - D_x\Lambda_{u_{xxx}}. \tag{2.143}$$

Note that Eq. (2.142) is a differential consequence of Eq. (2.143), while Eq. (2.141) combined with a differential consequence of Eq. (2.142) reduces to

$$0 = \Lambda_u. \tag{2.144}$$

Consider Eq. (2.143). Its highest-order term is $\Lambda_{u_{xxx}u_{xxx}}u_{xxxx}$, and hence $\Lambda_{u_{xxx}u_{xxx}} = 0$. This yields, using Eq. (2.144),

$$\Lambda = a(x, u_x, u_{xx})u_{xxx} + b(x, u_x, u_{xx}). \tag{2.145}$$

Then the remaining terms in Eq. (2.143) give

$$b_{u_{xx}} = a_x + a_{u_x}u_{xx}. \tag{2.146}$$

We now turn to the symmetry determining equation (2.126). This becomes, taking into account Eqs. (2.140) and (2.144),

$$0 = \mathscr{D}_t \Lambda_x + u_{xx} \mathscr{D}_t \Lambda_{u_x} + u_{xxx} \mathscr{D}_t \Lambda_{u_{xx}} + u_{xxxx} \mathscr{D}_t \Lambda_{u_{xxx}} + \Lambda_{u_x} D_x g + \Lambda_{u_{xx}} D_x^2 g + \Lambda_{u_{xxx}} D_x^3 g - \Lambda g'. \tag{2.147}$$

We substitute Eq. (2.145) into Eq. (2.147) and separate it into highest derivative terms in descending order. The terms of highest order involve u_{xxxx}, which yield

$$0 = a_{u_x} g + a_{u_{xx}} u_x g'. \tag{2.148}$$

Since g depends only on u, while a does not depend on u, the terms in Eq. (2.148) can balance only in two ways: either g and g' are proportional, or g and g' are linearly independent with vanishing coefficients. This leads to two cases to consider.

Case (i): g, g' linearly independent

In this case $a_{u_x} = a_{u_{xx}} = 0$, so $a = a(x)$. Consequently from Eq. (2.146), we have $b_{u_{xx}} = a_x$, and hence

$$b = a'(x) u_{xx} + c(x, u_x). \tag{2.149}$$

To proceed, we return to the symmetry determining equation (2.147). The highest order remaining terms now involve u_{xx}, which lead to

$$0 = a' g' + 3 a u_x g'' + c_{u_x u_x} g. \tag{2.150}$$

By taking a partial derivative with respect to u_x we obtain $0 = 3ag'' + c_{u_x u_x u_x} g$. Since g depends only on u, while a does not depend on u, this equation immediately yields

$$g'' = \sigma g, \sigma = \text{const}, \tag{2.151}$$

$$c_{u_x u_x u_x} = -3 a \sigma, \tag{2.152}$$

where $\sigma \neq 0$ in order for $g(u)$ to be nonlinear. From Eq. (2.152) we obtain

$$c = -\frac{1}{2} \sigma a(x) {u_x}^3 + c_2(x) {u_x}^2 + c_1(x) u_x + c_0(x). \tag{2.153}$$

Now we find that Eq. (2.150) reduces to $0 = a' g' + 2 c_2 g$, which immediately separates into the equations

$$a' = c_2 = 0. \tag{2.154}$$

Finally, the terms remaining in the symmetry determining equation (2.147) simplify to $0 = c_{1x} g - c_0 g'$, and thus we have

$$c_0 = c_{1x} = 0. \tag{2.155}$$

From Eq. (2.145), Eq. (2.149) and Eqs. (2.153) to (2.155), we obtain

$$\Lambda = a \left(u_{xxx} - \frac{1}{2} \sigma {u_x}^3 \right) + c_1 u_x, a = \text{const}, c_1 = \text{const}. \tag{2.156}$$

This multiplier is admitted for all nonlinear interactions $g(u)$ satisfying Eq. (2.151) with

$g(u)$ and $g'(u)$ being linearly independent. The general solution for $g(u)$ breaks into two forms:

$$g(u) = \alpha \sin(\sqrt{|\sigma|}u + \beta), \text{ if } \sigma < 0; \tag{2.157}$$

$$g(u) = \alpha(e^{\sqrt{\sigma}u+\beta} \pm e^{-\sqrt{\sigma}u-\beta}), \text{ if } \sigma > 0; \tag{2.158}$$

with arbitrary constants $\alpha, \beta, \sigma \neq 0$. By a scaling and a translation $\sqrt{|\sigma|}u + \beta \to u$, and a scaling $\alpha\sqrt{|\sigma|}t \to t$, we see that the resulting Klein-Gordon equation (2.105) becomes, respectively, the sine-Gordon equation (2.116) or the sinh-Gordon equation (2.117). This completes the classification in case (i).

Case (ii): g, g' proportional

In this case, we have

$$g' = \sigma g, \sigma = \text{const} \tag{2.159}$$

where $\sigma \neq 0$ for $g(u)$ to be nonlinear. Now Eq. (2.148) becomes

$$0 = a_{u_x} + \sigma a_{u_{xx}} u_x. \tag{2.160}$$

This is a linear first-order PDE for a in u_x, u_{xx}. The solution is

$$a = a(x, \xi), \quad \xi = u_{xx} - \frac{1}{2}\sigma {u_x}^2. \tag{2.161}$$

Then from Eq. (2.146) we obtain

$$b = \tilde{a}_x(x, \xi) + \sigma u_x(\tilde{a}(x, \xi) - a(x, \xi)u_{xx}) + c(x, u_x), \quad \tilde{a} = \int a d\xi. \tag{2.162}$$

To proceed, we return to the symmetry determining equation (2.147). We find that the highest order terms involving ${u_{xxx}}^2$ and u_{xxx} vanish as a consequence of Eq. (2.160). Then we find that the next highest order terms which involve u_{xx} in Eq. (2.147) yield $c_{u_x u_x} = 0$. Hence, we obtain

$$c = c_1(x)u_x + c_0(x). \tag{2.163}$$

Finally, the remaining terms in Eq. (2.147) reduce to

$$0 = c_1' - \sigma c_0. \tag{2.164}$$

Thus, from Eq. (2.145), Eqs. (2.161) and (2.162), Eqs. (2.163) and (2.164), we obtain

$$\Lambda = \tilde{a}_\xi u_{xxx} + \sigma u_x(\tilde{a} - u_{xx}\tilde{a}_\xi) + \tilde{a}_x + c_1 u_x + \frac{1}{\sigma}c_1', \quad \xi = u_{xx} - \frac{1}{2}\sigma {u_x}^2, \tag{2.165}$$

with arbitrary functions $c_1(x), \tilde{a}(x, \xi)$. This multiplier is admitted for all nonlinear interactions $g(u)$ satisfying Eq. (2.159). The general solution of this equation is

$$g(u) = e^{\sigma u + \beta} \tag{2.166}$$

with arbitrary constants $\beta, \sigma \neq 0$. By a scaling and a translation $\sigma u + \beta \to u$, and a scaling $\sigma t \to t$, we see that the resulting Klein-Gordon equation (2.105) becomes the Liouville equation. This completes the classification in Case (ii). □

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