Counting invariants for Calabi-Yau threefolds

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Calabi-Yau threefolds

Definition

A *Calabi-Yau threefold* is a complex projective manifold Y of dimension 3, endowed with a nowhere vanishing holomorphic volume form $\omega_Y \in \Gamma(Y, \Omega^3_Y)$.

Example. $Y = Z(x_0^5 + \ldots + x_4^5) \subset \mathbb{P}^4$ the *Fermat quintic*.

Example. More generally, $g(x_0, \ldots, x_4)$ a generic polynomial of degree 5 in 5 variables. $Y = Z(g) \subset \mathbb{P}^4$ the *quintic threefold*.

Example. Algebraic torus $\mathbb{C}^3/\mathbb{Z}^6$ (sometimes excluded, because it is not simply connected).

CY3: the compact part of 10-dimensional space-time according to superstring theory.

Moduli spaces of sheaves

Y: Calabi-Yau threefold.

Fix numerical invariants, and a stability condition.

X: associated moduli space of stable sheaves (derived category objects) on Y.

Example: Fix integer $n > 0$. $X = \mathrm{Hilb}^n(Y)$, *Hilbert scheme of n points on Y.* $E \in X \iff E$ is a (degenerate) set of n points in Y.

degenerate: $n = 2$: $E =$ (point P, tangent vector to Y at P)

$n = 3$: $E =$ (point P, two tangent vectors at P), or

$E =$ (2-jet of a curve in Y)

Example: Fix integers $n \in \mathbb{Z}$, $d > 0$. $X = I_{n,d}(Y)$, moduli space of (degenerate) curves of genus $1 - n$, degree d in Y.

$E \in X \iff E$ ideal sheaf of a 1-dimensional subscheme $Z \subset Y$.

Degenerate curves: singular curves, curve with several components, curves with clusters of points as in $\mathrm{Hilb}^n(Y)$.

Example: Fix $r > 0$, and $c_i \in H^{2i}(Y, \mathbb{Z})$. X: moduli space of stable sheaves (degenerate vector bundles) of rank r, with Chern classes c_i on Y.

Moduli spaces contd.

X: can be a finite set of points.
Example. Y: quintic 3-fold in $\mathbb{P}^4$.
$X = I_{1,1}(Y)$ moduli space of lines on Y. X: 2875 discrete points.
Example. Y: quintic 3-fold in $\mathbb{P}^4$.
$X = I_{1,2}(Y)$ moduli space of conics in Y. X: 609250 discrete points.
(First success of *mirror symmetry*: continue this sequence.)

Slogan. If the world were *without obstructions*, all instances of X would be finite sets of points.

X: almost always very singular.
X: quite often compact: always for examples $\mathrm{Hilb}^n(Y)$ and $I_{n,d}(Y)$, sometimes in the last example (depending on the c_i).

Gauge Theory: why X 'looks like' $\mathrm{Crit}\, f$

X is trying to look like the critical set of a holomorphic function:

X=complex structures on a fixed bundle E.

$L^1 = A^{0,1}(Y, \mathrm{End}\, E)$ almost complex structures on E.
$L^2 = A^{0,2}(Y, \mathrm{End}\, E)$.

Curvature: $F : L^1 \longrightarrow L^2, \quad F(\alpha) = \bar\partial\alpha + \alpha \wedge \alpha.$

α is a complex structure $\Longleftrightarrow F(\alpha) = 0.$ $\qquad X = \{F = 0\} \subset L^1.$

Serre duality pairing $\kappa(\alpha, \beta) = \int_Y \mathrm{tr}(\alpha \wedge \beta) \wedge \omega_Y$ makes L^2 dual to L^1. So F is a 1-form on L^1.

$f : L^1 \longrightarrow \mathbb{C}, \quad f(\alpha) = \frac{1}{2}\kappa(\alpha, \bar\partial\alpha) + \frac{1}{3}\kappa(\alpha, \alpha \wedge \alpha)$ holomorphic Chern-Simons. $df = F$. $X = \{F = 0\} = \mathrm{Crit}\, f \subset L^1$.

Warning: this is most definitely not rigorous.

The main theorem

Y: is a complex projective Calabi-Yau threefold.

X: a moduli space of sheaves on Y.

Theorem (B.)

Suppose that X is compact. Then

$$\int_{[X]^{\mathrm{virt}}} 1 = \chi(X, \nu_X)\,.$$

$[X]^{\mathrm{virt}}$	$\in H_0(X, \mathbb{Z})$. The *virtual fundamental class* of X. From deformation theory and intersection theory.
$\int_{[X]^{\mathrm{virt}}} 1$	$\in \mathbb{Z}$ *virtual number* of points of X, *Donaldson-Thomas counting invariant*. Needs X compact to be defined.
$\nu_X : X \to \mathbb{Z}$	a constructible function
$\nu_X(P) \in \mathbb{Z}$	an invariant of the singularity of X at $P \in X$.
$\chi(X, \nu_X)$	topological Euler characteristic of X, with respect to weight function ν_X.

Example: X smooth

Y: CY3 X: moduli space Theorem: $\int_{[X]^{\mathrm{virt}}} 1 = \chi(X, \nu_X)$.

Suppose X is smooth. Then

$$[X]^{\mathrm{virt}} = c_{\mathrm{top}} \Omega_X \cap [X] .$$

Hence,

$$\begin{aligned}
\int_{[X]^{\mathrm{virt}}} 1 &= \int_{[X]} c_{\mathrm{top}} \Omega_X \\
&= (-1)^{\dim X} \int_{[X]} c_{\mathrm{top}} T_X \\
&= (-1)^{\dim X} \chi(X) , && \text{by Gauß-Bonnet} \\
&= \chi(X, \nu_X) , && \text{with } \nu_X = (-1)^{\dim X} .
\end{aligned}$$

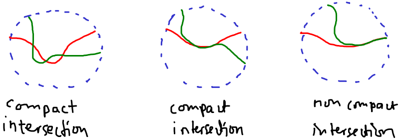
Remark: Moduli spaces X are almost never smooth.

Example: $X = \mathrm{Crit}\, f$

M	smooth complex manifold (not compact),
$f : M \to \mathbb{C}$	holomorphic function,
$X = \mathrm{Crit}\, f$	$\subset M$. X compact.

Then X is the intersection of two submanifolds in Ω_M:

$$\begin{array}{ccc} X & \longrightarrow & M \\ \downarrow & & \downarrow \Gamma_{df} \\ M & \xrightarrow{\ 0\ } & \Omega_M \end{array}$$



As X is compact, the intersection number $\int_{[X]^{\mathrm{virt}}} 1 = \mathcal{I}_{\Omega_M}(M, \Gamma_{df})$ is well-defined.

Theorem (Singular Gauß-Bonnet. From microlocal geometry)

$$\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) \quad = \quad \chi(X, \mu)$$

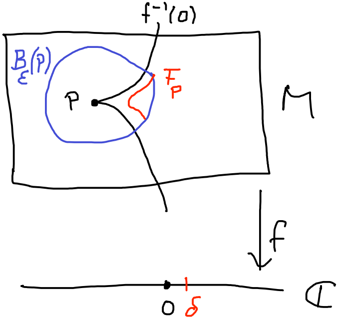
$\mu(P) =$ Milnor number of f at P $= (-1)^{\dim M}\Big(1 - \chi(F_P)\Big)$

$F_P =$ Milnor fibre of f at P

Milnor fibre

$X = \operatorname{Crit} f \subset M$ $f : M \to \mathbb{C}$ holomorphic Theorem: $\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) = \chi(X, \mu)$

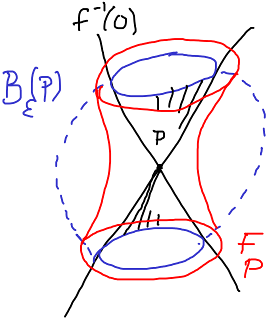
F_P: Milnor fibre of f at P: intersection of a nearby fibre of f with a small ball around P.

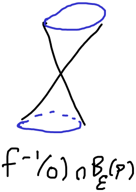


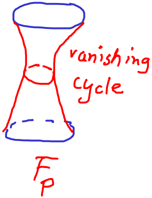
$\mu(P) = (-1)^{\dim M}\Big(1 - \chi(F_P)\Big)$

$\mu : X \to \mathbb{Z}$

Milnor fibre example. $f(x, y) = x^2 + y^2$

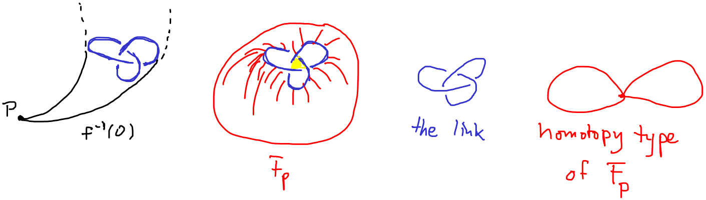






$X = \mathrm{Crit}(f) = \{P\}$. Isolated singularity.
Near P, the surface $f^{-1}(0)$ is a cone over the link of the singularity. The cone is contractible.
The Milnor fibre is a manifold with boundary. The boundary is the link.
The Milnor fibre supports the *vanishing cycles*. The Milnor number $\mu(P) = (-1)^{\dim M}\Big(1 - \chi(F_P)\Big)$ is the number of vanishing cycles.
In this example, $\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) = 1 = \chi(X, \mu)$.

Milnor fibre example $f(x,y) = x^2 + y^3$



$X = \mathrm{Crit}(f) = \{(x,y) \,|\, 2x = 0, 3y^2 = 0\} = \mathrm{Spec}\,\mathbb{C}[y]/y^2$. Isolated singularity of multiplicity 2. $\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) = 2$.

Link: $(2,3)$ torus knot (trefoil). The singularity is a cone over the knot. The link bounds the Milnor fibre. Homotopy type (Milnor fibre) = bouquet of 2 circles. $\chi(F_P) = 1 - 2 = -1$.

The Milnor number is $\mu(P) = (-1)^2\Big(1 - (-1)\Big) = 2$. There are 2 vanishing cycles.

In this example, $\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) = 2 = \chi(X, \mu)$.

For the case $\dim X = 0$ (isolated singularities) it is a theorem of Milnor that $\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) =$ Milnor number $= \chi(X, \mu)$.

Special case: $X = \operatorname{Crit} f$ concluded

$X = \operatorname{Crit} f \subset M$ $\quad$ $f : M \to \mathbb{C}$ holomorphic $\quad$ Theorem: $\mathcal{I}_{\Omega_M}(M, \Gamma_{df}) = \chi(X, \mu)$

Remark: The case X smooth is the special case $M = X$, and $f = 0$.

The intersection diagram is (self-intersection)

$$\begin{array}{ccc} X & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle 0} \\ X & \xrightarrow{0} & \Omega_X \end{array}$$

Hence we have $\mathcal{I}_{\Omega_X}(X, X) = \int_{[X]} c_{\mathrm{top}}(\Omega_X)$.
This explains why we took $[X]^{\mathrm{vir}} = c_{\mathrm{top}}(\Omega_X) \cap [X]$.
The Milnor fibre is empty. So $\mu(P) = (-1)^{\dim X}$
So in the case where $f = 0$, the theorem is Gauß-Bonnet.

The general case follows from the micro-local index theorem of Kashiwara-MacPherson, and the identification of the characteristic variety of a hypersurface in terms of the Jacobian ideal.

Lagrangian Intersections

Theorem

Suppose that X is compact. Then $\int_{[X]^{\mathrm{virt}}} 1 = \chi(X, \nu_X)$.

$[X]^{\mathrm{virt}} \in H_0(X\mathbb{Z})$.	X can locally be written as the critical set of a holomorphic function. Locally defined intersection classes glue. [B.-Fantechi], [Li-Tian], [Thomas]
$\int_{[X]^{\mathrm{virt}}} 1$	counting invariant. Is invariant under deformations of Y.
$\chi(X, \nu_X)$	can be computed by cutting up X into pieces.

In fact, $\nu_X(P)$ should be thought of as the contribution of $P \in X$ to the counting invariant. $\chi(X, \nu)$ makes sense, even when X is not compact. Unusual: in general, intersection points move away to infinity, when the intersection is not compact. This works because

$$\begin{array}{ccc} X & \longrightarrow & M \\ \downarrow & & \downarrow \Gamma_{df} \\ M & \xrightarrow{0} & \Omega_M \end{array}$$

is a *Lagrangian* intersection inside a *symplectic* manifold.

Application: Hilbert scheme of n points

Theorem (B.-Fantechi, Levine-Pandharipande, Li)

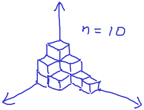
$$\sum_{n=0}^{\infty}\Big(\int_{[\mathrm{Hilb}^n Y]^{\mathrm{virt}}} 1\Big)t^n = \left(\prod_{n=1}^{\infty}\Big(\frac{1}{1-(-t)^n}\Big)^n\right)^{\chi(Y)}$$

This theorem makes sense even when Y is not compact, for example $Y=\mathbb{C}^3$. Then $\chi(Y)=1$, and

$$\sum_{n=0}^{\infty}\chi(\mathrm{Hilb}^n\,\mathbb{C}^3,\nu)\,t^n = \prod_{n=1}^{\infty}\Big(\frac{1}{1-(-t)^n}\Big)^n.$$

This is (up to signs) the generating function for 3-dimensional partitions [MacMahon]

$$\sum_{n=0}^{\infty}\#\{\text{3D partitions of } n\}\,t^n = \prod_{n=1}^{\infty}\Big(\frac{1}{1-t^n}\Big)^n$$



Application: wall crossing

We can define a number $\nu(E) \in \mathbb{Z}$, for every coherent sheaf E on Y. More generally, for any derived category object $E \in D(Y)$. Because the singularity at E is always the same, for every moduli space $E \in X$, independent of the stability condition.

Joyce, Kontsevich-Soibelman: Define invariants for every stability condition on a derived category $D(Y)$, where Y is a CY3. (No need even for moduli spaces.) Also, study how invariants change, under change of stability condition (wall crossing).

For example, if Y' is a CY3, birational to Y,

$$\text{moduli of sheaves on } Y' = \text{moduli of certain objects of } D(Y)\,.$$

Compare counting invariants for Y and Y' via wall crossing in $D(Y)$.

For example [Toda], for every *flop*, $\dfrac{\sum_{(n,\beta)}\left(\int_{[I_{n,\beta}(Y)]^{\mathrm{virt}}} 1\right)x^{\beta}q^{n}}{\sum_{(n,\beta),f_*\beta=0}\left(\int_{[I_{n,\beta}(Y)]^{\mathrm{virt}}} 1\right)x^{\beta}q^{n}}$ does not change.

Applications: motivic Donaldson-Thomas invariants

Motivated by our theorem: use more general kind of counting: not just numbers, but motivic counting: Instead of using Euler characteristic of the Milnor fibre of local Chern-Simons map $f : \mathrm{Ext}^1(E, E) \to \mathbb{C}$ to define $\nu(E)$, use its Poincaré polynomial $\in \mathbb{Q}[t]$, Hodge polynomial $\in \mathbb{Q}[u, v]$, or even its motive $\in K_0(\mathrm{Var})$. This is being done by [Kontsevich-Soibelman].

Theorem (B.-Bryan-Szendrői)

$$\sum_{n=0}^{\infty} [\mathrm{Hilb}^n\, Y]^{\mathrm{virt}}\, t^n = \left(\prod_{m=1}^{\infty} \prod_{k=1}^{m} \frac{1}{1 - q^{k-2-\frac{m}{2}}\, t^m} \right)^{[Y]}$$

$[\mathrm{Hilb}^n\, Y]^{\mathrm{virt}}$	virtual motive of $Hilb^n Y$, defined using motivic vanishing cycles of a suitable local Chern-Simons, which is a homogeneous polynomial of degree 3, in this simple case,
$q = [\mathbb{C}]$	the motive of the affine line,
$[Y]$	the motive of Y. The formula uses the *power structure* on $K_0(\mathrm{Var})$.