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Conservation laws for nonlinear telegraph equations

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Abstract

A complete conservation law classification is given for nonlinear telegraph (NLT) systems with respect to multipliers that are functions of independent and dependent variables. It turns out that a very large class of NLT systems admits four nontrivial local conservation laws. The results of this work are summarized in tables which display all multipliers, fluxes and densities for the corresponding conservation laws. A physical example is considered for possible applications.

Keywords: Conservation laws; Nonlinear telegraph equations

1. Introduction

In this paper we give the complete conservation law classification for nonlinear telegraph (NLT) systems of the form

$$\begin{aligned} H_1[u,v] &= v_t - F(u)u_x - G(u) = 0, \\ H_2[u,v] &= u_t - v_x = 0, \end{aligned} \tag{1}$$

with respect to a class of multipliers that are functions of the independent and dependent variables of the system.

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In [1], it is shown how to find the multipliers for conservation laws of a given system of PDEs. In [2] and [3], the direct construction method is presented to obtain the multipliers and, through an integral formula, the corresponding conservation laws for a given system of partial differential equations (PDEs). Connections between multipliers and the linearization of a given system of PDEs are presented in [4].

One physical example related to system (1) is represented by the equations of telegraphy of a two-conductor transmission line with v as the current in the conductors, u as the voltage between the conductors, $G(u)$ as the leakage current per unit length, $F(u)$ as the differential capacitance, t as a spatial variable and x as time [5]. Another physical example related to system (1) is the equation of motion of a hyperelastic homogeneous rod whose cross-sectional area varies exponentially along the rod. Here u is the displacement gradient related to the difference between a spatial Eulerian coordinate and a Lagrangian coordinate x, v is the velocity of a particle displaced by this difference, $G(u)$ is essentially the stress-tensor, $F(u) = \lambda G'(u)$ for some constant λ, and t is time (see [6,7]).

The complete point symmetry classification of system (1) is given in [8]. As far as the authors know, a classification of the conservation laws of (1) for various forms of $F(u)$ and $G(u)$ has not yet been presented.

Two functions $\xi = \xi(x, t, U, V)$ and $\phi = \phi(x, t, U, V)$ are *multipliers* of a conservation law of system (1) if they satisfy

$$\xi H_1[U, V] + \phi H_2[U, V] \equiv D_x X + D_t T \tag{2}$$

for all differentiable functions $U(x, t)$ and $V(x, t)$ and some differentiable functions $X = X(x, t, U, V)$ and $T = T(x, t, U, V)$. Consequently, the conservation law

$$D_x X + D_t T = 0 \tag{3}$$

holds for all solutions $U = u(x, t)$, $V = v(x, t)$ of NLT system (1) with *flux* $X(x, t, u, v)$ and *density* $T(x, t, u, v)$. Throughout this paper, (U, V) denotes arbitrary functions of x and t; (u, v) denotes solutions of PDE system (1).

The necessary and sufficient conditions for $\xi(x, t, U, V)$, $\phi(x, t, U, V)$ to yield multipliers for a conservation law of (1) are that the Euler operators E_U and E_V with respect to U and V, respectively, annihilate the left-hand side of (2), i.e.,

$$\begin{aligned} &E_U\big[\phi(x, t, U, V)(U_t - V_x) + \xi(x, t, U, V)\big(V_t - F(U)U_x - G(U)\big)\big] \equiv 0, \\ &E_V\big[\phi(x, t, U, V)(U_t - V_x) + \xi(x, t, U, V)\big(V_t - F(U)U_x - G(U)\big)\big] \equiv 0, \end{aligned}$$

for all differentiable functions $U(x, t)$ and $V(x, t)$, where

$$E_U = \frac{\partial}{\partial U} - D_x \frac{\partial}{\partial U_x} - D_t \frac{\partial}{\partial U_t}, \qquad E_V = \frac{\partial}{\partial V} - D_x \frac{\partial}{\partial V_x} - D_t \frac{\partial}{\partial V_t};$$

D_x, D_t are total derivative operators with respect to the independent variables x and t. In particular,

$$\begin{aligned} D_x = {} & \frac{\partial}{\partial x} + U_x \frac{\partial}{\partial U} + V_x \frac{\partial}{\partial V} + U_{xx} \frac{\partial}{\partial U_x} + U_{xt} \frac{\partial}{\partial U_t} + V_{xx} \frac{\partial}{\partial V_x} \\ & + V_{xt} \frac{\partial}{\partial V_t} + \cdots, \quad \text{etc.} \end{aligned}$$

Consequently, we obtain the following determining equations for ξ and ϕ:

$$\phi_V - \xi_U = 0, \tag{4a}$$

$$\phi_U - F(U)\xi_V = 0, \tag{4b}$$

$$\phi_x - \xi_t - G(U)\xi_V = 0, \tag{4c}$$

$$F(U)\xi_x - \phi_t - G(U)\xi_U - G'(U)\xi = 0, \tag{4d}$$

with x, t, U, V as independent variables and ξ, ϕ as dependent variables.

In Section 2, we give equivalence transformations of NLT system (1) in order to simplify subsequent calculations and presentations of results. In Section 3, we give the complete classification of NLT systems (1) that admit conservation laws resulting from multipliers of the form $\xi(x,t,U,V)$, $\phi(x,t,U,V)$. The pairs of $F(u), G(u)$ admitting conservation laws and the corresponding multipliers, fluxes and densities are given in Tables 1, 3–8. The second physical example is considered in Section 4. Further comments are presented in Section 5.

2. Equivalence transformations of NLT system (1)

To simplify our calculations, we use equivalence transformations of NLT system (1). In particular, one has the following theorem.

Theorem 1. *Any transformation of the form*

$$\begin{gathered} x = a\bar{x} + b, \qquad t = c\bar{t} + d, \qquad U = \alpha\bar{U} + \beta, \qquad V = \gamma\bar{V} + \rho\bar{t} + \delta, \\ \bar{F}(\bar{U}) = \frac{\alpha c}{a\gamma}F(U), \qquad \bar{G}(\bar{U}) = \frac{c}{\gamma}G(U) - \frac{\rho}{\gamma} \end{gathered} \tag{5}$$

with $a\alpha = \gamma c$, $a\alpha\gamma c \neq 0$, is an equivalence transformation of NLT *system* (1)*, i.e., transformation* (5) *maps the* NLT *system* (1) *to the* NLT *system* (H) *given by*

$$\begin{gathered} \bar{H}_1[\bar{u},\bar{v}] = \bar{v}_{\bar{t}} - \bar{F}(\bar{u})\bar{u}_{\bar{x}} - \bar{G}(\bar{u}) = 0, \\ \bar{H}_2[\bar{u},\bar{v}] = \bar{u}_{\bar{t}} - \bar{v}_{\bar{x}} = 0, \end{gathered} \tag{H}$$

where $H_1[U,V] = \frac{\gamma}{c}\bar{H}_1[\bar{U},\bar{V}]$, $H_2[U,V] = \frac{\gamma}{a}\bar{H}_2[\bar{U},\bar{V}]$.

The identity (2) and transformation (5) directly lead to the following proposition.

Proposition 1. *An equivalence transformation* (5) *induces a one-to-one mapping between conservation laws of system* (1) *and conservation laws of system* (H)*. In particular, if $X = X(x,t,u,v)$, $T = T(x,t,u,v)$ are the flux and density for a conservation law of* NLT *system* (1) *corresponding to the multipliers $\xi = \xi(x,t,U,V)$, $\phi = \phi(x,t,U,V)$, then the equivalence transformation* (5) *maps (X,T) and (ξ,ϕ) to the flux and density $X' = X'(\bar{x},\bar{t},\bar{u},\bar{v})$, $T' = T'(\bar{x},\bar{t},\bar{u},\bar{v})$ for a conservation law of system* (H) *corresponding to the multipliers $\xi' = \xi'(\bar{x},\bar{t},\bar{U},\bar{V})$, $\phi' = \phi'(\bar{x},\bar{t},\bar{U},\bar{V})$ with*

$$
\begin{aligned}
&X'(\bar{x},\bar{t},\bar{u},\bar{v}) = cX(a\bar{x}+b, c\bar{t}+d, \alpha\bar{u}+\beta, \gamma\bar{v}+\rho\bar{t}+\delta),\\
&T'(\bar{x},\bar{t},\bar{u},\bar{v}) = aT(a\bar{x}+b, c\bar{t}+d, \alpha\bar{u}+\beta, \gamma\bar{v}+\rho\bar{t}+\delta);\\
&\xi'(\bar{x},\bar{t},\bar{U},\bar{V}) = a\gamma\xi(a\bar{x}+b, c\bar{t}+d, \alpha\bar{U}+\beta, \gamma\bar{V}+\rho\bar{t}+\delta),\\
&\phi'(\bar{x},\bar{t},\bar{U},\bar{V}) = c\gamma\phi(a\bar{x}+b, c\bar{t}+d, \alpha\bar{U}+\beta, \gamma\bar{V}+\rho\bar{t}+\delta).
\end{aligned}
\tag{6}
$$

This proposition means that if the conservation laws of system (1) are known for a given pair $(F(u), G(u))$, then through the equivalence transformation (5) and corresponding relations (6), one can find the conservation laws of system (1) for all pairs $(\bar{F}, \bar{G})$ obtained from $(F(u), G(u))$ by an arbitrary translation and scaling of u, an arbitrary translation and scaling of G and arbitrary scaling of F. Throughout this paper we use such equivalence transformations to simplify our analysis.

An important use of the equivalence transformation is to get standard representatives for $F(u)$ and $G(u)$ in presenting results. For example, if $G(u) = Ae^{Bu} + C$ and $F(u) = \beta_1 e^{2Bu} + \beta_2 e^{Bu} + \beta_3$ for constants $A, B, C, \beta_1, \beta_2$ and β_3, then, without lost of generality, one can assume that $G(u) = e^u$ and $F(u) = e^{2u} + \beta_2 e^u + \beta_3$ if $\beta_1 \neq 0$.

Note that the parameters in an equivalence transformation can be complex. In particular, if t and V are imaginary, then the equivalence transformation (5) with $a = -1$, $c = i$, $\alpha = 1$, $\gamma = i$, $b = d = \beta = \rho = \delta = 0$, i.e.,

$$
x = -\bar{x}, \qquad t = i\bar{t}, \qquad U = \bar{U}, \qquad V = i\bar{V}
\tag{7}
$$

has real variables $\bar{x}, \bar{t}, \bar{U}, \bar{V}$. Transformation (7) maps NLT system (1) to system (H) with

$$
\bar{F}(\bar{U}) = -F(U), \qquad \bar{G}(\bar{U}) = G(U).
\tag{8}
$$

Consequently, one obtains

$$
\begin{aligned}
&X'(\bar{x},\bar{t},\bar{u},\bar{v}) = iX(-\bar{x}, i\bar{t}, \bar{u}, i\bar{v}),\\
&T'(\bar{x},\bar{t},\bar{u},\bar{v}) = -T(-\bar{x}, i\bar{t}, \bar{u}, i\bar{v});\\
&\xi'(\bar{x},\bar{t},\bar{U},\bar{V}) = -i\xi(-\bar{x}, i\bar{t}, \bar{U}, i\bar{V}),\\
&\phi'(\bar{x},\bar{t},\bar{U},\bar{V}) = -\phi(-\bar{x}, i\bar{t}, \bar{U}, i\bar{V}).
\end{aligned}
\tag{9}
$$

From identity (2), we see that the real and imaginary parts of (X', T') and (ξ', ϕ') in (9) yield real conservation laws of NLT system (H).

The equivalence transformation (5) includes the continuous symmetry

$$
x = \bar{x}, \qquad t = \bar{t} + d, \qquad u = \bar{u}, \qquad v = \bar{v} + \delta,
\tag{10}
$$

and the discrete symmetry

$$
x = \bar{x}, \qquad t = -\bar{t} + d, \qquad u = \bar{u}, \qquad v = -\bar{v} + \delta
\tag{11}
$$

of NLT system (1).

Later in this paper, we will see that the special equivalence transformations (7), (10) and (11) significantly simplify the classification of the conservation laws for NLT system (1).

3. Determination of multipliers, fluxes and densities

In this section, we consider the problem of solving the system of determining equations (4) and finding the corresponding fluxes and densities. In particular, we find compatibility conditions so that $F(u)$ and $G(u)$ yield solutions of (4). Then we determine all conservation laws of each NLT system (1) in terms of multipliers.

We first derive a formula to determine the flux and density (X, T) for a known set of multipliers (ξ, ϕ). Expanding the right-hand side of (2), one obtains

$$\begin{aligned}&\xi(x,t,U,V)\big(V_t - F(U)U_x - G(U)\big) + \phi(x,t,U,V)(U_t - V_x)\\ &\quad \equiv X_x + X_U U_x + X_V V_x + T_t + T_U U_t + T_V V_t,\end{aligned} \tag{12}$$

which must hold for *all* differentiable functions $U(x,t)$, $V(x,t)$. Comparing the coefficients of U_t, V_t, U_x, V_x and the remaining terms of both sides of (12), one obtains

$$\begin{aligned}&T_U = \phi, \qquad T_V = \xi, \qquad X_U = -F(U)\xi,\\ &X_V = -\phi, \qquad X_x + T_t = -G(U)\xi.\end{aligned} \tag{13}$$

By direct calculation, one can establish the following theorem.

Theorem 2. *For any set of multipliers $\xi(x,t,U,V)$, $\phi(x,t,U,V)$ solving the determining system* (4)*, the solution of system* (13) *is given by*

$$\begin{aligned}X &= -\int_a^U \xi(x,t,s,b)F(s)\,ds - \int_b^V \phi(x,t,U,s)\,ds - G(a)\int^x \xi(s,t,a,b)\,ds,\\ T &= \int_a^U \phi(x,t,s,b)\,ds + \int_b^V \xi(x,t,U,s)\,ds.\end{aligned} \tag{14}$$

Moreover, $X = X(x,t,u,v)$ and $T = T(x,t,u,v)$ defined by (14) *yield the flux and density of the corresponding conservation law of* NLT *system* (1) *for any solution $U = u(x,t)$, $V = v(x,t)$ of* (1)*. In* (14)*, constants a and b are chosen so that the integrals are not singular.*

Now we consider the problem of finding all pairs $(F(u), G(u))$ so that the determining system (4) for multipliers $\xi(x,t,U,V)$, $\phi(x,t,U,V)$ has a solution.

Note that the NLT system (1) is linearizable by a point transformation [4,9] if and only if

$$F(u) = \frac{c}{(au+b)^2}, \qquad G(u) = \frac{d}{au+b} + f$$

for arbitrary constants a, b, c, d, f, or

$$F(u) \text{ is any specific function}, \qquad G(u) = \text{const}.$$

Obviously, if

$$F(u) = \text{const}, \qquad G(u) = u,$$

then the NLT system (1) is linear. For the linearizable and linear cases, the NLT system (1) admits an infinite number of conservation laws [4,9]. In this paper we exclude the investigation of conservation laws for both the linearizable and linear cases.

Before proceeding further, note that for arbitrary $F(u)$ and $G(u)$, the determining system (4) has a particular solution

$$\xi = 0, \qquad \phi = \text{const.} \tag{15}$$

More generally, it is easy to see that if $\xi \equiv 0$, then $\phi = \text{const}$ is the only possible solution of system (4) for any specific $F(u)$ and $G(u)$. The set of multipliers (15) leads to an obvious conservation law with flux and density given by

$$X = -v, \qquad T = u. \tag{16}$$

For the rest part of this paper, we exclude this obvious case.

In solving system (4), we consider separately three possible cases: (a) $G(u)$ arbitrary; (b) $F(u)$ arbitrary; (c) $F(u)$ and $G(u)$ specific functions of u. In (b) and (c), further subcases arise.

3.1. $G(u)$ is arbitrary

It is easy to see that if $G(u)$ is arbitrary, then $\xi \equiv 0$ follows from Eq. (4d). This leads to solution (15) and flux and density (16). Hence the NLT system admits no additional conservation laws.

3.2. $F(u)$ is arbitrary

If $F(u)$ is arbitrary, then from Eqs. (4b, d), we have $\xi_x = \xi_V = 0$ and consequently, $\phi_U = 0$. Using $\phi_U = 0$ and taking $\partial/\partial U$ of (4a, d), we obtain

$$\xi_{UU} = 0, \qquad \big(G(U)\xi\big)_{UU} = 0. \tag{17}$$

Consequently, the NLT system (1) admits nontrivial conservation laws for three cases of $G(u)$ as summarized in the following theorem.

Theorem 3. *If $F(u)$ is arbitrary, then* NLT *system* (1) *admits nontrivial conservation laws if and only if $G(u) = 0$, $G(u) = u$ or $G(u) = 1/u$.*

The corresponding multipliers, fluxes and densities are given in Table 1.

3.3. $F(u)$ and $G(u)$ are specific functions

If neither $F(u)$ nor $G(u)$ is an arbitrary function, the determining system (4) is solvable if and only if the integrability conditions for ϕ and ξ are satisfied. The integrability conditions $\phi_{UV} = \phi_{VU}$, $\phi_{Vx} = \phi_{xV}$ and $\phi_{Ut} = \phi_{tU}$ yield

$$2G'(U)\xi_U - F'(U)\xi_x + G''(U)\xi = 0. \tag{18}$$

Table 1
$F(u)$ is arbitrary

$F(u)$	$G(u)$	Multipliers (ξ, ϕ); flux and density (X, T)
Arbitrary	0	$\xi_1 = U,\ \phi_1 = V$; $X_1 = -\int^u sF(s)\,ds - v^2/2,\ T_1 = v^2$. $\cdots\cdots$ $\xi_2 = t,\ \phi_2 = x$; $X_2 = -t\int^u F(s)\,ds - vx,\ T_2 = xu + tv$. $\cdots\cdots$ $\xi_3 = 1,\ \phi_3 = 0$; $X_3 = -\int^u F(s)\,ds,\ T_3 = v$.
	u	$\xi_1 = t,\ \phi_1 = x - t^2/2$; $X_1 = (t^2/2 - x)v - t\int^u F(s)\,ds,\ T_1 = (x - t^2/2)u + tv$. $\cdots\cdots$ $\xi_2 = 1,\ \phi_2 = -t$; $X_2 = tv - \int^u F(s)\,ds,\ T_2 = v - tu$.
	$1/u$	$\xi_1 = U,\ \phi_1 = V$; $X_1 = -x - v^2/2 - \int^u sF(s)\,ds,\ T_1 = uv$. $\cdots\cdots$ $\xi_2 = UV,\ \phi_2 = V^2/2 + x + \int^U sF(s)\,ds$; $X_2 = -v^3/6 - xv - v\int^u sF(s)\,ds$, $T_2 = \int^u (\int^s (zF(z))\,dz)\,ds + (x + v^2/2)u$.

Using the Characteristic Set algorithm for differential polynomial systems [10] and its implemented computer algebra system program, we obtain the following additional reduced equations for ξ:

$$a(U)\xi_{xx} + b(U)\xi_x + c(U)\xi = 0, \tag{19}$$

$$d(U)\xi_x - h(U)\xi = 0 \tag{20}$$

under the nondegeneracy condition

$$F(U)G'(U) \neq 0.$$

In (19) and (20),

$$\begin{aligned}
a(U) &= G'(U)F'(U)^2 + 2F(U)F'(U)G''(U) - 2F(U)G'(U)F''(U),\\
b(U) &= 2\big[-2G'(U)F'(U)G''(U) - F(U)G''(U)^2 + G'(U)^2F''(U) + F(U)G'(U)G'''(U)\big],\\
c(U) &= G'(U)\big[3G''(U)^2 - 2G'(U)G'''(U)\big],\\
d(U) &= G'(U)^2F^{(3)}(U) - 3G'(U)G''(U)F''(U) + \big[3G''(U)^2 - G'(U)G'''(U)\big]F'(U),\\
h(U) &= 3G''(U)^3 - 4G'(U)G''(U)G'''(U) + G'(U)^2G^{(4)}(U).
\end{aligned} \tag{21}$$

Note that if $G'(u) = 0$, then system (1) is linearizable.

System (18)–(20) significantly simplifies the analysis for solving the determining system (4) and leads to compatibility conditions for $F(u)$ and $G(u)$.

Table 2
$F(u)$ and $G(u)$ in case (C3) [$d(u)=0$, $h(u)=0$]

Case	$G(u)$	$F(u)$
A	u	$\beta_1 u^2+\beta_2 u+\beta_3$
B	$1/u$	$\beta_1/u^2+\beta_2/u+\beta_3$
C	e^u	$\beta_1 e^{2u}+\beta_2 e^u+\beta_3$
D	$\tanh u$	$\beta_1 \operatorname{sech}^2 u+\beta_2\tanh u+\beta_3$
E	$\tan u$	$\beta_1\sec^2 u+\beta_2\tan u+\beta_3$

Since $\xi\not\equiv 0$, from (20) it follows that if $d(U)=0$, then $h(U)=0$. Thus the solution of the determining system (4) naturally separates into three subcases:

$$d(U)\neq 0,\qquad h(U)\neq 0; \tag{C1}$$
$$d(U)\neq 0,\qquad h(U)=0; \tag{C2}$$
$$d(U)=0,\qquad h(U)=0. \tag{C3}$$

In subcase (C2), one can prove that the situation reduces to the cases listed in Table 1. In subcase (C3), $F(u)$ and $G(u)$ satisfy the coupled nonlinear system of ODEs

$$\begin{aligned} d(u)={}&G'(u)^2F^{(3)}(u)-3G'(u)G''(u)F''(u)\\ &+\big[3G''(u)^2-G'(u)G'''(u)\big]F'(u)=0, \end{aligned}\tag{22}$$

$$h(u)=3G''(u)^3-4G'(u)G''(u)G'''(u)+G'(u)^2G^{(4)}(u)=0. \tag{23}$$

The details for the solution of system (22), (23) are given in Appendix A. Its solutions are summarized in Table 2.

For all other solutions of the determining system (4), using (18)–(20), one can show that $F(u)$ and $G(u)$ must satisfy one of the following two conditions:

$$\gamma F(u)-G'(u)=\frac{\alpha}{\gamma}\big(G(u)+\beta\big)^2 \tag{D1}$$

for some constants α, β and γ with $\alpha\neq 0$, $\gamma\neq 0$;

$$\nu F(u)-G'(u)=\frac{\mu}{\nu} \tag{D2}$$

for some constants ν and μ with $\nu\neq 0$.

In summary, one has the following theorem.

Theorem 4. *Under the condition $F(u)G'(u)\neq 0$, the* NLT *system* (1) *admits nontrivial conservation laws if and only if $F(u)$ and $G(u)$ satisfy one of the following three conditions*:

(a) $\gamma F(u)-G'(u)=\frac{\alpha}{\gamma}(G(u)+\beta)^2$ *for some constants β, $\alpha\neq 0$, $\gamma\neq 0$;*
(b) $\nu F(u)-G'(u)=\frac{\mu}{\nu}$ *for some constants $\nu\neq 0$, μ;*
(c) *$F(u)$ and $G(u)$ take on one of the forms listed in Table* 2.

Table 3
Conditions (a) and (b) in Theorem 4 [$d(u) \neq 0$, $h(u) \neq 0$]

Case	Restrictions on $F(u)$ and $G(u)$	Subcase	Multipliers (ξ, ϕ); flux and density (X, T)
1	$\gamma F(u) - G'(u) = \frac{\alpha}{\gamma}(G(u)+\beta)^2$, [$\alpha \neq 0$, $\gamma \neq 0$, $\rho = \sqrt{\lvert\alpha\rvert}$]	$\alpha > 0$	$\xi_1 = A(x,U)e^{\rho(\beta t+V)}$, $\phi_1 = \frac{\rho}{\gamma}(G(U)+\beta)A(x,U)e^{\rho(\beta t+V)}$; $X_1 = -\frac{1}{\gamma}(G(u)+\beta)A(x,u)e^{\rho(\beta t+v)}$, $T_1 = \frac{1}{\rho}A(x,u)e^{\rho(\beta t+v)}$ $\left[A(x,U) = \exp(\gamma x + \frac{\alpha}{\gamma}\int(G(U)+\beta)\,dU)\right]$. .... $\xi_2 = \xi_1(x,-t,U,-V)$, $\phi_2 = -\phi_1(x,-t,U,-V)$; $X_2 = X_1(x,-t,u,-v)$, $T_2 = -T_1(x,-t,u,-v)$.
		$\alpha < 0$	Obtained from the subcase $\alpha > 0$ through transformation (7); (ξ, ϕ) and (X, T) are given by the real and imaginary parts of (9).
2	$\nu F(u) - G'(u) = \frac{\mu}{\nu}$, [$\nu \neq 0$, $\rho = \sqrt{\lvert\mu\rvert}$]	$\mu > 0$	$\xi_1 = \xi_1(x,t) = e^{\nu x+\rho t}$, $\phi_1 = \phi_1(x,t) = \frac{\rho}{\nu}e^{\nu x+\rho t}$; $X_1 = -e^{\nu x+\rho t}(\int F(u)\,du + \frac{\rho}{\gamma}v)$, $T_1 = e^{\nu x+\rho t}(\frac{\rho}{\nu}u + v)$. .... $\xi_2 = \xi_1(x,-t)$, $\phi_2 = -\phi_1(x,-t)$; $X_2 = X_1(x,-t,u,-v)$, $T_2 = -T_1(x,-t,u,-v)$.
		$\mu < 0$	Obtained from the subcase $\mu > 0$ through transformation (7); (ξ, ϕ) and (X, T) are given by the real and imaginary parts of (9).
		$\mu = 0$	$\xi_1 = te^{\nu x}$, $\phi_1 = \frac{1}{\nu}e^{\nu x}$; $X_1 = -\frac{1}{\nu}e^{\nu x}(tG(u) + v)$, $T_1 = e^{\nu x}(\frac{1}{\nu}u + tv)$. .... $\xi_2 = Ve^{\nu x}$, $\phi_2 = \frac{1}{\nu}G(U)e^{\nu x}$; $X_2 = -\frac{1}{\nu}vG(u)e^{\nu x}$, $T_2 = \frac{1}{\nu}e^{\nu x}(\int G(u)\,du + \frac{\nu}{2}v^2)$. .... $\xi_3 = e^{\nu x}$, $\phi_3 = 0$; $X_3 = -\frac{1}{\nu}G(u)e^{\nu x}$, $T_3 = ve^{\nu x}$.

Note that there exist solutions of (22), (23), i.e., case (C3), which satisfy (D1) or (D2).

The classifying functions $F(u)$ and $G(u)$ and their corresponding multipliers, fluxes and densities for conditions (a) and (b) in Theorem 4 (with the additional restriction that $d(u) \neq 0$, $h(u) \neq 0$, i.e., $F(u)$ and $G(u)$ are not one of the five classifying function pairs listed in Table 2) are presented in Table 3. The multipliers, fluxes and densities for the other five classifying function pairs listed in Table 2 are respectively given in Tables 4–8.

4. A physical example

We know specialize to the situation for the second physical example mentioned in the introduction. Here we have $G'(u) = \nu F(u)$, i.e., condition (b) of Theorem 4. This corre-

Table 4
$G(u) = u,\ F(u) = \beta_1 u^2 + \beta_2 u + \beta_3$

Case	Restrictions on constants β_i	Subcase	Multipliers (ξ, ϕ); flux and density (X, T) *in addition to* the two sets given in Table 1
1	$\beta_2^2 \neq 4\beta_1\beta_3$	$\beta_1 \neq 0$	Two more sets of (ξ, ϕ) and (X, T) are given by Case 1 in Table 3 with $\gamma = \frac{4\beta_1}{4\beta_1\beta_3 - \beta_2^2},\ \alpha = \beta_1\gamma^2,\ \beta = \frac{\beta_2}{2\beta_1}$.
		$\beta_1 = 0$	$\xi_3 = 4[6xt + 3\beta_2(Ut - V) - t^3]$, $\phi_3 = t^2(t^2 - 12x + 12\beta_3) + 12x^2 + 6\beta_2(2tV - \beta_2 U^2 - 2\beta_3 U)$; $X_3 = 2\beta_2[(t^3 - 6tx + 3\beta_2 v - 3\beta_3 t)u^2 - t(3v^2 + 2\beta_2 u^3)]$ $\quad + 4\beta_3(t^3 - 6xt + 3\beta_2 v)u - (t^4 - 12xt^2 + 12x^2 + 12\beta_3 t^2)v$, $T_3 = 4(6tx - t^3)v - 2\beta_2(3v^2 + \beta_2 u^3 + 3\beta_3 u^2)$ $\quad + (t^4 - 12xt^2 + 12x^2 + 12\beta_2 tv + 12\beta_3 t^2)u$. .... $\xi_4 = 3(2x + \beta_2 U - t^2)$, $\phi_4 = t^3 - 6tx + 3\beta_2 V + 6\beta_3 t$; $X_4 = 3\beta_3(ut^2 - 2tv - 2xu) - \beta_2^2 u^3 - (t^2 - 6x)tv$ $\quad - \frac{3}{2}\beta_2(v^2 - t^2u^2 + 2xu^2 + \beta_3 u^2)$, $T_4 = 3(\beta_2 u + 2x - t^2)v + t(t^2 - 6x + 6\beta_3)u$.
2	$\beta_2^2 = 4\beta_1\beta_3$	$\beta_1 \neq 0$	No additional conservation laws
		$\beta_1 = 0$, $\beta_2 = 0$, $\beta_3 \neq 0$	Linear case

sponds to $\mu = 0$ in Case 2 of Table 3, $\beta_1 = \beta_2 = 0$ in Case 2 of Table 4 (linear case), $\beta_2 = \beta_3 = 0$ in Case 3 of Table 5 (linearizable case), $\beta_1 = \beta_3 = 0$ in Case 4 of Table 6, $\beta_2 = \beta_3 = 0$ in Case 3 of Table 7 and $\beta_2 = \beta_3 = 0$ in Case 2 of Table 8. In Table 9, we give the multipliers and corresponding fluxes and densities for each of the resulting four cases (excluding the linear and linearizable cases).

5. Further discussion

In this paper we have given a classification of conservation laws for nonlinear telegraph systems of the form (1) in terms of classifying functions $F(u)$ and $G(u)$. In future work we intend to consider further classifications of such systems in terms of seeking conservation laws for equivalent scalar PDEs and systems of PDEs related to (1) under nonlocal transformations. The classification of (1) in terms of admitted pointed symmetries was given in [8]. In both classifications, the Characteristic Set algorithm for differential polynomial systems [10] efficiently solves the overdetermined systems of linear determining equations for multipliers or symmetries.

For a PDE system admitting a variational principle, multipliers for conservation laws are a subset of admitted point symmetries. In a future paper we will compare our classifications of point symmetries and conservation laws for systems of the form (1).

Table 5
$G(u) = 1/u,\ F(u) = \beta_1/u^2 + \beta_2/u + \beta_3$

Case	Restrictions on constants β_i	Subcase	Multipliers (ξ, ϕ); flux and density (X, T) *in addition to* the two sets given in Table 1
1	$\beta_2^2 \neq 4\beta_1\beta_3$	$\beta_2\beta_3 \neq 0$	Two more sets of (ξ, ϕ) and (X, T) are given by Case 1 in Table 3 with $\gamma = 4\beta_3/(\beta_2^2 - 4\beta_1\beta_3)$, $\alpha = (\gamma\beta_2)^2/(4\beta_3)$, $\beta = 2\beta_3/\beta_2$.
		$\beta_2 = 0$	Two more sets of (ξ, ϕ) and (X, T) are given by Case 2 in Table 3 with $\mu = \beta_3/\beta_1^2$, $\nu = -1/\beta_1$.
		$\beta_3 = 0$	$\xi_3 = 4UA(x,t,U,V)$, $\phi_3 = VA(x,t,U,V) + [2(x+\beta_1\ln U) + 3\beta_2 U + 4\beta_1]B(x,U,V) + \beta_2(3tV - 8\beta_1 U - \beta_2 U^2) - 8\beta_1(x + \beta_1\ln U)$; $X_3 = \frac{4}{3}(-\beta_1\ln u - \beta_1 - x)v^3 - 4(-\beta_1\ln u - x)^2 v - \frac{1}{15}v^5 - 2\beta_2(tv^2 + 2tx + 2\beta_1 t\ln u + \beta_2 u^2 v) - 4\beta_2[(2x - \beta_1 + 2\beta_1\ln u)v + \frac{1}{3}v^3 + \beta_2 t]u$, $T_3 = [\frac{1}{3}v^4 + 4(\beta_1\ln u + x)^2 + 8\beta_1^2(1 - \ln u) + 4\beta_2 tv]u - 8\beta_1 x + \frac{2}{3}\beta_2^2 u^3 + 2\beta_2[v^2 + 2x - 2\beta_1(1 - \ln u)]u^2 + 4(x + \beta_1\ln u)v^2 u$. . . . . $\xi_4 = UB(x,U,V)$, $\phi_4 = A(x,t,U,V) + \beta_2 UV + 2\beta_1 V$; $X_4 = -\beta_1[(1 + \ln u)v^2 + \beta_2(1 - u + 2u\ln u)] - xv^2 - \frac{1}{12}v^4 - (\beta_1\ln u + x)^2 - \frac{1}{2}\beta_2^2(u^2 - 1) - \beta_2(tv + uv^2 - x + 2xu)$, $T_4 = (\frac{1}{3}v^2 + 2x + 2\beta_1\ln u)uv + \beta_2[u^2 v + t(u - 1)]$ $\big[A(x,t,U,V) = V^3/3 + \beta_2(UV + t) + 2V(x + \beta_1\ln U)$, $B(x,U,V) = 2x + \beta_2 U + 2\beta_1\ln U + V^2\big]$.
2	$\beta_2^2 = 4\beta_1\beta_3$	$\beta_3 \neq 0$	No additional conservation laws
3	$\beta_2 = \beta_3 = 0$	$\beta_1 \neq 0$	Linearization case

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Appendix A. Solution of nonlinear ODE system (22), (23)

Here we show how to solve the coupled nonlinear system of ODEs (22), (23). After the obvious substitutions $g = G'$, $f = F'$, this system reduces to

$$g^2 f'' - 3gg'f' + (3g'^2 - gg'')f = 0, \tag{A.1}$$

$$3g'^3 - 4gg'g'' + g^2 g''' = 0. \tag{A.2}$$

Note that for any g solving (A.2), the ODE (A.1) becomes a second order linear ODE in terms of f. Moreover, observe that $f = g$ is a solution of (A.1). Hence the substitution

Table 6
$G(u)=e^u$, $F(u)=\beta_1 e^{2u}+\beta_2 e^u+\beta_3$

Case	Restrictions on constants β_i	Subcase	Multipliers (ξ,ϕ); flux and density (X,T)
1	$\beta_1\beta_3>0$, $\beta_2^2\neq 4\beta_1\beta_3$		Four sets of (ξ,ϕ) and (X,T) are given by Case 1 in Table 3 with $\alpha=\beta_1/(\beta_2\pm 2\sqrt{\beta_1\beta_3})^2$, $\gamma=1/(\beta_2\pm 2\sqrt{\beta_1\beta_3})$, $\beta=\mp\sqrt{\beta_3/\beta_1}$.
2	$\beta_1\beta_3<0$, $\beta_2^2\neq 4\beta_1\beta_3$ $\big[a=\frac{\beta_2}{(\beta_2^2-4\beta_1\beta_3)}$, $b=\frac{2\sqrt{\lvert\beta_1\beta_3\rvert}}{(\beta_2^2-4\beta_1\beta_3)}$, $r=\sqrt{\frac{\lvert\beta_3\rvert}{\lvert\beta_1\rvert}}$, $\alpha=b\sqrt{\lvert\beta_3\rvert}$, $\kappa=a\sqrt{\lvert\beta_3\rvert}$, $\rho=a\sqrt{\lvert\beta_1\rvert}$, $\gamma=b\sqrt{\lvert\beta_1\rvert}\big]$	$\beta_3>0$	$\xi_1=e^{A(x,t,U,V)}\sin B(x,t,U,V)$, $\phi_1=\sqrt{\lvert\beta_1\rvert}e^{A(x,t,U,V)}(e^U\cos B(x,t,U,V)-r\sin B(x,t,U,V))$; $X_1=\frac{\sqrt{\lvert\beta_1\rvert}}{\gamma^2+\rho^2}e^{A(x,t,u,v)}[(\alpha-\rho e^u)\sin B(x,t,u,v)$ $-(\kappa+\gamma e^u)\cos B(x,t,u,v)]$, $T_1=\frac{1}{\gamma^2+\rho^2}e^{A(x,t,u,v)}[\gamma\sin B(x,t,u,v)-\rho\cos B(x,t,u,v)]$ $\big[A(x,t,U,V)=ax-\kappa t+\gamma V+\frac{a\beta_2-1}{2}U+a\beta_1 e^U$, $B(x,t,U,V)=\alpha t+\rho V-b(x+\frac{\beta_2}{2}U+\beta_1 e^U)\big]$. ……… $\xi_2=\xi_1(x,t-\gamma\pi/2(\alpha\gamma+\rho\kappa),U,V-\kappa\pi/2(\alpha\gamma+\rho\kappa))$, $\phi_2=\phi_1(x,t-\gamma\pi/2(\alpha\gamma+\rho\kappa),U,V-\kappa\pi/2(\alpha\gamma+\rho\kappa))$; $X_2=X_1(x,t-\gamma\pi/2(\alpha\gamma+\rho\kappa),u,v-\kappa\pi/2(\alpha\gamma+\rho\kappa))$, $T_2=T_1(x,t-\gamma\pi/2(\alpha\gamma+\rho\kappa),u,v-\kappa\pi/2(\alpha\gamma+\rho\kappa))$. ……… $\xi_3=\xi_1(x,-t,U,-V)$, $\phi_3=-\phi_1(x,-t,U,-V)$; $X_3=X_1(x,-t,u,-v)$, $T_3=-T_1(x,-t,u,-v)$. ……… $\xi_4=\xi_1(x,-t-\gamma\pi/2(\alpha\gamma+\rho\kappa),U,-V-\kappa\pi/2(\alpha\gamma+\rho\kappa))$, $\phi_4=-\phi_1(x,-t-\gamma\pi/2(\alpha\gamma+\rho\kappa),U,-V-\kappa\pi/2(\alpha\gamma+\rho\kappa))$; $X_4=X_1(x,-t-\gamma\pi/2(\alpha\gamma+\rho\kappa),u,-v-\kappa\pi/2(\alpha\gamma+\rho\kappa))$, $T_4=-T_1(x,-t-\gamma\pi/2(\alpha\gamma+\rho\kappa),u,-v-\kappa\pi/2(\alpha\gamma+\rho\kappa))$.
		$\beta_3<0$	Obtained from the subcase $\beta_3>0$ through transformation (7); (ξ,ϕ) and (X,T) are given by the real and imaginary parts of (9).
3	$\beta_2^2=4\beta_1\beta_3$	$\beta_2\neq 0$	Two sets of (ξ,ϕ) and (X,T) are given by Case 1 in Table 3 with $\alpha=\beta_1/(4\beta_2^2)$, $\beta=-\beta_2/(2\beta_1)$, $\gamma=1/(2\beta_2)$.
4	$\beta_2\neq 0$, $\beta_1=0$ $\big[\kappa=\frac{\sqrt{\lvert\beta_3\rvert}}{\beta_2}\big]$	$\beta_3>0$	In addition to the two sets of Case 2 in Table 3 with $\mu=\beta_3/\beta_2^2>0$, $\nu=1/\beta_2$: $\xi_3=A(x,t,U,V)e^{\frac{x}{\beta_2}+\kappa t}$, $\phi_3=\beta_2(e^U+\kappa A(x,t,U,V))e^{\frac{x}{\beta_2}+\kappa t}$; $X_3=-\beta_2 e^{\kappa t+x/\beta_2}[(A(x,t,u,v)-\kappa\beta_2)e^u+\kappa A(x,t,u,v)^2/2]$, $T_3=e^{\kappa t+x/\beta_2}[\beta_2 e^u+A(x,t,u,v)^2/2]$ $\big[A(x,t,U,V)=2\kappa(\kappa\beta_2 t+x)+(V-\kappa\beta_2(1-U))\big]$. ……… $\xi_4=\xi_3(x,-t,U,-V)$, $\phi_4=-\phi_3(x,-t,U,-V)$; $X_4=X_3(x,-t,u,-v)$, $T_4=-T_3(x,-t,u,-v)$.
		$\beta_3<0$	Obtained from the subcase $\beta_3>0$ through transformation (7); (ξ,ϕ) and (X,T) are given by the real and imaginary parts of (9).
		$\beta_3=0$	In addition to the three sets of Case 2 in Table 3 with $\mu=0$, $\nu=1/\beta_2$: $\xi_4=e^{\frac{x}{\beta_2}}(2x+tV+\beta_2 U)$, $\phi_4=\beta_2 e^{\frac{x}{\beta_2}}(te^U+V)$; $X_4=-\beta_2 e^{\frac{x}{\beta_2}+u}[v^2e^{-u}/2+tv+2x+\beta_2(u-1)]$, $T_4=e^{\frac{x}{\beta_2}}[(tv/2+2x)v+\beta_2(uv+te^u)]$.

Table 6
(Continued)

Case	Restrictions on constants β_i	Subcase	Multipliers (ξ,ϕ); flux and density (X,T)
5	$\beta_2\neq 0$, $\beta_3=0$ $\left[\kappa=\frac{\sqrt{\lvert\beta_1\rvert}}{\beta_2}\right]$	$\beta_1>0$	In addition to the two sets of Case 1 in Table 3 with $\alpha=\beta_1/\beta_2^2>0$, $\beta=0$, $\gamma=1/\beta_2$: $\xi_3=\exp(\frac{x+\beta_1e^U}{\beta_2}+\kappa V)A(x,t,U,V)$, $\phi_3=\beta_2\exp(\frac{x+\beta_2U+\beta_1e^U}{\beta_2}+\kappa V)(e^{-U}+\kappa A(x,t,U,V))$; $X_3=-e^{\frac{x}{\beta_2}}\{e^{\kappa(v+\kappa\beta_2e^u)}[\beta_2(A(x,t,u,v)-2\kappa\beta_2]e^u+\beta_2/\kappa] - (\beta_2/\kappa)e^{\beta_1/\beta_2}\}$, $T_3=e^{\frac{x}{\beta_2}}\{e^{\beta_1/\beta_2}[2\kappa\beta_2(v+\kappa\beta_2e^u)+\beta_2u-A(x,t,u,v)/\kappa] + e^{\kappa(v+\kappa\beta_2e^u)}(A(x,t,u,v)/\kappa-2\beta_2)\}$ $\left[A(x,t,U,V)=t+2\beta_2\kappa^2(V+\kappa\beta_2e^U)+\kappa(2x-\beta_2(1-U))\right]$. .... $\xi_4=\xi_3(x,-t,U,-V)$, $\phi_4=-\phi_3(x,-t,U,-V)$; $X_4=X_3(x,-t,u,-v)$, $T_4=-T_3(x,-t,u,-v)$.
		$\beta_1<0$	Obtained from the subcase $\beta_1>0$ through transformation (7); (ξ,ϕ) and (X,T) are given by the real and imaginary parts of (9).

$f=zg$ reduces (A.1) to a first order linear ODE in terms of dependent variable z. In particular, $z'=\mu g$ for some arbitrary constant μ. Thus for any solution $G(u)$ of ODE (23), it follows that

$$F(u)=c_1G^2(u)+c_2G(u)+c_3 \tag{A.3}$$

yields the general solution of ODE (22).

Now we obtain the general solution of ODE (A.2). First note that the third order ODE (A.2) admits a solvable three parameter Lie group of point transformations corresponding to its invariance under translations in u, scalings in u, and scalings in g. Hence the general solution of this ODE can be obtained by successively reducing the order of this ODE to an integrable first order ODE in terms of variables chosen according to a correct solvability ordering. See [9] or [11] for details. A correct ordering for these symmetries arises from the corresponding point symmetry generators

$$X_1=\frac{\partial}{\partial u},\qquad X_2=u\frac{\partial}{\partial u},\qquad X_3=g\frac{\partial}{\partial g}.$$

We use the method of differential invariants. Obvious differential invariants for X_1 are given by

$$U=g,\qquad V=g'. \tag{A.4}$$

Then in terms of the variables given by (A.4), we have

$$g''=V\frac{dV}{dU},\qquad g'''=V^2\frac{d^2V}{dU^2}+V\left(\frac{dV}{dU}\right)^2.$$

Table 7
$G(u)=\tanh u,\ F(u)=\beta_1\mathrm{sech}^2 u+\beta_2\tanh u+\beta_3$

Case	Restrictions on constants β_i	Subcase	Multipliers (ξ,ϕ); flux and density (X,T)
1	$\lvert\beta_3\rvert>\lvert\beta_2\rvert$, $\big[c=\beta_1(\beta_1+\beta_3)+\frac{\beta_2^2}{4}\neq 0$, $\lambda^{\pm}=\frac{2\beta_1+\beta_3\pm\sqrt{\beta_3^2-\beta_2^2}}{2c}\big]$	$\beta_2\neq 0$	Four sets of (ξ,ϕ) and (X,T) are given by Case 1 in Table 3 with $\alpha=\lambda^{\pm}(1-\beta_1\lambda^{\pm})$, $\beta=\beta_2\lambda^{\pm}/2(1-\beta_1\lambda^{\pm})$, $\gamma=\lambda^{\pm}$.
		$\beta_2=0$	Four sets of (ξ,ϕ) and (X,T) are given by Case 1 in Table 3 with $\gamma=\frac{1}{\beta_1+\beta_3}$, $\alpha=\frac{\beta_3}{(\beta_1+\beta_3)^2}$, $\beta=0$ and by Case 2 in Table 3 with $\mu=\beta_3/\beta_1^2$, $\nu=1/\beta_1$.
2	$\lvert\beta_3\rvert<\lvert\beta_2\rvert$, $\big[c=\beta_1(\beta_1+\beta_3)+\frac{\beta_2^2}{4}\neq 0$, $a=(2\beta_1+\beta_3)/2c$, $b=-(\sqrt{\lvert\beta_3^2-\beta_2^2\rvert})/2c$, $r=\sqrt{\lvert\frac{\beta_3+\beta_2}{\beta_3-\beta_2}\rvert}$, $\alpha_1=-b\sqrt{\lvert\beta_2+\beta_3\rvert}/2$, $\kappa_1=-a\sqrt{\lvert\beta_2+\beta_3\rvert}/2$, $\rho=\frac{b(2\beta_1-\beta_2)}{2\sqrt{\lvert\beta_3+\beta_2\rvert}}$, $\gamma=-\frac{b(2\beta_1+\beta_2)}{2\sqrt{\lvert\beta_3-\beta_2\rvert}}$, $\alpha=-2\alpha_1-\rho$, $\kappa=-2\kappa_1-\gamma\big]$	$\beta_3>\beta_2$	$\xi_1=e^A(\cosh U)^{1-a\beta_1}\cos B$, $\phi_1=\frac{\sqrt{\beta_3-\beta_2}}{1+e^{2U}}e^A(\cosh U)^{1-a\beta_1}[\cos B-re^{2U}\sin B]$; $X_1=-\frac{\sqrt{\beta_3-\beta_2}}{2(\gamma^2+\rho^2)}e^A(\cosh u)^{-a\beta_1}[(r\gamma e^u+\rho e^{-u})\cos B$ $+(\gamma e^{-u}-r\rho e^u)\sin B]$, $T_1=\frac{1}{\gamma^2+\rho^2}e^A(\cosh u)^{1-a\beta_1}[\rho\cos B+\gamma\sin B]$ $\big[A=A(x,t,U,V)=ax+\alpha t+\rho V+\frac{1}{2}a\beta_2 U$, $B=B(x,t,U,V)=\kappa t+\gamma V-b(x-\beta_1\ln\cosh U+\frac{1}{2}\beta_2 U)\big]$. . . . . $\xi_2=\xi_1(x,t+\rho\pi/2(\alpha\gamma-\rho\kappa),U,V-\alpha\pi/2(\alpha\gamma-\rho\kappa))$, $\phi_2=\phi_1(x,t+\rho\pi/2(\alpha\gamma-\rho\kappa),U,V-\alpha\pi/2(\alpha\gamma-\rho\kappa))$; $X_2=X_1(x,t+\rho\pi/2(\alpha\gamma-\rho\kappa),u,v-\alpha\pi/2(\alpha\gamma-\rho\kappa))$, $T_2=T_1(x,t+\rho\pi/2(\alpha\gamma-\rho\kappa),u,v-\alpha\pi/2(\alpha\gamma-\rho\kappa))$. . . . . $\xi_3=\xi_1(x,-t,U,-V)$, $\phi_3=-\phi_1(x,-t,U,-V)$; $X_3=X_1(x,-t,u,-v)$, $T_3=-T_1(x,-t,u,-v)$. . . . . $\xi_4=\xi_1(x,-t+\rho\pi/2(\alpha\gamma-\rho\kappa),U,-V-\alpha\pi/2(\alpha\gamma-\rho\kappa))$, $\phi_4=-\phi_1(x,-t+\rho\pi/2(\alpha\gamma-\rho\kappa),U,-V-\alpha\pi/2(\alpha\gamma-\rho\kappa))$; $X_4=X(x,-t+\rho\pi/2(\alpha\gamma-\rho\kappa),u,-v-\alpha\pi/2(\alpha\gamma-\rho\kappa))$, $T_4=-T(x,-t+\rho\pi/2(\alpha\gamma-\rho\kappa),u,-v-\alpha\pi/2(\alpha\gamma-\rho\kappa))$.
		$\beta_3<\beta_2$	Obtained from the subcase $\beta_3>\beta_2$ through transformation (7); (ξ,ϕ) and (X,T) are given by the real and imaginary parts of (9).

Hence ODE (A.2) reduces to the second order ODE

$$3V^2-4UV\frac{dV}{dU}+U^2\left(V\frac{d^2V}{dU^2}+\left(\frac{dV}{dU}\right)^2\right)=0. \tag{A.5}$$

Next, obvious differential invariants for X_2 are given by

$$X=U=g,\qquad Y=V^{-1}\frac{dV}{dU}=(g')^{-2}g''.$$

Then

$$\frac{d^2V}{dU^2}=V\left(Y^2+\frac{dY}{dX}\right),$$

and ODE (A.5) transforms to

$$V^2\left(3-4XY+X^2\left(\frac{dY}{dX}+2Y^2\right)\right)=0.$$

Table 7
(Continued)

Case	Restrictions on constants β_i	Subcase	Multipliers (ξ,ϕ); flux and density (X,T)
3	$\beta_3=\beta_2,\ \beta_2\neq-2\beta_1$ $\left[\kappa=\frac{\sqrt{2\lvert\beta_2\rvert}}{2\beta_1+\beta_2}\right]$	$\beta_2>0$	In addition to the two sets of Case 1 in Table 3 with $\alpha=2\beta_2/(2\beta_1+\beta_2)^2>0$, $\beta=1$, $\gamma=2/(2\beta_1+\beta_2)$: $\xi_3=C(x,U)(1+e^{2U})Ae^{\kappa(t+V)}$, $\phi_3=(2\beta_2)^{1/2}C(x,U)(Ae^{2U}+2(\beta_1+\beta_2/2)^2)e^{\kappa(t+V)}$; $X_3=-(2\beta_1+\beta_2)C(x,u)e^{\kappa(t+v)}\{2(\beta_1+\beta_2/2)^2+e^{2u}[A+2\beta_1^2-\beta_2^2/2]\}$, $T_3=\frac{2\beta_1+\beta_2}{\sqrt{2\beta_2}}e^{\kappa(t+v)}(1+e^{2u})C(x,u)[A+2\beta_1^2-\beta_2^2/2]$ $\big[A=A(x,t,U,V)=(2\beta_2)^{\frac{1}{2}}[(\beta_2/2-\beta_1)V+(\frac{3}{2}\beta_2+\beta_1)t]+2\beta_2(\frac{1}{2}\beta_2+\beta_1)U+2\beta_2(x-\beta_1\log(1+e^{2U}))+\beta_1^2-\beta_2^2/4$, $C(x,U)=\exp(\frac{2x}{\beta_2+2\beta_1})(1+e^{2U})^{-\frac{2\beta_1}{\beta_2+2\beta_1}}\big]$. .... $\xi_4=\xi_3(x,-t,U,-V)$, $\phi_4=-\phi_3(x,-t,U,-V)$; $X_4=X_3(x,-t,u,-v)$, $T_4=-T_3(x,-t,u,-v)$.
		$\beta_2<0$	Obtained from the subcase $\beta_2>0$ through transformation (7); (ξ,ϕ) and (X,T) are given by the real and imaginary parts of (9).
		$\beta_2=0$	In addition to the three sets of Case 2 in Table 3 with $\mu=0$, $\nu=1/\beta_1$: $\xi_4=e^{x/\beta_1}[2x+t^2-V^2-2\beta_1\ln\cosh U]$, $\phi_4=2\beta_1e^{x/\beta_1}[t-V\tanh U]$; $X_4=\beta_1e^{x/\beta_1}\{-2(tv+\beta_1u)+[v^2-t^2-2x+2\beta_1(1+\ln\cosh u)]\tanh u\}$, $T_4=e^{x/\beta_1}[v(t^2-v^2/3+2x)+2\beta_1(tu-v\ln\cosh u)]$.
4	$\beta_3=-\beta_2$, $\beta_2\neq2\beta_1$	Obtained from Case 3 of this table through equivalence transformation $x=\bar{x}$, $t=-\bar{t}$, $U=-\bar{U}$, $V=\bar{V}$; (ξ,ϕ) and (X,T) are obtained from (6).	
5	$\beta_2^2=-4\beta_1(\beta_1+\beta_3)$, $\lvert\beta_3\rvert\neq\lvert\beta_2\rvert$	$\beta_2\neq0$ or $\beta_1=0$, $\beta_2=0$	Two sets of (ξ,ϕ) and (X,T) are given by Case 1 in Table 3 with $\gamma=\frac{1}{2\beta_1+\beta_3}$, $\alpha=(\beta_1+\beta_3)\gamma^2$, $\beta=\beta_2/2(\beta_1+\beta_3)$.
		$\beta_2=0$, $\beta_1\neq0$	Two sets of (ξ,ϕ) and (X,T) are given by Case 2 in Table 3 with $\mu=-1/\beta_1$, $\nu=1/\beta_1$.

Thus two cases arise according to whether or not $V=0$.

If $V\neq0$ the reduced ODE is

$$\frac{dY}{dX}=\frac{4XY-3-2X^2Y^2}{X^2}. \tag{A.6}$$

Now we focus on the first order ODE (A.6). Clearly

$$Z=XY=g(g')^{-2}g''$$

Table 8
$G(u) = \tan u,\ F(u) = \beta_1 \sec^2 u + \beta_2 \tan u + \beta_3$

Case	Restrictions on constants β_i	Subcase	Multipliers (ξ, ϕ); flux and density (X, T)
1	$\beta_2^2 \neq 4\beta_1(\beta_1 + \beta_3)$, $\left[\lambda^{\pm} = \frac{2(-2\beta_1 - \beta_3 \pm \sqrt{\beta_2^2 + \beta_3^2})}{\beta_2^2 - 4\beta_1(\beta_1+\beta_3)}\right]$	$\beta_2 \neq 0$	Four sets of (ξ, ϕ) and (X, T) are given by Case 1 in Table 3 with $\alpha = \lambda^{\pm}(\lambda^{\pm}\beta_1 - 1)$, $\beta = \frac{\beta_2 \lambda^{\pm}}{2(\lambda^{\pm}\beta_1 - 1)}$, $\gamma = \lambda^{\pm}$.
2	$\beta_2 = 0$, $\beta_1(\beta_1 + \beta_3) \neq 0$	$\beta_3 \neq 0$	Four sets of $(\xi, \phi)$and (X, T) are given by Case 1 in Table 3 with $\alpha = -\beta_3/(\beta_1 + \beta_3)^2$, $\gamma = 1/(\beta_1 + \beta_3)$, $\beta = 0$ and Case 2 in Table 3 with $\mu = \beta_3/\beta_1^2$, $\nu = 1/\beta_1$.
		$\beta_3 = 0$	In addition to the three sets of Case 2 in Table 3 with $\mu = 0$, $\nu = 1/\beta_1$: $\xi_4 = e^{\frac{x}{\beta_1}}(2x + t^2 + V^2 - 2\beta_1 \ln(\lvert\cos U\rvert))$, $\phi_4 = 2\beta_1 e^{\frac{x}{\beta_1}}(t + V \tan U)$; $X_4 = -\beta_1 e^{\frac{x}{\beta_1}}[2tv + (t^2 + v^2 + 2x)\tan u + 2\beta_1(u - (1 + \log\lvert\cos u\rvert)\tan u)]$, $T_4 = e^{\frac{x}{\beta_1}}[v(t^2 + \frac{v^2}{3} + 2x) + 2\beta_1(tu - v\log\lvert\cos u\rvert)]$.
3	$\beta_2^2 = 4\beta_1(\beta_1 + \beta_3)$	$\beta_2 \neq 0$ or $\beta_2 = 0$, $\beta_1 = 0$, $\beta_3 \neq 0$	Two sets of (ξ, ϕ) and (X, T) are given by Case 1 in Table 3 with $\gamma = \frac{1}{2\beta_1 + \beta_3}$, $\alpha = -\frac{\beta_1 + \beta_3}{(2\beta_1 + \beta_3)^2}$, $\beta = -\frac{\beta_2}{2(\beta_1 + \beta_3)}$.
		$\beta_2 = 0$, $\beta_1 \neq 0$	Two sets of (ξ, ϕ) and (X, T) are given by Case 2 in Table 3 with $\mu = -1/\beta_1$, $\nu = 1/\beta_1$.

is an invariant under X_3. Consequently, ODE (A.6) reduces to the separable ODE

$$\frac{dz}{(Z - \frac{3}{2})(Z - 1)} = -2\frac{dX}{X}. \tag{A.7}$$

Thus four cases arise for the solution of ODE (23):

Case I: $V = 0$.
This corresponds to $G''(u) = 0$, so that here

$$G(u) = \alpha_1 + \alpha_2 u.$$

Case II: $Z = 1$.
This corresponds to $YX = 1$, i.e., the second order ODE

$$(g')^{-2} g'' = 1. \tag{A.8}$$

It is easy to show that the general solution of (A.8) yields

$$G(u) = \alpha_1 + \alpha_2 e^{\alpha_3 u}.$$

Table 9
Physical example: $G'(u) = \nu F(u)$

Case	$G(u)$	Multipliers (ξ, ϕ); flux and density (X, T)
1	$\nu \int F(u)\,du$	$\xi_1 = te^{\nu x},\ \phi_1 = \frac{1}{\nu}e^{\nu x}$; $X_1 = -\frac{1}{\nu}e^{\nu x}(tG(u) + v),\ T_1 = e^{\nu x}(\frac{1}{\nu}u + tv)$. .... $\xi_2 = Ve^{\nu x},\ \phi_2 = \frac{1}{\nu}G(U)e^{\nu x}$; $X_2 = -\frac{1}{\nu}vG(u)e^{\nu x},\ T_2 = \frac{1}{\nu}e^{\nu x}(\int G(u)\,du + \frac{\nu}{2}v^2)$. .... $\xi_3 = e^{\nu x},\ \phi_3 = 0$; $X_3 = -\frac{1}{\nu}G(u)e^{\nu x},\ T_3 = ve^{\nu x}$.
2	e^u	In addition to the three sets of Case 1 in this table: $\xi_4 = e^{\nu x}(2x + tV + U/\nu),\ \phi_4 = \frac{1}{\nu}e^{\nu x}(te^U + V)$; $X_4 = -\frac{1}{\nu}e^{\nu x+u}[v^2e^{-u}/2 + tv + 2x + (u-1)/\nu]$, $T_4 = e^{\nu x}[(tv/2 + 2x)v + (uv + e^u)/\nu]$.
3	$\tanh u$	In addition to the three sets of Case 1 in this table: $\xi_4 = e^{\nu x}[2x + t^2 - V^2 - \frac{2}{\nu}\ln\cosh U]$, $\phi_4 = \frac{2}{\nu}e^{\nu x}[t - V\tanh U]$; $X_4 = \frac{1}{\nu}e^{\nu x}\{-2(tv + u/\nu) + [v^2 - t^2 - 2x + \frac{2}{\nu}(1 + \ln\cosh u)]\tanh u\}$, $T_4 = e^{\nu x}[v(t^2 - v^2/3 + 2x) + \frac{2}{\nu}(tu - v\ln\cosh u)]$.
4	$\tan u$	In addition to the three sets of Case 1 in this table: $\xi_4 = e^{\nu x}(2x + t^2 + V^2 - 2\ln(\lvert\cos U\rvert)/\nu)$, $\phi_4 = \frac{2}{\nu}e^{\nu x}(t + V\tan U)$; $X_4 = -\frac{1}{\nu}e^{\nu x}[2tv + (t^2 + v^2 + 2x)\tan u + \frac{2}{\nu}(u - (1 + \log\lvert\cos u\rvert)\tan u)]$, $T_4 = e^{\nu x}[v(t^2 + \frac{v^2}{3} + 2x) + \frac{2}{\nu}(tu - v\log\lvert\cos u\rvert)]$.

Case III: $Z = \frac{3}{2}$.

This case corresponds to the ODE

$$g(g')^{-2}g'' = \frac{3}{2},$$

and its general solution yields

$$G(u) = \alpha_1 + \frac{1}{\alpha_2 u + \alpha_3}.$$

Case IV arises from the solution of (A.7) with $Z \neq 1, \frac{3}{2}$. With $Z = YX$, the general solution of (A.7) yields

$$Y = \frac{\frac{3}{2}X - \alpha}{X^2 - \alpha X},$$

which, in turn, leads to

$$V = \beta\big(U(U-\alpha)^{1/2}\big),$$

and thus to the separable first order ODE

$$g' = \beta\big(g(g-\alpha)^{1/2}\big).$$

Two subcases arise according to the sign of α (the limiting cases $\alpha = 0,\ \alpha = \infty$, respectively, correspond to $Z = \frac{3}{2}$, $Z = 1$):

Case IVa: $\alpha < 0$.

Here it is easy to show that

$$G(u) = \alpha_1 + \frac{1}{\alpha_2 + \alpha_3 e^{\alpha_4 u}}.$$

Case IVb: $\alpha > 0$.

Here

$$G(u) = \alpha_1 + \alpha_2 \tan(\alpha_3 u + \alpha_4).$$

For each $G(u)$, there corresponds a three-parameter class of $F(u)$ given by the expression (A.3). For each $G(u)$, modulo translations in G and u, and scalings in G and u, the results are summarized in Table 3 in Section 3.

References

[1] P.J. Olver, Applications of Lie Groups to Differential Equations, Grad. Texts in Math., vol. 107, Springer-Verlag, New York, 1986.
[2] S.C. Anco, G. Bluman, Eur. J. Appl. Math. 13 (2002) 545.
[3] S.C. Anco, G. Bluman, Eur. J. Appl. Math. 13 (2002) 567.
[4] G. Bluman, P. Doran-Wu, Acta Appl. Math. 41 (1995) 21.
[5] I.G. Katayev, Electromagnetic Shock Waves, Iliffe, London, 1966.
[6] A. Jeffrey, Wave Motion 4 (1982) 173.
[7] A.C. Eringen, E.S. Suhubi, Elastodynamics 1: Finite Motions, Academic Press, New York, 1974.
[8] G. Bluman, Temuerchaolu, R. Sahadevan, Local and nonlocal symmetries for nonlinear telegraph equations, J. Math. Phys. (2005), in press.
[9] G. Bluman, S. Kumei, Symmetries and Differential Equations, Appl. Math. Sci., vol. 81, Springer-Verlag, New York, 1989.
[10] Temuerchaolu, Adv. Math. 32 (2003) 208.
[11] G. Bluman, S.C. Anco, Symmetry and Integration Methods for Differential Equations, Appl. Math. Sci., vol. 154, Springer-Verlag, New York, 2002.