JOURNAL OF MATHEMATICAL PHYSICS 46, 073513 (2005)

Comparing symmetries and conservation laws of nonlinear telegraph equations

G. W. Bluman[a] and Temuerchaolu[b]
Department of Mathematics, University of British Columbia, Vancouver, BC, Canada V6T 1Z2

(Received 8 February 2005; accepted 18 March 2005; published online 6 July 2005)

A comparison is made between the symmetries and conservation laws admitted by nonlinear telegraph (NLT) systems. Such systems are not variational. Unlike the situation for variational systems where all conservation laws arise from symmetries, there are many NLT systems that admit more conservation laws than symmetries. The results are summarized in a table which includes the numbers of symmetries and conservation laws for each NLT system. It is also indicated when symmetries map conservation laws to other conservation laws. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1915292]

I. INTRODUCTION

In this paper, we consider the problem of comparing the multipliers of conservation laws and the point symmetries of a given nonlinear system of partial differential equations (PDEs) that is not variational, i.e., whose associated Fréchet derivative is not self-adjoint. As a protypical example, we consider nonlinear telegraph (NLT) systems of the form

$$H_1[u,v] = v_t - F(u)u_x - G(u) = 0,$$

$$H_2[u,v] = u_t - v_x = 0. \tag{1}$$

One physical example related to system (1) is represented by the equations of telegraphy of a two-conductor transmission line with v as the current in the conductors, u as the voltage between the conductors, $G(u)$ as the leakage current per unit length, $F(u)$ as the differential capacitance, t as a spatial variable and x as time.[1] Another physical example related to system (1) is the equation of motion of a hyperelastic homogeneous rod whose cross-sectional area varies exponentially along the rod. Here u is the displacement gradient related to the difference between a spatial Eulerian coordinate and a Lagrangian coordinate x, v is the velocity of a particle displaced by this difference, $G(u)$ is essentially the stress-tensor, $F(u)=\lambda G'(u)$ for some constant λ, and t is time (see Refs. 2 and 3).

A point symmetry

$$x^* = x + \hat{\xi}(x,t,u,v)\varepsilon + O(\varepsilon^2),$$

$$t^* = t + \hat{\tau}(x,t,u,v)\varepsilon + O(\varepsilon^2),$$

$$u^* = u + \hat{\eta}(x,t,u,v)\varepsilon + O(\varepsilon^2),$$

[a]Electronic mail: bluman@math.ubc.ca
[b]Permanent address: Inner Mongolia University of Technology, Hohhot, People's Republic of China. Electronic mail: tmchaolu@impu.edu.cn

$$v^* = v + \hat{\zeta}(x,t,u,v)\varepsilon + O(\varepsilon^2), \tag{2}$$

with corresponding infinitesimal generator (in evolutionary form)

$$X = \eta\partial_u + \zeta\partial_v$$

is admitted by system (1)[4–6] if and only if

$$\begin{aligned} X^{(1)}H_1[u,v]\Big|_{\substack{H_1[u,v]=0,\\ H_2[u,v]=0}} &= 0,\\ X^{(1)}H_2[u,v]\Big|_{\substack{H_1[u,v]=0,\\ H_2[u,v]=0}} &= 0, \end{aligned} \tag{3}$$

where $X^{(1)}$ is the first extension of X and

$$\eta = \hat{\eta}(x,t,u,v) - \hat{\xi}(x,t,u,v)u_x - \hat{\tau}(x,t,u,v)u_t,$$

$$\zeta = \hat{\zeta}(x,t,u,v) - \hat{\xi}(x,t,u,v)v_x - \hat{\tau}(x,t,u,v)v_t.$$

The Fréchet derivative associated with system (1) is the linear operator

$$L[u] = \begin{bmatrix} D_t & -D_x \\ -F'(u)u_x - F(u)D_x - G'(u) & D_t \end{bmatrix}, \tag{4}$$

and it yields the linearized system of (1) given by

$$L[u]\begin{bmatrix}\Phi\\ \Psi\end{bmatrix}\Big|_{\substack{H_1[u,v]=0,\\ H_2[u,v]=0}} = 0 \tag{5}$$

in terms of total derivative operators D_x and D_t. It is easy to show that a point symmetry of system (3) is any solution $(\Phi,\Psi)=(\eta,\zeta)$ of the linearized system (5).

On the other hand, a set of multipliers

$$\xi = \xi(x,t,U,V), \quad \phi = \phi(x,t,U,V), \tag{6}$$

yields a conservation law of system (1) if and only if

$$\begin{aligned} E_U[\xi(x,t,U,V)H_1[U,V] + \phi(x,t,U,V)H_2[U,V]] &\equiv 0,\\ E_V[\xi(x,t,U,V)H_1[U,V] + \phi(x,t,U,V)H_2[U,V]] &\equiv 0, \end{aligned} \tag{7}$$

for *all* differentiable functions $U(x,t)$ and $V(x,t)$, where

$$E_U = \frac{\partial}{\partial U} - D_x\frac{\partial}{\partial U_x} - D_t\frac{\partial}{\partial U_t}, \quad E_V = \frac{\partial}{\partial V} - D_x\frac{\partial}{\partial V_x} - D_t\frac{\partial}{\partial V_t}$$

are Euler operators. One can show that a necessary condition for $\{\xi(x,t,U,V),\phi(x,t,U,V)\}$ to be a set of multipliers for a conservation law of system (1) is that

$$L^*[u]\begin{bmatrix}\xi(x,t,u,v)\\ \phi(x,t,u,v)\end{bmatrix}\Big|_{\substack{H_1[u,v]=0,\\ H_2[u,v]=0}} = 0, \tag{8}$$

where in terms of the Fréchet derivative operator $L[u]$, the adjoint operator $L^*[u]$ is the unique operator having the property that

$$[\alpha,\beta]L[u]\begin{bmatrix}\Phi\\ \Psi\end{bmatrix}-[\Phi,\Psi]L^{*}[u]\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

is a divergence expression for any differentiable functions α, β, Φ, and Ψ.[4,7–9] It is easy to show that for the Fréchet derivative operator $L[u]$ defined by (4), one has

$$L^{*}[u]=\begin{bmatrix}-D_t & F(u)D_x-G'(u)\\ D_x & -D_t\end{bmatrix}.$$

In the study of physical systems, the importance of conservation laws is well known, including connections with integrability, linearization and modern numerical methods. Conservation laws are intrinsic properties of field equations since they must hold for any posed data. Familiarly, conservation laws are derived from variational principles through Noether's theorem.[4,5] A given system of PDEs (linear or nonlinear) can be directly obtained from a variational principle if and only if its Fréchet derivative is *self-adjoint*,[4,10] i.e., $L^{*}[u]=L[u]$. Noether's theorem yields a conservation law for any point symmetry (2) that leaves invariant the action functional of the variational principle. This is equivalent to the Euler operator annihilating the scalar product of the multipliers and the given system of PDEs' functions in which solutions (u,v) are replaced by arbitrary differentiable functions (U,V), i.e., system (7). Consequently, any set of multipliers for a conservation law of a system of PDEs having a variational principle must also be admitted symmetries of the given system. Hence, when a system of PDEs has an associated self-adjoint Fréchet derivative, its multipliers for conservation laws are a subset of its admitted symmetries.

It is well known that for a linear system of PDEs, any solution of its adjoint system yields a set of multipliers for a conservation law since for any linear operator L, a divergence expression is yielded by $vLu-uL^{*}v$. Moreover, from this it follows that if a given nonlinear system of PDEs can be mapped into a linear system by a point or contact transformation, then its infinite number of admitted symmetries exhibit this linear system and its infinite number of multipliers for conservation laws exhibit the adjoint of this linear system.[5,11,12]

It is easy to show that for any $F(u)$ and $G(u)$, the NLT system (1) is *not* self-adjoint. The extra conditions beyond the necessary condition (8) for $\xi(x,t,U,V)$, $\phi(x,t,U,V)$ to be multipliers for a conservation law of system (1) are given in Refs. 7–9. At first sight, one might think that a given system (1) admits more point symmetries (2) which involve four unknowns $\{\hat{\xi},\hat{\tau},\hat{\eta},\hat{\zeta}\}$ than sets of multipliers of the form (6) which involve only two unknowns $\{\xi,\phi\}$. However, we will show that for many NLT systems (1), there are more admitted sets of multipliers of the form (6) than admitted point symmetries (2).

The point symmetry and conservation law classifications of the NLT system (1) have been separately investigated in Refs. 13 and 14, respectively. In Ref. 15, it is shown how to obtain new conservation laws from the action of an admitted symmetry on a known conservation law.

In Sec. II, we give the determining equations for point symmetries and multipliers admitted by NLT systems (1). We present the Symmetry and Conservation Law Classification Table for NLT systems (1) and show that when (1) is not linearizable, there are many cases where it can admit, nontrivially, one symmetry and four, three, two or zero conservation laws as well as cases where it can admit, nontrivially, zero symmetries and four or two conservation laws. We also comment on situations when a symmetry maps a conservation law into another conservation law(s). Further comments are presented in Sec. III.

II. COMPARISON OF SYMMETRIES AND CONSERVATION LAWS FOR NLT SYSTEMS

By inspection, any NLT system (1) obviously admits, as point symmetries, translations $t\to t+\varepsilon_1$, $v\to v+\varepsilon_2$, and $x\to x+\varepsilon_3$, corresponding to admitted infinitesimal generators $X_1=u_t\partial_u+v_t\partial_v$, $X_2=\partial_v$, and $X_3=u_x\partial_u+v_x\partial_v$, respectively, as well as a set of multipliers $(\xi,\phi)=(0,1)$ since the second PDE of any NLT system (1) is written as a conservation law. Any additional admitted point symmetries or sets of multipliers for conservation laws are considered to be nontrivial.

A point symmetry (2) admitted by an NLT system (1) is represented by any solution $\{\hat{\xi}(x,t,u,v),\hat{\tau}(x,t,u,v),\hat{\eta}(x,t,u,v),\hat{\zeta}(x,t,u,v)\}$ of the linear determining system of PDEs[4–6]

$$\hat{\xi}_v - \hat{\tau}_u = 0,$$

$$\hat{\eta}_u - \hat{\psi}_v + \xi_x - \tau_t = 0,$$

$$G(u)\hat{\eta}_v + \hat{\eta}_t - \hat{\psi}_x + G(u)\hat{\tau}_x = 0,$$

$$\hat{\xi}_u - F(u)\hat{\tau}_v = 0,$$

$$\hat{\psi}_u - G(u)\hat{\tau}_u - F(u)\hat{\eta}_v = 0,$$

$$G(u)\hat{\xi}_v + \hat{\xi}_t - F(u)\hat{\tau}_x = 0,$$

$$[\hat{\psi}_v - \hat{\tau}_t - 2G(u)\hat{\tau}_v - \hat{\eta}_u + \hat{\xi}_x]F(u) - F'(u)\hat{\eta} = 0,$$

$$[\hat{\psi}_v - \hat{\tau}_t - G(u)\hat{\tau}_v]G(u) - F(u)\hat{\eta}_x - G'(u)\hat{\eta} + \hat{\psi}_t = 0. \tag{9}$$

The complete solution of the determining system (9) is presented in Ref. 13.

A set of multipliers for a conservation law of an NLT system (1) is represented by any solution $\{\xi(x,t,U,V),\phi(x,t,U,V)\}$ of the linear determining system of PDEs[7,9]

$$\phi_V - \xi_U = 0,$$

$$\phi_U - F(U)\xi_V = 0,$$

$$\phi_x - \xi_t - G(U)\xi_V = 0,$$

$$F(U)\xi_x - \phi_t - G(U)\xi_U - G'(U)\xi = 0. \tag{10}$$

The complete solution of the determining system (10) and the corresponding conservation laws are presented in Ref. 14.

Equivalence transformations[10,11] simplify the symmetry and conservation law classifications of NLT systems (1). In particular, in Refs. 13 and 14, it is shown how to obtain the corresponding conservation law for any $(\bar{F}(u),\bar{G}(u))$ pair related to the conservation law for a given $(F(u),G(u))$ pair through any similarity transformation

$$\bar{F}(u) = \gamma F(\alpha u + \beta), \quad \bar{G}(u) = \delta G(\alpha u + \beta) + \rho. \tag{11}$$

In the following Symmetry and Conservation Law Classification Table (Table I), we list and compare the additional admitted nontrivial symmetries and nontrivial conservation laws for all possible pairs $(F(u),G(u))$, modulo any similarity transformation (11). For each such pair, we indicate the number of additional admitted point symmetries, the number of additional admitted conservation laws, list all such admitted point symmetries in evolutionary form, show where to find such admitted conservation laws in Ref. 14, and state pertinent comments. Most important, in the comments column we indicate where an admitted symmetry can map a conservation law to

TABLE I. Symmetry and conservation law classification table.

$G(u)$	$F(u)$	Number of additional point symmetries; listing of symmetries		Number of additional conservation laws; see Ref. 14 for conservation laws		Comments; symmetry mappings of conservation laws (Ref. 15)
0	Arbitrary	∞	$X=[-A(u,v)u_x+B(u,v)u_t]\partial_u$ $+[-A(u,v)v_x+B(u,v)v_t]\partial_v$ with $A_u=-F(u)B_v, A_v=-B_u$	∞	Multipliers $\xi=a(U,V)$, $\phi=b(U,V)$ with $a_U=b_V$, $F(U)a_V=b_U$	Linearizable by a hodograph transformation; (a,b) system is adjoint of (A,B) system
u	1	∞	$X_\infty=B(x,t)\partial_u-A(x,t)\partial_v$ with $A_x+B_t=0$, $A_t+B_x+B=0$	∞	Multipliers $\xi=b(x,t)$, $\phi=a(x,t)$ with $a_x=b_t$, $a_t=b_x-b$	Linear system; (a,b) system is adjoint of (A,B) system
	u	1	$X=(2u-2xu_x-tu_t)\partial_u$ $+(3v-2xv_x-tv_t)\partial_v$	4	Table 4: Case 1 with $\beta_2=1$, $\beta_1=\beta_3=0$	$t\to t+\varepsilon$ maps (ξ_3,ϕ_3) to additional three (ξ_i,ϕ_i), $i=1,2,4$
	$u^\alpha[\alpha\neq 0,1]$	1	$X=(2u-2\alpha xu_x-\alpha tu_t)\partial_u$ $+[(2+\alpha)v-2\alpha xv_x-\alpha tv_t]\partial_v$	2	Table 1	$t\to t+\varepsilon$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)
	$e^{\alpha u}[\alpha\neq 0]$	1	$X=(2-2\alpha xu_x-\alpha tu_t)\partial_u$ $+(\alpha v+2t-2\alpha xv_t)\partial_v$	2	″	″
	$u^2+\alpha_1 u+\alpha_2$ $[\alpha_1^2\neq 4\alpha_2]$	0		4	Table 4: Case 1 with $\beta_1=1$, $\beta_2=\alpha_1,\beta_3=\alpha_2$	In Ref. 14: in Table 1, $t\to t+\varepsilon$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2); in Table 3, $(t,V)\to(-t,-V)$ maps a *different* (ξ_1,ϕ_1) to a *different* (ξ_2,ϕ_2) than for Table 1
	All other $F(u)$	0		2	Table 1	$t\to t+\varepsilon$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)
u^{-1}	u^{-2}	∞	$X=[u^{-1}A(\hat{u},v)u_x$ $-B(\hat{u},v)u_t+A(\hat{u},v)]\partial_u$ $+[u^{-1}A(\hat{u},v)v_x$ $-B(\hat{u},v)v_t]\partial_u$ with $A_v+B_{\hat{u}}=0$, $A_{\hat{u}}+B_v-A=0$ $[\hat{u}=x+\ln u]$	∞	Multipliers $\xi=e^{-x}b(\hat{U},V)$, $\phi=a(\hat{U},V)$ with $a_V-e^{-\hat{U}}b_{\hat{U}}=0$, $a_{\hat{U}}-e^{-\hat{U}}b_V=0$ $[\hat{U}=x+\ln U]$	Linearizable; (a,b) system is adjoint of (A,B) system
	u^{-1}	1	$X=(2xu_x+3tu_t-2u)\partial_u$ $+(2xu_x+3tu_t-v)\partial_v$	4	Table 5: Case 1 with $\beta_2=1$, $\beta_1=\beta_3=0$	$V\to V+\varepsilon$ maps (ξ_3,ϕ_3) to additional three (ξ_i,ϕ_i), $i=1,2,4$

TABLE I. (*Continued.*)

$G(u)$	$F(u)$	Number of additional point symmetries; listing of symmetries		Number of additional conservation laws; see Ref. 14 for conservation laws		Comments; symmetry mappings of conservation laws (Ref. 15)
u^{-1}	$(u+1)/u^2$	1	$X=[2(x+\ln u-1/u)u_x$ $+(3t+2v)u_t-2(u+1)]\partial_u$ $+[2(x+\ln u-1/u)v_x$ $+(3t+2v)v_t-v]\partial_v$	4	Table 5: Case 1 with $\beta_3=0$, $\beta_1=\beta_2=1$	X maps (ξ_3,ϕ_3) to (ξ_2,ϕ_2); $V\to V+\varepsilon$ maps (ξ_3,ϕ_3) to additional three (ξ_i,ϕ_i), $i=1,2,4$
	$(u\pm 1)^{\beta}/u^2$ $[\beta\neq 0,1]$	1	$X=[2(\beta x\pm\int F(u)du)u_x$ $+((\beta+2)t\pm 2v)u_t$ $-2(u\pm 1)]\partial_u$ $+[2(\beta x\pm\int F(u)du)v_x$ $+((\beta+2)t\pm 2v)v_t$ $-\beta v]\partial_v$	2	Table 1	$V\to V+\varepsilon$ maps (ξ_2,ϕ_2) to (ξ_1,ϕ_1)
	u^{α} $[\alpha\neq -1,-2]$	1	$X=[2(2+\alpha)xu_x$ $+(4+\alpha)tu_t-2u]\partial_u$ $+[2(2+\alpha)xu_x$ $+(4+\alpha)tu_t-(2+\alpha)v]\partial_u$	2	″	″
	$u^{-2}e^{\alpha u}$ $[\alpha\neq 0]$	1	$X=[(2\alpha x+2\int F(u)du)u_x$ $+(\alpha t+2v)u_t-2]\partial_u$ $[(2\alpha x+2\int F(u)du)v_x$ $+(\alpha t+2v)v_t-\alpha v]\partial_v$	2	″	″
	$u^{-2}+\alpha_1 u^{-1}+\alpha_2$ $[\alpha_1^2\neq 4\alpha_2$, $\alpha_2\neq 0]$	0		4	Table 5: Case 1 with $\beta_1=1$, $\beta_2=\alpha_1$, $\beta_3=\alpha_2$	In Ref. 14: in Table 1, $V\to V+\varepsilon$ maps (ξ_2,ϕ_2) to (ϕ_1,ϕ_1); in Table 3, $(t,V)\to(-t,-V)$ maps a *different* (ξ_1,ϕ_1) to a *different* (ξ_2,ϕ_2) than for Table 1
	All other $F(u)$	0		2	Table 1	$v\to v+\varepsilon$ maps (ξ_2,ϕ_2) to (ξ_1,ϕ_1)
u^{δ} $[\delta\neq 0,\pm 1]$	$u^{\delta-1}$	1	$X=[(\delta-1)tu_t+2u]\partial_u$ $+[(\delta-1)tv_t$ $+(1+\delta)v]\partial_v$	3	Table 3: Case 2 with $\nu=\delta$, $\mu=0$	$t\to t+\varepsilon$ maps (ξ_1,ϕ_1) to (ξ_3,ϕ_3); $V\to V+\varepsilon$ maps (ξ_3,ϕ_2) to (ξ_3,ϕ_3)
	$u^{\beta}[\beta\neq\delta-1]$	1	$X=[2(\delta-\beta-1)xu_x$ $+(2\delta-\beta-2)tu_t+2u]\partial_u$ $+[2(\delta-\beta-1)xu_x$ $+(2\delta-\beta-2)tv_t$ $+(2+\beta)]\partial_u$	0		
	$u^{\delta-1}+\beta$ $[\beta\neq 0]$	0		2	Table 3: Case 2 with $\nu=\delta$, $\mu=\delta^2\beta$	$(t,V)\to(-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)k

TABLE I. (*Continued.*)

$G(u)$	$F(u)$	Number of additional point symmetries; listing of symmetries		Number of additional conservation laws; see Ref. 14 for conservation laws		Comments; symmetry mappings of conservation laws (Ref. 15)
u^{δ} $[\delta\neq 0,\pm 1]$	$u^{\delta-1}+\kappa(u^{\delta}+\rho)^2$ $[\kappa\neq 0]$	0		2	Table 3: Case 1 with $\gamma=\delta$, $\alpha=\delta^2\kappa$, $\beta=\rho$	〃
$\ln u$	u^{-1}	1	$X=(tu_t-2u)\partial_u+(tv_t-v-2t)\partial_v$	3	Table 3: Case 2 with $\nu=1$, $\mu=0$	X maps (ξ_2,ϕ_2) to (ξ_1,ϕ_1); $t\rightarrow t+\varepsilon$ maps (ξ_1,ϕ_1) to (ξ_3,ϕ_3)
	u^{α} $[\alpha\neq -1]$	1	$X=[2(\alpha+1)xu_x+(\alpha+2)tu_t-2u]\partial_u+[2(\alpha+1)xv_x+(\alpha+2)tv_t-(2t+(2+\alpha)v)]\partial_v$	0		
	$u^{-1}+\alpha$ $[\alpha\neq 0]$	0		2	Table 3: Case 2 with $\nu=1$, $\mu=\alpha$	$(t,V)\rightarrow(-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)
	$u^{-1}+\kappa(\ln u+\rho)^2$ $[\kappa\neq 0]$	0		2	Table 3: Case 1 with $\gamma=1$, $\alpha=\kappa$, $\beta=\rho$	〃
$\ln^{-1} u$	$1/(u\ln^2 u)$	1	$X=[-2\ln^{-1}uu_x+(t+2v)u_t-2u]\partial_u+[-2\ln^{-1}uv_x+(t+2v)v_t-v]\partial_v$	3	Table 3: Case 2 with $\nu=-1$, $\mu=0$	X maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2); $t\rightarrow t+\varepsilon$ maps (ξ_1,ϕ_1) to (ξ_3,ϕ_3)
	$u^{\beta}/\ln^2 u$ $[\beta\neq -1]$	1	$X=2[((\beta+1)x+\int F(u)du)u_x+((\beta+2)t+2v)u_t-2u]\partial_u+2[((\beta+1)x+\int F(u)du)v_x+((\beta+2)t+2v)v_t-(\beta+2)v]\partial_v$	0		
	$1/(u\ln^2 u)+\alpha$ $[\alpha\neq 0]$	0		2	Table 3: Case 2 with $\nu=-1$, $\mu=\alpha$	$(t,V)\rightarrow(-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)
	$1/(u\ln^2 u)+\kappa(\ln^{-1}u+\rho)^2$ $[\kappa\neq 0]$	0		2	Table 3: Case 1 with $\gamma=-1$, $\alpha=\kappa$, $\beta=\rho$	〃
e^u	e^u	1	$X=(2+tu_t)\partial_u+(v+tv_t)\partial_v$		Table 6: Case 4 with $\beta_1=\beta_3=0$, $\beta_2=1$	$V\rightarrow V+\varepsilon$ maps (ξ_4,ϕ_4) to (ξ_1,ϕ_1); $t\rightarrow t+\varepsilon$ maps (ξ_4,ϕ_4) to (ξ_2,ϕ_2); X maps (ξ_4,ϕ_4) to (ξ_3,ϕ_3)

TABLE I. (*Continued.*)

$G(u)$	$F(u)$	Number of additional point symmetries; listing of symmetries		Number of additional conservation laws; see Ref. 14 for conservation laws		Comments; symmetry mappings of conservation laws (Ref. 15)
e^u	$e^{\alpha u}$ $[\alpha \neq 0,1]$	1	$X=[2(\alpha-1)xu_x + (\alpha-2)tu_t - 2]\partial_u + [2(\alpha-1)xv_x + (\alpha-2)tv_t - \alpha v]\partial_v$	0		
	$e^u+\alpha$ $[\alpha \neq 0]$	0		4	Table 6: Case 4 with $\beta_1=0$, $\beta_2=1$, $\beta_3=\alpha$	$t \to t+\varepsilon$ maps (ξ_3,ϕ_3) to (ξ_1,ϕ_1); $(t,V)\to(-t,-V)$ maps (ξ_3,ϕ_3) to (ξ_4,ϕ_4) and maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)
	$e^{2u}+\alpha_1 e^u+\alpha_2$ $[\alpha_1^2 \neq 4\alpha_2]$	0		4	Table 6: Case 1 with $\beta_1=1$, $\beta_2=\alpha_1$, $\beta_3=\alpha_2>0$	$(t,V)\to(-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)
					$\xi_1=e^A \sin B$, $\phi_1=e^A(r\cos B - e^U \sin B)$ with $A=a(x+e^U) + {}_{1/2}(a\alpha_1 - 1)U + \alpha t - \rho V$, $B=\kappa t+\gamma V - b(x + e_U + {}_{1/2}\alpha_1 U)$; $a,b,r,\alpha,\gamma,\rho,\kappa$ are given in Table 6: Case 2 with $\beta_1=1$, $\beta_2=\alpha_1$, $\beta_3=\alpha_2<0$	$(t,V)\to(t+c_1,\ V+c_2)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2); $(t,V)\to(-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_3,ϕ_3); $(t,V)\to(-t+c_1,-V+c_2)$ maps (ξ_1,ϕ_1) to (ξ_4,ϕ_4) $[c_1=\alpha_1\pi/2\sqrt{\lvert\alpha_2\rvert}$, $c_2=\sqrt{\lvert\alpha_2\rvert}\pi]$
					Table 6: Case 5 with $\beta_1=1$, $\beta_2=\alpha_1$, $\beta_3=\alpha_2=0$	$t\to t+\varepsilon$ maps (ξ_3,ϕ_3) to (ξ_1,ϕ_1); $(t,V)\to(-t,-V)$ maps (ξ_3,ϕ_3) to (ξ_4,ϕ_4) and maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)
	$(e^u+\alpha)^2$ $[\alpha \neq 0]$	0		2	Table 6: Case 3 with $\beta_1=1$, $\beta_2=2\alpha$, $\beta_3=\alpha^2$	$(t,V)\to(-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)

other conservation laws through the methods presented in Ref. 15. In particular, we show that in many cases the obvious admitted point symmetries $t \to t+\varepsilon_1$ and $v \to v+\varepsilon_2$ are very useful to obtain new conservation laws from a known conservation law.

TABLE I. (*Continued.*)

$G(u)$	$F(u)$	Number of additional point symmetries; listing of symmetries		Number of additional conservation laws; see Ref. 14 for conservation laws		Comments; symmetry mappings of conservation laws (Ref. 15)
$\frac{u^{\delta}\pm 1}{u^{\delta}\mp 1}$ $[\delta\neq 0,\pm 1]$	$\frac{u^{\delta-1}}{(u^{\delta}\mp 1)^2}$	1	$X=\left[\mp\frac{1}{2}\frac{u^{\delta}\pm 1}{u^{\delta}\mp 1}u_x+(t+\delta v)u_t-2u\right]\partial_u+\left[\mp\frac{1}{2}\frac{u^{\delta}\pm 1}{u^{\delta}\mp 1}v_x+(t+\delta v)v_t-(\delta t+v)\right]\partial_v$	3	Table 3: Case 2 with $\nu=\mp 2\delta$, $\mu=0$	X maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2); $t\rightarrow t+\varepsilon$, maps (ξ_1,ϕ_1) to (ξ_3,ϕ_3); $V\rightarrow V+\varepsilon$, maps (ξ_2,ϕ_2) to (ξ_3,ϕ_3)
	$\frac{u^{\delta+\beta-1}}{(u^{\delta}\mp 1)^2}$ $[\beta\neq 0]$	1	$X=[(2\beta x+\delta\int F(u)du)u_x+((\beta+1)t+\delta v)u_t-2u]\partial_u+[(2\beta x+\delta\int F(u)du)v_x+((\beta+1)t+\delta v)v_t-(\delta t+(\beta+1)v\ ]\partial_v$	0		
	$\frac{u^{\delta-1}}{(u^{\delta}\mp 1)^2}+\alpha$ $[\alpha\neq 0]$	0		2	Table 3: Case 2 with $\nu=\mp 2\delta$, $\mu=(2\delta)^2\alpha$	$(t,V)\rightarrow(-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)
	$\frac{u^{\delta-1}}{(u^{\delta}\mp 1)^2}+\kappa\left(\frac{u^{\delta}\pm 1}{u^{\delta}\mp 1}+\rho\right)^2$ $[\kappa\neq 0]$	0		2	Table 3: Case 1 with $\gamma=\mp 2\delta$, $\alpha=(2\delta)^2\kappa$, $\beta=\rho$	″
$\tan(\delta\ln u)$ $[\delta\neq 0]$	$u^{-1}\sec^2(\delta\ln u)$	1	$X=[-2\tan(\delta\ln u)u_x+(t-2\delta v)u_t-2u]\partial_u+[-2\tan(\delta\ln u)v_x+(t-2\delta v)v_t-(2\delta t+v)]\partial_v$	3	Table 3: Case 2 with $\nu=\delta$, $\mu=0$	X maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2); $t\rightarrow t+\varepsilon$, maps (ξ_1,ϕ_1) to (ξ_3,ϕ_3); $V\rightarrow V+\varepsilon$, maps (ξ_2,ϕ_2) to (ξ_3,ϕ_3)
	$u^{\beta}\sec^2(\delta\ln u)$ $[\beta\neq -1]$	1	$X=[2((\beta+1)x-\delta\int F(u)du)u_x+((\beta+2)t-2\delta u)u_t-2u]\partial_u+[2((\beta+1)x-\delta\int F(u)du)v_x+((\beta+2)t-2\delta v)v_t+(2\delta t+(\beta+2)v)]\partial_v$	0		
	$\left(\sec^2(\delta\ln u)/u\right)+\alpha$ $[\alpha\neq 0]$	0		2	Table 3: Case 2 with $\nu=\delta$, $\mu=\delta^2\alpha$	$(t,V)\rightarrow(-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)
	$\left(\sec^2(\delta\ln u)/u\right)+\kappa[\rho+\tan(\delta\ln u)]^2$ $[\kappa\neq 0]$	0		2	Table 3: Case 1 with $\gamma=\delta$, $\alpha=\delta^2\kappa$, $\beta=\rho$	″

III. FURTHER DISCUSSION

(1) As can be seen in the Symmetry and Conservation Law Classification Table (Table I), for each $(F(u),G(u))$ pair where the NLT system (1) admits nontrivial conservation laws, there exists

TABLE I. (*Continued.*)

$G(u)$	$F(u)$	Number of additional point symmetries; listing of symmetries		Number of additional conservation laws; see Ref. 14 for conservation laws		Comments; symmetry mappings of conservation laws (Ref. 15)
$\tanh u$	$\operatorname{sech}^2 u$	1	$X=(\tanh u u_x+v u_t-1)\partial_u+(\tanh u v_x+v v_t-t)\partial_v$	4	Table 7: Case 3 with $\beta_1=1$, $\beta_2=\beta_3=0$	$t\to t+\varepsilon$ maps (ξ_4,ϕ_4) to (ξ_i,ϕ_i), $i=1,3$; $V\to V+\varepsilon$, maps (ξ_4,ϕ_4) to (ξ_i,ϕ_i), $i=2,3$; X maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)
	$e^{\beta u}\operatorname{sech}^2 u$ $[\beta\neq 0]$	1	$X=[2(\beta x+\int F(u)du)u_x+(\beta t+2v)u_t-2]\partial_u+[2(\beta x+\int F(u)du)v_x+(\beta t+2v)v_t-(2t+\beta v)]\partial_v$	0		
	$\tanh u+\delta$	0		4	Table 7: Case 1 with $\beta_1=0$, $\beta_2=1$, $\beta_3=\delta$, $\lvert\delta\rvert>1$	$(t,V)\to(-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)
					$\xi_1=e^A\cosh U\cos B$, $\phi_1=\left[(1-\delta)^{1/2}/1+e^{2U}\right]e^A\cosh U\times[\sin B-re^{2U}\cos B]$ with $A=a\left(x+\frac{1}{2}U\right)-(\kappa t+\gamma V)$, $B=-b\left(x+\frac{1}{2}U\right)-(\alpha t+\rho V)$, $a,b,r,\alpha,\gamma,\rho,\kappa$ are given in Table 7: Case 2 with $\beta_1=0$, $\beta_2=1$, $\beta_3=\delta$, $\lvert\delta\rvert<1$	$(t,V)\to(t+c_1,V+c_2)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2); $(t,V)\to(-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_3,ϕ_3); $(t,V)\to(-t+c_1,-V+c_2)$ maps (ξ_1,ϕ_1) to (ξ_4,ϕ_4) $[c_1=-\pi/4\sqrt{1-\delta}$, $c_2=(2\delta-1)\pi/4\sqrt{1-\delta}]$
					Table 7: Case 3 with $\beta_1=0$, $\beta_2=1$, $\beta_3=\delta$, $\lvert\delta\rvert=1$	$t\to t+\varepsilon$ maps (ξ_3,ϕ_3) to (ξ_1,ϕ_1); $(t,V)\to(-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2) and maps (ξ_3,ϕ_3) to (ξ_4,ϕ_4)
	$\tanh^2 u+\delta$ $[\delta\neq-1,0]$	0		4	Table 7: Case 1 with $\beta_1=-1$, $\beta_2=0$, $\beta_3=1+\delta$	$(t,V)\to(-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)

TABLE I. (*Continued.*)

$G(u)$	$F(u)$	Number of additional point symmetries; listing of symmetries	Number of additional conservation laws; see Ref. 14 for conservation laws	Comments; symmetry mappings of conservation laws (Ref. 15)
$\tanh u$	$\tanh^2 u + \alpha_1 \tanh u + \alpha_2$ $[\alpha_1^2 \neq 4\alpha_2, \alpha_1 \neq 0]$	0	4 Table 7: Case 1 with $\beta_1 = -1$, $\beta_2 = \alpha_1$, $\beta_3 = 1 + \alpha_2$, $\lvert 1+\alpha_2 \rvert > \lvert \alpha_1 \rvert$	$(t,V) \to (-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)
			Table 7: Case 2 with $\beta_1 = -1$, $\beta_2 = \alpha_1 < 0$, $\beta_3 = 1 + \alpha_2$, $\lvert 1+\alpha_2 \rvert < -\alpha_1$	$(t,V) \to (t+c_1, V+c_2)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2); $(t,V) \to (-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_3,ϕ_3); $(t,V) \to (-t+c_1, -V+c_2)$ maps (ξ_1,ϕ_1) to (ξ_4,ϕ_4) $\left[c_1 = \frac{(2+\alpha_1)\pi}{4\sqrt{-(1+\alpha_1+\alpha_2)}}, c_2 = -\frac{(\alpha_1+2\alpha_2)\pi}{4\sqrt{-(1+\alpha_1+\alpha_2)}}\right]$
			$\xi_1 = e^A \cosh^{1+a} u \cos B$, $\phi_1 = \left[\frac{\sqrt{\alpha_1-(1+\alpha_2)}}{1+e^{2U}}\right] e^A \times \cosh^{1+a} U[\sin B - re^{2U} \cos B]$ with $A = a(x - \frac{1}{2}U) - (\kappa t + \gamma V)$, $B = -b(x + \ln \cosh U + \frac{1}{2}\alpha_1 U) - (\alpha t + \rho V)$; $a, b, r, \alpha, \gamma, \rho, \kappa$ are given in Table 7: Case 2 with $\beta_1 = -1$, $\beta_2 = \alpha_1 > 0$, $\beta_3 = 1 + \alpha_2$, $\lvert 1+\alpha_2 \rvert < \alpha_1$	$(t,V) \to (t+c_1, V+c_2)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2); $(t,V) \to (-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_3,ϕ_3); $(t,V) \to (-t, +c_1, -V+c_2)$ maps (ξ_1,ϕ_1) to (ξ_4,ϕ_4) $\left[c_1 = \frac{(2-\alpha_1)\pi}{4\sqrt{\alpha_1-(1+\alpha_2)}}, c_2 = \frac{(2\alpha_2-\alpha_1)\pi}{4\sqrt{\alpha_1-(1+\alpha_2)}}\right]$

an admitted group of point transformations (discrete and/or continuous) that maps a conservation law to one or more (up to three) new conservation laws for the same $(F(u), G(u))$ pair through use of the work presented in Ref. 15.

TABLE I. (*Continued.*)

$G(u)$	$F(u)$	Number of additional point symmetries; listing of symmetries		Number of additional conservation laws; see Ref. 14 for conservation laws		Comments; symmetry mappings of conservation laws (Ref. 15)
$\tanh u$	$\tanh^2 u$ $+\alpha_1 \tanh u+\alpha_2$ $[\alpha_1^2 \neq 4\alpha_2, \alpha_1 \neq 0]$	0		4	Table 7: Case 3 with $\beta_1=-1$, $\beta_2=\alpha_1$, $\beta_3=1$ $+\alpha_2$, $\lvert\beta_3\rvert=\lvert\alpha_1\rvert$	$t\rightarrow t+\varepsilon$ maps (ξ_3,ϕ_3) to (ξ_1,ϕ_1); $(t,V)\rightarrow(-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2) and maps (ξ_3,ϕ_3) to (ξ_4,ϕ_4)
	$(\tanh u+\delta)^2$ $[\delta \neq \pm 1]$	0		2	Table 7: Case 5 with $\beta_1=-1$, $\beta_2=2\delta$, $\beta_3=1+\delta^2$	$(t,V)\rightarrow(-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)
	1	0		2	Table 7: Case 5 with $\beta_3=1$ $\beta_1=\beta_2=0$	"
$\coth u$	$e^{\beta u}\cosh^2 u$	1	See comments	0 or 4	See comments	Symmetries and conservation laws obtained from the the cases where $G(u)=\tanh u$ through equivalence transformation $(x,t,u,v)\rightarrow\left(x,t,u+\frac{\pi}{2}i,v\right)$
	$\gamma_1 \coth^2 u$ $+\gamma_2 \coth u+\gamma_3$	0		2 or 4	See comments	
$\tan u$	$\sec^2 u$	1	$X=(1+\tan u u_x+v u_t)\partial_u$ $+(t+\tan u v_x+v v_t)\partial_v$	4	Table 8: Case 2 with $\beta_1=1$, $\beta_2=\beta_3=0$	$V\rightarrow V+\varepsilon$ maps (ξ_1,ϕ_1) to (ξ_i,ϕ_i), $i=2,3$; $t\rightarrow t+\varepsilon$ maps (ξ_1,ϕ_1) to ξ_i,ϕ_i, $i=1,3$; X maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)
	$e^{\beta u}\sec^2 u$ $[\beta \neq 0]$	1	$X=[2(\beta x-\int F(u)du)u_x$ $+(\beta t-2v)u_t-2]\partial_u$ $+[2(\beta x-\int F(u)du)v_x$ $+(\beta t-2v)v_t$ $-(2t+\beta v)]\partial_v$	0		
	$\tan u+\delta$	0		4	Table 8: Case 1 with $\beta_1=0$, $\beta_2=1$, $\beta_3=\delta$	$(t,V)\rightarrow(-t,-V)$ maps (ξ_1,ϕ_1) to (ξ_2,ϕ_2)

(2) It can happen that a nonvariational system of PDEs can become variational through a differential substitution. For example, the Korteweg–de Vries equation

$$u_t + uu_x + u_{xxx} = 0$$

is not variational but becomes variational after the differential substitution $u=\Gamma_x$. One can show that the differential substitution $(u,v)=(\Gamma_x,\Delta_x)$, leads to an NLT system (1) that is variational if

TABLE I. (*Continued.*)

$G(u)$	$F(u)$	Number of additional point symmetries; listing of symmetries	Number of additional conservation laws; see Ref. 14 for conservation laws	Comments; symmetry mappings of conservation laws (Ref. 15)
$\tan u$	$\tan^2 u+\delta$ $[\delta\neq 0,1]$	0	4 Table 8: Case 2 with $\beta_1=1$, $\beta_2=0$, $\beta_3=\delta-1$	"
	$\tan^2 u$ $+\alpha_1 \tan u+\alpha_2$ $[\alpha_1^2\neq 4\alpha_2, \alpha_1\neq 0]$	0	4 Table 8: Case 1 with $\beta_1=1$, $\beta_2=\alpha_1$, $\beta_3=\alpha_2-1$	"
	$(\tan u+\delta)^2$	0	2 Table 8: Case 3 with $\beta_1=1$, $\beta_2=2\delta$, $\beta_3=\delta^2-1$	"
	1	0	2 Table 8: Case 3 with $\beta_3=1$, $\beta_1=\beta_2=0$	"

and only if $G(u)=\text{const}$ [in this variational case, the NLT system (1) is linearizable by a hodograph transformation].

(3) In general, suppose a system of PDEs with two dependent variables (u,v) and two independent variables (x,t) is not variational but becomes variational through the differential substitution $(u,v)=(\Gamma_x,\Delta_x)$, then an admitted point symmetry

$$\eta=\hat{\eta}(x,t,\Gamma,\Delta)-\hat{\xi}(x,t,\Gamma,\Delta)\Gamma_x-\hat{\tau}(x,t,\Gamma,\Delta)\Gamma_t,$$

$$\zeta=\hat{\eta}(x,t,\Gamma,\Delta)-\hat{\xi}(x,t,\Gamma,\Delta)\Delta_x-\hat{\tau}(x,t,\Gamma,\Delta)\Delta_t,$$

of the variational system would yield multipliers of the form $\{\xi(x,t,U,V),\phi(x,t,U,V)\}$ of the given system, if and only if $\hat{\tau}(x,t,\Gamma,\Delta)=0$ and $\hat{\eta}$, $\hat{\zeta}$ and $\hat{\xi}$ do not depend explicitly on Γ and Δ (otherwise, such a set of multipliers yields a set of nonlocal multipliers and nonlocal symmetries of the given system). Conversely, suppose the given system admits a conservation law resulting from a set of multipliers of the form $\{\xi(x,t,U,V),\phi(x,t,U,V)\}$, then such a set of multipliers yields a point symmetry admitted by the variational system if and only if $\xi_V=\xi_{UU}=\phi_U=\phi_{VV}=0$ [otherwise, such a set of multipliers would yield a local (but not point) symmetry admitted by the variational system].

(4) In general, for a nonvariational system of PDEs, there is a direct connection between conservation laws and symmetries if the system is linear or directly linearizable by a point or contact transformation. For the other exhibited cases, in view of the previous two remarks it would be interesting to investigate if there exist nonlocal symmetries directly connected to the conservation laws.

ACKNOWLEDGMENT

The authors acknowledge financial support from the National Sciences and Engineering Research Council of Canada and also the second author (T.) is thankful for partial support from NSF of China (Grant No. 10461005).

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