The Product Formula for Gromov-Witten Invariants

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Abstract

We prove that the Gromov-Witten CohFT of the product of two
varieties is equal to the tensor product of the Gromov-Witten CohFTs
of the two factors.

Contents

Introduction

Let V and W be smooth and projective varieties over the field k. In this
article we treat the question how to express the Gromov-Witten invariants
of $V \times W$ in terms of the Gromov-Witten invariants of V and W.

On an intuitive level, the answer is quite obvious. For example, assume
$V = W = \mathbb{P}^1$ and let us ask the question how many curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of
genus g and bidegree (d_1, d_2) pass through $n = 2(d_1 + d_2) + g - 1$ given
points $P_1, \ldots, P_n$ of $\mathbb{P}^1 \times \mathbb{P}^1$ in general position. The answer is given by the
Gromov-Witten invariant

$$I_{g,n}^{\mathbb{P}^1 \times \mathbb{P}^1}(d_1, d_2)(\gamma^{\otimes n}),$$

where $\gamma \in H^4(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q})$ is the cohomology class Poincaré dual to a point.

We rephrase the question by asking how many triples $(C, x_1, \ldots, x_n, f)$,
where C is a curve of genus g, $x_1, \ldots, x_n$ are marked points on C and $f : C \to$

$\mathbb{P}^1 \times \mathbb{P}^1$ is a morphism of bidegree (d_1, d_2) exist (up to isomorphism) which satisfy $f(x_i) = P_i$, for all $i = 1, \ldots, n$. Now a morphism $f : C \to \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (d_1, d_2) is given by two morphisms $f_1 : C \to \mathbb{P}^1$ and $f_2 : C \to \mathbb{P}^1$ of degrees d_1 and d_2, respectively. The requirement that $f(x_i) = P_i$ translates into $f_1(x_i) = Q_i$ and $f_2(x_i) = R_i$, if we write the components of P_i as $P_i = (Q_i, R_i)$. The family of all marked curves $(C, x_1, \ldots, x_n)$ admitting such an f_1 is some cycle, say Γ_1, in $\overline{M}_{g,n}$. Of course, the family of all curves $(C, x_1, \ldots, x_2)$ admitting an f_2 as above is another cycle Γ_2 in $\overline{M}_{g,n}$ and the family of all $(C, x_1, \ldots, x_n)$ admitting an f_1 and an f_2 is the intersection $\Gamma_1 \cdot \Gamma_2$. So the Gromov-Witten number we are interested in is

$$I_{g,n}^{\mathbb{P}^1 \times \mathbb{P}^1}(d_1, d_2)(\gamma^{\otimes n}) = \Gamma_1 \cdot \Gamma_2.$$

In fact, the dual cohomology classes of Γ_1 and Γ_2 are Gromov-Witten invariants themselves, namely $I_{g,n}^{\mathbb{P}^1}(d_1)(\tilde{\gamma}^{\otimes n})$ and $I_{g,n}^{\mathbb{P}^1}(d_2)(\tilde{\gamma}^{\otimes n})$, where $\tilde{\gamma} \in H^2(\mathbb{P}^1, \mathbb{Q})$ is the cohomology class dual to a point. Thus we have

$$I_{g,n}^{\mathbb{P}^1 \times \mathbb{P}^1}(d_1, d_2)(\gamma^{\otimes n}) = I_{g,n}^{\mathbb{P}^1}(d_1)(\tilde{\gamma}^{\otimes n}) \cup I_{g,n}^{\mathbb{P}^1}(d_2)(\tilde{\gamma}^{\otimes n})$$

in $H^*(\overline{M}_{g,n}), \mathbb{Q})$. This is the simplest instance of the product formula, which we shall prove in this article. (Note that we have identified, as usual, top degree cohomology classes on $\overline{M}_{g,n}$ with their integrals over the fundamental cycle $[\overline{M}_{g,n}]$.)

We get a more general statement by letting V and W be arbitrary smooth projective varieties over k. We fix cohomology classes $\gamma_1, \ldots, \gamma_n \in H^*(V)$ and $\epsilon_1, \ldots, \epsilon_n \in H^*(W)$, which we assume to be homogeneous, for simplicity. Then the product formula says that

$$
\begin{aligned}
&I_{g,n}^{V \times W}(\beta)(\gamma_1 \otimes \epsilon_1 \otimes \ldots \otimes \gamma_n \otimes \epsilon_n) \\
&= (-1)^s I_{g,n}^V(\beta_V)(\gamma_1 \otimes \ldots \otimes \gamma_n) \cup I_{g,n}^W(\beta_W)(\epsilon_1 \otimes \ldots \otimes \epsilon_n) \qquad (1)
\end{aligned}
$$

in $H^*(\overline{M}_{g,n}, \mathbb{Q})$. Here $\beta \in H_2(V \times W)^+$ and $\beta_V = p_{V*}\beta$, $\beta_W = p_{W*}\beta$, where p_V and p_W are the projections onto the factors of $V \times W$. The sign is given by

$$s = \sum_{i>j} \deg \gamma_i \deg \epsilon_j.$$

This formula is already stated in [6] as a property expected of Gromov-Witten invariants. In the case of $g = 0$ and V and W (and hence $V \times W$) convex, it is not difficult to prove, once the properties of stacks of stable

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maps are established, as they are, for example, in [3]. Essentially, the above intuitive argument can then be translated into a rigorous proof. In the general case, the enumerative meaning of Gromov-Witten invariants is much less clear, since one has to use 'virtual' fundamental classes to define them. (This is done in [2] and [1] or [7].) So the theorem follows from properties of virtual fundamental classes. This is what we prove in the present paper.

Formula (1) has been used by various authors to understand the quantum cohomology of a product. (See [8], [5] and [4].) By Formula (1), the codimension zero Gromov-Witten invariants (ie. those that are numbers, like $I_{g,n}^{\mathbb{P}^1 \times \mathbb{P}^1}(d_1, d_2)(\gamma^{\otimes n})$, above) of a product are determined by the Gromov-Witten invariants of higher codimension of the factors and by the intersection theory of $\overline{M}_{g,n}$.

To explain the treatment in this article, let us reformulate (1) by saying that

$$
\begin{array}{ccc}
h(V \times W)^{\otimes n} & \xrightarrow{I_{g,n}^{V \times W}(\beta)} & h(\overline{M}_{g,n}) \\
\| & & \uparrow \Delta^* \\
h(V)^{\otimes n} \otimes h(W)^{\otimes n} & \xrightarrow{I_{g,n}^V(\beta_V) \otimes I_{g,n}^W(\beta_W)} & h(\overline{M}_{g,n}) \otimes h(\overline{M}_{g,n}),
\end{array}
$$

where $\Delta : \overline{M}_{g,n} \to \overline{M}_{g,n} \times \overline{M}_{g,n}$ is the diagonal, commutes. Here we have passed to the motivic Gromov-Witten invariants. These are homomorphisms between DMC-motives. (These are like Chow motives, except that they are made from smooth and proper Deligne-Mumford stacks, instead of varieties. For details see [3], Section 8.)

To summarize all of their functorial properties, Gromov-Witten invariants where defined in [3] as natural transformations between the functors $h(V)^{\otimes S}$ and $h(\overline{M})$, which are functor from a certain graph category $\widetilde{\mathfrak{G}}_s(V)_{\mathrm{cart}}$ to the categroy of graded DMC-motives. To explain, let us start by reviewing some graph theory. The category $\widetilde{\mathfrak{G}}_s = \widetilde{\mathfrak{G}}_s(0)$ is the category of stable modular graphs (graphs whose vertices are labelled with genuses; see [3], Definition 1.5) whith so called extended isogenies as morphisms. An *extended isogeny* is either a morphism glueing two tails to an edge, or it is a proper isogeny (or a composition of the two). An *isogeny* is a morphism which contracts various edges or tails or both. (The name isogeny comes from the fact that such morphisms do not affect the genus of the components of the graphs involved.) For the definition of composition of extended isogenies, see [3], Page 36.

The category $\widetilde{\mathfrak{G}}_s(V)_{\mathrm{cart}}$ is called the *cartesian extended isogeny category over V*. The most important objects of $\widetilde{\mathfrak{G}}_s(V)_{\mathrm{cart}}$ are pairs $(\tau, (\beta_i)_{i \in I})$, where

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τ is a stable modular graph and $(\beta_i)_{i \in I}$ is a family of $H_2(V)^+$ markings on τ. This means that each β_i is a function $\beta_i : V_\tau \to H_2(V)^+$, where V_τ is the set of vertices of τ. (The indexing set I is finite.) The fundamental property of $\widetilde{\mathfrak{G}}_s(V)_{\mathrm{cart}}$ is that it is *fibered* over $\widetilde{\mathfrak{G}}_s$. This means that there is a functor $\widetilde{\mathfrak{G}}_s(V)_{\mathrm{cart}} \to \widetilde{\mathfrak{G}}_s$ (projection onto the first component) and that given an object $(\tau, (\beta_i)_{i \in I})$ of $\widetilde{\mathfrak{G}}_s(V)_{\mathrm{cart}}$ and a morphism $\phi : \sigma \to \tau$ there exists, up to isomorphism, a unique object $(\sigma, (\gamma_j)_{j \in J})$ of $\widetilde{\mathfrak{G}}_s(V)_{\mathrm{cart}}$ together with a morphism $\Phi : (\sigma, (\gamma_j)_{j \in J}) :\to (\tau, (\beta_i)_{i \in I})$ covering $\phi : \sigma \to \tau$. When constructing Φ, the basic non-obvious case is that where ϕ contracts a non-looping edge of σ and I has only one element. Then we have the graph τ with an $H_2(V)^+$-marking β and there are two vertices v_1, v_2 of σ corresponding to one vertex w of τ. Then $(\sigma, (\gamma_j)_{j \in J})$ is defined such that J counts the ways to write $\beta(w) = \beta_1 + \beta_2$ in $H_2(V)^+$ and γ_j assigns β_1 to v_1 and β_2 to v_2, and otherwise does not differ from β.

Things get more complicated, if one also considers the less important objects of $\widetilde{\mathfrak{G}}_s(V)_{\mathrm{cart}}$. These are of the form $(\tau, (\tau_i)_{i \in I})$, where, as above, τ is a stable modular graph, but now each τ_i is a stable $H_2(V)^+$-marked graph (as opposed to an $H_2(V)^+$-marked stable graph), together with a *stabilizing morphism* $\tau_i \to \tau$. For the complete picture, see [3], Definition 5.9.

On objects, the morphsims $h(V)^{\otimes S}$ and $h(\overline{M})$ from $\widetilde{\mathfrak{G}}_s(V)_{\mathrm{cart}}$ to (graded DMC-motives) are defined as follows: For an object $(\tau, (\beta_i)_{i \in I})$ of $\widetilde{\mathfrak{G}}_s(V)_{\mathrm{cart}}$ we have

$$h(V)^{\otimes S}(\tau, (\beta_i)_{i \in I}) = h(V)^{\otimes S_\tau},$$

where S_τ is the set of tails of τ and

$$h(\overline{M})(\tau, (\beta_i)_{i \in I}) = h(\overline{M}(\tau)),$$

where

$$\overline{M}(\tau) = \prod_{v \in V_\tau} \overline{M}_{g(v), F_\tau(v)}$$

and $F_\tau(v)$ is the set of flags meeting the vertex v of τ. So both of these functors only depend on the first component τ of $(\tau, (\beta_i)_{i \in I})$. For the definition of these functors on morphisms, see [3], Section 9. Note that $h(V)^{\otimes S}$ actually comes with a twist (ie. a degree shift) $\chi \dim V$. This we shall ignore here, to shorten notation and since nothing interesting happens to it, anyway.

The *Gromov-Witten transormation* of V is now defined as a natrual transformation

$$I^V : h(V)^{\otimes S} \longrightarrow h(\overline{M})$$

of functors from $\widetilde{\mathfrak{G}}_s(V)_{\text{cart}}$ to (graded DMC-motives). In this paper (Theorem 9), we shall prove that

$$I^{V \times W} = I^V \cup I^W,$$

where $I^V \cup I^W$ is defined as $\Delta^*(I^V \otimes I^W)$.

Since Gromov-Witten invariants are defined in terms of virtual fundamental classes on moduli stacks of stable maps, this theorem follows from a certain compatability between virtual fundamental classes. This is our main result (Theorm 1) and takes up most of this paper.

Virtual Fundamental Classes

Fix a ground field k. For a smooth projective k-variety V let $\widetilde{\mathfrak{G}}_s(V)$ be the category of extended isogenies of stable $H_2(V)^+$-graphs bounded by the characteristic of k (see [3], Definition 5.6 and Example II following Definition 5.11). Let $J(V, \tau) \in A_{\dim(V,\tau)}(\overline{M}(V, \tau))$, for $\tau \in \text{ob}\,\widetilde{\mathfrak{G}}_s(V)$, be the 'virtual fundamental class', or orientation ([3], Definition 7.1) of $\overline{M}$ over $\widetilde{\mathfrak{G}}_s(V)$ constructed in [1], Theorem 6, using the techniques from [2].

Now let us consider two smooth projective k-varieties V and W; denote the two projections by $p_V : V \times W \to V$ and $p_W : V \times W \to W$. If τ is a stable $H_2(V \times W)^+$-graph, we denote by $p_{V*}(\tau)$ and $p_{W*}(\tau)$ the stabilizations of τ with respect to $p_{V*} : H_2(V \times W)^+ \to H_2(V)^+$ and $p_{W*} : H_2(V \times W)^+ \to H_2(W)^+$ (see [3], Remark 1.15), by τ^s the absolute stabilization of τ.

Applying the functor $\overline{M}$ to the commutative diagram

$$
\begin{array}{ccc}
(V \times W, \tau) & \longrightarrow & (W, p_{W*}(\tau)) \\
\downarrow & & \downarrow \\
(V, p_{V*}(\tau)) & \longrightarrow & (\text{Spec}\,k, \tau^s)
\end{array}
$$

in $\mathfrak{VG}_s$ (see [3], Remark 3.1 and the remark following Theorem 3.14) we get a commutative diagram of proper Deligne-Mumford stacks

$$
\begin{array}{ccc}
\overline{M}(V \times W, \tau) & \longrightarrow & \overline{M}(W, p_{W*}(\tau)) \\
\downarrow & & \downarrow \\
\overline{M}(V, p_{V*}(\tau)) & \longrightarrow & \overline{M}(\tau^s).
\end{array}
$$

In general, this diagram is not cartesian; let P be the cartesian product

$$
\begin{array}{ccc}
P & \longrightarrow & \overline{M}(W, p_{W*}(\tau)) \\
\downarrow & & \downarrow \\
\overline{M}(V, p_{V*}(\tau)) & \longrightarrow & \overline{M}(\tau^s).
\end{array}
$$

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Rewrite these diagrams as follows:

$$\begin{array}{ccccc}
\overline{M}(V\times W,\tau) & \xrightarrow{\ h\ } & P & \longrightarrow & \overline{M}(V,p_{V*}\tau)\times\overline{M}(W,p_{W*}\tau) \\
 & \searrow & \downarrow & & \downarrow \\
 & & \overline{M}(\tau^s) & \xrightarrow{\ \Delta\ } & \overline{M}(\tau^s)\times\overline{M}(\tau^s).
\end{array}$$

To shorten notation, write $J(V\times W)=J(V\times W,\tau)$, $J(V)=J(V,p_{V*}\tau)$ and $J(W)=J(W,p_{W*}\tau)$.

Theorem 1 *We have*

$$\Delta^!(J(V)\times J(W))=h_*(J(V\times W)).$$

For a stable A-graph τ $(A=H_2(V\times W)^+,\ H_2(V)^+$ etc.) we denote by $\mathfrak{M}(\tau)$ the algebraic k-stack of τ-marked prestable curves, forgetting the A-structure, and thinking of τ simply as a (possibly not stable) modular graph. We consider the diagram

$$\begin{array}{ccccccc}
 & \overline{M}(V\times W,\tau) & \xrightarrow{\ h\ } & P & \longrightarrow & \overline{M}(V,p_{V*}\tau)\times\overline{M}(W,p_{W*}\tau) \\
\diagup & \downarrow{\scriptstyle c} & & \downarrow & & \downarrow{\scriptstyle a} \\
b\mid & \mathfrak{D}(\tau) & \xrightarrow{\ l\ } & \mathfrak{P} & \xrightarrow{\ \phi\ } & \mathfrak{M}(p_{V*}\tau)\times\mathfrak{M}(p_{W*}\tau) \\
\searrow & \downarrow{\scriptstyle e} & & \downarrow & & \downarrow{\scriptstyle s\times s} \\
 & \mathfrak{M}(\tau) & & \overline{M}(\tau^s) & \xrightarrow{\ \Delta\ } & \overline{M}(\tau^s)\times\overline{M}(\tau^s).
\end{array}$$
$$(2)$$

Here $s\times s$ is given by stabilizations and $\mathfrak{P}$ is defined as the fibered product of Δ and $s\times s$. The morphisms a and b are given by forgetting maps, retaining only marked curves.

The algebraic stack $\mathfrak{D}(\tau)$ is defined as follows. For a k-scheme T the groupoid $\mathfrak{D}(\tau)(T)$ has as objects diagrams

$$\begin{array}{ccc}
(C,x) & \longrightarrow & (C'',x'') \\
\downarrow & & \\
(C',x') & &
\end{array}$$
$$(3)$$

where (C,x) is a τ-marked prestable curve over T, (C',x') a $p_{V*}(\tau)$-marked prestable curve over T and (C'',x'') a $p_{W*}(\tau)$-marked prestable curve over T. The arrow $(C,x)\to(C',x')$ is a morphism of marked prestable curves covering the morphism $\tau\to p_{V*}(\tau)$ of modular graphs. Similarly, $(C,x)\to(C'',x'')$ is a morphism of marked prestable curves covering $\tau\to p_{W*}(\tau)$. This concept has not been defined in [3]; the definition (in this special case)

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is as follows. Let us explain it for the case of W instead of V, since this will lead to less confusion of notation with the set of vertices of a graph. The morphism $\tau \to p_{W*}(\tau)$ is given by a combinatorial morphism of 0-marked graphs $a : p_{W*}(\tau) \to \tau$ (see [3], Definition 1.7). So there are maps $a : V_{p_{W*}(\tau)} \to V_\tau$ and $a : F_{p_{W*}(\tau)} \to F_\tau$. The morphism $(C, x) \to (C'', x'')$ is given by a family $p = (p_v)_{v \in V_{p_{W*}(\tau)}}$ of morphisms of prestable curves ([3], Definition 2.1) $p_v : C_{a(v)} \to C_v''$ such that for every $i \in F_{p_{W*}(\tau)}$ we have $p_{\partial(i)}(x_{a(i)}) = x_i''$.

There are morphisms of stacks $e : \mathfrak{D}(\tau) \to \mathfrak{M}(\tau)$, $\mathfrak{D}(\tau) \to \mathfrak{M}(p_{V*}\tau)$ and $\mathfrak{D}(\tau) \to \mathfrak{M}(p_{W*}\tau)$, given, respectively, by mapping Diagram (3) to (C, x), (C', x') and (C'', x''). Let us denote the product of the latter two by

$$\widetilde{\Delta} : \mathfrak{D}(\tau) \longrightarrow \mathfrak{M}(p_{V*}\tau) \times \mathfrak{M}(p_{W*}\tau).$$

Lemma 2 *In Diagram (3) both morphisms induce isomorphisms on stabilizations.*

PROOF. This follows from the fact that any morphism of stable marked curves (with identical dual graphs) is an isomorphism. This fact is proved in [3], at the very end of the proof of Theorem 3.6, which immediately precedes Definition 3.13. $\square$

By this lemma there is a commutative diagram

$$
\begin{array}{ccc}
\mathfrak{D}(\tau) & \overset{\widetilde{\Delta}}{\longrightarrow} & \mathfrak{M}(p_{V*}\tau) \times \mathfrak{M}(p_{W*}\tau) \\
e \downarrow & & \\
\mathfrak{M}(\tau) & & \downarrow s \times s \\
s \downarrow & & \\
\overline{M}(\tau^s) & \overset{\Delta}{\longrightarrow} & \overline{M}(\tau^s) \times \overline{M}(\tau^s),
\end{array}
$$

which gives rise to the morphism $l : \mathfrak{D}(\tau) \to \mathfrak{P}$ of Diagram (2).

Proposition 3 *The morphisms Δ and $\widetilde{\Delta}$ are proper regular local immersions. Their natural orientations satisfy*

$$l_*[\widetilde{\Delta}] = (s \times s)^*[\Delta].$$

PROOF. Let S_1 and S_2 be finite sets, set $S = S_1 \amalg S_2$. Let the modular graph $p_{V*}(\tau)'$ be obtained from $p_{V*}(\tau)$ by adding (in any fashion) S_1 to the set of tails of $p_{V*}(\tau)$. Similarly, let $p_{W*}(\tau)'$ be obtained form $p_{W*}(\tau)$ by adding

S_2 to the set of tails, arbitrarily. Now let τ' be obtained from τ by adding the set S to the tails of τ in the unique way such that $\tau \to p_{V*}(\tau)$ induces a morphism $\tau' \to p_{V*}(\tau)'$, which gives the inclusion $S_1 \subset S$ on tails, and $\tau \to p_{W*}(\tau)$ induces a morphism $\tau' \to p_{W*}(\tau)'$, which gives the inclusion $S_2 \subset S$ in tails.

With these choices we have a cartesian diagram of k-stacks

$$
\begin{array}{ccc}
\overline{M}(\tau') & \xrightarrow{\delta} & \overline{M}(p_{V*}(\tau)') \times \overline{M}(p_{W*}(\tau)') \\
\downarrow & & \downarrow \chi \\
\mathfrak{D}(\tau) & \xrightarrow{\widetilde{\Delta}} & \mathfrak{M}(p_{V*}(\tau)) \times \mathfrak{M}(p_{W*}(\tau)).
\end{array}
\tag{4}
$$

The proof that this is the case is similar to the proof of Proposition 5, below. The morphism χ in Diagram (4) is a local presentation of $\mathfrak{M}(p_{V*}(\tau)) \times \mathfrak{M}(p_{W*}(\tau))$, (see [1], remarks following Lemma 1). Moreover, by choosing S and the primed graphs correctly, any point of $\mathfrak{M}(p_{V*}(\tau)) \times \mathfrak{M}(p_{W*}(\tau))$ can be assumed to be in the image of χ. So to prove that $\widetilde{\Delta}$ is a proper regular local immersion, it suffices to prove that δ is a proper regular local immersion. Properness is clear; the stacks $\overline{M}(\tau')$, $\overline{M}(p_{V*}(\tau)')$ and $\overline{M}(p_{W*}(\tau)')$ are proper. The regular local immersion property follows from injectivity on tangent spaces which can be proved by a deformation theory argument.

The proof for Δ is comparatively trivial.

To prove the fact about the orientations, first note that $s \times s$ is flat (see [1], Proposition 3) and so ϕ is a regular local immersion and $(s \times s)^*[\Delta] = [\phi]$. To prove that $l_*[\widetilde{\Delta}] = [\phi]$, it suffices to identify dense open substacks $\mathfrak{D}(\tau)' \subset \mathfrak{D}(\tau)$ and $\mathfrak{P}' \subset \mathfrak{P}$ such that l induces an isomorphism $\mathfrak{D}(\tau)' \to \mathfrak{P}'$. We define $\mathfrak{D}(\tau)'$ to be the open substack of $\mathfrak{D}(\tau)$ over which $C_v \to C_v'$ is an isomorphism for all $v \in V_{p_{V*}(\tau)}$ and $C_v \to C_v''$ is an isomorphism for all $v \in V_{p_{W*}(\tau)}$. We define $\mathfrak{P}'$ to be the pullback via Δ of $\mathfrak{M}(p_{V*}(\tau))' \times \mathfrak{M}(p_{W*}(\tau))'$, where $\mathfrak{M}(p_{V*}(\tau))'$ is the open substack over which $(C_v, (x_i)_{i \in F(v)})$ is stable, for all $v \in V_{\tau^s}$, similarly for $\mathfrak{M}(p_{W*}(\tau))'$. Note the slight abuse of notation; we have denoted vertices of different graphs by the same letter. $\square$

Lemma 4 *The morphism $e : \mathfrak{D}(\tau) \to \mathfrak{M}(\tau)$ is étale.*

PROOF. Similar to the proof of [1], Lemma 7. $\square$

To complete Diagram (2), define a morphism $\overline{M}(V \times W, \tau) \to \mathfrak{D}(\tau)$ by mapping a stable $(V \times W, \tau)$-map (C, x, f) first to the diagram

$$
\begin{array}{ccc}
(C, x, f) & \longrightarrow & (C, x, p_W \circ f)^{\mathrm{stab}} \\
\downarrow & & \\
(C, x, p_V \circ f)^{\mathrm{stab}} & &
\end{array}
$$

and then passing to the underlying prestable curves.

Proposition 5 *The diagram*

$$\begin{array}{ccc} \overline{M}(V \times W, \tau) & \longrightarrow & \overline{M}(V, p_{V*}\tau) \times \overline{M}(W, p_{W*}\tau) \\ c \downarrow & & \downarrow a \\ \mathfrak{D}(\tau) & \xrightarrow{\widetilde{\Delta}} & \mathfrak{M}(p_{V*}\tau) \times \mathfrak{M}(p_{W*}\tau) \end{array}$$

is cartesian.

PROOF. We have to construct a morphism from the fibered product of $\widetilde{\Delta}$ and a to $\overline{M}(V \times W, \tau)$. So let there be given a Diagram (3), representing an object of $\mathfrak{D}(\tau)(T)$, for a k-scheme T. Moreover, let there be given families of maps $(f')_{v \in V_{p_{V*}\tau}}$, $f'_v : C'_v \to V$ and $(f'')_{v \in V_{p_{W*}\tau}}$, $f''_v : C''_v \to W$, making (C', x', f') and (C'', x'', f'') stable maps. We need to construct a stable map from (C, x) to $V \times W$. So let $v \in V_\tau$ be a vertex of τ.

Let us construct a map $h_v : C_v \to W$. In case v is in the image of $V_{p_{W*}\tau} \to V_\tau$, and $w \mapsto v$ under this map, we take h_v to be the composition

$$C_v \xrightarrow{p_W} C''_w \xrightarrow{f''_w} W.$$

In case v is not in the image of $V_{p_{W*}\tau} \to V_\tau$, then v partakes in a long edge or a long tail associated to and edge $\{i, \bar{i}\}$ or a tail i of $p_{W*}\tau$ (see the discussion of *stabilizing morphisms*, Definition 5.7, in [3] for this terminology). Then we define $f_v : C_v \to W$ to be the composition

$$C_v \longrightarrow T \xrightarrow{x''_i} C''_{\partial(i)} \xrightarrow{f''_{\partial(i)}} W.$$

In the same manner, construct a map $g_v : C_v \to V$. Finally, let $f_v : C_v \to V \times W$ be the product $g_v \times h_v$. Then the family $(f_v)_{v \in V_\tau}$ makes (C, x, f) a stable map over T to $V \times W$. One checks that $(C, x, p_V \circ f)^{\mathrm{stab}} = (C', x', f')$ and $(C, x, p_W \circ f)^{\mathrm{stab}} = (C'', x'', f'')$, using the universal mapping property of stabilization and the fact already alluded to in the proof of Lemma 2. □

Let $E^\bullet(V) = E^\bullet(V, p_{V*}\tau)$ and $E^\bullet(W) = E^\bullet(W, p_{W*}\tau)$ denote the relative obstruction theories for $\overline{M}(V, p_{V*}\tau) \to \mathfrak{M}(p_{V*}\tau)$ and $\overline{M}(W, p_{W*}\tau) \to \mathfrak{M}(p_{W*}\tau)$, respectively, which were defined in [1]. As in [2] Proposition 7.4 there is an induced obstruction theory $E^\bullet(V) \boxplus E^\bullet(W)$ for the morphism a. Pulling back via $\widetilde{\Delta}$ (as in [2] Proposition 7.1) we get an induced obstruction theory $\widetilde{\Delta}^*(E^\bullet(V) \boxplus E^\bullet(W))$ for the morphism c.

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On the other hand, we have the relative obstruction theory $E^\bullet(V \times W) = E^\bullet(V \times W, \tau)$ for the morphism b. Since $e : \mathfrak{D}(\tau) \to \mathfrak{M}(\tau)$ is étale, we may think of $E^\bullet(V \times W)$ as a relative obstruction theory for c.

Proposition 6 *The two relative obstruction theories $\widetilde{\Delta}^*(E^\bullet(V) \boxplus E^\bullet(W))$ and $E^\bullet(V \times W)$ for the morphism c are naturally isomorphic.*

PROOF. Let $\mathcal{C}(V \times W, \tau) \to \overline{M}(V \times W, \tau)$, $\mathcal{C}(V, p_{V*}\tau) \to \overline{M}(V, p_{V*}\tau)$ and $\mathcal{C}(W, p_{W*}\tau) \to \overline{M}(W, p_{W*}\tau)$ be the universal curves. Recall from [1] that they are constructed by gluing the curves associated to the vertices of a graph according the the edges of that graph. Let us denote the pullbacks of the latter two universal curves to $\overline{M}(V \times W, \tau)$ by $\mathcal{C}(V)$ and $\mathcal{C}(W)$, respectively. We have maps $f_{V \times W}$, f_V and f_W, constructed from the universal stable maps, which fit into the following commutative diagram:

$$
\begin{array}{ccccc}
V & \xleftarrow{p_V} & V \times W & \xrightarrow{p_W} & W \\
f_V \uparrow & & \uparrow f_{V \times W} & & \uparrow f_W \\
\mathcal{C}(V) & \xleftarrow{q_V} & \mathcal{C}(V \times W, \tau) & \xrightarrow{q_W} & \mathcal{C}(W) \\
& \pi_V \searrow & \downarrow \pi_{V \times W} & \swarrow \pi_W & \\
& & \overline{M}(V \times W, \tau) & &
\end{array}
$$

By base change it is clear that

$$
\widetilde{\Delta}^*(E^\bullet(V)^\vee \boxplus E^\bullet(W)^\vee) = R\pi_{V*} f_V^* T_V \oplus R\pi_{W*} f_W^* T_W.
$$

For any vector bundle F on $\mathcal{C}(W)$ the canonical homomorphism $F \to Rq_{W*} q_W^* F$ is an isomorphism. Of course, the same property is enjoyed by q_V. Hence we have a canonical isomorphism

$$
\begin{aligned}
R\pi_{V*} f_V^* T_V \oplus R\pi_{W*} f_W^* T_W & \xrightarrow{\sim} \\
R\pi_{V*} Rq_{V*} q_V^* f_V^* T_V & \oplus R\pi_{W*} Rq_{W*} q_W^* f_W^* T_W \\
= \quad R\pi_{V \times W*} f_{V \times W}^* & (T_V \boxplus T_W) \\
= \quad E^\bullet(V \times W)^\vee. &
\end{aligned}
$$

To conclude, we have a canonical isomorphism

$$
\widetilde{\Delta}^*(E^\bullet(V)^\vee \boxplus E^\bullet(W)^\vee) \xrightarrow{\sim} E^\bullet(V \times W)^\vee
$$

and by dualizing

$$
E^\bullet(V \times W) \xrightarrow{\sim} \widetilde{\Delta}^*(E^\bullet(V) \boxplus E^\bullet(W)).
$$

$\square$

By this proposition, we have

$$
\begin{aligned}
J(V \times W) &= [\overline{M}(V \times W, \tau), E^\bullet(V \times W)] \\
&= [\overline{M}(V \times W, \tau), \widetilde{\Delta}^*(E^\bullet(V) \boxplus E^\bullet(W))] \\
&= \widetilde{\Delta}^![\overline{M}(V, p_{V*}\tau) \times \overline{M}(W, p_{W*}\tau), E^\bullet(V) \boxplus E^\bullet(W)] \\
&\qquad \text{(by [2] Proposition 7.2)} \\
&= \widetilde{\Delta}^!([\overline{M}(V, p_{V*}\tau), E^\bullet(V)] \times [\overline{M}(W, p_{W*}\tau), E^\bullet(W)]) \\
&\qquad \text{(by [2] Proposition 7.4)} \\
&= \widetilde{\Delta}^!(J(V) \times J(W)).
\end{aligned}
$$

So we may now calculate as follows:

$$
\begin{aligned}
\widetilde{\Delta}^!(J(V) \times J(W)) &= a^*(s \times s)^*[\Delta](J(V) \times J(W)) \\
&= a^* l_*[\widetilde{\Delta}](J(V) \times J(W)) \\
&\qquad \text{(by Proposition 3)} \\
&= h_* \widetilde{\Delta}^!(J(V) \times J(W)) \\
&= h_* J(V \times W),
\end{aligned}
$$

which is the product property. This finishes the proof of Theorem 1.

Gromov-Witten Transformations

Theorem 1 easily implies that the system of Gromov-Witten invariants for $V \times W$ is equal to the tensor product (see [6], 2.5) of the systems of Gromov-Witten invariants for V and W, respectively. To get the full 'operadic' picture, we need a few graph theoretic preparations.

Proposition 7 *There is a natural functor of categories fibered over $\widetilde{\mathfrak{G}}_s(0)$*

$$
\Psi : \widetilde{\mathfrak{G}}_s(V \times W)_{\mathrm{cart}} \longrightarrow \widetilde{\mathfrak{G}}_s(V)_{\mathrm{cart}} \times_{\widetilde{\mathfrak{G}}_s(0)} \widetilde{\mathfrak{G}}_s(W)_{\mathrm{cart}}.
$$

This functor is cartesian.

PROOF. Let $p : V \to W$ be a morphism of smooth projective varieties over k. We shall construct a natural functor of fibered categories over $\widetilde{\mathfrak{G}}_s(0)$

$$
\Psi_p : \widetilde{\mathfrak{G}}_s(V)_{\mathrm{cart}} \longrightarrow \widetilde{\mathfrak{G}}_s(W)_{\mathrm{cart}}.
$$

This functor will be cartesian.

For an object $(\tau, (\overline{a}_i, \tau_i)_{i \in I})$ of $\widetilde{\mathfrak{G}}_s(V)_{\text{cart}}$ let the image under Ψ_p be $(\tau, (\overline{b}_i, p_*(\tau_i))_{i \in I})$. Here $p_*(\tau_i)$ is the stabilization of τ_i covering p_* : $H_2(V)^+ \to H_2(W)^+$. It comes with a natural morphism $\overline{b}_i : p_*(\tau_i) \to \tau$. To make $\overline{b}_i$ a stabilizing morphism, we have to endow it with an orbit map (see [3], Definition 5.7). Let $\overline{a}_i^m : E_\tau \cup S_\tau \to E_{\tau_i} \cup S_{\tau_i}$ be the orbit map of $\overline{a}_i : \tau_i \to \tau$. Let $\{f, \overline{f}\}$ be an edge of τ. Then there exists a unique factor $\{f', \overline{f}'\}$ of the long edge of $p_*(\tau_i)$ associated to $\{f, \overline{f}\}$, such that $\overline{a}_i^m(\{f, \overline{f}\})$ is a factor of the long edge of τ_i associated to $\{f', \overline{f}'\}$. We set $\overline{b}_i^m(\{f, \overline{f}\}) = \{f', \overline{f}'\}$. This defines the orbit map $\overline{b}_i^m$ of $\overline{b}_i$ on edges. On tails it is defined entirely analogously.

This achieves the definition of Ψ_p on objects. We leave it to the reader to explicate the action of Ψ_p on morphisms; it boils down to checking that the *pullback* of [3], Definition 5.8 is compatible with applying p_*.

Now we define Ψ by

$$\Psi : \widetilde{\mathfrak{G}}_s(V \times W)_{\text{cart}} \longrightarrow \widetilde{\mathfrak{G}}_s(V)_{\text{cart}} \times_{\widetilde{\mathfrak{G}}_s(0)} \widetilde{\mathfrak{G}}_s(W)_{\text{cart}}$$

$$(\tau, (\tau_i)) \longmapsto (\Psi_{p_V}(\tau, (\tau_i)), \Psi_{p_W}(\tau, (\tau_i))).$$

$\square$

Let us denote, for any V, the Gromov-Witten transformation for V (see [3], Theorem 9.2), by

$$I^V : h(V)^{\otimes S}(\chi \dim V) \longrightarrow h(\overline{M}).$$

Recall that I^V is a natural transformation between functors

$$\widetilde{\mathfrak{G}}_s(V)_{\text{cart}} \longrightarrow (\text{graded DMC-motives}),$$

where the two functors $h(V)^{\otimes S}(\chi \dim V)$ and $h(\overline{M})$ are induced from functors (with the same names) $\widetilde{\mathfrak{G}}_s(0) \to (\text{DMC-motives})$, constructed in [3], Section 9.

Remark 8 By our various definitions we have

$$h(V)^{\otimes S}(\chi \dim V) \otimes h(W)^{\otimes S}(\chi \dim W) = h(V \times W)^{\otimes S}(\chi \dim V \times W)$$

as functors $\widetilde{\mathfrak{G}}_s(0) \to (\text{DMC-motives})$.

The transformations I^V and I^W induce a transformation

$$I^V \otimes I^W : h(V)^{\otimes S}(\chi \dim V) \otimes h(W)^{\otimes S}(\chi \dim W) \longrightarrow h(\overline{M}) \otimes h(\overline{M})$$

between functors

$$\widetilde{\mathfrak{G}}_s(V)_{\mathrm{cart}} \times_{\widetilde{\mathfrak{G}}_s(0)} \widetilde{\mathfrak{G}}_s(W)_{\mathrm{cart}} \longrightarrow (\text{graded DMC-motives}).$$

It is defined as follows. Let $((\tau, (\tau_i)_{i \in I}), (\tau, (\sigma_j)_{j \in J}))$ be an object of $\widetilde{\mathfrak{G}}_s(V)_{\mathrm{cart}} \times_{\widetilde{\mathfrak{G}}_s(0)} \widetilde{\mathfrak{G}}_s(W)_{\mathrm{cart}}$. The value of $I^V \otimes I^W$ on this object is the morphism

$$I^V(\tau, (\tau_i)) \otimes I^W(\tau, (\sigma_j)) :$$
$$h(V)^{\otimes S_\tau}(\chi(\tau) \dim V) \otimes h(W)^{\otimes S_\tau}(\chi(\tau) \dim W) \longrightarrow h(\overline{M}(\tau)) \otimes h(\overline{M}(\tau)).$$

Composing with $\Delta^* : h(\overline{M}) \otimes h(\overline{M}) \to h(\overline{M})$ we get the transformation

$$\Delta^*(I^V \otimes I^W) : h(V)^{\otimes S}(\chi \dim V) \otimes h(W)^{\otimes S}(\chi \dim W) \longrightarrow h(\overline{M}),$$

which we shall also denote by $I^V \cup I^W = \Delta^*(I^V \otimes I^W)$.

Pulling back via the functor Ψ of Proposition 7 and using Remark 8, we may think of $I^V \cup I^W$ as a natural transformation

$$I^V \cup I^W : h(V \times W)^{\otimes S}(\chi \dim V \times W) \longrightarrow h(\overline{M})$$

between functors $\widetilde{\mathfrak{G}}_s(V \times W)_{\mathrm{cart}} \to (\text{graded DMC-motives})$.

Theorem 9 *We have*
$$I^V \cup I^W = I^{V \times W}.$$

PROOF. This follows from the identity principle for DMC-motives, Proposition 8.2 of [3]. □

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