On the Virtual Fundamental Class

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Overview

Donaldson-Thomas theory: counting invariants of sheaves on Calabi-Yau threefolds.

Symmetric obstruction theories and the **virtual fundamental class**.

Additivity (over stratifications) of Donaldson-Thomas invariants.

Motivic Donaldson-Thomas invariants.

Categorification of Donaldson-Thomas invariants.

Begin with: review of the local case: **critical loci**.

The singular Gauß-Bonnet theorem

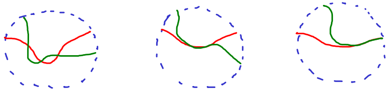
M smooth complex manifold (not compact),

$f : M \to \mathbb{C}$ holomorphic function,

$X = \operatorname{Crit} f \subset M$. Assume X compact.

Then X is the intersection of two Lagrangian submanifolds in Ω_M (complementary dimensions):

$$\begin{tikzcd} X \arrow[r] \arrow[d] & M \arrow[d, "\Gamma_{df}"] \\ M \arrow[r, "0"] & \Omega_M \end{tikzcd}$$



compact intersection   compact intersection   non compact intersection

X is compact:

intersection number $\#^{\mathrm{virt}}(X) = \mathcal{I}_{\Omega_M}(M, \Gamma_{df}) = \int_{[X]^{\mathrm{virt}}} 1$ well-defined.

$[X]^{\mathrm{virt}} \in A_0(X)$ virtual fundamental class of the intersection scheme X.

Theorem (Singular Gauß-Bonnet)

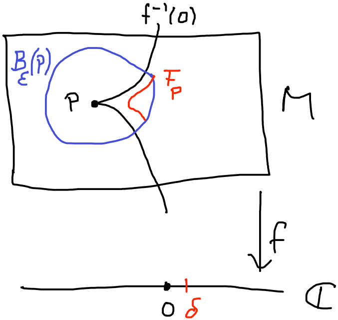
$$\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) \quad = \quad \chi(X, \mu)$$

$\mu : X \to \mathbb{Z}$ constructible function.

Milnor fibre

$X = \mathrm{Crit}\, f \subset M$ $f: M \to \mathbb{C}$ holomorphic Theorem: $\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) = \chi(X, \mu)$

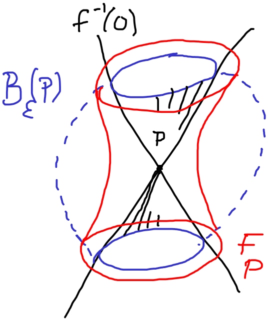
F_P: **Milnor fibre** of f at P: intersection of a nearby fibre of f with a small ball around P.

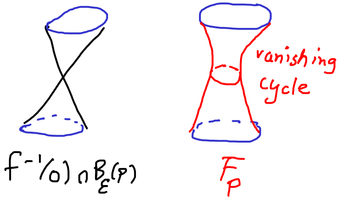


$\mu(P) = (-1)^{\dim M}\Big(1 - \chi(F_P)\Big)$: **Milnor number** of f at $P \in X = \mathrm{Crit}\, f$.

Consider two cases of the theorem: $\dim X = 0$, X smooth.

Milnor fibre example: $f(x,y) = x^2 + y^2$





$X = \mathrm{Crit}(f) = \{P\}$. Isolated singularity. $\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) = 1$.

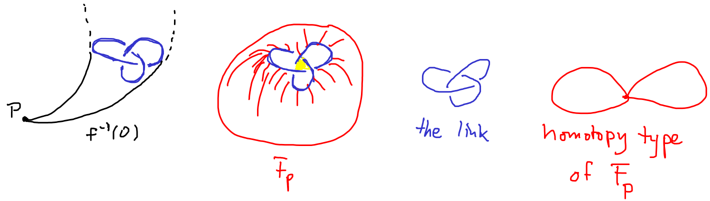
Near P, the surface $f^{-1}(0)$ is a cone over the link of the singularity. The cone is contractible.

The Milnor fibre is a manifold with boundary. The boundary is the link.

The Milnor fibre supports the *vanishing cycles*.The Milnor number $\mu(P) = (-1)^{\dim M}\Big(1 - \chi(F_P)\Big)$ is the number of vanishing cycles.

Here, $\chi(X, \mu) = \mu(P) = 1$, and hence $\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) = \chi(X, \mu)$.

Milnor fibre example: $f(x, y) = x^2 + y^3$



$X = \mathrm{Crit}(f) = \{(x, y) \mid 2x = 0, 3y^2 = 0\} = \mathrm{Spec}\, \mathbb{C}[y]/y^2$. Isolated singularity of multiplicity 2. $\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) = 2$.

Link: $(2, 3)$ torus knot (trefoil). The singularity is a cone over the knot.

The link bounds the Milnor fibre. Homotopy type (Milnor fibre) = bouquet of 2 circles. $\chi(F_P) = 1 - 2 = -1$.

The Milnor number is $\mu(P) = (-1)^2(1 - (-1)) = 2$. There are 2 vanishing cycles.

In this example, $\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) = 2 = \chi(X, \mu)$.

Theorem (Milnor, 1969)

For the case $\dim X = 0$ *(isolated singularities)*
$\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) =$ *Milnor number* $= \chi(X, \mu)$.

The excess bundle

$X = \mathrm{Crit}\, f \subset M$ $\quad f : M \to \mathbb{C}$ holomorphic $\quad$ Theorem: $\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) = \chi(X, \mu)$

Suppose X is smooth. So $\mathscr{N}^\vee_{X/M} = \mathscr{I}_X/\mathscr{I}_X^2$ is a vector bundle on X.

Epimorphism $T_M \xrightarrow{df} \mathscr{I}_X \subset \mathscr{O}_M$.

Restrict to X:

$$\begin{tikzcd} T_M|_X \arrow[r, two heads, "df"] \arrow[rr, bend right, "H(f)"'] & \mathscr{I}/\mathscr{I}^2 \arrow[r, hook, "d"] & \Omega_M|_X . \end{tikzcd}$$

(Recall: $\mathscr{I}/\mathscr{I}^2 \xrightarrow{d} \Omega_M|_X \longrightarrow \Omega_X \longrightarrow 0$.)

The Hessian matrix $H(f)$ is symmetric, so taking duals we get the *same diagram*, so that $\mathscr{I}/\mathscr{I}^2 = \mathscr{N}_{X/M}$.

So the **excess bundle** (or **obstruction bundle**) is

$$\begin{tikzcd} X \arrow[r] \arrow[d] & M \arrow[d, "\Gamma_{df}"] \\ M \arrow[r, "0"] & \Omega_M \end{tikzcd} \qquad \frac{\Omega_M|_X}{\mathscr{N}_{X/M}} = \frac{\Omega_M|_X}{\mathscr{I}/\mathscr{I}^2} = \Omega_X .$$

Intrinsic to the intersection X. $\quad$ Always so for Lagrangian intersections.

The excess bundle

$X = \mathrm{Crit}\, f \subset M$ $\quad f : M \to \mathbb{C}$ holomorphic $\quad$ Theorem: $\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) = \chi(X, \mu)$

In the case of clean intersection, the virtual fundamental class is the top Chern class of the excess bundle, so:

Proposition

$[X]^{\mathrm{virt}} = c_{\mathrm{top}} \Omega_X \cap [X]$

Hence,

$$\begin{aligned}
\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) &= \int_{[X]} c_{\mathrm{top}} \Omega_X \\
&= (-1)^{\dim X} \int_{[X]} c_{\mathrm{top}} T_X \\
&= (-1)^{\dim X} \chi(X)\,, && \text{by Gauß-Bonnet} \\
&= \chi(X, \mu_X)\,, && \text{with } \mu_X = (-1)^{\dim X}\,.
\end{aligned}$$

and for smooth X it turns out the $\mu_X \equiv (-1)^{\dim X}$.

For X smooth, the theorem is equivalent to Gauß-Bonnet.

Additive nature of $\#^{\mathrm{virt}}(X)$.

$f : M \to \mathbb{C}, \quad X = \mathrm{Crit}\, f.$

Theorem (Singular Gauß-Bonnet)

$$\#^{\mathrm{virt}}(X) \quad = \quad \chi(X, \mu)$$

is a result of microlocal geometry, in the 1970s.

Main ingredient in proof: microlocal index theorem of Kashiwara, MacPherson.

Also: determination of the characteristic variety of the perverse sheaf of vanishing cycles.

Major significance: intersection number is *motivic*, i.e.,

- intersection number makes sense for non-compact schemes: $\#^{\mathrm{virt}}(X) = \chi(X, \mu)$,
- intersection number is additive over stratifications: $\chi(X, \mu_X) = \chi(X \backslash Z, \mu_X) + \chi(Z, \mu_X)$, if $Z \hookrightarrow X$ is closed.

This is unusual for intersection numbers,
only true for Lagrangian intersections.

Motivic critical loci

Group of **motivic weights** $K(\mathrm{Var})$: Grothendieck group of $\mathbb{C}$-varieties modulo scissor relations: $[X] = [X \setminus Z] + [Z]$, whenever $Z \to X$ is a closed immersion.

There exists a lift (**motivic virtual count of critical loci**):

$$\begin{tikzcd} & K(\mathrm{Var}) \arrow[d, "\chi"] \\ \text{critical loci} \arrow[ur, "\Phi"] \arrow[r, "\#^{\mathrm{virt}}"'] & \mathbb{Z} \end{tikzcd}$$

where $\Phi(M, f) = -q^{-\frac{\dim M}{2}}[\phi_f]$,

$q = [\mathbb{C}]$ motivic weight of the affine line,

$[\phi_f]$ *motivic vanishing cycles* of Denef-Loeser (2000): motivic version of Milnor fibres. (From their work on motivic integration.)

Example: $\mathrm{Hilb}^n(\mathbb{C}^3)$

$\mathrm{Hilb}^n(\mathbb{C}^3)$: scheme of three commuting matrices: critical locus of
$\left(M_{n\times n}(\mathbb{C})^3 \times \mathbb{C}^n\right)^{\mathrm{stab}}/GL_n \longrightarrow \mathbb{C}, \qquad (A,B,C,v) \longmapsto \mathrm{tr}([A,B]C).$

Theorem (B.-Bryan-Szendrői)

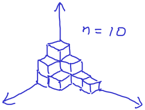
$$\sum_{n=0}^{\infty} \Phi\left(\mathrm{Hilb}^n(\mathbb{C}^3)\right) t^n = \prod_{m=1}^{\infty}\prod_{k=1}^{m} \frac{1}{1-q^{k+1-\frac{m}{2}}t^m}$$

Specialize: $q^{\frac{1}{2}} \to -1$, get

$$\sum_{n=0}^{\infty} \#^{\mathrm{virt}}\left(\mathrm{Hilb}^n(\mathbb{C}^3)\right) t^n = \prod_{m=1}^{\infty}\left(\frac{1}{1-(-t)^m}\right)^m.$$

This is (up to signs) the generating function for 3-dimensional partitions

$$\sum_{n=0}^{\infty} \#\{\text{3D partitions of } n\}\, t^n = \prod_{m=1}^{\infty}\left(\frac{1}{1-t^m}\right)^m$$



Categorified critical locus

$f : M \to \mathbb{C}, \quad X = \mathrm{Crit}\, f.$

Let $\widetilde{\Phi}_f = \Phi_f[\dim M - 1] \in D_c(\mathbb{C}_X)$ be the **perverse sheaf** of (shifted) **vanishing cycles** for f (Deligne, 1967).

Φ_f globalizes the reduced cohomology of the Milnor fibre: $H^i(\Phi_f|_P) = \overline{H}^i(F_P)$.

Theorem

$$\textstyle\sum_i (-1)^i \dim H^i(X, \widetilde{\Phi}_f) = \chi(X, \mu)$$

So $(X, \widetilde{\Phi}_f)$ **categorifies** the virtual count.

De Rham model: **twisted de Rham complex**. Pass to ground field $\mathbb{C}(\!(\hbar)\!)$.

Theorem (Sabbah, 2010)

$\big(\Omega^\bullet_M(\!(\hbar)\!), df + \hbar d\big)[\dim M] \in D_c(\mathbb{C}(\!(\hbar)\!)_X)$ *is a perverse sheaf, and*

$$\textstyle\sum_i (-1)^i \dim_{\mathbb{C}(\!(\hbar)\!)} H^i\big(\Omega^\bullet_M(\!(\hbar)\!), df + \hbar d\big) = \chi(X, \mu)\,.$$

Calabi-Yau threefolds

Definition

A *Calabi-Yau threefold* is a complex projective manifold Y of dimension 3, endowed with a nowhere vanishing holomorphic volume form $\omega_Y \in \Gamma(Y, \Omega^3_Y)$.

Example. $Y = Z(x_0^5 + \ldots + x_4^5) \subset \mathbb{P}^4$ the *Fermat quintic*.

Example. More generally, $g(x_0, \ldots, x_4)$ a generic polynomial of degree 5 in 5 variables. $Y = Z(g) \subset \mathbb{P}^4$ the *quintic threefold*.

Example. Algebraic torus $\mathbb{C}^3/\mathbb{Z}^6$ (sometimes excluded, because it is not simply connected).

CY3: the compact part of 10-dimensional space-time according to superstring theory.

Moduli spaces of sheaves

Y: Calabi-Yau threefold.

Fix numerical invariants, and a stability condition.

X: associated moduli space of stable sheaves (derived category objects) on Y.

Example: Fix integer $n > 0$. $X = \mathrm{Hilb}^n(Y)$, *Hilbert scheme of n points on Y.*
$E \in X \iff E$ is the ideal sheaf of a (degenerate) set of n points in Y.

Example: Fix integers $n \in \mathbb{Z}$, $d > 0$. $X = I_{n,d}(Y)$, [MNOP] moduli space of (degenerate) curves of genus $1 - n$, degree d in Y.
$E \in X \iff E$ ideal sheaf of a 1-dimensional subscheme $Z \subset Y$.

Example: Fix $r > 0$, and $c_i \in H^{2i}(Y, \mathbb{Z})$. X: moduli space of stable sheaves (degenerate vector bundles) of rank r, with Chern classes c_i on Y.

Donaldson-Thomas theory

X: can be a finite set of points.

Example. Y: quintic 3-fold in $\mathbb{P}^4$.

$X = I_{1,1}(Y)$ moduli space of lines on Y. X: 2875 discrete points.
$X = I_{1,2}(Y)$ moduli space of conics in Y. X: 609250 discrete points.

Slogan. If the world were *without obstructions*, all instances of X would be finite sets of points.

Goal (of Donaldson-Thomas theory)

Count the (virtual) number of points of X.

Bad news. X almost never zero-dimensional, almost always very singular.

Good news. X is quite often compact: always for examples $\mathrm{Hilb}^n(Y)$ and $I_{n,d}(Y)$, sometimes in the last example (depending on the c_i).

Thomas: constructs a virtual fundamental class $[X]^{\mathrm{virt}} \in A_0(X)$, and defines $\#^{\mathrm{virt}}(X) = \int_{[X]^{\mathrm{virt}}} 1 \in \mathbb{Z}$, if X compact.

Kuranishi: X is locally isomorphic to $\mathrm{Crit}\, f$, for suitable f (restrict Chern-Simons to local Kuranishi slices).

Derived schemes: virtual fundamental class

More fundamental geometrical object, the *derived moduli scheme* $X \hookrightarrow \mathfrak{X}$.

Induces morphism $\mathbb{T}_X \to \mathbb{T}_{\mathfrak{X}}|_X$ in $D(\mathscr{O}_X)$ of tangent complexes.
This morphism is an **obstruction theory** for X.

All derived schemes come with an *amplitude of smoothness*:

$$\mathbb{T}_{\mathfrak{X}}|_X \in D^{[0,n]}(X) \quad \Longleftrightarrow \quad \text{amplitude} \leq n.$$

(e.g. classical smooth schemes are derived schemes of amplitude 0)

Derived schemes $\mathfrak{X}$ of amplitude ≤ 1 have a
virtual fundamental class $[X]^{\mathrm{virt}} \in A_{\mathrm{rk}\,\mathbb{T}_{\mathfrak{X}}|_X}(X)$.

$$[X]^{\mathrm{virt}} = 0^!_{\mathfrak{V}}[\mathfrak{C}]\,,$$

$\mathfrak{V}$: the **vector bundle stack** associated to the obstruction theory $\mathbb{T}_{\mathfrak{X}}|_X$,
if $\mathbb{T}_{\mathfrak{X}}|_X = [V^0 \to V^1]$, $\quad \mathfrak{V} = [V^1/V^0]$,

$\mathfrak{C}$: the **intrinsic normal cone** of X, $[\mathfrak{C}]$ its fundamental cycle $\in A_0(\mathfrak{E})$,
$\mathfrak{C} = [C_{X/M}/T_M|_X]$, $\quad$ if $X \hookrightarrow M$,

$\mathfrak{C} \hookrightarrow \mathfrak{V}$ (cone stack in vector bundle stack) comes from $\mathbb{T}_X \hookrightarrow \mathbb{T}_{\mathfrak{X}}|_X$,

$[X]^{\mathrm{virt}} = 0^!_{\mathfrak{V}}[\mathfrak{C}]$ the Gysin pullback, via $\quad 0_{\mathfrak{V}} : X \to \mathfrak{V}$, of $[\mathfrak{C}]$.

Shifted symplectic structures

X: moduli space $\quad \pi : X \times Y \to X \quad \mathscr{E}$ on $X \times Y$ universal sheaf.

$\mathbb{T}_{\mathfrak{X}}|_X = \big(\tau_{[1,2]} R\pi_* R\,\mathscr{H}om(\mathscr{E},\mathscr{E})\big)[1] \in D^{[0,1]}(X)$.

If $P = [E]$, $\quad H^0(\mathbb{T}_{\mathfrak{X}}|_P) = \mathrm{Ext}^1_{\mathscr{O}_Y}(E,E) = T_X|_P$, *deformation space*,
$H^1(\mathbb{T}_{\mathfrak{X}}|_P) = \mathrm{Ext}^2_{\mathscr{O}_Y}(E,E)$, *obstruction space*.

Serre duality: Deformation space dual of obstruction space
$H^0(\mathbb{T}_{\mathfrak{X}}|_P) = H^1(\mathbb{T}_{\mathfrak{X}}|_P)^\vee$.

$X = \mathrm{Crit}\, f, \quad \mathbb{T}_{\mathfrak{X}}|_X = [\, T_M|_X \xrightarrow{H(f)} \Omega_M|_X \,]$.

In both cases, $\mathbb{T}_{\mathfrak{X}}|_X$ is a **symmetric** obstruction theory,

i.e., $\quad$ isomorphism $\theta : \mathbb{T}_{\mathfrak{X}}|_X \xrightarrow{\sim} (\mathbb{T}_{\mathfrak{X}}|_X)^\vee[-1]$, such that $\theta^\vee[-1] = -\theta$.

As a pairing: $\theta : \Lambda^2 \mathbb{T}_{\mathfrak{X}}|_X \to \mathscr{O}_X[-1]$.

This is the ‘classical shadow’ on the classical locus $X \hookrightarrow \mathfrak{X}$ of a **shifted symplectic structure** on $\mathfrak{X}$.

Shifted Darboux theorem. Every -1 shifted symplectic structure is locally a derived critical locus.

Global version and generalization of singular Gauß-Bonnet

Y: is a complex projective Calabi-Yau threefold.

X: a moduli space of sheaves on Y,
or any scheme endowed with a symmetric obstruction theory.

Theorem (B.)

Suppose that X is compact. Then the Donaldson-Thomas virtual count is

$$\int_{[X]^{\mathrm{virt}}} 1 = \chi(X, \nu_X)\,.$$

$\nu_X : X \to \mathbb{Z}$	constructible function
$\nu_X(P) \in \mathbb{Z}$	invariant of the singularity of X at $P \in X$. Contribution of $P \in X$ to the virtual count
$\nu_X(P) = \mu(P)$	if there exists a holomorphic function $f : M \to \mathbb{C}$, such that $X = \mathrm{Crit}\, f$, near P.

Construction. ν_X is the local Euler obstruction of the image of $[\mathfrak{C}]$ in X.

Proof. Globally embed $X \hookrightarrow M$. When performing deformation to the normal cone (locally) inside Ω_M, you get *Lagrangian* cone. Then use K-M.

Additivity of DT invariants

Now Donaldson-Thomas invariants exist for X not compact, and are additive over stratifications.

Example: global version of
$\sum_{n=0}^{\infty}\left(\#^{\mathrm{virt}}\,\mathrm{Hilb}^n(\mathbb{C}^3)\right)t^n=\prod_{m=1}^{\infty}\left(\frac{1}{1-(-t)^m}\right)^m$

Theorem (B.-Fantechi, Levine-Pandharipande, Li, 2008)

Y: Calabi-Yau threefold.

$$\sum_{n=0}^{\infty}\left(\#^{\mathrm{virt}}\,\mathrm{Hilb}^n\,Y\right)t^n=\Big(\prod_{m=1}^{\infty}\big(\frac{1}{1-(-t)^m}\big)^m\Big)^{\chi(Y)}$$

Simplest non-trivial computation of Donaldson-Thomas invariants using additive nature of the invariants.

More in Toda's talk.

Motivic Donaldson-Thomas invariants

X moduli space of sheaves on Calabi-Yau threefold Y.

To define motivic Donaldson-Thomas invariants, use

- X is locally $\mathrm{Crit}\, f$,
- motivic vanishing cycles
- orientation data

$$\begin{tikzcd} & K(\mathrm{Var}) \arrow[d, "\chi"] \\ \text{moduli spaces} \arrow[ur, "\Phi"] \arrow[r, "\#^{\mathrm{virt}}"'] & \mathbb{Z} \end{tikzcd}$$

Theorem (B.-Bryan-Szendrői, 2013)

$$\sum_{n=0}^{\infty} \Phi(\mathrm{Hilb}^n Y)\, t^n = \Big(\prod_{m=1}^{\infty} \prod_{k=1}^{m} \frac{1}{1 - q^{k-2-\frac{m}{2}} t^m} \Big)^{[Y]}$$

This formula uses the *power structure* on $K(\mathrm{Var})$.

Elaborate theory of motivic invariants by Kontsevich-Soibelman.

Categorification by gluing perverse sheaves

Kiem-Li, Joyce et al, (2013) have constructed a perverse sheaf Φ on X, such that

$$\#^{\mathrm{virt}}(X) = \chi(X, \nu_X) = \sum (-1)^i \dim H^i(X, \Phi)\,,$$

by gluing the locally defined perverse sheaves of vanishing cycles for locally existing Chern-Simons potentials.

Categorification via quantization

To globalize the de Rham categorification to moduli spaces X, expect to need derived geometry, not just its 'classical shadows', such as $\mathbb{T}_{\mathfrak{X}}|_X$. (More: see Toën's talk.)

Consider the local case $X = \mathrm{Crit}\, f$, $\quad f : M \to \mathbb{C}$, $\quad M$ smooth.

The derived critical locus:

$\mathscr{A}$, with $\mathscr{A}^{-i} = \Lambda^i T_M$, the *graded algebra of polyvector fields*.

Contraction with df defines a derivation $Q : \mathscr{A}^i \to \mathscr{A}^{i+1}$, such that $Q \circ Q = \frac{1}{2}[Q, Q] = 0$.

The differential graded scheme $\mathfrak{X} = (M, \mathscr{A}, Q)$ is one model of the derived scheme $\mathfrak{X}$.

$\mathfrak{X}$ has a -1-shifted symplectic structure on it, of which $[\, T_M|_X \xrightarrow{H(f)} \Omega_M|_X\,]$ is the classical shadow.

$\mathscr{A}$ has the *Lie Schouten bracket* $\{\,,\}$ of degree $+1$ on it. This is the Poisson bracket on the algebra of functions of the shifted symplectic scheme $\mathfrak{X}$. (Q is a derivation with respect to this bracket.)

Categorification via quantization

$\mathfrak{X} = (M, \mathscr{A}, Q)$ dg scheme, $\mathscr{A} = \Lambda T_M$, $Q = \lrcorner\, df$ $\{\,,\}$

Suppose given a **volume form** on M, (or just a **flat connection** on the canonical line bundle on M.)

This defines a **divergence operator** $\Delta : T_M \to \mathscr{O}_M$, wich extends to $\Delta : \mathscr{A} \to \mathscr{A}[1]$, such that $\Delta^2 = 0$.

Δ *generates* the bracket $\{\,,\}$

$$\Delta(xy) - (-1)^x x\Delta(y) - \Delta(x)y = \{x, y\}$$

and commutes with Q. (Batalin-Vilkovisky operator).

Then $(\mathscr{A}((\hbar)), Q + \hbar\Delta)$ categorifies $\#^{\mathrm{virt}}(\mathrm{Crit}\, f)$.

(Using a volume form on M, giving rise to the divergence Δ, we can identify $\Lambda T_M = \Omega^\bullet_M[\dim M]$, and then $(\mathscr{A}((\hbar)), Q + \hbar\Delta) = (\Omega^\bullet_M((\hbar)), df + \hbar d)[\dim M]$ the *twisted de Rham complex* from above.)

For Lagrangian intersections in complex symplectic manifolds, Kashiwara-Schapira (2007) globalized this construction, thus categorifying Lagrangian intersection numbers.

Thanks!

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