Differential Graded Schemes I:
Perfect Resolving Algebras

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Abstract

We introduce *perfect resolving algebras* and study their fundamental properties. These algebras are basic for our theory of differential graded schemes, as they give rise to *affine* differential graded schemes. We also introduce *étale morphisms*. The purpose for studying these, is that they will be used to glue differential graded schemes from affine ones with respect to an étale topology.

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Introduction

This is the first in a series of papers devoted to establishing a workable theory of differential graded schemes.

Here we lay the necessary algebraic foundations for this theory. Differential graded schemes will be glued with respect to an étale topology from affine differential graded schemes, very much like usual schemes are glued with respect to the Zariski topology from usual affine schemes, or algebraic spaces are glued with respect to the étale topology from affine schemes.

Thus our goal in this paper is twofold: we introduce an appropriate class of differential graded algebras providing us with a good class of affine differential graded schemes, and we introduce the notion of étale morphism between such differential graded algebras.

The first principle we follow, is that all differential graded algebras which represent geometric objects, are

(i) graded commutative with 1, over a field of characteristic 0,

(ii) graded *in non-positive degrees*, if the differential has degree +1, which is the convention we will follow.

A consequence of this is that every such differential graded algebra A has a morphism of differential graded algebras $A \to h^0(A)$, where $h^0(A)$ is the 0-th cohomology module of A.

A *quasi-isomorphism* of differential graded algebras is a morphism $A \to B$ which induces an isomorphism on cohomology modules $h^i(A) \to h^i(B)$, for all $i \leq 0$.

Another principle is, that quasi-isomorphic differential graded algebras should give rise to identical geometric objects. Thus we may replace the arbitrary differential graded algebra A (concentrated in non-positive degree) by a quasi-isomorphic differential graded algebra A', which is free as a graded algebra, disregarding its differential (a property which has been referred to as *quasi-free* in the literature). (See Scholum 1.10 for a proof of the existence of *resolutions* $A' \to A$.)

Thus, we are able to restrict our attention do differential graded algebras which are free as graded commutative algebras with 1, free on a set of generators all of which have non-positive degree. We call algebras satisfying this property *resolving algebras*, because their purpose is to resolve more general differential graded algebras (see Definition 1.4). We also required a term which is shorter than 'quasi-free on a set of generators in non-positive degree', and can more easily be qualified.

For purposes of our geometric theory, we need finiteness assumptions on resolving algebras. Thus we call a resolving algebra *finite* if we can find a finite set of quasi-free generators for it. The finite resolving algebras will serve as a category of local models for differential graded schemes. In other words, every differential graded scheme will be locally (with respect to the étale topology) given by a finite resolving algebra. Put another way, every differential graded scheme will locally determine up to quasi-isomorphism a finite resolving algebra as an analogue of 'affine coordinate ring'.

On the other hand, it turns out that (the differential graded schemes associated to) finite resolving algebras form too small a class to be considered as *affine* differential graded scheme. More specifically, a fundamental (and too useful to forgo) property of affine morphisms of usual schemes is, that the affine property is *local* in the base, or target, of the morphism. (If local means local with respect to the étale topology, this is one of the most important results of étale descent theory.)

Thus we are led to relax the requirement of finiteness on resolving algebras. We call a resolving algebra A *perfect* (see Definition 3.1) if

(i) it is *quasi-finite*, i.e., we can find a set of quasi-free generators, with finitely many elements of every degree,

(ii) its differential graded module of differentials Ω_A (see Section 1.5) tensored with $h^0(A)$ is a perfect complex of $h^0(A)$-modules, i.e., is Zariski locally in $\operatorname{Spec} h^0(A)$ quasi-isomorphic to a finite complex of finite rank free modules.

Note that every finite resolving algebras is perfect.

Using perfect resolving algebras to define affine differential graded schemes, the notion of affine morphism of differential graded schemes is local in the étale topology on the base (see [1]). Thus we choose perfect resolving algebras as our affine models for differential graded schemes.

We prove two fundamental facts about perfect resolving algebras, showing that they are, in fact, not very far from finite resolving algebras:

(i) Every perfect resolving algebra A is locally with respect to the Zariski topology on $\operatorname{Spec} h^0(A)$ quasi-isomorphic to a finite resolving algebra (see Theorem 3.8).

(ii) The derivations of a perfect resolving algebra are in a certain sense compatible with the derivations of its truncations (the subalgebras generated by finite subsets of a generating set). See Theorem 3.13 for the precise statement. A similar result also holds for homotopy groups (see Corollary 4.12).

The main results

There is a natural structure of simplicial closed model category on the category of differential graded algebras. Resolving algebras are cofibrant objects for this closed model category structure and for any two resolving algebras A, B, the simplicial set of morphisms from B to A, denoted $\operatorname{Hom}^{\triangle}(B, A)$, is fibrant, i.e., has the Kan property, and can thus be considered as a (topological) space. In particular, we have homotopy groups $\pi_\ell \operatorname{Hom}^{\triangle}(B, A)$, for $\ell > 0$. These facts are reviewed in 1.3. In 1.4, we prove that a morphism of resolving algebras is a quasi-isomorphism if and only if it is a homotopy equivalence.

Given two resolving algebras A, B, we also have the differential graded A-module $\underline{\operatorname{Der}}(B, A)$: an element $D : B \to A$ of degree n in $\underline{\operatorname{Der}}(B, A)$ is a degree n homomorphism of graded vector spaces, which satisfies the graded Leibniz rule (see Definition 1.27).

The main results of this paper relate the homotopy groups $\pi_\ell \operatorname{Hom}^{\triangle}(B, A)$ to the $h^0(A)$-modules $h^{-\ell} \underline{\operatorname{Der}}(B, A)$:

Theorem *Let A and B be perfect resolving algebras. Then, for every $\ell > 0$, there is a canonical bijection*

$$\Xi_\ell : h^{-\ell}\,\underline{\mathrm{Der}}(B, A) \longrightarrow \pi_\ell \operatorname{Hom}^\vartriangle(B, A)\,.$$

The bijection Ξ_ℓ is an isomorphism of groups in the following two cases:

(i) if $\ell \geq 2$,

(ii) if B is generated by a set of homogeneous generators, all of which have the same degree.

There is also a relative version of this theorem, making Case (ii) more interesting and useful. In fact, the relative version and Case (ii) are the key results, as they allow proofs by induction. The relative version of Case (ii) also extends to $\ell = 0$ (see Corollary 4.9).

The map Ξ_ℓ is defined by Formula (14) in 4.2. The theorem is Theorem 4.11 and its Corollary 4.12.

Overview

In Section 1 we start by reviewing basic definitions involving differential graded algebras and differential graded modules. We introduce resolving algebras. We also study derivations, differentials and the cotangent complex.

In Section 2 we introduce the notion of étale morphism between quasi-finite resolving algebras (see Definition 2.8). There are many different ways to characterize étale morphisms. The most important are as follows. A morphism $A \to B$ is étale, if and only if any of the following equivalent conditions holds:

(i) the relative cotangent complex $L_{B/A}$ is acyclic,

(ii) $\operatorname{Spec} h^0(B) \to \operatorname{Spec} h^0(A)$ is an étale morphism of usual affine schemes and $h^i(A) \otimes_{h^0(A)} h^0(B) \to h^i(B)$ is an isomorphism, for all $i \leq 0$, (in particular, any quasi-isomorphism is étale),

(iii) (if A and B are perfect) the induced morphism of completions $\widehat{A} \to \widehat{B}$ is a quasi-isomorphism, for the completions at every augmentation of B (see Theorem 2.14),

(iv) (if A and B are perfect) for every $B \to C$, the map $\operatorname{Hom}^\vartriangle(B, C) \to \operatorname{Hom}^\vartriangle(A, C)$ induces isomorphisms on homotopy groups π_ℓ, for all, or for one fixed $\ell > 0$ (see Proposition 4.18).

The most important example of an étale morphism is the *standard* étale morphism $A \to B$, where B is generated over A by formal variables $x_1, \ldots, x_r$ in degree 0 and $\xi_1, \ldots, \xi_r$ in degree -1. The differential on B is given by $dx_i = 0$ and $d\xi_j = f_j \in A^0[x_1, \ldots, x_r]$. Moreover, $\det(\frac{\partial f_j}{\partial x_i})$ is a unit in $h^0(A)[x_1, \ldots, x_r]/(f_1, \ldots, f_r)$.

We prove that every étale morphism is locally standard (see Proposition 2.18, for the exact statement). Thus the study of étale morphisms can often be reduced to the study of standard étale morphisms.

As a byproduct of the equivalence of (i) and (ii) we get a very useful quasi-isomorphism criterion: $A \to B$ is a quasi-isomorphism if and only if $L_{B/A}$ is acyclic and $h^0(A) \to h^0(B)$ an isomorphism. (See Corollary 2.9.)

In Section 3, we introduce perfect resolving algebras and prove the two fundamental facts alluded to, above.

Section 4, is devoted to the proof of the main theorem mentioned above, to the effect that we can 'linearize' homotopy groups.

Finally, in Section 5, we prove that any morphism $A \to B$ between finite resolving algebras admits a factorization $A \to B \to B'$, where $A \to B$ makes B a finite resolving algebra over A and $B' \to B$ is a quasi-isomorphism. As an application, we prove that the fibered homotopy coproduct (or derived tensor product, as we prefer to call it) of a diagram of finite (perfect) resolving algebras can be represented, again, by a finite (perfect) resolving algebra.

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1 Differential Graded Algebras

1.1 Review and terminology

We will fix a base field of characteristic zero, denoted k, throughout. All rings and algebras are assumed to be commutative with unit (if they are graded, then commutative means graded commutative). Algebra without qualifier means k-algebra. If a is a homogeneous element of a graded k-vector space, we denote by $\bar{a}$ its degree.

Differential graded algebras A *differential graded* k-algebra A is a graded k-algebra

$$A^\natural = \bigoplus_{n \in \mathbb{Z}} A^n$$

endowed with a differential $d : A^\natural \to A^\natural$ of degree 1, i.e., a degree 1 homomorphism of graded k-vector spaces satisfying $d^2 = 0$ and the graded Leibniz rule

$$d(ab) = (da)b + (-1)^{\bar{a}} a\, db.$$

We always use the notation $A^\natural$ for the underlying graded algebra of the differential graded algebra $A = (A^\natural, d)$.

For a differential graded algebra A, the cohomology $h^*(A) = \ker d / \operatorname{im} d$ is a graded k-algebra. The degree 0 cohomology $h^0(A)$ is a subalgebra. If A is concentrated in non-positive degrees, then there is a morphism of differential graded algebras $A \to h^0(A)$.

Modules A *differential graded module* M over the differential graded algebra A is a (left) graded $A^\natural$-module $M^\natural$ endowed with a differential $d_M : M^\natural \to M^\natural$ of degree 1, i.e., a degree 1 homomorphism of graded k-vector spaces satisfying $d_M^2 = 0$ and the graded Leibniz rule

$$d_M(am) = (da)m + (-1)^{\bar{a}} a\, d_M m,$$

for $a \in A$ and $m \in M$. Note that $h^*(M)$ is a graded $h^*(A)$-module.

Sometimes, we will write the action of A on the right. In this case the notation ma is understood to mean

$$ma := (-1)^{\bar{a}\bar{m}} am \,. \tag{1}$$

Note that every differential graded A-module has an underlying complex of k-vector spaces.

Let M and N be differential graded A-modules. Tensor product and internal hom are defined as follows. The *tensor product* $M \otimes_A N$ has underlying graded $A^\natural$-module

$$M^\natural \otimes_{A^\natural} N^\natural$$

and the differential is given by

$$d_{M\otimes N}(m\otimes n) = d_M m\otimes n + (-1)^{\bar{m}} m\otimes d_N n.$$

The *internal hom* $\underline{\mathrm{Hom}}_A(M,N)$ has underlying graded $A^\natural$-module

$$\underline{\mathrm{Hom}}_A(M,N)^\natural = \bigoplus_n \mathrm{Hom}^n_{A^\natural}(M^\natural, N^\natural)$$

and differential given by

$$d_N\phi(m) = d_{\underline{\mathrm{Hom}}(M,N)}(\phi)(m) + (-1)^{\bar{\phi}}\phi(d_M m),$$

for $m\in M$ and $\phi\in\underline{\mathrm{Hom}}_A(M,N)$. The k-vector space of differential graded A-module homomorphisms $M\to N$ is the set of 0-cocycles in $\underline{\mathrm{Hom}}_A(M,N)$:

$$\mathrm{Hom}_A(M,N) = Z^0\,\underline{\mathrm{Hom}}_A(M,N)$$

Note that differential graded A-modules form an abelian category with kernels, cokernels, images and direct sums taken degree-wise. In fact, the category of differential graded A-modules is an abelian subcategory of the category of complexes of k-vector spaces.

We have the following formulas:

$$\underline{\mathrm{Hom}}_A(M,N)\otimes_A P = \underline{\mathrm{Hom}}_A(M, N\otimes_A P)\,,$$

$$\underline{\mathrm{Hom}}_A(M\otimes_A N, P) = \underline{\mathrm{Hom}}_A\big(M, \underline{\mathrm{Hom}}_A(N,P)\big)\,,$$

for differential graded modules M, N, P over the differential graded algebra A,

$$\underline{\mathrm{Hom}}_A(M,N) = \underline{\mathrm{Hom}}_B(M\otimes_A B, N)\,,$$

for a morphism of differential graded algebras $A\to B$ and a differential graded modules M over A and N over B.

Cones Given a differential graded A-module M, the shift $M[1]$ is defined by shifting the underlying complex of k-vector spaces:

$$(M[1])^i = M^{i+1};\quad d_{M[1]} = -d_M.$$

The shift is again a differential graded A-module in a natural way.

The cone over the homomorphism $\phi : M\to N$ of differential graded A-modules is defined by

$$C(\phi) = C(M\to N) = N\oplus M[1].$$

Thus $C(M\to N)$ is again a differential graded A-module. There is a canonical triangle

$$M \xrightarrow{\phi} N \longrightarrow C(\phi)\longrightarrow M[1]$$

of differential graded A-modules, which induces a long exact sequence of cohomology groups.

Given two homomorphisms of differential graded A-modules

$$M \xrightarrow{\phi} N \xrightarrow{\rho} P,$$

then, if $\rho \circ \phi = 0$, there is a canonical homomorphism $C(\phi) \to P$ (induced by the second projection) making

$$\begin{array}{ccc} N & \longrightarrow & C(\phi) \\ & \overset{\rho}{\searrow} & \downarrow \\ & & P \end{array}$$

commute. If

$$0 \longrightarrow M \xrightarrow{\phi} N \longrightarrow P \longrightarrow 0$$

is exact, then $C(\phi) \to P$ is a quasi-isomorphism.

Spectral Sequence Let M be a differential graded A-module. Then M is filtered by differential graded submodules $F^p M = A \cdot M^{\geq p}$. Thus we get an associated spectral sequence, which converges if both A and M are bounded above.

Proposition 1.1 *Assume the differential graded algebra A is concentrated in non-positive degrees. Let M be a differential graded A-module which is bounded above, and such that $M^\natural$ is a free $A^\natural$-module. Then we have a spectral sequence of $h^0(A)$-modules*

$$E_2^{p,q} = h^q(A) \otimes_{h^0(A)} h^p\big(M \otimes_A h^0(A)\big) \Longrightarrow h^{p+q}(M)\,. \quad \square$$

As an application, we may for example deduce, that M is acyclic if and only if $M \otimes_A h^0(A)$ is acyclic.

1.2 Resolving algebras

Symmetric algebras Given a complex of k-vector spaces V, the n-th symmetric power, notation $S^n V$, is defined to be the quotient of $V^{\otimes n}$ by all relations of the form

$$x_1 \otimes \ldots \otimes x_p \otimes x \otimes y \otimes y_1 \otimes \ldots \otimes y_q = (-1)^{\overline{xy}} x_1 \otimes \ldots \otimes x_p \otimes y \otimes x \otimes y_1 \otimes \ldots \otimes y_q\,,$$

for x, y, $x_1, \ldots, x_p$, $y_1 \ldots, y_q$ homogeneous elements of V and $p + q + 2 = n$. Thus $S^n V$ is again a complex of k-vector spaces, for all $n \geq 0$. The direct sum $SV = \bigoplus_{n\geq 0} S^n V$ is a differential graded algebra. The functor $V \mapsto SV$ is a left adjoint for the forgetful functor

$$(\text{differential graded algebras}) \longrightarrow (\text{complexes of } k\text{-vector spaces})\,.$$

Definition 1.2 A differential graded algebra A which is isomorphic to SV, for some complex of k-vector spaces V, is called **free**. If $V \subset A$ is a subcomplex inducing an isomorphism $SV \to A$, we call V a **free basic complex** for A.

If $A \to B$ is a morphism of differential graded algebras and there exists a subcomplex $V \subset B$ such that $A \otimes_k SV \to B$ is an isomorphism, then we call $A \to B$ **free**, or we say that B is free over A. Moreover, V is called a **free basic complex** for B over A.

The importance of free differential graded algebras for us is that they occur as tangent spaces of differential graded schemes.

Quasi-free algebras As a special case of symmetric algebras, we may consider symmetric algebras SV, on a graded k-vector spaces V. This is the case of complexes V with zero differential. In this case SV is simply a graded k-algebra, as it has vanishing differential, too. The functor $V \mapsto SV$ is a left adjoint for the forgetful functor

$$(\text{graded } k\text{-algebras}) \to (\text{graded } k\text{-vector spaces})\,.$$

Another common notation for SV is $k[V]$. If (x_i) is a homogeneous basis for the graded k-vector space V, then we denote $k[V]$ also by $k[x]$.

Definition 1.3 Let A be a differential graded algebra. Suppose that $V \subset A$ is a graded sub-k-vector space, such that $SV \to A^\natural$ is an isomorphism of graded k-algebras. Then we say that the differential graded algebra A is **quasi-free** and we call V a **basic** space for A.

Let $V \subset A$ be a basic space for the quasi-free differential graded algebra A. If (x_i) is a homogeneous k-basis for the graded k-vector space V, then we call (x_i) a **basis** for A.

Let $A \to B$ be a morphism of differential graded algebras. If $V \subset B$ is a graded subspace such that $A^\natural \otimes_k SV \to B^\natural$ is an isomorphism, then B is **quasi-free** over A and V is a **basic** space for B over A. If (x_i) is a homogeneous basis for V, then it is called a **basis** for B over A.

Thus any basis (x_i) of a quasi-free differential graded algebra A defines an isomorphism $k[x] \to A^\natural$.

We have chosen the terms *basic* and *basis*, rather than the more logical terms *quasi-basic* and *quasi-basis*, because these terms will be used much more often than the terms defined in Definition 1.2, and we do not want the prefix 'quasi' to take over the paper.

Let B be quasi-free over A and (x_i) a basis. If $A \to C$ is a morphism of differential graded algebras, then any family (c_i) of homogeneous elements of C, such that $\deg c_i = \deg x_i$ for all i, induces a unique morphism of graded algebras $B^\natural \to C^\natural$, extending $A^\natural \to C^\natural$. This morphism $B \to C$ is a morphism of differential graded algebras if and only if it maps dx_i to dc_i for all i.

The notion of quasi-freeness itself is not very useful for us. Its main purpose is to enable the definition of *resolving algebra*, which is next.

Resolving algebras We now come to be most important set of concepts for this work.

Definition 1.4 We call a differential graded algebra A a **resolving algebra**, if it is quasi-free and there exists a basis (x_i) for A, such that $\deg x_i \leq 0$, for all i.

A morphism of differential graded algebras $A \to B$ is called a **resolving morphism**, if B is quasi-free over A and there exists a basis (x_i) for B over A, such that $\deg x_i \leq 0$, for all i.

If we speak of a **basis** for a resolving algebra or a resolving morphism, it is understood that it consists of elements x_i such that $\deg x_i \leq 0$, for all i.

The main purpose of resolving morphisms for us is that they are *cofibrations* for the natural simplicial closed model category structure on the category of differential graded algebras. For geometric purposes, we have to put some additional finiteness assumptions:

Definition 1.5 A resolving algebra A is called **quasi-finite**, if there exists a basis (x_i) for A satisfying

(i) for every $n > 0$, the set $\{i \mid \deg x_i = n\}$ is empty,

(ii) for every $n \leq 0$, the set $\{i \mid \deg x_i = n\}$ is finite.

Any basis for A satisfying these two properties is called a **quasi-finite** basis or a **coordinate system** for A.

A resolving morphism $A \to B$ is called **quasi-finite**, if there exists a **quasi-finite** basis for B over A, i.e., a basis (x_i), satisfying (i) and (ii).

Definition 1.6 A resolving algebra A is called **finite**, if there exists a finite basis (x_i) for A such that $\deg x_i \leq 0$, for all i. Any such basis for A is called a **finite** basis, or a **finite coordinate system**.

A resolving morphism $A \to B$ is called **finite**, if there exists a **finite** basis for B over A, i.e., a basis (x_i) which is finite and satisfies $\deg x_i \leq 0$, for all i.

The purpose of finite resolving algebras for us is, that they provide local models for differential graded schemes. There is another important class of resolving algebras which are somewhere between finite and quasi-finite resolving algebras. These are the *perfect* resolving algebras introduced in Definition 3.1. Perfect resolving algebras serve as affine differential graded schemes.

Definition 1.7 Let A be a finite resolving algebra. If we can find a finite basis (x_i) for A such that $\deg x_i \geq -n$, for all i, then we say that A is of **amplitude** n.

If $A \to B$ is a finite resolving morphism and we can find a finite basis (x_i) for B over A such that $\deg x_i \in [-n, 0]$, for all i, then we say that $A \to B$ has **amplitude** n.

The reason for the terminology is that resolving algebras serve as resolutions:

Definition 1.8 If C is a differential graded algebra and $A \to C$ is a morphism of differential graded algebras, where A is a resolving algebra, then we call $A \to C$ a **resolution** of C, if $A \to C$ is a quasi-isomorphism.

If $A \to C$ is a morphism of differential graded algebras, $A \to B$ a resolving morphism, and $B \to C$ a quasi-isomorphism over A, then $A \to B \to C$ is a **resolution** of $A \to C$.

Let us show that morphisms between resolving algebras always admit resolutions:

Proposition 1.9 *For any morphism of resolving algebras $A \to C$, there exists a resolution $A \to B \to C$. If A and C are quasi-finite, we may choose $A \to B$ quasi-finite.*

Proof. We construct inductively a sequence of partial resolutions $A \to B_{(n)} \to C$ with the properties:

(i) $A \to B_{(n)}$ is resolving and the composition $A \to B_{(n)} \to C$ is equal to $A \to C$,

(ii) $h^{-n}(B_{(n)}) \to h^{-n}(C)$ is surjective,

(ii) $h^{1-n}(B_{(n)}) \to h^{1-n}(C)$ is bijective.

We may take A for $B_{(-1)}$. Once we have constructed $B_{(n)} \to C$, we choose elements $b_i \in Z^{-n}(B_{(n)})$ generating the kernel of $h^{-n}(B_{(n)}) \to h^{-n}(C)$, and $e_i \in C^{-n-1}$ such that de_i is the image of b_i under $B_{(n)} \to C$, for all i. We also choose $c_j \in Z^{-n-1}(C)$ generating $h^{-n-1}(C)$. Then we define $B_{(n+1)}$ by adjoining to $B_{(n)}$ formal variables x_i and y_j of degree $-n-1$ and setting $dx_i = b_i$ and $dy_j = 0$. We define the morphism $B_{(n+1)} \to C$ by extending $B_{(n)} \to C$ by $x_i \mapsto e_i$ and $y_j \mapsto c_j$.

Finally, we let $B = \varinjlim B_{(n)}$.

To deal with the quasi-finite case, let us make the remark that if A is a quasi-finite resolving algebra, then $h^0(A)$ is a finite type k-algebra and $h^i(A)$ is a finitely generated $h^0(A)$-module, for all i. Now we examine the above proof more closely, and specify more carefully what we mean by 'generating'. In fact, to construct $B_{(0)}$ we choose generators for $h^0(C)$ as an $h^0(A)$-algebra. To construct all $B_{(n+1)}$ for $n \geq 0$, we choose generators of $\ker\big(h^{-n}(B_{(n)}) \to h^{-n}(C)\big)$ and $h^{-n-1}(C)$ as $h^0(B_{(n)})$-modules. Since $h^0(B_{(n)}) \to h^0(C)$ is onto if $n \geq 0$, we see that in all cases we have finite generation. □

Scholum 1.10 *If A is a differential graded algebra concentrated in non-positive degrees, such that $h^0(A)$ is a k-algebra of finite type and $h^i(A)$ is a finitely generated $h^0(A)$-module, for every i, then there exists a resolution $A' \to A$, where A' is quasi-finite.*

1.3 The simplicial closed model category structure

For the definitions and properties of simplicial sets and closed model categories, the reader may consult, for example, the recent text book by Goerss-Jardine [5]. We will commit the common abuse of calling a simplicial set a *space*.

Definition 1.11 A **simplicial category** is a category enriched over simplicial sets. Thus a simplicial category $\mathfrak{S}$ is given by

(i) a class of objects $\operatorname{ob}\mathfrak{S}$,

(ii) for any two objects U, V of $\mathfrak{S}$ a simplicial set $\mathrm{Hom}^{\triangle}(U,V)$,

(iii) for any three objects U, V, W of $\mathfrak{S}$ a simplicial map

$$\begin{aligned} \circ : \mathrm{Hom}^{\triangle}(V,W) \times \mathrm{Hom}^{\triangle}(U,V) &\longrightarrow \mathrm{Hom}^{\triangle}(U,W) \\ (f,g) &\longmapsto f \circ g \end{aligned}$$

(iv) for every object U of $\mathfrak{S}$, a 0-simplex id_U in $\mathrm{Hom}^{\triangle}(U,U)$, which we can also view as a simplicial map

$$* \xrightarrow{\mathrm{id}_U} \mathrm{Hom}^{\triangle}(U,U),$$

such that

(i) the composition $\circ$ is associative, i.e., for four objects U, V, W, Z of $\mathfrak{S}$ the diagram of simplicial sets

$$\begin{array}{ccc} \mathrm{Hom}^{\triangle}(W,Z) \times \mathrm{Hom}^{\triangle}(W,V) \times \mathrm{Hom}^{\triangle}(V,U) & \longrightarrow & \mathrm{Hom}^{\triangle}(V,Z) \times \mathrm{Hom}^{\triangle}(U,V) \\ \downarrow & & \downarrow \\ \mathrm{Hom}^{\triangle}(W,Z) \times \mathrm{Hom}^{\triangle}(U,W) & \longrightarrow & \mathrm{Hom}^{\triangle}(U,Z) \end{array}$$

commutes,

(ii) the objects id_U act as identities for $\circ$, i.e., the diagrams

$$\begin{array}{ccc} \mathrm{Hom}^{\triangle}(U,V) & & \\ \downarrow & \searrow & \\ \mathrm{Hom}^{\triangle}(U,V) \times \mathrm{Hom}^{\triangle}(U,U) & \longrightarrow & \mathrm{Hom}^{\triangle}(U,V) \end{array} \qquad \begin{array}{ccc} \mathrm{Hom}^{\triangle}(U,V) & \longrightarrow & \mathrm{Hom}^{\triangle}(V,V) \times \mathrm{Hom}^{\triangle}(U,V) \\ & \searrow & \downarrow \\ & & \mathrm{Hom}^{\triangle}(U,V) \end{array}$$

commute.

Passing from $\mathrm{Hom}^{\triangle}(U,V)$ to the set of 0-simplices $\mathrm{Hom}^{\triangle}_0(U,V)$ we get the underlying category $\mathfrak{S}_0$ of the simplicial category $\mathfrak{S}$.

Assume that the underlying category of the simplicial category $\mathfrak{S}$ has a closed model category structure. The following property (which $\mathfrak{S}$ might enjoy, or not) is called the *simplicial model category axiom*

Axiom 1.12 If $j : A \to B$ is a cofibration and $q : X \to Y$ a fibration then

$$\mathrm{Hom}^{\triangle}(B,X) \longrightarrow \mathrm{Hom}^{\triangle}(A,X) \times_{\mathrm{Hom}^{\triangle}(A,Y)} \mathrm{Hom}^{\triangle}(B,Y)$$

is a fibration of simplicial sets, which is trivial if j or q is trivial.

Definition 1.13 If the underlying category of the simplicial category $\mathfrak{S}$ is a closed model category and the simplicial model category axiom is satisfied, then we call $\mathfrak{S}$ a **simplicial closed model category**. (Note that this notion is weaker than the one treated in [5].)

Differential graded algebras form a simplicial closed model category: Let $\mathfrak{A}$ be the category of all differential graded k-algebras.

Proposition 1.14 *Call a morphism $f : A \to B$ in $\mathfrak{A}$ a* **fibration** *if f is degree-wise surjective. Call f a* **weak equivalence** *if it is a quasi-isomorphism, i.e., if it induces bijections on cohomology groups. Finally, call f a* **cofibration** *if it satisfies the left lifting property with respect to all trivial fibrations. With these definitions $\mathfrak{A}$ is a closed model category.*

Let $\Omega_n = \Omega(\Delta^n)$ be the algebraic de Rham complex of the algebraic n-simplex

$$\operatorname{Spec} k[x_0, \ldots, x_n]/\textstyle\sum x_i = 1,$$

(which is a differential graded k-algebra). For two differential graded algebras A, B define the simplicial set $\operatorname{Hom}^{\Delta}(A, B)$ to have the n-simplices

$$\operatorname{Hom}^{\Delta}_n(A, B) = \operatorname{Hom}(A, B \otimes \Omega_n).$$

With this definition, $\mathfrak{A}$ is a simplicial closed model category.

Proof. This (and much more) is proved in [6]. □

Remark 1.15 By definition, we have

$$\operatorname{Hom}\left(\Delta^n, \operatorname{Hom}^{\Delta}(A, B)\right) = \operatorname{Hom}\left(A, B \otimes \Omega(\Delta^n)\right).$$

More generally, for every finite simplicial set K we have an algebraic de Rham complex $\Omega(K)$ and there is a natural bijection

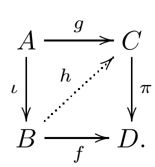
$$\operatorname{Hom}\left(K, \operatorname{Hom}^{\Delta}(A, B)\right) = \operatorname{Hom}\left(A, B \otimes \Omega(K)\right).$$

For the definition of $\Omega(K)$, see [3].

We shall now identify a class of cofibrations in $\mathfrak{A}$.

Proposition 1.16 *Any resolving morphism $\iota : A \to B$ of differential graded algebras is a cofibration.*

Proof. We have to show that $\iota : A \to B$ satisfies the left lifting property with respect to all surjective quasi-isomorphisms. So let $\pi : C \to D$ be a surjective quasi-isomorphism of differential graded algebras. Assume given the commutative diagram of solid arrows



We need to show the existence of the dotted arrow h.

Let $(y_i)_{i\in I}$ be a basis for B over A. By induction, we may assume that all y_i, $i\in I$, have the same degree, say $n\leq 0$. This implies that $dy_i\in A$, for all $i\in I$, or more precisely, that there exist $a_i\in A$ such that $\iota a_i = dy_i$. Moreover, $da_i = 0$. (Note that this is where we use the assumption of non-positive degree.)

Since $C^\bullet\to D^\bullet$ is a quasi-isomorphism, the diagram

$$\begin{array}{ccc} C^n/dC^{n-1} & \xrightarrow{d} & Z^{n+1}(C^\bullet) \\ \downarrow & & \downarrow \\ D^n/dD^{n-1} & \xrightarrow{d} & Z^{n+1}(D^\bullet) \end{array}$$

of k-vector spaces is cartesian. So by surjectivity of $\pi : C^{n-1}\to D^{n-1}$ we have that C^n maps onto the fibered product

$$\begin{array}{ccccc} C^n & \twoheadrightarrow & \cdot & \xrightarrow{d} & Z^{n+1}(C^\bullet) \\ & & {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ & & D^n & \xrightarrow{d} & Z^{n+1}(D^\bullet). \end{array}$$

Thus we can choose elements $h(y_i)\in C^n$ such that

1. $d\,h(y_i) = g(a_i)$,
2. $\pi\,h(y_i) = f(y_i)$,

for all $i\in I$. (Note that $\pi\big(g(a_i)\big) = f\big(\iota(a_i)\big) = f(dy_i) = d\big(f(y_i)\big)$.) By the freeness of $B^\natural$ over $A^\natural$ on $(y_i)_{i\in I}$ this defines a morphism of graded $A^\natural$-algebras $h : B^\natural\to C^\natural$. In particular, $h\circ\iota = g$. Property 2 implies $\pi\circ h = f$. Finally, $hd = dh$ follows from Property 1. □

Corollary 1.17 *If $A\to B$ is a resolving morphism of differential graded algebras, then*

$$\operatorname{Hom}^\vartriangle(B,C)\longrightarrow\operatorname{Hom}^\vartriangle(A,C) \qquad (2)$$

is a fibration of simplicial sets, for all differential graded algebras C.

Proof. Take $X = C$ and $Y = 0$ in Axiom 1.12. □

Corollary 1.18 *Every resolving algebra if fibrant-cofibrant in $\mathfrak{A}$.*

Let $A\to B$ be a resolving morphism of differential graded algebras and $A\to C$ a fixed morphism of differential graded algebras. Then we denote the fiber of the fibration (2) over the point $A\to C$ of $\operatorname{Hom}^\vartriangle(A,C)$ by $\operatorname{Hom}^\vartriangle_A(B,C)$. This fiber is a fibrant simplicial set, and we may think of it as the space of A-algebra morphisms from B to C.

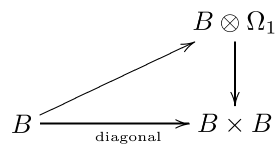
If $A\to B'\to B$ are two resolving morphisms and $A\to C$ any fixed morphism of differential graded algebras, then we have a fibration

$$\operatorname{Hom}^\vartriangle_A(B,C)\longrightarrow\operatorname{Hom}^\vartriangle_A(B',C)\,.$$

1.4 Homotopies

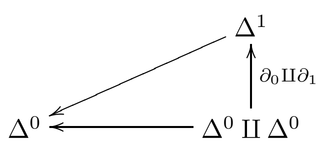
The following expresses another compatibility between the closed model category structure and the simplicial category structure on differential graded algebras.

Lemma 1.19 *Let B be a differential graded algebra. The canonical commutative diagram*



(3)

which is induced by the commutative diagram of algebraic simplices



is a path object for B.

Proof. By the definition of path object (see [5], Section II.1.) we need only check that

(i) $B \to B \otimes \Omega_1$ is a weak equivalence,

(ii) $B \otimes \Omega_1 \to B \times B$ is a fibration.

Both claims reduce immediately to the case $B = k$ (by flatness of B over k). Then (i) is the algebraic de Rham theorem and (ii) is obvious. □

Corollary 1.20 *Assume that A is a cofibrant differential graded algebra and B a fibrant differential graded algebra. Two morphisms of differential graded algebras $f, g : A \to B$ are homotopic (with respect to the closed model category structure on $\mathfrak{A}$) if and only if there exists a 1-simplex $h \in \mathrm{Hom}_1^{\Delta}(A, B)$ such that $\partial_0 h = f$ and $\partial_1 h = g$.*

Proof. By Corollary 1.9 of [5, Chapter II], the notion of homotopy between f and g is well-defined. We may use the path object of Lemma 1.19 to check if f and g are homotopic. □

In particular, for morphisms between resolving algebras the two notions of homotopic are equivalent.

Corollary 1.21 *Let $A \to B$ be a quasi-isomorphism of resolving algebras. Then $A \to B$ is a homotopy equivalence.*

Proof. By the Theorem of Whitehead (Theorem 1.10 in [5, Chapter II]) $A \to B$ is a homotopy equivalence with respect to the closed model category structure on $\mathfrak{A}$. □

Proposition 1.22 *If two morphisms of resolving algebras $f, g : A \to B$ are homotopic, then they induce the same homomorphisms on cohomology $h^*(A) \to h^*(B)$.*

Proof. Let $H : A \to B \otimes \Omega_1$ be a homotopy from f to g. Then we have a commutative diagram of differential graded algebras

$$\begin{array}{ccccc} & & B \otimes \Omega_1 & & \\ & \overset{H}{\nearrow} & \downarrow & \nwarrow & \\ A & \xrightarrow[f \times g]{} & B \times B & \longleftarrow & B\,, \end{array}$$

whose right half is our path object (3). Applying h^*, we get the commutative diagram

$$\begin{array}{ccccc} & & h^*(B \otimes \Omega_1) & & \\ & \overset{h^*(H)}{\nearrow} & \downarrow & \overset{\text{isomorphism}}{\nwarrow} & \\ h^*(A) & \xrightarrow[h^*(f) \times h^*(g)]{} & h^*(B) \times h^*(B) & \longleftarrow & h^*(B)\,, \end{array}$$

which implies that, indeed, $h^*(f) = h^*(g)$. $\square$

Corollary 1.23 *Let $f : A \to B$ be a morphism of resolving algebras. If f is a homotopy equivalence, then f is a quasi-isomorphism.*

1.5 Derivations and differentials

Let $B \to A$ be a morphism of differential graded k-algebras.

Definition 1.24 For an A-module M, a **B-derivation** $D : A \to M$ is a homomorphism of complexes of k-vector spaces vanishing on B and satisfying the Leibniz rule (see (1))

$$D(ab) = Da\, b + a\, Db,$$

for all $a, b \in A$.

Lemma 1.25 *Assume that A is resolving over B on the basis $(x_i)_{i \in I}$. Then there exists a* universal *B-derivation $\mathfrak{d} : A \to \Omega_{A/B}$. It may be constructed as follows:*

(i) As underlying graded $A^\natural$-module take

$$\Omega^\natural_{A/B} = \bigoplus_{i \in I} A^\natural \mathfrak{d} x_i,$$

for formal generators $\mathfrak{d} x_i$, which have the same degrees as the x_i.

(ii) Construct the unique B-derivation $\mathfrak{d} : A^\natural \to \Omega^\natural_{A/B}$, satisfying $\mathfrak{d}(x_i) = \mathfrak{d} x_i$, for all $i \in I$.

(iii) Define the differential d *on* $\Omega^\natural_{A/B}$ *by*

$$d(a\mathfrak{d}x_i) = da\,\mathfrak{d}x_i + (-1)^{\bar{a}}a\,\mathfrak{d}(dx_i).$$

Then

(i) $\Omega_{A/B}$ *is an* A*-module, i.e.,*

$$d(a\omega) = da\,\omega + (-1)^{\bar{a}}a\,d\omega,$$

(ii) $\mathfrak{d} : A \to \Omega_{A/B}$ *is a derivation, i.e.,* $d\mathfrak{d} = \mathfrak{d}d$*,*

(iii) $\mathfrak{d} : A \to \Omega_{A/B}$ *is a universal derivation.* □

Corollary 1.26 *If* $C \to B$ *is a resolving morphism of differential graded algebras, then for every differential graded algebra in non-positive degrees* A*, we have a convergent spectral sequence*

$$h^q(A) \otimes_{h^0(A)} h^p\big(\Omega_{B/C} \otimes_B h^0(A)\big) \Longrightarrow h^{p+q}(\Omega_{B/C} \otimes_B A)\,.$$

Proof. By Lemma 1.25 we know that $\Omega^\natural_{B/C}$ is free over $B^\natural$. Since $\Omega_{B/C}$ is automatically bounded above, we can apply Proposition 1.1. □

Definition 1.27 The **internal module of derivations** of A over B, notation $\underline{\mathrm{Der}}_B(A, M)$, is defined to be the subcomplex of $\underline{\mathrm{Hom}}_B(A, M)$, consisting of all elements $D : A \to M$, vanishing on B, and satisfying the graded Leibniz rule

$$D(ab) = Da\,b + (-1)^{\bar{a}\bar{D}}a\,Db,$$

for all $a, b \in A$. Thus a B-derivation $D : A \to M$ is a 0-cocycle in the complex $\underline{\mathrm{Der}}_B(A, M)$.

Lemma 1.28 *The internal module of derivations* $\underline{\mathrm{Der}}_B(A, M)$ *is a differential graded* A*-module. If* A *is resolving over* B*, we have a natural isomorphism*

$$\underline{\mathrm{Der}}_B(A, M) = \underline{\mathrm{Hom}}_A(\Omega_{A/B}, M).$$

Derivations correspond to homomorphisms under this isomorphism. □

Remark 1.29 Let $A \to C$ be a morphism of differential graded algebras. Then $\underline{\mathrm{Der}}_B(A, C)$ is a differential graded C-module via the action $(cD)(a) = c\big(D(a)\big)$.

Example 1.30 If A is a resolving algebra, we set

$$\Theta_A := \underline{\mathrm{Der}}_k(A, A) = \underline{\mathrm{Hom}}_A(\Omega_A, A)\,.$$

This has the additional structure of a differential graded Lie algebra. The differential on Θ_A is given by bracket with $d : A \to A \in \underline{\mathrm{Der}}^1_k(A, A)$.

If A is resolving over the differential graded algebra B, we set

$$\Theta_{A/B} = \underline{\mathrm{Der}}_B(A, A) = \underline{\mathrm{Hom}}(\Omega_{A/B}, A)\,.$$

Now assume given morphisms of differential graded algebras $C \to B \to A$. Let both B and A be resolving over C. We get a homomorphism of B-modules $\Omega_{B/C} \to \Omega_{A/C}$ (since a C-derivation on A restricts to a C-derivation on B), and hence a homomorphism of A-modules $\Omega_{B/C} \otimes_B A \to \Omega_{A/C}$.

Lemma 1.31 *If $B \to A$ is a quasi-isomorphism, then*

$$\Omega_{B/C} \otimes_B A \to \Omega_{A/C}$$

is a quasi-isomorphism.

PROOF. See [6]. $\square$

Assume, in addition, that A is also resolving over B. Then we get a homomorphism of A-modules $\Omega_{A/C} \to \Omega_{A/B}$ (since a B-derivation of A is also a C-derivation). The sequence of differential graded A-modules

$$0 \longrightarrow \Omega_{B/C} \otimes_B A \longrightarrow \Omega_{A/C} \longrightarrow \Omega_{A/B} \longrightarrow 0 \tag{4}$$

is exact.

1.6 The cotangent complex

Let us define the cotangent complex of a morphism between resolving algebras.

Definition 1.32 Let B and A be resolving algebras over k, and $f : B \to A$ a morphism. We define the **cotangent complex** $L_{A/B}$ to the differential graded A-module defined as the cone over the homomorphism $\Omega_B \otimes_B A \to \Omega_A$:

$$L_{A/B} = \mathcal{C}(\Omega_B \otimes_B A \to \Omega_A).$$

If A is resolving over B, then by (4) there is a canonical quasi-isomorphism

$$L_{A/B} \longrightarrow \Omega_{A/B}\,.$$

Lemma 1.33 *(i) If $B \to A$ is a quasi-isomorphism, then $L_{A/B}$ is acyclic.*

(ii) Given morphisms of resolving algebras $C \to B \to A$, there exists a natural distinguished triangle of differential graded A-modules

$$L_{B/C} \otimes_B A \longrightarrow L_{A/C} \longrightarrow L_{A/B} \longrightarrow L_{B/C} \otimes_B A\,[1]\,.$$

(iii) Let A, B and C be resolving algebras and $B \to A$ a resolving morphism. Consider the tensor product

$$\begin{array}{ccc} A \otimes_B C & \longleftarrow & C \\ \uparrow & & \uparrow \\ A & \longleftarrow & B. \end{array}$$

So $A \otimes_B C$ is also a resolving algebra. Then $L_{C/B} \otimes_B A \to L_{A \otimes_B C/A}$ and $L_{A/B} \otimes_B C \to L_{A \otimes_B C/C}$ are quasi-isomorphisms. $\square$

Occasionly, we will use the cotangent complex of a morphism between differential graded algebras which are not resolving. The necessary theory is developed in [6].

Note 1.34 If $C \to B$ is a morphism of resolving algebras, then we have a convergent spectral sequence

$$h^q(A) \otimes_{h^0(A)} h^p\big(L_{B/C} \otimes_B h^0(A)\big) \Longrightarrow h^{p+q}(L_{B/C} \otimes_B A)\,,$$

for every differential graded algebra in non-positive degrees A. $\square$

Definition 1.35 Let $C \to B$ be a morphism of resolving algebras. Then we call

$$T_{B/C} = \underline{\mathrm{Hom}}_B(L_{B/C}, B)$$

the **tangent complex** of B over C.

If $C \to B$ is a resolving morphism of resolving algebras, we have a canonical quasi-isomorphism of differential graded B-modules

$$\Theta_{B/C} \longrightarrow T_{B/C}\,.$$

Acyclicity criteria

In the following proposition we use the differential graded algebras Λ_n, defined for every $n > 0$ as follows:

If n is even, the underlying graded k-algebras is given by $\Lambda_n^\natural = k[x, \xi]$, where $\deg x = -n$ and $\deg \xi = -2n - 1$, and the differential is given by $dx = 0$ and $d\xi = x^2$.

If n is odd, $\Lambda_n = k[x]$, with $\deg x = -n$ and $d(x) = 0$.

Note that

$$h^i(\Lambda_n) = \begin{cases} k & \text{if } k = 0, -n, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, Λ_n is quasi-isomorphic to $h^*(\Lambda_n)$.

Proposition 1.36 *Let $B \to A$ be a morphism of resolving algebras and $r \leq 0$ an integer. Fix a morphism $A \to k$ of differential graded algebras. The following are equivalent:*

(i) $\underline{\mathrm{Der}}(A, k) \to \underline{\mathrm{Der}}(B, k)$ is a quasi-isomorphism,

(ii) $L_{A/B} \otimes_A k$ is acyclic,

(iii) $h^r\, \underline{\mathrm{Der}}(A, C) \to h^r\, \underline{\mathrm{Der}}(B, C)$ is an isomorphism for all (finite) resolving algebras C,

(iv) $h^r\, \underline{\mathrm{Der}}(A, \Lambda_n) \to h^r\, \underline{\mathrm{Der}}(B, \Lambda_n)$ is an isomorphism for all $n > 0$.

In (iii) and (iv) the A-module structure on C and Λ_n is given via $A \to k$.

If, moreover, $B \to A$ is a resolving morphism, further equivalent conditions are

(iii′) $h^r\, \underline{\mathrm{Der}}_B(A, C) = 0$, for all (finite) resolving algebras C,

(iv′) $h^r\, \underline{\mathrm{Der}}_B(A, \Lambda_n) = 0$, for all $n > 0$.

Proof. By definition, we have a distinguished triangle

$$\underline{\mathrm{Hom}}_A(L_{A/B},k) \longrightarrow \underline{\mathrm{Der}}(A,k) \longrightarrow \underline{\mathrm{Der}}(B,k) \longrightarrow \underline{\mathrm{Hom}}_A(L_{A/B},k)[1]\,.$$

Thus (i) implies that $\underline{\mathrm{Hom}}_A(L_{A/B},k) = \underline{\mathrm{Hom}}_k(L_{A/B}\otimes_A k,k)$ is acyclic, which implies (ii). Let us now assume that (ii) holds. Then we use the distinguished triangle

$$\underline{\mathrm{Hom}}_A(L_{A/B},C) \longrightarrow \underline{\mathrm{Der}}(A,C) \longrightarrow \underline{\mathrm{Der}}(B,C) \longrightarrow \underline{\mathrm{Hom}}_A(L_{A/B},C)[1]$$

and the fact that $\underline{\mathrm{Hom}}_A(L_{A/B},C) = \underline{\mathrm{Hom}}_k(L_{A/B}\otimes_A k,C)$, to conclude that (iii) holds. The fact that (iii) implies (iv) is trivial, because Λ_n is a finite resolving algebra, for all n.

Finally, assume that (iv) holds. Note that we have

$$\begin{aligned} h^r\,\underline{\mathrm{Der}}(A,\Lambda_n) &= h^r\big(\underline{\mathrm{Der}}(A,k)\otimes\Lambda_n\big)\\ &= h^r\,\underline{\mathrm{Der}}(A,k)\oplus h^{r+n}\,\underline{\mathrm{Der}}(A,k)\,. \end{aligned}$$

Thus we may conclude that $h^\ell\,\underline{\mathrm{Der}}(A,k)\to h^\ell\,\underline{\mathrm{Der}}(A,k)$ is an isomorphism for all $\ell\geq r$, hence for all $\ell\geq 0$. Then (i) follows, because for $\ell<0$ we have $h^\ell\,\underline{\mathrm{Der}}(A,k) = h^\ell\,\underline{\mathrm{Der}}(B,k) = 0$. □

Proposition 1.37 *Let $B\to A$ be a morphism of quasi-finite resolving algebras and $r\leq 0$ and integer. The following are equivalent:*

(i) $L_{A/B}$ is acyclic,

(ii) $\underline{\mathrm{Der}}(A,C)\to\underline{\mathrm{Der}}(B,C)$ is a quasi-isomorphism, for all morphisms of differential graded algebras $A\to C$, where C is a (finite) resolving algebra,

(iii) $h^r\,\underline{\mathrm{Der}}(A,C)\to h^r\,\underline{\mathrm{Der}}(B,C)$ is an isomorphism, for all $A\to C$ as in (ii).

If, moreover, $B\to A$ is a resolving morphism, then a further equivalent conditions is

(iii') $h^r\,\underline{\mathrm{Der}}_B(A,C)=0$, for all $A\to C$ as in (ii).

Proof. The claim that (i) implies (ii) follows as in the proof of Proposition 1.36 from a distinguished triangle. Then (ii) implies (iii) trivially. So let us assume that (iii) holds. To prove (i), we may assume without loss of generality that k is algebraically closed. We may conclude from Proposition 1.36 that $L_{A/B}\otimes_A k$ is acyclic, for all $A\to k$. This implies that $L_{A/B}$ is acyclic, because A is quasi-finite, and so $h^\ell(L_{A/B})$ is a finitely generated $h^0(A)$-module, for all ℓ, and hence Nakayama's lemma applies. □

2 Étale Morphisms

2.1 Augmentations

Let A be a resolving algebra. An *augmentation* of A is a morphism of differential graded k-algebras $A \to k$. Note that every augmentation $A \to k$ is induced (in a unique way) from a k-algebra morphism $h^0(A) \to k$. Moreover, every augmentation $A \to k$ induces a morphism of k-algebras $A^0 \to k$.

The *augmentation ideal* $\mathfrak{m}$ is the kernel of $A \to k$. It is a differential graded ideal in A. The augmentation ideal of A^0 is the degree zero component $\mathfrak{m}^0$ and the augmentation ideal of $h^0(A)$ is $\mathfrak{m}h^0(A) = \mathfrak{m}^0 h^0(A)$.

Note that any basis (x_i) of A defines an augmentation $A \to k$ by the rule $x_i \mapsto 0$. Conversely, for any augmentation of A, we can find a basis (x_i) for A, with x_i in the kernel of $A \to k$, for all i. Such a basis is called *compatible* with the augmentation $A \to k$. If (x_i) is a basis compatible with the augmentation $A \to k$, then we get an induced isomorphism of augmented graded algebras $k[x] \to A^\natural$.

Lemma 2.1 *Let A be a resolving algebra and $A \to k$ an augmentation with ideal $\mathfrak{m}$. Then $\mathfrak{m}/\mathfrak{m}^2 = \Omega_A \otimes_A k$.*

Proof. Consider the canonical map

$$\begin{aligned} \mathfrak{m} &\longrightarrow \Omega_A \otimes_A k \\ x &\longmapsto \mathfrak{d}x \otimes 1 \,, \end{aligned}$$

and prove that it induces an isomorphism of complexes of k-vector spaces. For this purpose it is useful to choose a basis (x_i) for A, compatible with the augmentation. Then $\Omega_A \otimes_A k$ is free as a graded k-vector space on the basis $(\mathfrak{d}x_i \otimes 1)$, so surjectivity is clear. For injectivity, use that we have

$$f(x) \equiv f(0) + \sum_i x_i \frac{\partial f}{\partial x_i}(0) \mod \mathfrak{m}^2 \,,$$

for every $a = f(x) \in A$. □

Given an augmented resolving algebra $A \to k$, all powers $\mathfrak{m}^n$ of the augmentation ideal are differential ideals. Hence all $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ are complexes of k-vector spaces. The direct sum

$$\operatorname{gr} A = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

of these complexes has an induced multiplication, making it a differential graded algebra. It is free:

Lemma 2.2 *Let A be a resolving algebra and $\mathfrak{m}$ an augmentation ideal. Then*

$$\operatorname{gr} A = S(\mathfrak{m}/\mathfrak{m}^2) \,,$$

as differential graded algebras.

Proof. The inclusion of the subcomplex $\mathfrak{m}/\mathfrak{m}^2 \to \operatorname{gr} A$ induces the canonical morphism of differential graded algebras $S(\mathfrak{m}/\mathfrak{m}^2) \to \operatorname{gr} A$. One checks that for every n the induced map $S^n(\mathfrak{m}/\mathfrak{m}^2) \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an isomorphism by considering a basis (x_i) for A. □

2.2 Point-wise étale morphisms

If $A \to k$ is a quasi-finite augmented resolving algebra, we let $\widehat{A}^0$ be the completion of A^0 at its augmentation ideal $\mathfrak{m}^0$ and $h^0(A)^\wedge$ the completion of $h^0(A)$ at its augmentation ideal $\mathfrak{m}h^0(A)$. Note that we have $h^0(A)^\wedge = h^0(A) \otimes_{A^0} \widehat{A}^0$.

We let $\widehat{A}^r$ be the completion of the A^0-module A^r at $\mathfrak{m}^0$ and $h^r(A)^\wedge$ the completion of the $h^0(A)$-module $h^r(A)$ at $\mathfrak{m}h^0(A)$. Again, we have $h^r(A)^\wedge = h^r(A) \otimes_{A^0} \widehat{A}^0$. The direct sum

$$\widehat{A}^* = \bigoplus_{r \leq 0} \widehat{A}^r = A \otimes_{A^0} \widehat{A}^0$$

is a differential graded algebra. We have

$$h^r(\widehat{A}^*) = h^r(A \otimes_{A^0} \widehat{A}^0) = h^r(A)^\wedge \,,$$

for every $r \leq 0$.

If $A \to B$ is a morphism of resolving algebras, then any augmentation $B \to k$ induces an augmentation $A \to k$. If A is endowed with the induced augmentation, then $A \to B$ is a *morphism* of augmented resolving algebras. A morphism of quasi-finite augmented resolving algebras $A \to B$ induces morphisms of k-algebras $\widehat{A}^0 \to \widehat{B}^0$ and $h^0(A)^\wedge \to h^0(B)^\wedge$. For every r, it induces a homomorphism of $\widehat{A}^0$-modules $\widehat{A}^r \to \widehat{B}^r$ and a homomorphism of $h^0(A)^\wedge$-modules $h^r(A)^\wedge \to h^r(B)^\wedge$. Finally, it induces a morphism of differential graded algebras $\widehat{A}^* \to \widehat{B}^*$.

Proposition 2.3 *Let $A \to B$ be a morphism of quasi-finite augmented resolving algebras. The following are equivalent:*

(i) $L_{B/A} \otimes_B k$ is acyclic,

(ii) $L_{B/A} \otimes_{B^0} \widehat{B}^0$ is acyclic,

(iii) $A/\mathfrak{m}_A^n \to B/\mathfrak{m}_B^n$ is a quasi-isomorphism for all n,

(iv) $\widehat{A}^* \to \widehat{B}^*$ is a quasi-isomorphism,

(v) For all $r \leq 0$, the homomorphism $h^r(A)^\wedge \to h^r(B)^\wedge$ of $h^0(A)^\wedge$-modules is bijective,

(vi) $h^0(A) \to h^0(B)$ is étale at $h^0(B) \to k$ and $h^r(A) \otimes_{h^0(A)} h^0(B) \to h^r(B)$ is an isomorphism in a Zariski neighborhood of $h^0(B) \to k$, for all $r \leq 0$.

For more equivalent statements, see Proposition 1.36.

Proof. The equivalence of (iv) and (v) follows immediately from the preceding remarks. Let us prove the equivalence of (v) and (vi): By properties of étaleness for usual finite type k-algebras, we know that $h^0(A) \to h^0(B)$ is étale at $h^0(B) \to k$ if and only if $h^0(A)^\wedge \to h^0(B)^\wedge$ is an isomorphism of k-algebras. Let us assume this to be the case. Then we have

$$h^r(A)^\wedge = h^r(A) \otimes_{h^0(A)} h^0(A)^\wedge = \big(h^r(A) \otimes_{h^0(A)} h^0(B)\big) \otimes_{h^0(B)} h^0(B)^\wedge$$

and

$$h^r(B)^\wedge = h^r(B) \otimes_{h^0(B)} h^0(B)^\wedge \,,$$

because A and B are quasi-finite. Now using the fact that for finitely generated $h^0(B)$-modules M and N, a homomorphism $M \to N$ is an isomorphism in a Zariski-open neighbourhood of $h^0(B) \to k$ if and only if $(M \to N) \otimes_{h^0(B)} h^0(B)^\wedge$ is an isomorphism of $h^0(B)^\wedge$-modules, we conclude the proof that (v) and (vi) are equivalent.

The fact that (ii) implies (i) is clear.

Let us now prove that (i) implies (iii). Assume that $L_{B/A} \otimes_B k$ is acyclic. Then $\Omega_A \otimes_A k \to \Omega_B \otimes_B k$ is a quasi-isomorphism of complexes of k-vector spaces. By Lemma 2.1, $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is a quasi-isomorphism. Since taking symmetric powers preserves quasi-isomorphisms, Lemma 2.2 implies that we have a quasi-isomorphism $\mathfrak{m}_A^n/\mathfrak{m}_A^{n+1} \to \mathfrak{m}_B^n/\mathfrak{m}_B^{n+1}$, for all $n \geq 0$. By induction, this implies that $A/\mathfrak{m}_A^n \to B/\mathfrak{m}_B^n$ is a quasi-isomorphism, for all n, proving (iii).

Assuming (iii), we will now prove (iv). Let us fix $r \leq 0$ and consider the limit

$$\varprojlim (A/\mathfrak{m}^n)^r \,.$$

(The superscript n denotes a power, the superscript r denotes the component of degree r.) Note that this limit is equal to $\widehat{A}^r$, because the topology on A^r defined by the descending sequence of subspaces $(\mathfrak{m}^n)^r$ is equal to the $\mathfrak{m}^0$-adic topology: for every n we have

$$(\mathfrak{m}^0)^n A^r \subset (\mathfrak{m}^n)^r \subset (\mathfrak{m}^0)^{n-r} A^r \,.$$

Thus Lemma 2.4 implies that we have an isomorphism $h^r(\widehat{A}^*) \to h^r(\widehat{B}^*)$.

Finally, let us prove that (iv) implies (ii).

$$\begin{aligned} \widehat{A}^* \to \widehat{B}^* \quad \text{qis} \quad &\Longrightarrow L_{\widehat{A}^*} \otimes_{\widehat{A}^*} \widehat{B}^* \to L_{\widehat{B}^*} \quad \text{qis} \\ &\Longrightarrow \Omega_A \otimes_A \widehat{B}^* \to \Omega_B \otimes_B \widehat{B}^* \quad \text{qis}\,. \end{aligned}$$

For the necessary facts about cotangent complexes we refer to [6]. □

Lemma 2.4 *Let M be a complex of k-vector spaces and $N_n \subset M$ a descending sequence of subcomplexes. Then for every r there is a natural exact sequence*

$$\prod_n h^{r-1}(M/N_n) \xrightarrow{\partial} \prod_n h^{r-1}(M/N_n) \xrightarrow{\sigma}$$
$$\xrightarrow{\sigma} h^r(\varprojlim M/N_n) \xrightarrow{\iota} \prod_n h^r(M/N_n) \xrightarrow{\partial} \prod_n h^r(M/N_n)\,.$$

Here $\varprojlim M/N_n$ denotes the componentwise projective limit of complexes.

Proof. This is straightforward to check from the definitions: The map ∂ maps the sequence $a = (a_n)$ to ∂a, with $(\partial a)_n = a_n - a_{n+1}$. The map σ sends the sequence a to $\sigma(a)$ with $\sigma(a)_n = \sum_{i=1}^{n-1} da_i$. The map ι sends the sequence a to a. One can also start with the short exact sequence of complexes whose degree r term is

$$0 \longrightarrow \varprojlim (M/N_n)^r \xrightarrow{\iota} \prod_n (M/N_n)^r \xrightarrow{\partial} \prod_n (M/N_n)^r \longrightarrow 0\,,$$

and then pass to the long exact cohomology sequence. $\square$

Definition 2.5 Let $A \to B$ be a morphism of quasi-finite resolving algebras. Let $B \to k$ be an augmentation. If the equivalent conditions of Corollary 2.3 are satisfied, then we call $A \to B$ **étale** at the augmentation $B \to k$.

If k is not algebraically closed, there might not exist very many augmentations. Thus the following generalization: Let K be a finite extension field of k. A morphism of differential graded algebras $A \to K$ is called a *K-valued* augmentation of A.

Corollary 2.6 *Let $A \to B$ be a morphism of quasi-finite resolving algebras and $B \to K$ a K-valued augmentation of B. Then the following are equivalent:*

(i) $L_{B/A} \otimes_B K$ is acyclic,

(ii) $h^0(A) \to h^0(B)$ is étale at $h^0(B) \to K$ and $h^r(A) \otimes_{h^0(A)} h^0(B) \to h^r(B)$ is an isomorphism in a Zariski neighborhood of $h^0(B) \to K$, for all $r \geq 0$.

(iii) $A \otimes_k K \to B \otimes_k K$ is étale at the augmentation $B \otimes_k K \to K$.

2.3 Étale morphisms

Corollary 2.7 *Let $A \to B$ be a morphism of quasi-finite resolving algebras. The following are equivalent:*

(i) $L_{B/A}$ is acyclic,

(ii) $h^0(A) \to h^0(B)$ is étale and $h^*(A) \otimes_{h^0(A)} h^0(B) \to h^*(B)$ is an isomorphism,

(iii) $A \to B$ is étale at every K-valued augmentation $B \to K$, for all finite extension K/k.

For more equivalent statements, see Proposition 1.37.

Proof. Proposition 2.3 implies directly that Conditions (ii) and (iii) are equivalent. To deal with Condition (i), we may assume that k is algebraically closed. We notice that $h^r(L_{B/A})$ is a finitely generated $h^0(B)$-module, for every r. Thus, $L_{B/A}$ is acyclic, if and only if $L_{B/A} \otimes_{B^0} \widehat{B}^0$ is acyclic for all augmentations $h^0(B) \to k$. $\square$

Definition 2.8 If the equivalent conditions of Corollary 2.7 are satisfied, we call the morphism of quasi-finite resolving algebras $A \to B$ **étale**.

Corollary 2.9 *Let $A \to B$ be a morphism of quasi-finite resolving algebras. Then $A \to B$ is a quasi-isomorphism if and only if $h^0(A) \to h^0(B)$ is an isomorphism and $L_{B/A}$ is acyclic.* □

Proposition 2.10 *Let $A \to B \to C$ be morphisms of quasi-finite resolving algebras, where $A \to B$ is étale. Then $B \to C$ is étale if and only if $A \to B$ is étale.* □

Proposition 2.11 *Let $A \to A'$ be an étale morphism of quasi-finite resolving algebras. Let M be a differential graded A-module, bounded from above and such that $M^\natural$ is flat over $A^\natural$. Then we have*

$$h^p(M \otimes_A A') = h^p(M) \otimes_{h^0(A)} h^0(A'),$$

for all p.

Proof. Since usual étale morphisms of finite type k-algebras are flat, $h^0(A) \to h^0(A')$ is flat. Since flatness is preserved under base change, this implies that $h^*(A) \to h^*(A')$ is flat. Hence

$$\operatorname{Tor}_i^{h^*(A)}\big(h^*(M), h^*(A')\big) = 0,$$

for all $i > 0$. By the Eilenberg-Moore spectral sequence (see [4]) this implies that

$$h^*(M \otimes_A A') = h^*(M) \otimes_{h^*(A)} h^*(A') = h^*(M) \otimes_{h^0(A)} h^0(A'),$$

which is what we wanted to prove. □

The following special case of this proposition will be essential for descent theory. Once we have defined *perfect resolving morphism* (see Section 3), it is clear that this corollary generalizes to the case that $C \to B$ is a perfect resolving morphism.

Corollary 2.12 *Let $C \to B \to A \to A'$ be morphisms of quasi-finite resolving algebras over k. Assume that $C \to B$ is a finite resolving morphism and $A \to A'$ is étale. Then we have*

$$h^p\big(\underline{\operatorname{Der}}_C(B, A')\big) = h^p\big(\underline{\operatorname{Der}}_C(B, A)\big) \otimes_{h^0(A)} h^0(A'),$$

for all p. □

Definition 2.13 We call $A \to B$ an **open immersion**, if
(i) $A \to B$ is étale,
(ii) $\operatorname{Spec} h^0(B) \to \operatorname{Spec} h^0(A)$ is an open immersion of affine k-schemes.

2.4 Completions

Here we would like to characterize étaleness in terms of completions. This section is not used in the rest of the paper. We include it here, even though it contains a forward reference to the notion of perfect resolving algebra.

Let A be a quasi-finite resolving algebra and $A \to k$ be an augmentation. Let $\mathfrak{m}$ be the augmentation ideal. the various quotients $A/\mathfrak{m}^n$, form an inverse system of differential graded k-algebras. We define

$$\widehat{A} = \varprojlim A/\mathfrak{m}^n,$$

as a projective limit of k-algebras and call it the *completion* of A at $\mathfrak{m}$ or at the augmentation $A \to k$. The completion $\widehat{A}$ inherits the structure of differential k-algebra, but it will lose its grading.

The differential algebra $\widehat{A}$ comes with a natural filtration. The components of the filtration are the differential ideals $\widehat{\mathfrak{m}}_n = \ker(\widehat{A} \to A/\mathfrak{m}^n)$ and the associated graded pieces are the $\widehat{A}$-modules $\widehat{\mathfrak{m}}_n/\widehat{\mathfrak{m}}_{n+1}$. Note that

$$\widehat{\mathfrak{m}}_n/\widehat{\mathfrak{m}}_{n+1} = \mathfrak{m}^n/\mathfrak{m}^{n+1}\,.$$

Choose a coordinate system (x_i) for A, compatible with the augmentation. Let $\langle x\rangle$ be the graded vector space generated by the x_i and let $k[[x]] = \widehat{S}\langle x\rangle = \prod_{n\geq 0} S^n\langle x\rangle$ be the completion of the symmetric algebra at the augmentation ideal $(x) \subset k[x]$. We may view the elements of $k[[x]] = \widehat{A}^\natural$ as formal power series in the variables x_i.

Note that

$$\widehat{A}^r = \{\hat{a} \in \widehat{A} \mid \text{for all } n, \text{ the projection of } \hat{a} \text{ into } A/\mathfrak{m}^n \text{ has degree } r\}\,.$$

Thus we have an inclusion of differential algebras

$$\widehat{A}^* \subset \widehat{A}\,,$$

which is not an isomorphism unless $A = k$.

From the various projections $\widehat{A} \to \widehat{A}^r$, we get a canonical inclusion

$$\widehat{A} \hookrightarrow \prod_{r\leq 0} \widehat{A}^r\,. \tag{5}$$

This is to be understood as a morphism of differential k-algebras. Multiplication on the right hand side is given by considering elements as formal infinite sums $\hat{a} = \sum_{r\leq 0} \hat{a}^r$ and multiplying using the distributive law. The differential on the right had side is given by $d\hat{a} = \sum d(\hat{a}^r)$. Note that $d(\hat{a}^r) = (d\hat{a})^{r+1}$, for all $\hat{a} \in \widehat{A}$.

The image of (5) consists of all formal series $\hat{a} = \sum_{r\leq 0} \hat{a}^r$ which converge in the $\mathfrak{m}$-adic topology, i.e., such that

$$\forall n \quad \exists R \quad \forall r > R\colon \quad \hat{a}^r \in \widehat{\mathfrak{m}}_n\,.$$

This condition is automatically satisfied if A a finite resolving algebra. So in this case, (5) is an isomorphism of differential algebras.

If we take cohomology of (5), we obtain an algebra morphism

$$h(\widehat{A}) \longrightarrow \prod_{r \leq 0} h^r(\widehat{A}^*)\,, \tag{6}$$

which is an isomorphism in the finite case.

Theorem 2.14 *Let $A \to B$ be a morphism of quasi-finite resolving algebras and $B \to k$ an augmentation. If $A \to B$ is étale at $B \to k$, then we have a quasi-isomorphism of completions $\widehat{A} \to \widehat{B}$. The converse is true if A and B are perfect.*

Proof. Assume that $A \to B$ is étale at $B \to k$. Then for all n we have that $A/\mathfrak{m}^n \to B/\mathfrak{m}^n$ is a quasi-isomorphism. We now employ the non-graded version of Lemma 2.4: Let M be a differential k-vector space and $N_n \subset M$ a descending sequence of subspaces respecting the differential. Then we have an exact triangle of k-vector spaces

$$\begin{array}{ccc} \prod_n h(M/N_n) & \xrightarrow{\ \partial\ } & \prod_n h(M/N_n) \\ & {}_{\iota}\nwarrow \quad \swarrow_{\sigma} & \\ & h(\varprojlim M/N_n) & \end{array}$$

where the maps are defined by the same formulas as those of Lemma 2.4. We may obtain this exact triangle from the short exact sequence of differential vector spaces

$$0 \longrightarrow \varprojlim M/N_n \longrightarrow \prod M/N_n \xrightarrow{\partial} \prod M/N_n \longrightarrow 0\,.$$

Applying this to our situation, we obtain the desired result that $h(\widehat{A}) \to h(\widehat{B})$ is an isomorphism.

Conversely, assume that we have a quasi-isomorphism $\widehat{A} \to \widehat{B}$. To prove that $A \to B$ is étale at $B \to k$, we may localize A and B and thus assume that they are finite. Then we use (6) to conclude that we have an isomorphism

$$\prod_{r \leq 0} h^r(\widehat{A}^*) \longrightarrow \prod_{r \leq 0} h^r(\widehat{B}^*)\,,$$

which implies that $h^r(\widehat{A}^*) \to h^r(\widehat{B}^*)$ is bijective for all r, hence that $\widehat{A}^* \to \widehat{B}^*$ is a quasi-isomorphism. □

Remark Note that unlike the non-differential graded case, we do not have that $\widehat{A} = \widehat{S}(\mathfrak{m}/\mathfrak{m}^2)$. By assumption on A, we can always find a subcomplex

$V \subset \mathfrak{m}$, such that $V \to \mathfrak{m}/\mathfrak{m}^2$ is an isomorphism and $A^\natural = S(V)^\natural$. Hence we also have that the filtered algebras $\widehat{A}^\natural$ and $\widehat{S}(\mathfrak{m}/\mathfrak{m}^2)^\natural$ are isomorphic. But the differentials are different, as soon as the differential d on A has any higher order (≥ 2) terms.

We may say that the tangent cone and the tangent space are isomorphic, but the completion of the differential graded algebra itself is not isomorphic to the tangent space. Thus quasi-free differential graded algebras have some, but not other properties of smooth non-differential graded algebras.

2.5 Local structure of étale morphisms

Given a differential graded algebra A, assume that the differential graded algebra B is quasi-free over A on the basis $x_1, \ldots, x_r$ in degree 0 and $\xi_1, \ldots, \xi_s$ in degree -1. Denote $d\xi_i$ by $f_i \in A^0[x_1, \ldots, x_r]$. In this case we write

$$B = A[x_1, \ldots, x_r]\{\xi_1, \ldots, \xi_s\}/d\xi_i = f_i.$$

Lemma 2.15 *Let A be a quasi-finite resolving algebra. Let $f_1, \ldots, f_r \in A^0[x_1, \ldots, x_r]$ be polynomials such that $\det(\frac{\partial f_i}{\partial x_j})$ is a unit in $h^0(A)[x_1, \ldots, x_r]/(f_1, \ldots, f_r)$. Then*

$$A \longrightarrow A[x_1, \ldots, x_r]\{\xi_1, \ldots, \xi_r\}/d\xi_i = f_i$$

is étale.

PROOF. Let $B = A[x]\{\xi\}/dx = f$. By usual facts about étale morphisms between k-algebras, we know that $h^0(A) \to h^0(B)$ is étale. The assumption on the Jacobian immediately implies that $L_{B/A} \otimes_B K$ is acyclic, for all K-valued augmentations $B \to K$. □

Definition 2.16 We call an étale morphism

$$A \longrightarrow A[x_1, \ldots, x_r]\{\xi_1, \ldots, \xi_r\}/d\xi_i = f_i$$

as in Lemma 2.15 a **standard étale** morphism.

Note A composition of standard étale morphisms is standard étale.

Let $g \in A^0$ and consider $f(x) = xg - 1 \in A^0[x]$. Then $\frac{\partial f}{\partial x} = g$ is a unit in $A^0[x]/xg - 1 = A^0_g$. Thus $A \to A[x]\{\xi\}/d\xi = xg - 1$ is étale. In fact, it is an open immersion, as it induces $h^0(A) \to h^0(A)_g$ on the h^0-level. We will abbreviate $A[x]\{\xi\}/d\xi = xg - 1$ by $A_{\{g\}}$.

Definition 2.17 An open immersion $A \to A_{\{g\}}$ is called an **elementary open immersion**.

Proposition 2.18 *Let $A \to B$ be a morphism of quasi-finite resolving algebras which is étale in a Zariski neighborhood of the augmentation $h^0(B) \to k$. Then there exists a commutative diagram of differential graded algebras*

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle \text{standard étale}}\downarrow & & \downarrow{\scriptstyle \text{elem. open}} \\ B'' & \xrightarrow{\text{qis}} & B' \end{array}$$

and an augmentation $B' \to k$ compatible with the given augmentation $B \to k$.

Proof. We use the local structure theory of usual étale morphisms: there exists $g \in h^0(B)$, further polynomials $f_1, \ldots, f_p \in h^0(A)[x_1, \ldots, x_p]$ and an $h^0(A)$-isomorphism $h^0(A)[x]/(f) \to h^0(B)_g$, such that $\det(\frac{\partial f}{\partial x})$ is a unit in $h^0(A)[x]/(f)$. Moreover, we may assume that g does not map to zero under the augmentation $h^0(B) \to k$ and that $A \to B$ is étale over $h^0(B)_g$

We choose a lifting $g \in B^0$ and consider the localization $B \to B_{\{g\}} = B[y]\{\eta\}/(d\eta = 1 - yg)$. We define an augmentation $B_{\{g\}} \to k$ by sending y to the inverse of the image of g in k and η to 0 and making sure that it restricts to the given augmentation on B. Note that $A \to B_{\{h\}}$ is étale.

Now lift $f_1, \ldots, f_p$ to elements of $A^0[x_1, \ldots, x_p]$, denoted by the same letters. Also lift the images of $x_1, \ldots, x_p$ in $h^0(B_{\{g\}})$ to elements $b_1, \ldots, b_p \in B^0_{\{g\}}$. Because $f_i(b)$ represents 0 in $h^0(B_{\{g\}})$ we may also choose $\beta_i \in B^{-1}_{\{g\}}$ such that $d\beta_i = f_i(b)$, for all $i = 1, \ldots, p$.

Having made these choices, we can define a morphism of differential graded algebras

$$\begin{aligned} A[x]\{\xi\}/(d\xi = f) &\longrightarrow B_{\{g\}} \\ x_i &\longmapsto b_i \\ \xi_i &\longmapsto \beta_i\,. \end{aligned}$$

Note that this is a quasi-isomorphism by Corollary 2.9. Thus the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A[x]\{\xi\}/(d\xi = f) & \xrightarrow{\text{qis}} & B_{\{g\}} \end{array}$$

finishes the proof. $\square$

Corollary 2.19 *Let $A \to B$ be an étale morphism of quasi-finite resolving algebras. Then there exists an integer n, and for every $i = 1, \ldots, n$ a commutative diagram of differential graded algebras*

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle \text{standard étale}}\downarrow & & \downarrow{\scriptstyle \text{elem. open}} \\ B'_i & \xrightarrow[\text{qis}]{} & B_i \end{array}$$

such that

(i) all $B \to B_i$ *are elementary open immersions and* $\coprod_i \operatorname{Spec} h^0(B_i) \to \operatorname{Spec} h^0(B)$ *is surjective,*

(ii) all $A \to B_i'$ *are standard étale,*

(iii) all $B_i' \to B_i$ *are quasi-isomorphisms.*

PROOF. Immediate from Proposition 2.18. □

3 Perfect resolving algebras

Perfect resolving algebras are between finite and quasi-finite resolving algebras. Their importance lies in the fact that they provide us with a good notion of affine differential graded schemes.

Definition 3.1 A **perfect** resolving algebra is a quasi-finite resolving algebra A, such that $\Omega_A \otimes_A h^0(A)$ is a perfect complex of $h^0(A)$ modules.

If $\Omega_A \otimes_A h^0(A)$ has perfect amplitude contained in $[-N, 0]$, then we say that A has **perfect amplitude** N.

The full sub-2-category of $\mathfrak{R}$, consisting of perfect resolving algebras will be denoted by $\mathfrak{R}_{\text{pf}}$.

Recall that a complex M of $h^0(A)$-modules is *perfect* if Zariski locally in $h^0(A)$, there exists a finite complex of finite rank free modules E, and a quasi-isomorphism $E \to M$. If all locally defined E can be chosen such that $E^i = 0$ for $i \notin [a, b]$, then M is of *perfect amplitude* contained in $[a, b]$. For details on perfect complexes see Exposés I and II of [2].

For example, any finite resolving algebra A is perfect, because if $x_1, \ldots, x_n$ is a basis for A, then $\mathfrak{d}x_1 \otimes 1, \ldots, \mathfrak{d}x_n \otimes 1$ is a basis for $\Omega_A \otimes_A h^0(A)$. If A has amplitude N as a finite resolving algebra, it has perfect amplitude N. The converse is true 'locally', if $N \geq 2$; see Theorem 3.8.

Definition 3.2 A **perfect** resolving morphism of differential graded algebras $C \to B$ is a quasi-finite resolving morphism such that $\Omega_{B/C} \otimes_B h^0(B)$ is a perfect complex of $h^0(B)$-modules.

If $\Omega_{B/C} \otimes_B h^0(B)$ has perfect amplitude contained in $[-N, 0]$, then $C \to B$ has **perfect amplitude** N.

Definition 3.3 A morphism $C \to B$ of quasi-finite resolving algebras is called **perfect**, if $L_{B/C} \otimes_B h^0(B)$ is a perfect complex of $h^0(B)$-modules.

If $L_{B/C} \otimes_B h^0(B)$ has perfect amplitude contained in $[-N, 0]$, we call $C \to B$ of **perfect amplitude** N.

For example, any étale morphism of quasi-finite resolving algebras if perfect, of perfect amplitude 0. Any morphism between finite resolving algebras is perfect. If A has amplitude N and B has amplitude M, then $A \to B$ has perfect amplitude $\max(N + 1, M)$. More precisely, we have:

Remark 3.4 Let $C \to B \to A$ be morphisms of differential graded algebras. Assume either that both $C \to B$ and $B \to A$ are quasi-finite resolving morphisms, or that all three of C, B, A are quasi-finite resolving algebras.

$$\begin{array}{ccc} C & \xrightarrow{f} & B \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle g} \\ & & A \end{array}$$

If any two of f, g, h are perfect, then so is the third. If we denote the amplitudes of f, g and h, by M, N and P, respectively, then we have

$$\begin{aligned} P &= \max(M, N) \\ N &= \max(M+1, P) \\ M &= \max(N-1, P) \end{aligned}$$

For a reference, see Complément 4.11 in Exposé I of [2].

The first basic result about perfect morphisms is that we have a convergent spectral sequence for the tangent complex.

Proposition 3.5 *Let $C \to B$ be a perfect resolving morphism of differential graded algebras and $B \to A$ a morphism to a differential graded algebra A. Assume that A is concentrated in non-positive degrees (it is sufficient that the cohomology of A be concentrated in non-positive degrees). Then, for all p sufficiently large, we have that $h^p\big(\Theta_{B/C} \otimes h^0(A)\big) = 0$. Thus there is a convergent third third quadrant spectral sequence*

$$E_2^{p,q} = h^q(A) \otimes_{h^0(A)} h^p\big(\Theta_{B/C} \otimes_B h^0(A)\big) \Longrightarrow h^{p+q}(\Theta_{B/C} \otimes_B A)\,.$$

Rewriting in terms of derivations yields

$$E_2^{p,q} = h^q(A) \otimes_{h^0(A)} h^p \, \underline{\mathrm{Der}}_C\big(B, h^0(A)\big) \Longrightarrow h^{p+q}\, \underline{\mathrm{Der}}_C(B, A)\,. \qquad (7)$$

More precisely, if $C \to B$ has perfect amplitude N, then $\Theta_{B/C} \otimes_B h^0(A)$ has perfect amplitude contained in $[0, N]$, and hence $h^p\big(\Theta_{B/C} \otimes_B h^0(A)\big) = 0$, for all $p > N$.

Let $C \to B$ be a perfect morphism of quasi-finite resolving algebras. Then we have a convergent third quadrant spectral sequence

$$E_2^{p,q} = h^q(A) \otimes_{h^0(A)} h^p\big(T_{B/C} \otimes_B h^0(A)\big) \Longrightarrow h^{p+q}(T_{B/C} \otimes_B A)\,.$$

If $C \to B$ has perfect amplitude N, then $T_{B/C} \otimes_B h^0(A)$ has perfect amplitude contained in $[0, N]$ and $h^p\big(T_{B/C} \otimes_B h^0(A)\big) = 0$, for all $p > N$.

Proof. We note that

$$\Theta_{B/C} \otimes_B h^0(A) = \underline{\mathrm{Hom}}_{h^0(A)}\big(\Omega_{B/C} \otimes_B h^0(A), h^0(A)\big)\,.$$

Thus the perfection of $\Omega_{B/C} \otimes_B h^0(A)$ implies that of $\Theta_{B/C} \otimes_B h^0(A)$. Then the proposition follows. □

Corollary 3.6 *If $C \to B$ is a perfect resolving morphism and A is concentrated in non-positive degrees, then $h^\ell \, \underline{\mathrm{Der}}_C(B, A)$ is a finite rank $h^0(A)$-module.*

Application to the behaviour of $\underline{\mathrm{Der}}$ with respect to homotopies

Let $C \to B$ be a perfect resolving morphism of differential graded algebras. Let $f : B \to A$ and $g : B \to A$ be morphisms, where the cohomology of A is concentrated in non-positive degrees. Let $\theta : f \Rightarrow g$ be a homotopy, i.e., a morphism $\theta : B \to A \otimes \Omega_1$, such that $\partial_0 \theta = f$ and $\partial_1 \theta = g$.

$$B \underset{g}{\overset{f}{\rightrightarrows}} A \qquad (\Downarrow \theta)$$

In this situation, there are two B-module structures on A, one given by f, the other by g. Let us denote them by ${}_fA$ and ${}_gA$. The algebra $A \otimes \Omega_1$ has three B-module structures. Let us denote the one induced by θ by ${}_\theta A \otimes \Omega_1$.

We get induced homomorphisms of B-modules

$$\underline{\mathrm{Der}}(B, {}_fA) \xleftarrow{\partial_{0*}} \underline{\mathrm{Der}}(B, {}_\theta A \otimes \Omega_1) \xrightarrow{\partial_{1*}} \underline{\mathrm{Der}}(B, {}_gA)$$

By Proposition 3.5, both of these are quasi-isomorphisms. Thus we can make the following definition:

Definition 3.7 The isomorphism of $h^0(B)$-modules

$$h_\ell \, \underline{\mathrm{Der}}(B, {}_fA) \xrightarrow{\theta_*} h_\ell \, \underline{\mathrm{Der}}(B, {}_gA)$$

obtained as the composition of $(\partial_{0*})^{-1}$ with ∂_{1*}, is called the **canonical** isomorphism induced by θ, and is denoted by θ_*.

Note that the canonical isomorphism induced by θ depends only on the homotopy class of θ and is functorial: $\theta_* \eta_* = (\theta\eta)_*$.

Remark In fact, θ_* is entirely independent of θ.

is this true? I don't think we need it. It should be clear for $\ell < -1$, by comparison with homotopy.

3.1 Local finiteness

We will prove two fundamental results on perfect resolving algebras. The first one says that every perfect resolving algebra is locally finite:

Theorem 3.8 *Let $C \to B$ be a perfect resolving morphism of quasi-finite resolving algebras. For every K-valued augmentation $B \to K$, there exists a $g \in B^0$, such that $g(K) \neq 0$ and a resolution $C \to A \to B_{\{g\}}$, of $C \to B_{\{g\}}$, where $C \to A$ is a finite resolving morphism.*

$$\begin{array}{ccc} C & \longrightarrow & B \\ {\scriptstyle \text{finite}}\downarrow & & \downarrow \\ A & \xrightarrow{\text{qis}} & B_{\{g\}} \end{array}$$

If $C \to B$ has perfect amplitude N, then we can choose A such that $C \to A$ has amplitude $\max(2, N)$.

PROOF. Since $C \to B$ is a resolving morphism, we have a quasi-isomorphism $L_{B/C} \to \Omega_{B/C}$, and so we know that $\Omega_{B/C} \otimes_B h^0(B)$ is perfect. Suppose that $n \geq 2$ is an integer such that $\Omega_{B/C} \otimes_B h^0(B)$ has perfect amplitude contained in $[-n, 0]$. Then $\tau_{\geq -n}\big(\Omega_{B/C} \otimes_B h^0(B)\big)$ has perfect amplitude contained in $[-n, 0]$, too, because $\Omega_{B/C} \otimes_B h^0(B) \to \tau_{\geq -n}\big(\Omega_{B/C} \otimes_B h^0(B)\big)$ is a quasi-isomorphism. Now, since $\tau_{\geq -n}\big(\Omega_{B/C} \otimes_B h^0(B)\big)^i$ is a free $h^0(B)$-module, for all $i > -n$ (a basis if provided by $(\mathfrak{d}x_j \otimes 1)$, where (x_j) is the degree i part of a basis for B over C), it follows that

$$\tau_{\geq -n}\big(\Omega_{B/C} \otimes_B h^0(B)\big)^{-n} = \\ = \operatorname{cok}\Big(\big(\Omega_{B/C} \otimes_B h^0(B)\big)^{-n-1} \longrightarrow \big(\Omega_{B/C} \otimes_B h^0(B)\big)^{-n}\Big)$$

is a locally free $h^0(B)$-module (for a reference, see Lemme 4.16, Exposé I in [2]). Let $h^0(B)_g$ be a Zariski neighborhood of the K-valued point $h^0(B) \to K$ of $\operatorname{Spec} h^0(B)$ over which the above cokernel is free. Lifting g to B^0, we get the elementary open immersion $B \to B_{\{g\}}$ through which $B \to K$ factors. Then $\Omega_{B_{\{g\}}/C} \otimes_{B_{\{g\}}} h^0(B_{\{g\}})$ has perfect amplitude contained in $[-n, 0]$, because $n \geq 1$, and

$$\tau_{\geq -n}\big(\Omega_{B_{\{g\}}/C} \otimes_{B_{\{g\}}} h^0(B_{\{g\}})\big)^{-n} = \tau_{\geq -n}\big(\Omega_{B/C} \otimes_B h^0(B)\big)^{-n} \otimes_{h^0(B)} h^0(B)_g$$

is a free $h^0(B_{\{g\}}) = h^0(B)_g$-module. Thus, to simplify notation, we may replace B by $B_{\{g\}}$, and assume that $\tau_{\geq -n}\big(\Omega_{B/C} \otimes_B h^0(B)\big)^{-n}$ is free.

Choose elements $b_1, \ldots, b_r \in B^{-n}$, such that the images of $\mathfrak{d}b_j \otimes 1 \in \big(\Omega_{B/C} \otimes_B h^0(B)\big)^{-n}$ in $\tau_{\geq -n}\big(\Omega_{B/C} \otimes_B h^0(B)\big)^{-n}$ give a basis. Then define $A = B_{(n-1)}[\xi_1, \ldots, \xi_r]$, where $\xi_1, \ldots, \xi_r$ are formal variables of degree $-n$ and $B_{(n-1)} \subset B$ is the differential graded C-subalgebra generated over C by the part of degree $> -n$ of a basis for B over C. Set $d\xi_j = db_j$, which is allowed, because $db_j \in Z^{1-n}B_{(n-1)}$. Extend the inclusion $B_{(n-1)} \subset B$ to a morphism of differential graded algebras $A \to B$ by $\xi_j \mapsto b_j$. Note that $C \to B_{(n-1)}$, and hence $C \to A$ are finite resolving morphisms. The amplitude of A is n.

Now, all that is left, is to check that $A \to B$ is a quasi-isomorphism. We use Corollary 2.9: $h^0(A) \to h^0(B)$ is an isomorphism since $n \geq 2$. Moreover, we have a commutative diagram of complexes of $h^0(B)$-modules

$$\begin{array}{ccc} \Omega_{A/C} \otimes_A h^0(B) & \longrightarrow & \Omega_{B/C} \otimes_B h^0(B) \\ & \searrow^{\text{isom.}} & \Big\downarrow \text{qis} \\ & & \tau_{\geq -n}\big(\Omega_{B/C} \otimes_B h^0(B)\big), \end{array}$$

where the diagonal map is an isomorphism by construction. This proves that $L_{B/A} \otimes_B h^0(B)$, and hence $L_{B/A}$, is acyclic. $\square$

Corollary 3.9 *Let $C \to B$ be a perfect morphism of quasi-finite resolving algebras. Then for every K-valued augmentation $B \to K$ there exists an open immersion $B \to B'$, such that $B \to K$ factors through $B \to B'$, and a commutative diagram*

$$\begin{array}{ccc} C & \longrightarrow & B \\ {\scriptstyle\text{finite}}\big\downarrow & & \big\downarrow{\scriptstyle\text{open imm.}} \\ A & \xrightarrow{\text{qis}} & B' \end{array}$$

where $C \to A$ is a finite resolving morphism and $A \to B'$ a quasi-isomorphism. If $C \to B$ has perfect amplitude N, then $C \to A$ can be chosen to have amplitude $\max(2, N)$.

Proof. Start by choosing (using Proposition 1.9) a quasi-finite resolution $C \to B'' \to B$ of $C \to B$, and choose a homotopy inverse for $B'' \to B$, to obtain the homotopy commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & B \\ & \searrow \quad \swarrow & \big\downarrow{\scriptstyle\text{qis}} \\ & & B'' \end{array}$$

Apply Theorem 3.8 to $C \to B''$, to obtain the homotopy commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & B \\ {\scriptstyle\text{finite}}\big\downarrow & \swarrow & \big\downarrow{\scriptstyle\text{open imm.}} \\ A & \xrightarrow[\text{qis}]{} & B' \end{array}$$

Finally, use the fact that $C \to A$ is resolving, to eliminate the homotopy. □

By weakening the conclusion somewhat, we can improve on the estimate for the amplitude of the resolution A:

Scholum 3.10 *Let $C \to B$ be a perfect morphism of quasi-finite resolving algebras. Then for every K-valued augmentation $B \to K$ there exists an open immersion $B \to B'$, such that $B \to K$ factors through $B \to B'$, and a commutative diagram*

$$\begin{array}{ccc} C & \longrightarrow & B \\ {\scriptstyle\text{finite}}\big\downarrow & & \big\downarrow{\scriptstyle\text{open imm.}} \\ A & \xrightarrow{\text{étale}} & B' \end{array}$$

where $C \to A$ is a finite resolving morphism and $A \to B'$ étale. If $C \to B$ has perfect amplitude N, then $C \to A$ can be chosen to have amplitude $\max(1, N)$.

Proof. By examining the proof of Theorem 3.8, we see that the only place we used that $n \geq 2$ was to conclude that $h^0(A) \to h^0(B_{\{g\}})$ was an isomorphism, in the proof that $A \to B_{\{g\}}$ was a quasi-isomorphism. If we drop the assumption that $n \geq 2$, but assume that $n \geq 1$, we can still conclude that $L_{B_{\{g\}}/A}$ is acyclic, which proves that $A \to B_{\{g\}}$ is étale. □

To show that the class of perfect resolving algebras is larger than the class of finite resolving algebras, we provide an example of a perfect resolving algebra which has no resolution by a finite resolving algebra:

Example 3.11 let X be a smooth affine scheme, E a vector bundle over X and $s : X \to E$ a section. To this data we associate the Koszul complex

$$\Lambda_A M : \quad \ldots \longrightarrow \Lambda_A^2 M \longrightarrow M \longrightarrow A\,. \tag{8}$$

Here A is the affine coordinate ring of X and M is the projective A-module corresponding to E, i.e., the module of sections of the dual of E. The homomorphism $M \to A$ comes from the section s. We consider (8) as a differential graded algebra B by placing $\Lambda_A^r M$ in degree $-r$, so that $B^{-r} = \Lambda_A^r M$. We note that B has a universal derivation $\mathfrak{d} : B \to \Omega_B$, and we have

$$\Omega_B \otimes_B h^0(B) = \big[\ldots \longrightarrow \Lambda_A^2 M \longrightarrow M \longrightarrow \Omega_A\big] \otimes_A A/I\,,$$

where I is the ideal of A defined by the image of $M \to A$, in other words the ideal defining the zero locus of s. The homomorphism $M \to \Omega_A$ is defined as the composition of $M \to A$ and the universal derivation $A \to \Omega_A$. We note that $\Omega_B \otimes_B h^0(B)$ is perfect. Moreover, it admits a determinant, defined as

$$\begin{aligned} \det\big(\Omega_B \otimes_B h^0(B)\big) &= \big(\det \Omega_A \otimes \det M^{-1} \otimes \ldots\big) \otimes_A A/I \\ &= \big(\omega_A \otimes \det(\Lambda_A M)\big) \otimes_A A/I\,. \end{aligned} \tag{9}$$

By Scholum 1.10, we may find a quasi-finite resolving algebra B', together with a quasi-isomorphism $B' \to B$. Then we get a quasi-isomorphism $\Omega_{B'} \otimes_{B'} A/I \to \Omega_B \otimes_B A/I$, and so B' is a perfect resolving algebra. If B was quasi-isomorphic to a finite resolving algebra, then (9) would have to be a free A/I-module of rank one. Of course, there are many examples where (9) is non-trivial.

Example 3.12 For a concrete example of such a perfect resolving algebra, consider the affine cubic curve $y^2 = 4(x^3 - x)$. The projective completion has one additional point (a flex), called P_0. We consider the projective completion as an elliptic curve X with base point P_0. Let $X' = X - \{P_0\}$.

Let $P = (0, 0)$ be the origin, which is a 2-division point in X and consider the line bundle $L = \mathcal{O}(-P)$ on X'. This is a non-trivial line bundle on the affine curve X'. Because P is a 2-division point, there exists a regular function s on X', with a double zero at P, but no other zeroes in X', in other words, $\operatorname{div}(s) = 2P - 2P_0$. To be specific, we may take $s = x$.

Choose points Q, R, Q' and R' on X' such that $Q + R = Q' + R' = P$ in the group X, but $\{Q, R\} \cap \{Q', R'\} = \varnothing$. Then there exist regular functions a and

b on X', such that $\operatorname{div}(a) = P + Q + R - 3P_0$ and $\operatorname{div}(b) = P + Q' + R' - 3P_0$. To be specific, we may choose $a = x + \frac{1}{2}y$ and $b = x - \frac{1}{2}y$, which determines the points Q, R, Q' and R' as intersections of $y^2 = 4(x^3 - x)$ with two lines of slope ± 2 through the origin.

Consider the matrix

$$M = \frac{1}{s}\begin{pmatrix} -ab & -b^2 \\ a^2 & ab \end{pmatrix} \quad \in \quad M\big(2 \times 2, \Gamma(X', \mathcal{O})\big)\,.$$

One checks that

$$\ldots \xrightarrow{M} \mathcal{O}^2 \xrightarrow{M} \mathcal{O}^2 \xrightarrow{(a\ b)} L \to 0 \tag{10}$$

is an infinite resolution of L by free modules of rank 2.

Now let us construct a perfect resolving algebra B. Let $f(x, y) = y^2 - 4(x^3 - x)$. We start with the finite resolving algebra $A = k[x, y]\{\xi\}/(d\xi = f)$, which resolves the affine coordinate ring $h^0(A) = k[x, y]/f$ of X'.

Take sequences $\theta_1, \theta_2, \ldots$ and $\eta_1, \eta_2, \ldots$ of formal variables such that $\deg \theta_i = \deg \eta_i = -i$, for all $i \in \mathbb{Z}_{\geq 1}$. We let $B^\natural = A[\theta, \eta]$ be the graded algebra, free over A on all θ_i and η_i. We turn $B^\natural$ into a differential graded algebra B by defining a differential d by $d\theta_1 = d\eta_1 = 0$ and

$$\begin{aligned} d\theta_{i+1} &= \alpha\theta_i + \gamma\eta_i - \xi\theta_{i-1} \\ d\eta_{i+1} &= \beta\theta_i + \delta\eta_i - \xi\eta_{i-1}\,, \end{aligned}$$

for all $i \geq 1$, where we use the convention $\theta_0 = \eta_0 = 0$. Here, $\alpha, \beta, \gamma, \delta \in k[x, y]$ are polynomials such that the image of $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ in $h^0(A)$ is equal to M and $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)^2 = f(x, y)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. In fact, we may take

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} x^2 - x - 1 & y - (x^2 + x - 1) \\ y + (x^2 + x - 1) & -(x^2 - x - 1) \end{pmatrix}$$

Then B is a perfect resolving algebra: the complex $\Omega_B \otimes_B h^0(B) = \Omega_B \otimes_B h^0(A)$ is the direct sum of the resolution (10) of L and the two term resolution

$$h^0(A)\mathfrak{d}\xi \longrightarrow h^0(A)\mathfrak{d}x \oplus h^0(A)\mathfrak{d}y$$

of $\Omega_{h^0(A)}$. Thus $\Omega_B \otimes_B h^0(B)$ is perfect, of amplitude contained in $[-1, 0]$ and B is perfect of amplitude 1. On the other hand, B cannot be quasi-isomorphic to a finite resolving algebra, because

$$\det\big(\Omega_B \otimes_B h^0(A)\big) = L^{-1}$$

is not trivial.

This is the special case of Example 3.11, where E is a line bundle and s is the zero section.

3.2 Compatibility with limits over truncations

The second fundamental result on perfect resolving algebras expresses a certain compatibility between the derivations of a perfect resolving algebra B and its truncations $B_{(n)}$. Here $B_{(n)}$, for $n \geq -1$, denotes the differential graded subalgebra generated by $B^{\geq -n}$. Every truncation $B_{(n)}$ is a finite resolving algebra of amplitude n.

More generally, if $C \to B$ is a quasi-finite resolving morphism, then we define $B_{(n)}$ to be the differential graded subalgebra of B generated by all of C and $B^{\geq -n}$. Then every $C \to B_{(n)}$ if a finite resolving morphism of amplitude n. We hope that it is always clear from the context, which definition of $B_{(n)}$ applies.

Theorem 3.13 *Let $C \to B$ be a perfect resolving morphism of differential graded algebras. Then for every r we have*

$$\varprojlim_n{}^1 h^r \,\underline{\mathrm{Der}}_C(B_{(n)}, A) = 0$$

and

$$\varprojlim_n h^r \,\underline{\mathrm{Der}}_C(B_{(n)}, A) = h^r \,\underline{\mathrm{Der}}_C(B, A)$$

for all quasi-finite resolving algebras A endowed with a morphism $B \to A$.

Proof. Assume that $C \to B$ has perfect amplitude N. Consider the quasi-finite resolving morphism $B_{(n)} \to B$, for $n \geq N$. By Remark 3.4, we know that $B_{(n)} \to B$ has perfect amplitude $n+1$. By Proposition 3.5, the spectral sequence (7), computing $h^r \,\underline{\mathrm{Der}}_{B_{(n)}}(B, A)$, has exactly one non-zero column, the column $p = n+1$.

Now consider the canonical homomorphism $\Omega_{B/B_{(n)}} \to \Omega_{B/B_{(n+1)}}$ of differential graded B-modules. It induces a homomorphism of spectral sequences (7). Since both spectral sequences have only one non-zero column, but this column is off by 1, the homomorphism of spectral sequences vanishes. We conclude that the canonical homomorphism

$$h^r \,\underline{\mathrm{Der}}_{B_{(n+1)}}(B, A) \longrightarrow h^r \,\underline{\mathrm{Der}}_{B_{(n)}}(B, A)$$

is zero.

Next we note that we have a morphisms of short exact sequences of differential graded A-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{\mathrm{Der}}_{B_{(n+1)}}(B, A) & \longrightarrow & \underline{\mathrm{Der}}_C(B, A) & \longrightarrow & \underline{\mathrm{Der}}_C(B_{(n+1)}, A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{\mathrm{Der}}_{B_{(n)}}(B, A) & \longrightarrow & \underline{\mathrm{Der}}_C(B, A) & \longrightarrow & \underline{\mathrm{Der}}_C(B_{(n)}, A) & \longrightarrow & 0
\end{array}$$

We consider the following extract from the induced morphism of long exact cohomology sequences:

$$\begin{array}{ccccc}
\longrightarrow & h^r\big(\underline{\mathrm{Der}}_{B_{(n+1)}}(B,A)\big) & \longrightarrow & h^r\big(\underline{\mathrm{Der}}_C(B,A)\big) & \longrightarrow \\
& \downarrow & & \downarrow & \\
\longrightarrow & h^r\big(\underline{\mathrm{Der}}_{B_{(n)}}(B,A)\big) & \longrightarrow & h^r\big(\underline{\mathrm{Der}}_C(B,A)\big) & \longrightarrow
\end{array}$$

Since the left vertical arrow is zero, and the right vertical arrow is bijective, we conclude that the upper horizontal map is zero. Thus the whole long exact sequence on the $n+1$ level breaks up into short exact sequences

$$0 \longrightarrow h^r\,\underline{\mathrm{Der}}_C(B,A) \longrightarrow h^r\,\underline{\mathrm{Der}}_C(B_{(n+1)},A) \longrightarrow h^{r+1}\,\underline{\mathrm{Der}}_{B_{(n+1)}}(B,A) \longrightarrow 0\,.$$

Letting $n+1$ vary, we get an inverse system of short exact sequences. Passing to the limit, we get the following six term exact sequence

$$\begin{aligned}
0 \longrightarrow \varprojlim h^r\,\underline{\mathrm{Der}}_C(B,A) \longrightarrow \varprojlim h^r\,\underline{\mathrm{Der}}_C(B_{(n)},A) \longrightarrow \\
\longrightarrow \varprojlim h^{r+1}\,\underline{\mathrm{Der}}_{B_{(n)}}(B,A) \longrightarrow {\varprojlim}^1 h^r\,\underline{\mathrm{Der}}_C(B,A) \longrightarrow \\
\longrightarrow {\varprojlim}^1 h^r\,\underline{\mathrm{Der}}_C(B_{(n)},A) \longrightarrow {\varprojlim}^1 h^{r+1}\,\underline{\mathrm{Der}}_{B_{(n)}}(B,A) \longrightarrow 0
\end{aligned}$$

Since the connecting morphisms of the third projective system are zero for sufficiently large n, the associated $\varprojlim$ and ${\varprojlim}^1$ vanish. All connecting morphisms in the first projective system are bijective, so the associated ${\varprojlim}^1$ vanishes. In other words, the third, fourth and sixth term in our six term sequence vanish. This implies the theorem. □

Scholum 3.14 *The inverse system of groups $h^r\,\underline{\mathrm{Der}}_C(B_{(n)},A)$ has the property that if an element lifts one step, then it lifts two steps.*

Remark If an inverse system of groups has this property, then its ${\varprojlim}^1$ vanishes.

4 Linearization of homotopy groups

Occasionally, it seems unnatural to work with negative indices for the cohomology spaces of certain differential graded modules. We thus adopt the usual notation $h_i = h^{-i}$ for lowering indices.

4.1 Preliminaries

We start with three fundamental lemmas.

Lemma 4.1 *Let $C \to B \to A$ be morphisms of differential graded algebras. Let $D \in \underline{\mathrm{Der}}_C(B, A)$ and $\epsilon \in A$ be homogeneous elements of complementary degrees, i.e., such that $\deg D + \deg \epsilon = 0$. Assume $\epsilon^2 = 0$. Then $h = \phi + \epsilon D$ is a morphism of graded algebras, where $\phi : B \to A$ is the structure morphism. If D and ϵ are cocycles, then h is a morphism of differential graded algebras.* $\square$

Lemma 4.2 *Let A be a differential graded algebra and $\eta \in A \otimes \Omega_\ell$ a homogeneous element, where $\ell \geq 1$. Let $\Lambda \subset \Delta^\ell$ be a horn. If $\eta\,|_{\Lambda} = 0$ and $d\eta = 0$, then there exists $\theta \in A \otimes \Omega_\ell$ such that $\theta\,|_{\Lambda} = 0$ and $\eta = d\theta$.*

Proof. Without loss of generality, let $\Lambda \hookrightarrow \Delta^\ell$ be the horn defined by $\Lambda = \{t_1 = 0\} \cup \ldots \cup \{t_\ell = 0\}$.

Note that $A \otimes \Omega_\ell \to A; \omega \mapsto \omega(0)$ is a quasi-isomorphism, by the algebraic de Rham theorem. Thus we can certainly find $\theta_0 \in A \otimes \Omega_\ell$ such that $d\theta_0 = \eta$. We define θ_i inductively by $\theta_i = \theta_{i-1} - \theta_{i-1}\,|_{t_i=0}$. Then $\theta = \theta_\ell$ fits our requirements. $\square$

Lemma 4.3 *Let A be a differential graded algebra and let $\psi \in A \otimes \Omega_{\ell-1}$ be a homogeneous element, where $\ell \geq 1$. Assume that $\psi\,|_{\partial\Delta^{\ell-1}} = 0$. Then there exists $\Psi \in A \otimes \Omega_\ell$ such that $\partial_\ell \Psi = \psi$ and $\Psi\,|\,\Lambda = 0$, where $\Lambda \subset \Delta^\ell$ is the horn opposite to the face $\partial^\ell(\Delta^{\ell-1}) \subset \Delta^\ell$.*

Proof. Note that the face map $\partial^\ell : \Delta^{\ell-1} \to \Delta^\ell$ is given in inhomogeneous coordinates by $(t_1, \ldots, t_{\ell-1}) \mapsto (t_1, \ldots, t_{\ell-1}, 0)$. Define

$$\Psi(t_1, \ldots, t_\ell) = (1 - t_\ell)^{N+1} \psi\left(\tfrac{t_1}{1-t_\ell}, \ldots, \tfrac{t_{\ell-1}}{1-t_\ell}\right),$$

where N is large enough such that $(1 - t_\ell)^N \psi(\frac{t_1}{1-t_\ell}, \ldots, \frac{t_{\ell-1}}{1-t_\ell})$ has no denominators. $\square$

Remark The latter two lemmas can be interpreted in terms of the differential graded algebra $B_r = k[x, \xi]/(d\xi = x)$, which is quasi-free on the basis (x, ξ), where $\deg x = r$ and $\deg \xi = r-1$, and where the differential is defined by $d\xi = x$, $dx = 0$. Lemma 4.2 expresses the fact that $\mathrm{Hom}^\vartriangle(B_r, A) \to \mathrm{Hom}^\vartriangle(k[x], A)$ is a fibration, which also follows from Corollary 1.17. Lemma 4.3 says that $\pi_{\ell-1} \mathrm{Hom}^\vartriangle(B_r, A) = 0$. This also follows from the fact that $\mathrm{Hom}^\vartriangle(B_r, A) \to \mathrm{Hom}^\vartriangle(k, A) = *$ is a trivial fibration, since $k \to B_r$ is a quasi-isomorphism, which follows most easily from Corollary 2.9.

Let A be a differential graded algebra.

Consider the simplicial differential graded algebra $A \otimes \Omega_\bullet$. Let $N(A \otimes \Omega_\bullet)$ be the associated normalized chain complex. Thus $N(A \otimes \Omega_\ell) \subset A \otimes \Omega_\ell$ is the subcomplex defined by

$$\omega \in N(A \otimes \Omega_\ell) \quad \Longleftrightarrow \quad \text{for all } i = 0, \ldots, \ell - 1 \text{ we have } \partial_i \omega = 0,$$

where $\partial_i \omega = (\partial^i)^* \omega$ is restriction via the i-th inclusion map $\partial^i : \Delta^{\ell-1} \longrightarrow \Delta^\ell$. The boundary map of the normalized chain complex is defined to be

$$\widetilde{\partial}_\ell = (-1)^\ell \partial_\ell : N(A \otimes \Omega_\ell) \longrightarrow N(A \otimes \Omega_{\ell-1}) \,.$$

Written in such a way, $\widetilde{\partial}_\ell$ is a morphism of complexes, i.e., it commutes with the coboundary map d of $N(A \otimes \Omega_\bullet)$. Since we would like to think of $\widetilde{\partial}$ as being of degree -1, we have to change the sign of d on $N(A \otimes \Omega_\ell)$ for odd ℓ. We call the result $\widetilde{d}$. Thus we have

$$\widetilde{d}\omega = (-1)^\ell d\omega, \quad \text{for } \omega \in N(A \otimes \Omega_\ell).$$

Now we truncate each of the cochain complexes $N(A \otimes \Omega_\ell)$ at r. We obtain a chain complex of cochain complexes, whose boundary maps are given by

$$\widetilde{\partial}_\ell = (-1)^\ell \partial_\ell : \big(\tau_{\leq r} N(A \otimes \Omega_\ell), \widetilde{d}\big) \longrightarrow \big(\tau_{\leq r} N(A \otimes \Omega_{\ell-1}), \widetilde{d}\big) \,.$$

Thus we have defined a double complex, which we write in the third quadrant, with $\widetilde{d}$ vertical and $\widetilde{\partial}$ horizontal:

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{-\partial_3} & Z^r N(A \otimes \Omega_2) & \xrightarrow{\partial_2} & Z^r N(A \otimes \Omega_1) & \xrightarrow{-\partial_1} & Z^r A \\
 & & \uparrow d & & \uparrow -d & & \uparrow d \\
\cdots & \xrightarrow{-\partial_3} & N(A \otimes \Omega_2)^{r-1} & \xrightarrow{\partial_2} & N(A \otimes \Omega_1)^{r-1} & \xrightarrow{-\partial_1} & A^{r-1} \\
 & & \uparrow d & & \uparrow -d & & \uparrow d \\
\cdots & \xrightarrow{-\partial_3} & N(A \otimes \Omega_2)^{r-2} & \xrightarrow{\partial_2} & N(A \otimes \Omega_1)^{r-2} & \xrightarrow{-\partial_1} & A^{r-2} \\
 & & \uparrow d & & \uparrow -d & & \uparrow d \\
 & & \vdots & & \vdots & & \vdots
\end{array}$$

By Lemma 4.2, vertical cohomology vanishes everywhere, except in the last column. As for the horizontal cohomology, it vanishes everywhere except for in the first row, by Lemma 4.3. Therefore, our double complex induces an isomorphism of k-vector spaces

$$h^{r-\ell}(A) \xrightarrow{\sim} h_\ell\big(Z^r N(A \otimes \Omega_\bullet), \widetilde{\partial}\big) \,. \tag{11}$$

We will fix notation:

$$\omega_\ell = \ell!\, dt_1 \wedge \ldots \wedge dt_\ell \,. \tag{12}$$

Note the properties $d\omega_\ell = 0$ and $\omega_\ell\,|_{\partial\Delta^\ell} = 0$. Also, $\omega_0 = 1$.

Proposition 4.4 *Every element of* $h_\ell\big(Z^r N(A\otimes\Omega_\bullet),\widetilde{\partial}\big)$ *may be represented as* $\omega_\ell a$*, where* $a\in Z^{r-\ell}(A)$*. Changing* a *by a coboundary only changes* $\omega_\ell a$ *by a boundary.*

The canonical isomorphism (11) is given by

$$a \longmapsto (-1)^{\frac{1}{2}\ell(\ell-1)}\omega_\ell a\,.$$

PROOF. We shall introduce differential forms (for $\ell\geq 1$)

$$\tau_\ell = -(\ell-1)!\sum_{i=1}^{\ell}(-1)^i t_i\, dt_1\wedge\ldots\wedge\widehat{dt_i}\wedge\ldots\wedge dt_\ell$$

Note the following properties of τ_ℓ:

(i) $d\tau_\ell=\omega_\ell$,

(ii) for all $i=1,\ldots,\ell$ we have $\tau_\ell\,\big|_{t_i=0}=0$, i.e., $\partial_i\tau_\ell=0$,

(iii) $\partial_0\tau_\ell=\omega_{\ell-1}$.

To see (iii), recall that $\partial^0(t_1,\ldots,t_{\ell-1})=(1-\sum_{i=1}^{\ell-1}t_i,t_1,\ldots,t_{\ell-1})$.

For the present proof, it is convenient to consider the following variation:

$$\sigma_\ell=\tau_\ell+(-1)^\ell\omega_{\ell-1}\,.$$

These forms have the three properties:

(i) $d\sigma_\ell=\omega_\ell$,

(ii) for all $i=0,\ldots,\ell-1$ we have $\partial_i\sigma_\ell=0$,

(iii) $\partial_\ell\sigma_\ell=(-1)^\ell\omega_{\ell-1}$, so that $\widetilde{\partial}_\ell\sigma_\ell=\omega_{\ell-1}$.

Let $a\in Z^{r-\ell}(A)$ and consider the sum

$$\sum_{i=1}^{\ell}(-1)^{\frac{1}{2}i(i+1)}\sigma_i a\,. \tag{13}$$

Note that $\sigma_i a\in N(A\otimes\Omega_i)^{r-\ell+i-1}$, so that (13) is an element of total (cochain) degree $r-\ell-1$ in our double complex. Applying the total coboundary $\widetilde{d}+\widetilde{\partial}$ to (13) we get

$$\begin{aligned}(\widetilde{d}+\widetilde{\partial})\sum_{i=1}^{\ell}(-1)^{\frac{1}{2}i(i+1)}\sigma_i a &= \sum_{i=1}^{\ell}(-1)^{\frac{1}{2}i(i+1)+i}d\sigma_i a+\sum_{i=1}^{\ell}(-1)^{\frac{1}{2}i(i+1)}\widetilde{\partial}_i\sigma_i a\\ &= \sum_{i=1}^{\ell}(-1)^{\frac{1}{2}i(i+1)+i}\omega_i a+\sum_{i=1}^{\ell}(-1)^{\frac{1}{2}i(i+1)}\omega_{i-1}a\\ &= \sum_{i=1}^{\ell}(-1)^{\frac{1}{2}i(i-1)}\omega_i a+\sum_{i=0}^{\ell-1}(-1)^{\frac{1}{2}(i+1)(i+2)}\omega_i a\\ &= (-1)^{\frac{1}{2}\ell(\ell-1)}\omega_\ell a-a\,.\end{aligned}$$

This proves the formula of the proposition. It also proves the first claim, because (11) is surjective. For the second claim let $b\in A^{r-\ell-1}$. Then

$$\psi=\sigma_{\ell+1}db+(-1)^\ell\omega_{\ell+1}b$$

is an element of $Z^rN(A\otimes\Omega_{\ell+1})$ and $\widetilde{\partial}\psi=\omega_\ell db$. □

4.2 Linearization of homotopy groups

Let $C \to B$ be a fixed resolving morphism of resolving algebras over k. Let A be an arbitrary differential graded algebra and denote by $P : B \to A$ a fixed morphism. Denote by P also the restriction to C. We use P as a base point for the spaces $\mathrm{Hom}^{\Delta}(B, A)$ and $\mathrm{Hom}^{\Delta}(C, A)$ and for the fiber $\mathrm{Hom}^{\Delta}_C(B, A)$ of the fibration $\mathrm{Hom}^{\Delta}(B, A) \to \mathrm{Hom}^{\Delta}(C, A)$.

Consider, for every $\ell \geq 1$, the map

$$\begin{aligned} \Xi_\ell : h^{-\ell}\, \underline{\mathrm{Der}}_C(B, A) &\longrightarrow \pi_\ell \,\mathrm{Hom}^{\Delta}_C(B, A) \\ D &\longmapsto P + (-1)^{\frac{1}{2}\ell(\ell-1)}\omega_\ell D\,. \end{aligned} \tag{14}$$

Recall that ω_ℓ has been defined in (12). By Lemma 4.1, for given $D \in Z^{-\ell}\, \underline{\mathrm{Der}}_C(B, A)$, the map

$$h = P + (-1)^{\frac{1}{2}\ell(\ell-1)}\omega_\ell D$$

is a morphism of differential graded C-algebras $B \to A \otimes \Omega_\ell$, hence an ℓ-simplex in $\mathrm{Hom}^{\Delta}_C(B, A)$. Since $\omega_\ell|_{\partial\Delta^\ell} = 0$, it defines, in fact, an element of $\pi_\ell \,\mathrm{Hom}^{\Delta}_C(B, A)$.

Let us check that the homotopy class of h depends only on the cohomology class of D. So let D' be another element of $Z^{-\ell}\, \underline{\mathrm{Der}}_C(B, A)$, differing from D by a coboundary and let $h' = P + (-1)^{\frac{1}{2}\ell(\ell-1)}\omega_\ell D'$. Thus there exists $E \in \underline{\mathrm{Der}}_C^{-\ell-1}(B, A)$ such that

$$D' = D + dE\,.$$

Then one checks that

$$\Phi = P + (-1)^{\frac{1}{2}\ell(\ell-1)}\omega_\ell\big((1-s)D + sD'\big) + (-1)^{\frac{1}{2}\ell(\ell-1)}\omega_\ell ds\, E$$

defines a homotopy from h to h', i.e., an element

$$\Phi \in \mathrm{Hom}\left(I \times \Delta^\ell, \mathrm{Hom}^{\Delta}_C(B, A)\right),$$

such that $\Phi|_{s=0} = h$, $\Phi|_{s=1} = h'$ and $\Phi|_{I\times\partial\Delta^\ell} = P$. Here s is the coordinate on the 'interval' $I = \mathbb{A}^1$.

Thus the map Ξ_ℓ is well-defined, for all $\ell \geq 1$.

Remark 4.5 (Naturality) (i) Let $C' \to B'$ be a another resolving morphism of resolving algebras. Assume, moreover, given a commutative diagram of differential graded algebras

$$\begin{array}{ccc} C' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ C & \longrightarrow & B\,. \end{array}$$

Then we have, for all $\ell \geq 1$, an induced commutative diagram

$$\begin{array}{ccc} h^{-\ell}\, \underline{\mathrm{Der}}_C(B, A) & \longrightarrow & h^{-\ell}\, \underline{\mathrm{Der}}_{C'}(B', A) \\ \Big\downarrow{\scriptstyle \Xi_\ell} & & \Big\downarrow{\scriptstyle \Xi_\ell} \\ \pi_\ell \,\mathrm{Hom}^{\Delta}_C(B, A) & \longrightarrow & \pi_\ell \,\mathrm{Hom}^{\Delta}_{C'}(B', A)\,. \end{array}$$

(ii) Let $A \to A'$ be an arbitrary morphism of differential graded algebras. Then we have an induced commutative diagram

$$\begin{array}{ccc} h^{-\ell}\,\underline{\mathrm{Der}}_C(B,A) & \longrightarrow & h^{-\ell}\,\underline{\mathrm{Der}}_C(B,A') \\ \downarrow{\scriptstyle \Xi_\ell} & & \downarrow{\scriptstyle \Xi_\ell} \\ \pi_\ell \operatorname{Hom}^{\vartriangle}_C(B,A) & \longrightarrow & \pi_\ell \operatorname{Hom}^{\vartriangle}_C(B,A')\,. \end{array}$$

(iii) As a special case of (ii), applied to $\partial_0, \partial_1 : A \otimes \Omega_1 \rightrightarrows A$, we get the following compatibility of Ξ_ℓ with change of base point: let A be a differential graded C-algebra (whose cohomology is concentrated in non-positive degrees). Let $P, Q : B \to A$ be two base points for $\operatorname{Hom}^{\vartriangle}_C(B,A)$, giving two different B-modules structures on A, denoted ${}_PA$, ${}_QA$. Then every path $h : P \to Q$ in $\operatorname{Hom}^{\vartriangle}_C(B,A)$ gives rise to a commutative diagram

$$\begin{array}{ccc} h^{-\ell}\,\underline{\mathrm{Der}}_C(B,{}_PA) & \xrightarrow{h_*} & h^{-\ell}\,\underline{\mathrm{Der}}_C(B,{}_QA) \\ \downarrow{\scriptstyle \Xi_\ell} & & \downarrow{\scriptstyle \Xi_\ell} \\ \pi_\ell\big(\operatorname{Hom}^{\vartriangle}_C(B,A),P\big) & \xrightarrow{h(\,\cdot\,)} & \pi_\ell\big(\operatorname{Hom}^{\vartriangle}_C(B,A),Q\big) \end{array}$$

where h_* denotes the change of base point map of Definition 3.7.

Lemma 4.6 (Homomorphism) *Let $C \to B \to A$ be as above.*

(i) For $\ell \geq 2$, the map Ξ_ℓ is a group homomorphism.

(ii) If there exists a basis (x_ν) for B over C, such that $dx_\nu \in C$, for all ν, then this is also true for $\ell = 1$.

(iii) More generally, suppose that $B' \subset B$ is a subalgebra containing C, such that $C \to B'$ and $B' \to B$ are resolving and there exists a basis (x_ν) for B over B', such that $dx_\nu \in B'$, for all ν. Let $D \in h^{-1}\,\underline{\mathrm{Der}}_C(B,A)$ and $D' \in h^{-1}\,\underline{\mathrm{Der}}_{B'}(B,A)$. Then we have $\Xi_\ell(D + D') = \Xi_\ell(D) * \Xi_\ell(D')$.

Proof. First assume that $\ell \geq 2$. Let D, D' be two elements of $Z^{-\ell}\,\underline{\mathrm{Der}}_C(B,A)$. Let h, h' and g be the images of D, D' and $D+D'$ under Ξ_ℓ. An $(\ell+1)$-simplex in $\operatorname{Hom}^{\vartriangle}_C(B,A)$ showing that $h * h' = g$ is given by

$$\begin{aligned} \Phi = P + &\epsilon(dt_1 \wedge \ldots \wedge dt_{\ell-2} \wedge dt_\ell \wedge dt_{\ell+1} + dt_1 \wedge \ldots \wedge dt_{\ell-1} \wedge dt_{\ell+1})\,D \\ &\epsilon(dt_1 \wedge \ldots \wedge dt_{\ell-1} \wedge dt_{\ell+1} + dt_1 \wedge \ldots \wedge dt_\ell)\,D', \end{aligned}$$

where $\epsilon = (-1)^{\frac{1}{2}\ell(\ell-1)}\ell!$ is the necessary multiplier. Note that we need $\ell \geq 2$ to apply Lemma 4.1. We see directly from the definition that $\Phi \,|\, \{t_{\ell-1} = 0\} = h$, that $\Phi \,|\, \{t_{\ell+1} = 0\} = h'$, that $\Phi \,|\, \{t_\ell = 0\} = g$ and that Φ restricted to all the other faces of $\Delta^{\ell+1}$ is equal to P.

Now let us consider the case $\ell = 1$. Since (iii) implies (ii), let us prove (iii). So assume given $C \to B' \to B$. We define Φ to be the unique morphism of graded algebras $\Phi : B \to A \otimes \Omega_2$ such that $\Phi \,|\, B' = P + dt_2\, D$ and

$$\Phi(x_\nu) = P(x_\nu) + dt_2\, D(x_\nu) + (dt_2 + dt_1) D'(x_\nu)\,,$$

for all ν. One checks that Φ respects the differential, so that Φ is a 2-simplex in $\operatorname{Hom}^{\Delta}_C(B,A)$. Moreover, $\Phi\,|_{\substack{t_1=1-t\\ t_2=t}} = h$, $\Phi\,|_{t_1=0} = g$ and $\Phi\,|_{t_2=0} = h'$, so that, indeed, $h * h' = g$. $\square$

Remark 4.7 (Degree zero) Suppose B admits a basis (x_i) over C such that $dx_i \in C$, for all i. Then we may also define

$$\Xi_0 : h^0\,\underline{\operatorname{Der}}_C(B,A) \longrightarrow \pi_0 \operatorname{Hom}^{\Delta}_C(B,A)\,.$$

Given a C-derivation $D : B \to A$, we map D to the unique morphism of graded algebras $h : B \to A$ such that $h\,|\,C = P$ and $h(x_i) = P(x_i) + D(x_i)$. To check that h respects the differential, it suffices to prove that $h(dx_i) = d\big(h(x_i)\big)$, for all i, which is easily done using that $h(dx_i) = P(dx_i)$ and $D(dx_i) = 0$, which follows from our assumption that $dx_i \in C$. If $D' = D + dE$ is in the same cohomology class as D, then the image of D, which we call h', is homotopic to h via the homotopy Φ defined by $\Phi\,|\,C = P$ and $\Phi(x_i) = P(x_i) + (1-s)D(x_i) + sD'(x_i) + ds\,E(x_i)$. Thus Ξ_0 is indeed well-defined.

The question of whether or not Ξ_0 depends on the choice of generators (x_i) is slightly more subtle. The most common reason why a set of generators should satisfy $dx_i \in C$ is that they all have the same degree $r = \deg x_i$, for all i. So let us suppose that this is the case, and that the total number of generators x_i if finite.

In the case $r < 0$ it is easy to see that Ξ_0 is independent of the choice of the generators: if (y_j) is another family of generators, then $x_i = \sum_j c_{ij} y_j$, for a family of elements c_{ij} of degree zero, and hence necessarily contained in C. So if h' is defined by $h'(y_j) = P(y_j) + D(y_j)$, then $h'(x_i) = h'(\sum_j c_{ij} y_j) = \sum_j P(c_{ij})\big(P(y_j) + D(y_j)\big) = P(\sum_j c_{ij} y_j) + D(\sum_j c_{ij} y_j) = h(x_i)$ and so $h' = h$.

On the other hand, in the case $r = 0$, the map Ξ_0 depends on the choice of generators. For example, let $C = k$ and $B = k[x_1, x_2]$. Then another set of free generators for B is given by $y_1 = x_1$ and $y_2 = x_2 + x_1^2$. The fact that the change of coordinates is not linear is responsible for the fact that h and h' will be different. This is particularly easy to see if A is also concentrated in degree zero, because then $\pi_0 \operatorname{Hom}^{\Delta}(B,A) = \operatorname{Hom}(B,A)$, the set of k-algebra morphisms.

The reason for the sign in the definition of Ξ_ℓ becomes clear from the following fundamental lemma:

Lemma 4.8 (Main) *Suppose that B has a basis over C consisting of one homogeneous generator x. Let $r = \deg x$. Then for all $\ell \geq 0$ the diagram*

$$\begin{array}{ccc} h^{-\ell}\,\underline{\operatorname{Der}}_C(B,A) & \xrightarrow{\;D\mapsto D(x)\;} & h^{r-\ell}(A) \\ {\scriptstyle \Xi_\ell}\downarrow & & \downarrow{\scriptstyle \text{can. hom. (11)}} \\ \pi_\ell \operatorname{Hom}^{\Delta}_C(B,A) & \xrightarrow[\;h\mapsto h(x)-P(x)\;]{} & h_\ell\big(Z^r N(A\otimes\Omega_\bullet)\big) \end{array}$$

it seems that maybe the upper horizontal arrow should be endowed with a sign. The way it's written, it does not come from a morphism of differential graded A-modules! I think the sign is correct. The upper arrow shouldn't be a morphism of graded modules, as we evaluate on the right, whereas we write homomorphisms on the left *commutes. Moreover, all the maps are bijections of pointed sets, hence group isomorphisms for $\ell \geq 1$.*

PROOF. It is straightforward to check that the upper horizontal map is well-defined and an isomorphism of $h^0(A)$-modules. As for the lower horizontal map, note that we have in fact an isomorphism of pointed simplicial sets

$$\begin{aligned} \operatorname{Hom}^{\vartriangle}_C(B,A) &\longrightarrow Z^r(A\otimes\Omega_\bullet) \\ h &\longmapsto h(x)-P(x)\,. \end{aligned}$$

Moreover, the pointed simplicial set $Z^r(A\otimes\Omega_\bullet)$ is a simplicial k-vector space and so its homotopy groups are equal to the homology groups of the associated normalized chain complex. So we have

$$\pi_\ell\big(Z^r(A\otimes\Omega_\bullet)\big)=h_\ell\big(NZ^r(A\otimes\Omega_\bullet)\big)=h_\ell\big(Z^rN(A\otimes\Omega_\bullet)\big)\,.$$

So, indeed, the lower horizontal map is an isomorphism of groups (or pointed sets, if $\ell=0$). Commutativity of the diagram follows directly from the definitions. □

Corollary 4.9 *Suppose that B has a C-basis (x_ν), where all x_ν have the same degree r. Then Ξ_ℓ induces canonical isomorphisms*

$$h^{-\ell}\,\underline{\mathrm{Der}}_C(B,A)=\pi_\ell\operatorname{Hom}^{\vartriangle}_C(B,A)=h^{r-\ell}(A)^{\#\nu}\,,$$

for all $\ell \geq 0$.

Let $C\to B'\to B$ be resolving morphisms of resolving algebras. Consider the fibration

$$\operatorname{Hom}^{\vartriangle}_C(B,A)\longrightarrow\operatorname{Hom}^{\vartriangle}_C(B',A)\,, \tag{15}$$

whose fiber over P is $\operatorname{Hom}^{\vartriangle}_{B'}(B,A)$. There is also a short exact sequence of complexes of k-vector spaces

$$0\longrightarrow\underline{\mathrm{Der}}_{B'}(B,A)\longrightarrow\underline{\mathrm{Der}}_C(B,A)\longrightarrow\underline{\mathrm{Der}}_C(B',A)\longrightarrow 0\,. \tag{16}$$

Proposition 4.10 (Boundary) *Assume, moreover, that there exists a basis (x_ν) for B over B' such that $dx_\nu\in B'$, for all ν. Then for all $\ell\geq 1$, the diagram*

$$\begin{array}{ccc} h^{-\ell}\,\underline{\mathrm{Der}}_C(B',A) & \xrightarrow{\ \delta\ } & h^{1-\ell}\,\underline{\mathrm{Der}}_{B'}(B,A) \\ \downarrow{\scriptstyle \Xi_\ell} & & \downarrow{\scriptstyle \Xi_{\ell-1}} \\ \pi_\ell\operatorname{Hom}^{\vartriangle}_C(B',A) & \xrightarrow{\ \partial\ } & \pi_{\ell-1}\operatorname{Hom}^{\vartriangle}_{B'}(B,A) \end{array}$$

commutes. Here δ is the boundary map of the long exact cohomology sequence associated to (16) and ∂ is the boundary map of the long exact homotopy sequence associated to (15).

Proof. Consider $D \in Z^{-\ell}\,\underline{\mathrm{Der}}_C(B', A)$. Lift D to an internal derivation $\widetilde{D} : B \to A$ by setting $\widetilde{D}(x_\nu) = 0$, for all ν. Then δD is equal to $d\widetilde{D}$. Thus δD is given by

$$(\delta D)(x_\nu) = (d\widetilde{D})(x_\nu) = (-1)^{\ell-1} D(dx_\nu)\,.$$

Applying $\Xi_{\ell-1}$ we obtain the $\ell - 1$ simplex

$$g = P + (-1)^{\frac{1}{2}(\ell-1)(\ell-2)}\omega_{\ell-1}\delta D \in \pi_{\ell-1}\operatorname{Hom}^{\vartriangle}_{B'}(B, A)\,.$$

Note that

$$g(x_\nu) = P(x_\nu) + (-1)^{\frac{1}{2}\ell(\ell-1)}\omega_{\ell-1} D(dx_\nu)\,. \tag{17}$$

If, on the other hand, we apply Ξ_ℓ to D, we obtain the ℓ-simplex

$$h = P + (-1)^{\frac{1}{2}\ell(\ell-1)}\omega_\ell D \in \pi_\ell \operatorname{Hom}^{\vartriangle}_C(B', A)\,.$$

We lift h to $h' : B \to A \otimes \Omega_\ell$ by

$$h'(x_\nu) = P(x_\nu) + (-1)^{\frac{1}{2}\ell(\ell-1)}\tau_\ell D(dx_\nu)\,,$$

where τ_ℓ is the $\ell - 1$ form defined in the proof of Proposition 4.4. Note that $d\big(h'(x_\nu)\big) = h(dx_\nu)$, so that this formula defines a morphism of differential graded algebras $B \to A\otimes\Omega_\ell$. Note also that $h' \,|\, \{t_i = 0\} = P$, for all $i = 1, \ldots, \ell$. Thus the image of h under ∂ is equal to $\partial_0(h')$. We obtain $\partial h \,|\, B' = P$ and

$$\begin{aligned}\partial h(x_\nu) &= \partial_0 h'(x_\nu)\\ &= P(x_\nu) + (-1)^{\frac{1}{2}\ell(\ell-1)}\partial_0\tau_\ell D(dx_\nu) = P(x_\nu) + (-1)^{\frac{1}{2}\ell(\ell-1)}\omega_{\ell-1} D(dx_\nu)\,,\end{aligned}$$

which agrees with (17), so we see that $\partial h = g$. □

Theorem 4.11 *Assume that B is finite as a resolving algebra over C. Then for $\ell \geq 1$, the map*

$$\Xi_\ell : h^{-\ell}\,\underline{\mathrm{Der}}_C(B, A) \longrightarrow \pi_\ell \operatorname{Hom}^{\vartriangle}_C(B, A)$$

is bijective (so for $\ell \geq 2$ an isomorphism).

Proof. Induction on the number n of elements in a basis of B over C. If this number is zero, then $B = C$ and so the claim is trivial. Otherwise, there exists a differential graded subalgebra $B' \subset B$, such that B has a basis over B' consisting of one homogeneous generator x, and B' has a C-basis with $n - 1$ elements. Note that then $dx \in B'$.

Consider the fibration (15) and the short exact sequence of complexes of k-vector spaces (16). We have the associated long exact sequences of homotopy groups and cohomology spaces. The maps Ξ_ℓ relate the two:

$$\begin{array}{ccccccc} \cdots \longrightarrow & h^{-\ell}\,\underline{\mathrm{Der}}_{B'}(B, A) & \longrightarrow & h^{-\ell}\,\underline{\mathrm{Der}}_C(B, A) & \longrightarrow \\ & \downarrow{\scriptstyle \Xi_\ell} & & \downarrow{\scriptstyle \Xi_\ell} & \\ \cdots \longrightarrow & \pi_\ell \operatorname{Hom}^{\vartriangle}_{B'}(B, A) & \longrightarrow & \pi_\ell \operatorname{Hom}^{\vartriangle}_C(B, A) & \longrightarrow \end{array}$$

$$
\begin{array}{ccccccc}
\longrightarrow & h^{-\ell}\,\underline{\mathrm{Der}}_C(B',A) & \xrightarrow{\ \delta\ } & h^{1-\ell}\,\underline{\mathrm{Der}}_{B'}(B,A) & \longrightarrow & \cdots \\
& \Big\downarrow{\scriptstyle \Xi_\ell} & & \Big\downarrow{\scriptstyle \Xi_{\ell-1}} & & \\
\longrightarrow & \pi_\ell \operatorname{Hom}^{\vartriangle}_C(B',A) & \xrightarrow{\ \partial\ } & \pi_{\ell-1} \operatorname{Hom}^{\vartriangle}_{B'}(B,A) & \longrightarrow & \cdots
\end{array}
$$

By naturality and Proposition 4.10 all squares which are defined commute. Let us first consider the part up to $\Xi_1 : h^{-1}\,\underline{\mathrm{Der}}_{B'}(B,A) \to \pi_1 \operatorname{Hom}^{\vartriangle}_{B'}(B,A)$. This is, by Lemma 4.6, a homomorphism of long exact sequences of abelian groups. By induction and Lemma 4.8, all Ξ_ℓ adjacent to a square involving δ and ∂ are isomorphisms. Applying the 5-lemma proves the theorem in the $\ell \geq 2$ case.

For the $\ell = 1$ case, we look at the last five terms that still have a map Ξ_ℓ defined. This part ends at $\Xi_0 : h^0\,\underline{\mathrm{Der}}_{B'}(B,A) \to \pi_0 \operatorname{Hom}^{\vartriangle}_{B'}(B,A)$. By induction and Lemma 4.8 all five maps are bijective, except for the one in the middle, which is $\Xi_1 : h^{-1}\,\underline{\mathrm{Der}}_C(B,A) \to \pi_1 \operatorname{Hom}^{\vartriangle}_C(B,A)$. To show that this map is also bijective, note that is is equivariant for the action of $h^{-1}\,\underline{\mathrm{Der}}_{B'}(B,A)$, because of Lemma 4.6 (iii). This is sufficient to prove our claim, by a suitably generalized 5-lemma. □

Recall from Section 3.2 the truncations $B_{(n)}$ of the perfect resolving morphism $C \to B$.

Corollary 4.12 *Let $C \to B$ be a perfect resolving morphism of resolving algebras. Then for any differential graded B-algebra A, the canonical homomorphism*

$$\pi_\ell \operatorname{Hom}^{\vartriangle}_C(B,A) \xrightarrow{\ \sim\ } \varprojlim_n \pi_\ell \operatorname{Hom}^{\vartriangle}_C(B_{(n)},A)$$

is an isomorphism, for all $\ell \geq 0$. Moreover,

$$\Xi_\ell : h^{-\ell}\,\underline{\mathrm{Der}}_C(B,A) \longrightarrow \pi_\ell \operatorname{Hom}^{\vartriangle}_C(B,A)$$

is bijective, for all $\ell \geq 1$.

Proof. Directly from the definitions, it follows that $\operatorname{Hom}^{\vartriangle}_C(B_{(n)},A)$ is a tower of fibrations and that

$$\operatorname{Hom}^{\vartriangle}_C(B,A) = \varprojlim_n \operatorname{Hom}^{\vartriangle}_C(B_{(n)},A)\,.$$

From Theorem 4.11 and Theorem 3.13 we get that

$$\varprojlim_n{}^1 \pi_\ell \operatorname{Hom}^{\vartriangle}_C(B_{(n)},A) = 0\,, \tag{18}$$

for all $\ell \geq 1$. In fact, for the case $\ell = 1$, we have to be a little careful, because Ξ_1 is not a group homomorphism, and in general, $\varprojlim^1$ depends on the group structure, not only on the inverse system structure. But in Scholum 3.14 we mentioned a property of inverse systems which $h^{-1}\,\underline{\mathrm{Der}}_C(B_{(n)},A)$ enjoys, and which does not depend on the group structure, and is thus inherited by $\pi_1 \operatorname{Hom}_C^\vartriangle(B_{(n)}, A)$. As we remarked above (following the scholum), this property is sufficient to assure that $\varprojlim^1$ vanishes.

So the first claim follows immediately from the Milnor exact sequence (see [5], Chapter VI, Proposition 2.15). The second claim then follows by taking the limit over Theorem 4.11, applied to the various truncations $B_{(n)}$. □

Let us introduce the notation

$$\widetilde{\omega}_\ell = (-1)^{\frac{1}{2}\ell(\ell-1)}\omega_\ell\,.$$

Let B be a finite resolving algebra over k. Let A be an arbitrary differential graded k-algebra and $P : B \to A$ a base point for $\operatorname{Hom}^\vartriangle(B, A)$. Let $\ell \geq 1$.

By the theorem, every element of $\pi_\ell \operatorname{Hom}^\vartriangle(B, A)$ may be written as

$$h = P + \widetilde{\omega}_\ell D\,, \tag{19}$$

for a unique $D \in h^{-\ell}\,\underline{\operatorname{Der}}(B, A)$. We may call (19) the *standard* form of h. For $\ell \geq 2$, we have

$$(P + \widetilde{\omega}_\ell D) * (P + \widetilde{\omega}_\ell D') = P + \widetilde{\omega}_\ell(D + D')\,,$$

but we have no such simple formula in the case $\ell = 1$.

It is natural to ask if there is nonetheless some way to describe the composition in $\pi_1 \operatorname{Hom}^\vartriangle(B, A)$ in terms of $h^{-1}\,\underline{\operatorname{Der}}(B, A)$.

4.3 What can we say about π_0?

We will study more carefully $\pi_0 \operatorname{Hom}^\vartriangle(B, A)$, in particular how it behaves under change of B.

Let $C \to B' \to B$ be resolving morphisms of resolving algebras. Let A be a differential graded algebra and let us fix a morphism $P : B' \to A$, but let us not fix any morphism $B \to A$. Thus, in the fibration $\operatorname{Hom}_C^\vartriangle(B, A) \to \operatorname{Hom}_C^\vartriangle(B', A)$, only the base is pointed, the total space is not. We still have a well-defined fiber $\operatorname{Hom}_{B'}^\vartriangle(B, A)$, but this fiber is not pointed either. We write the tail end of the long exact homotopy sequence as

$$\pi_1 \operatorname{Hom}_C^\vartriangle(B', A) \multimap \pi_0 \operatorname{Hom}_{B'}^\vartriangle(B, A) \twoheadrightarrow \pi_0 \operatorname{Hom}_C^\vartriangle(B, A) \twoheadrightarrow \pi_0 \operatorname{Hom}_C^\vartriangle(B', A)\,. \tag{20}$$

This means that $\pi_1 \operatorname{Hom}_C^\vartriangle(B', A)$ acts on $\pi_0 \operatorname{Hom}_{B'}^\vartriangle(B, A)$ in such a way that two elements are in the same orbit if and only if they map to the same element of $\pi_0 \operatorname{Hom}_C^\vartriangle(B, A)$. Moreover, an element of $\pi_0 \operatorname{Hom}_C^\vartriangle(B, A)$ lifts to $\pi_0 \operatorname{Hom}_{B'}^\vartriangle(B, A)$ if and only if it maps to the base point P of $\pi_0 \operatorname{Hom}_C^\vartriangle(B', A)$.

Our first goal is an amplification of (20). For this we assume that B has a basis over B' consisting of one element x of degree r, which we shall fix throughout our discussion. Note that $dx \in B'$. We will also assume that B' is finite over C, so that we may apply Theorem 4.11.

Proposition 4.13 *The k-vector space $h^r(A)$ acts transitively on the fiber of*

$$\pi_0 \operatorname{Hom}^{\Delta}_{C}(B, A) \longrightarrow \pi_0 \operatorname{Hom}^{\Delta}_{C}(B', A)$$

over P. The stabilizer of this action is the image of the homomorphism

$$\begin{aligned} \xi_P : h^{-1} \underline{\operatorname{Der}}_C(B', A) &\longrightarrow h^r(A) \\ D &\longmapsto D(dx)\,. \end{aligned}$$

Proof. Let us start by defining an action of $h^r(A)$ on $\pi_0 \operatorname{Hom}^{\Delta}_{B'}(B, A)$. Given $a \in h^r(A)$ and $Q \in \pi_0 \operatorname{Hom}^{\Delta}_{B'}(B, A)$, we define $a * Q \in \pi_0 \operatorname{Hom}^{\Delta}_{B'}(B, A)$ to the the unique element represented by the morphism $h : B \to A$, such that $h\,|_{B'} = P$ and $h(x) = Q(x) + a$.

To check that this is well-defined, let $a' = a + db$ and $Q' : B \to A$ be homotopic to Q. Choose a homotopy $H : B \to A \otimes \Omega_1$ such that $H\,|_{B'} = P$, $H(0) = Q$, $H(1) = Q'$. Let $h : B \to A$ satisfy $h\,|_{B'} = P$ and $h(x) = Q(x) + a$. Let $h' : B \to A$ satisfy $h'\,|_{B'} = P$ and $h'(x) = Q'(x) + a'$. Then define a homotopy $H' : B \to A \otimes \Omega_1$ from h to h' by $H'\,|_{B'} = P$ and $H'(x) = H(x) + (1-t)a + ta' + (dt)b$.

For any $Q \in \pi_0 \operatorname{Hom}^{\Delta}_{B'}(B, A)$, the orbit map $h^r(A) \to \pi_0 \operatorname{Hom}^{\Delta}_{B'}(B, A)$ is equal to our earlier bijection described in Lemma 4.8, defined using Q as a base point for $\operatorname{Hom}^{\Delta}_{B'}(B, A)$. Thus our action is simply transitive. Note that it is given by

$$(a * Q)(x) = Q(x) + a\,.$$

Let us turn our attention to ξ_P. The fact that ξ_P is well-defined and a homomorphism is easily checked.

Define a homomorphism

$$\begin{aligned} \pi_1 \operatorname{Hom}^{\Delta}_{C}(B', A) &\longrightarrow h_1\big(Z^{r+1} N(A \otimes \Omega_\bullet)\big) \\ h &\longmapsto h(dx) - P(dx)\,. \end{aligned} \tag{21}$$

To see that this is well-defined, let $h : B' \to A \otimes \Omega_1$ and $h' : B' \to A \otimes \Omega_1$ represent loops in $\pi_1 \operatorname{Hom}^{\Delta}_{C}(B', A)$. Let $\Psi : B' \to A \otimes \Omega_2$ be a homotopy between them, i.e., let us assume that $\partial_0 \Psi = P$, $\partial_1 \Psi = h$ and $\partial_2 \Psi = h'$. Then $\psi = \Psi(dx) - P(dx)$ is an element of $Z^{r+1}(A \otimes \Omega_2)$ such that $\partial_0 \psi = 0$, $\partial_1 \psi = h(dx) - P(dx)$ and $\partial_2 \psi = h'(dx) - P(dx)$. Then $\widetilde{\psi}(t_1, t_2) = \psi(t_1, t_2) - \big(h(dx) - P(dx)\big)(t_1 + t_2)$ is an element $\widetilde{\psi} \in Z^{r+1} N(A \otimes \Omega_2)$ such that $\widetilde{\partial}(\widetilde{\psi}) = \partial_2(\widetilde{\psi}) = \big(h'(dx) - P(dx)\big) - \big(h(dx) - P(dx)\big)$, showing that $h(dx) - P(dx)$ and $h'(dx) - P(dx)$ are equal in $h_1\big(Z^{r+1} N(A \otimes \Omega_\bullet)\big)$.

We can show that (21) is a homomorphism by a similar argument: let h and h' be as before, except for not necessarily homotopic. Let $\Psi : B' \to A \otimes \Omega_2$ be any 2-simplex such that $\partial_0 \Psi = h$ and $\partial_2 \Psi = h'$. Then $\partial_1 \Psi$ represents $h * h'$, so without loss of generality we may set $h * h' = \partial_1 \Psi$. Let us abbreviate notation to $\alpha = h(dx) - P(dx)$, $\beta = h'(dx) - P(dx)$ and $\gamma = (h * h')(dx) - P(dx)$. We need to show that $\alpha + \beta - \gamma$ represents zero in $h_1\big(Z^{r+1} N(A \otimes \Omega_\bullet)\big)$. We again use the notation $\psi = \Psi(dx) - P(dx)$. Then we have $\partial_0 \psi = \alpha$, $\partial_1 \psi = \gamma$ and

$\partial_2\psi = \beta$. This time we set $\widetilde{\psi}(t_1,t_2) = \psi(t_1,t_2) + (\alpha-\gamma)(t_1+t_2) - \alpha(t_2)$. Then $\widetilde{\psi} \in Z^{r+1}N(A\otimes\Omega_2)$ and $\widetilde{\partial}(\widetilde{\psi}) = \alpha+\beta-\gamma$, showing that $\alpha+\beta-\gamma$ is a boundary, as required.

Now consider the diagram

$$\begin{array}{ccc} h^{-1}\,\underline{\mathrm{Der}}_C(B',A) & \xrightarrow{\quad \xi_P \quad} & h^r(A) \\ \Big\downarrow{\scriptstyle \Xi_1} & & \Big\downarrow{\scriptstyle \text{can. isom. (11)}} \\ \pi_1 \operatorname{Hom}^{\Delta}_C(B',A) & \xrightarrow{\quad (21) \quad} & h_1\big(Z^{r+1}N(A\otimes\Omega_\bullet)\big) \end{array}$$

It is easily checked that the diagram is commutative. We will eliminate $h_1\big(Z^{r+1}N(A\otimes\Omega_\bullet)\big)$ from this diagram and replace it by

$$\begin{array}{ccc} h^{-1}\,\underline{\mathrm{Der}}_C(B',A) & \searrow^{\xi_P} & \\ \Big\downarrow{\scriptstyle \Xi_1} & & h^r(A) \\ \pi_1 \operatorname{Hom}^{\Delta}_C(B',A) & \nearrow_{\rho} & \end{array} \qquad (22)$$

where we have given the composition of (21) with the inverse of (11) the name ρ. Let us emphasize again that the two sloped maps in this diagram are homomorphisms, whereas Ξ_1 is probably not a homomorphism, but it is bijective, because of Theorem 4.11.

Note that the action of $\pi_1 \operatorname{Hom}^{\Delta}_C(B',A)$ on $\pi_0 \operatorname{Hom}^{\Delta}_{B'}(B,A)$ defined via ρ is equal to the monodromy action indicated in (20). To see this, let $h \in \pi_1 \operatorname{Hom}^{\Delta}_C(B',A)$ and $Q \in \pi_0 \operatorname{Hom}^{\Delta}_{B'}(B,A)$. We need to prove that $h * Q = \rho(h) * Q$. We may take Q as base point for $\pi_0 \operatorname{Hom}^{\Delta}_{B'}(B,A)$. Then we obtain the boundary map $\partial : \pi_1 \operatorname{Hom}^{\Delta}_C(B',A) \to \pi_0 \operatorname{Hom}^{\Delta}_{B'}(B,A)$ and we have $h * Q = \partial h$. Moreover, because Ξ_1 is bijective, we may assume that $h = P + \omega_1 D = P + dtD$, for $D \in h^{-1}\,\underline{\mathrm{Der}}_C(B',A)$. We have seen in the proof of Proposition 4.10 that then $\partial h(x) = Q(x) + \omega_0 D(dx) = Q(x) + D(dx)$. Thus we have $(h*Q)(x) = Q(x) + D(dx)$. On the other hand, by the commutativity of (22), we have $\rho(h) = D(dx)$. So by the definition of the action of $h^r(A)$ on $\pi_0 \operatorname{Hom}^{\Delta}_{B'}(B,A)$, we have $\big(\rho(h)*Q\big)(x) = Q(x) + \rho(h) = Q(x) + D(dx)$. Since $h*Q$ and $\rho(h)*Q$ agree on x, they are, indeed, equal.

By the exactness properties of (20), $h^r(A)$ acts transitively on the fiber of $\pi_0 \operatorname{Hom}^{\Delta}_C(B,A) \to \pi_0 \operatorname{Hom}^{\Delta}_C(B',A)$ and the stabilizer is equal to the image of ρ. Because of Diagram (22), and the bijectivity of Ξ_1, this image is equal to the image of ξ_P. $\square$

Corollary 4.14 *The vector space* $\operatorname{cok}\xi_P$ *acts simply transitively on the fiber of* $\pi_0 \operatorname{Hom}^{\Delta}_C(B,A) \to \pi_0 \operatorname{Hom}^{\Delta}_C(B',A)$ *over* P.

Both in the proposition and the corollary, there is no reason why the fiber in question should be non-empty. We will address the question of when this fiber is non-empty next.

Let us fix $C \to B' \to B$ and A as above, but let us forget about P.

Proposition 4.15 *Define a map*

$$\begin{aligned} \pi_0 \operatorname{Hom}^{\vartriangle}_C(B', A) &\longrightarrow h^{r+1}(A) \\ h &\longmapsto h(dx)\,. \end{aligned}$$

Then the sequence

$$\pi_0 \operatorname{Hom}^{\vartriangle}_C(B, A) \longrightarrow \pi_0 \operatorname{Hom}^{\vartriangle}_C(B', A) \longrightarrow h^{r+1}(A)$$

is exact in the middle.

PROOF. Let $h : B' \to A$ represent an element of $\pi_0 \operatorname{Hom}^{\vartriangle}_C(B', A)$. It lifts to $\pi_0 \operatorname{Hom}^{\vartriangle}_C(B, A)$ if and only if there exists a morphism of differential graded algebras $h' : B \to A$, such that $h'\,|_{B'} = h$, because of the fact that $\operatorname{Hom}^{\vartriangle}_C(B, A) \to \operatorname{Hom}^{\vartriangle}_C(B', A)$ is a fibration.

Suppose an extension h' exists. Then we have $h(dx) = h'(dx) = d\big(h'(x)\big)$, so that $h(dx) = 0$ in $h^{r+1}(A)$.

Conversely, assume that $h(dx) = da$, for some $a \in A^r$. Then we may define an extension h' of h to B to be the unique extension of h as a morphism of graded k-algebras satisfying $h'(x) = a$. This respects the differential, because $d\big(h'(x)\big) = da = h(dx) = h'(dx)$. □

Corollary 4.16 *The map*

$$\pi_0 \operatorname{Hom}^{\vartriangle}_C(B, A) \longrightarrow \ker\big(\pi_0 \operatorname{Hom}^{\vartriangle}_C(B', A) \to h^{r+1}(A)\big)$$

is surjective. For all P in the kernel, the fiber over P is a principal homogeneous $\operatorname{cok}\xi_P$-space.

Remark We may also summarize the results of Propositions 4.13 and 4.15 in the exact sequence

$$\begin{aligned} &h^{-1}\,\underline{\operatorname{Der}}_C(B', A) \longrightarrow h^r(A) \longrightarrow\!\circ \\ &\qquad \longrightarrow\!\circ\, \pi_0 \operatorname{Hom}^{\vartriangle}_C(B, A) \longrightarrow \pi_0 \operatorname{Hom}^{\vartriangle}_C(B', A) \longrightarrow h^{r+1}(A)\,. \end{aligned}$$

This means that there is an action of $h^r(A)$ on $\pi_0 \operatorname{Hom}^{\vartriangle}_C(B, A)$, such that the orbits are equal to the fibers of the fibration $\pi_0 \operatorname{Hom}^{\vartriangle}_C(B, A) \to \pi_0 \operatorname{Hom}^{\vartriangle}_C(B', A)$. The stabilizers depend only on the orbit. For the orbit which is equal to the fiber over $P : B' \to A$, the stabilizer is equal to the image of $h^{-1}\,\underline{\operatorname{Der}}_C(B', A)$ (whose definition depends on P) in $h^r(A)$. Finally, a point of $\pi_0 \operatorname{Hom}^{\vartriangle}_C(B', A)$ has a non-empty fiber over it, if and only if it maps to zero in $h^{r+1}(A)$.

4.4 Applications to étale morphisms

Let $C \to B$ be a morphism of quasi-finite resolving algebras and $B \to k$ an augmentation. The augmentation provides is with a canonical base point for $\operatorname{Hom}^{\vartriangle}(B, A)$ and $\operatorname{Hom}^{\vartriangle}(C, A)$, for all differential graded algebras A.

Proposition 4.17 *Let $r > 0$ be an integer. If B and C are perfect resolving algebras then the following are equivalent:*

(i) $C \to B$ is étale at the augmentation $B \to k$,

(ii) the map of pointed spaces $\operatorname{Hom}^{\vartriangle}(B, A) \to \operatorname{Hom}^{\vartriangle}(C, A)$ induces an isomorphism on homotopy groups π_ℓ, for all $\ell \geq 1$, and all (finite) resolving algebras A,

(iii) $\pi_r \operatorname{Hom}^{\vartriangle}(B, A) \to \pi_r \operatorname{Hom}^{\vartriangle}(C, A)$ an isomorphism, for all (finite) resolving algebras A.

If, on the other hand, $C \to B$ is a perfect resolving morphism, then $\operatorname{Hom}^{\vartriangle}(B, A) \to \operatorname{Hom}^{\vartriangle}(C, A)$ is a fibration, for every (finite) resolving algebra A, and the following are equivalent:

(i) $C \to B$ is étale at the augmentation $B \to k$,

(ii) the fiber $\operatorname{Hom}^{\vartriangle}_C(B, A)$ is acyclic for all (finite) resolving algebras A,

(iii) $\pi_r \operatorname{Hom}^{\vartriangle}_C(B, A) = 0$, for all (finite) resolving algebras A.

Proof. Simply combine Corollary 4.12 with Proposition 1.36. □

Let us now forget the augmentation $B \to k$.

Proposition 4.18 *Let $r > 0$ be an integer. If B and C are perfect resolving algebras then the following are equivalent:*

(i) $C \to B$ is étale,

(ii) $\operatorname{Hom}^{\vartriangle}(B, A) \to \operatorname{Hom}^{\vartriangle}(C, A)$ induces isomorphisms on homotopy groups π_ℓ, for all $\ell \geq 1$, and all (finite) resolving algebras A and all base points $B \to A$ for $\operatorname{Hom}^{\vartriangle}(B, A)$,

(iii) $\pi_r \operatorname{Hom}^{\vartriangle}(B, A) \to \pi_r \operatorname{Hom}^{\vartriangle}(C, A)$ an isomorphism, for all $B \to A$ as in (ii).

If, on the other hand, $C \to B$ is a perfect resolving morphism, then the following are equivalent:

(i) $C \to B$ is étale,

(ii) the fiber $\operatorname{Hom}^{\vartriangle}_C(B, A)$ is acyclic for all (finite) resolving algebras A and all base points $B \to A$,

(iii) $\pi_r \operatorname{Hom}^{\vartriangle}_C(B, A) = 0$, for all $B \to A$ as in (ii).

Proof. This time, combine Corollary 4.12 with Proposition 1.37. □

5 Finite resolutions

Our goal in this section is to prove that any morphism of finite resolving algebras admits a finite resolution. This is a significant strengthening of Proposition 1.9 in the finite case. As an application, we can prove that the derived tensor product of finite resolving algebras may be represented by a finite resolving algebra.

The existence of resolutions is reduced to the existence of 'cylinder objects' by a formal (and standard) argument. A cylinder object is nothing other than a resolution of the diagonal $A \otimes A \to A$.

Let A be a finite resolving algebra. Let $(x_i)_{i=1,\ldots,n}$ be a basis for A with the property that $i \leq j$ implies $\deg x_i \geq \deg x_j$. For all $i = 0, \ldots, n$ let $A_{(i)}$ denote the subalgebra of A generated by $x_1, \ldots, x_i$. Because of our assumption on the degrees of the x_i, we have that $A_{(i)}$ is a differential graded subalgebra of A, which is itself a finite resolving algebra with basis $(x_1, \ldots, x_i)$. Notation: $dx_i = f_i(x)$. Note that $f_i(x) = f_i(x_1, \ldots, x_{i-1}) \in A_{(i-1)}$.

The differential graded algebra $A \otimes A$ is a finite resolving algebra with basis consisting of $y_i = x_i \otimes 1$ and $z_i = 1 \otimes x_i$.

Let ξ_i, for $i = 1, \ldots, n$, be a formal variable of degree $\deg \xi_i = \deg x_i - 1$, and let us consider the symmetric algebra $k[\xi]$ on the graded vector space with basis (ξ_i). We also consider the graded algebra $A \otimes A \otimes k[\xi] = A \otimes A[\xi]$, with its subalgebra $A \otimes A$.

Proposition 5.1 *There is a way to extend the differential from the subalgebra $A \otimes A$ to all of $A \otimes A[\xi]$, and to extend the diagonal morphism $\Delta : A \otimes A \to A$ to all of $A \otimes A[\xi]$ in such a way that $A \otimes A[\xi]$ becomes a differential graded algebra and $A \otimes A[\xi] \to A$ a quasi-isomorphism of differential graded algebras.*

In other words, $A \otimes A[\xi]$ will be a finite resolution of the diagonal $A \otimes A \to A$.

Proof. The proof is by induction on n, the case $n = 0$ serving as trivial base case. We assume that the proposition has been proved in the case of n generators, and we wish to prove that it is also true for the case of $n+1$ generators. Thus $d\xi_i$ and $\Delta(\xi_i)$, for $i = 1, \ldots, n$, have already been found, making $\Delta : A_{(n)} \otimes A_{(n)}[\xi_1, \ldots, \xi_n] \to A_{(n)}$ a quasi-isomorphism. We have $A = A_{(n+1)}$ and we need to find suitable values for $d\xi_{n+1}$ and $\Delta(\xi_{n+1})$. Let $r = \deg x_{n+1}$.

Claim. There exist $h(y, z, \xi) \in A_{(n)} \otimes A_{(n)}[\xi_1, \ldots, \xi_n]$ of degree r and $g(x) \in A_{(n)}^{r-1}$ such that

(i) $dh(y, z, \xi) = f_{n+1}(y) - f_{n+1}(z)$,

(ii) $\Delta(h) = h(x, x, \Delta\xi) = dg(x)$.

To prove that claim, we start by observing that $f_{n+1}(x)$ is a cocycle, because it is equal to dx_{n+1} and $d^2 = 0$. Hence $f_{n+1}(y) - f_{n+1}(z)$ is a cocycle, too. This cocycle maps to 0 under Δ, and so by the injectivity of the induced map

$$h^{r+1}\big(A_{(n)} \otimes A_{(n)}[\xi_1, \ldots, \xi_n]\big) \to h^{r+1}(A_{(n)}),$$

we can find $g(y,z,\xi) \in A_{(n)} \otimes A_{(n)}[\xi_1,\ldots,\xi_n]$ of degree r, such that

$$dg(y,z,\xi) = f_{n+1}(y) - f_{n+1}(z)\,.$$

Now we observe that $g(x,x,\Delta\xi) \in A$ is a cocycle, and so by the surjectivity of

$$h^r\big(A_{(n)} \otimes A_{(n)}[\xi_1,\ldots,\xi_n]\big) \to h^r(A_{(n)})\,,$$

there exists a cocycle $\widetilde{g}(y,z,\xi) \in A_{(n)} \otimes A_{(n)}[\xi_1,\ldots,\xi_n]$, such that

$$\widetilde{g}(x,x,\Delta\xi) = g(x,x,\Delta\xi) \in h^r(A_{(n)})\,.$$

Then $h = g - \widetilde{g}$ satisfies Condition (i) of the claim. Moreover, $\Delta(h)$ is a coboundary, so that we may choose $g(x) \in A_{(n)}$ such that Condition (ii) of the claim is also true.

Now we set

$$d\xi_{n+1} = z_{n+1} - y_{n+1} + h(y,z,\xi)\,.$$

This turns $A \otimes A[\xi]$ into a differential graded algebra, because $z_{n+1} - y_{n+1} + h$ is a cocycle:

$$d\big(z_{n+1} - y_{n+1} + h(y,z,\xi)\big) = f_{n+1}(z) - f_{n+1}(y) + f_{n+1}(y) - f_{n+1}(z) = 0\,.$$

Now we define

$$\Delta(\xi_{n+1}) = g(x)\,.$$

This defines a morphism of differential graded algebras $\Delta : A \otimes A[\xi] \to A$, because

$$\Delta(d\xi_{n+1}) = \Delta(z_{n+1} - y_{n+1} + h) = \Delta(h) = dg(x) = d(\Delta\xi_{n+1})\,.$$

We claim that Δ is a quasi-isomorphism. We will prove this using the criterion of Corollary 2.9.

Let us first check that $h^0(\Delta)$ is an isomorphism. This is trivial if $r \leq -2$, because in that case h^0 is not affected when passing from n to $n+1$. Let us consider the case $r = 0$. Then $h^0(A_{(n)}) = A_{(n)}$ is a polynomial ring in $x_1,\ldots,x_n$ and $h^0(A_{(n+1)}) = A_{(n+1)}$ is a polynomial ring in $x_1,\ldots,x_{n+1}$. On the other hand, $h^0(A_{(n)} \otimes A_{(n)}[\xi_1,\ldots,\xi_n])$ is a quotient of the polynomial ring $k[y_1,z_1,\ldots,y_n,z_n]$ by n relations $h_1(y,z),\ldots,h_n(y,z)$ and by the induction hypothesis we know that $y_i \mapsto x_i$ and $z_i \mapsto x_i$ induces an isomorphism

$$k[y_1,z_1,\ldots,y_n,z_n]/(h_1,\ldots,h_n) \longrightarrow k[x_1,\ldots,x_n]\,.$$

This means that as ideals we have $(h_1,\ldots,h_n) = (z_1 - y_1,\ldots,z_n - y_n)$. Now when adjoining y_{n+1}, z_{n+1} and ξ_{n+1} to $A_{(n)} \otimes A_{(n)}[\xi_1,\ldots,\xi_n]$, we see that

$$h^0(A \otimes A[\xi]) = k[y_1,z_1,\ldots,y_{n+1},z_{n+1}]/(z_1 - y_1,\ldots,z_n - y_n, z_{n+1} - y_{n+1} + h)\,,$$

where $h(y_1,z_1,\ldots,y_n,z_n)$ satisfies $h(x_1,x_1,\ldots,x_n,x_n) = 0$, and hence $h(y_1,z_1,\ldots,y_n,z_n) \in (z_1 - y_1,\ldots,z_n - y_n)$. Therefore we have

$$h^0(A \otimes A[\xi]) = k[y_1,z_1,\ldots,y_{n+1},z_{n+1}]/(z_1 - y_1,\ldots,z_n - y_n, z_{n+1} - y_{n+1} + h)$$

$$
\begin{aligned}
&= k[y_1, z_1, \ldots, y_{n+1}, z_{n+1}]/(z_1 - y_1, \ldots, z_{n+1} - y_{n+1}) \\
&= k[x_1, \ldots, x_n] \\
&= h^0(A) .
\end{aligned}
$$

In the case $r = -1$ we have that

$$
h^0(A) = k[x_1, \ldots, x_m]/\big(f_{m+1}(x), \ldots, f_{n+1}(x)\big) , \tag{23}
$$

where $m \leq n$ is the total number of degree 0 generators among the x_i. On the other hand, by the above results in the $r = 0$ case, we have

$$
\begin{aligned}
h^0(A \otimes A[\xi]) = k[y_1, z_1, \ldots, y_m, z_m]/ \\
/\big(y_1 - z_1, \ldots, y_m - z_m, f_{m+1}(y), f_{m+1}(z), \ldots, f_{n+1}(y), f_{n+1}(z)\big) ,
\end{aligned}
$$

which is clearly equal to (23).

Finally, we need to check that the cotangent complex of Δ is acyclic. Let us abbreviate $B = A \otimes A[\xi]$ and $B_{(n)} = A_{(n)} \otimes A_{(n)}[\xi_1, \ldots, x_n]$.

Note that $\Omega_{B/B_{(n)}} \otimes_B h^0(B)$ is freely generated over $h^0(B)$ by $\mathfrak{d} y_{n+1}$, $\mathfrak{d} z_{n+1}$ in degree r and $\mathfrak{d}\xi_{n+1}$ in degree $r-1$. Moreover,

$$
d(\mathfrak{d}\xi_{n+1}) = \mathfrak{d}(d\xi_{n+1}) = \mathfrak{d} z_{n+1} - \mathfrak{d} y_{n+1} + \mathfrak{d} h(y, z, \xi) = \mathfrak{d} z_{n+1} - \mathfrak{d} y_{n+1} ,
$$

because $h(y, z, \xi) \in B_{(n)}$.

On the other hand, $\Omega_{A/A_{(n)}} \otimes_A h^0(A)$ is freely generated over $h^0(A) = h^0(B)$ in degree r by $\mathfrak{d} x_{n+1}$.

The map $\mathfrak{d}\Delta$ maps $\mathfrak{d} y_{n+1}$ and $\mathfrak{d} z_{n+1}$ to $\mathfrak{d} x_{n+1}$ and it maps $\mathfrak{d}\xi_{n+1}$ to $\mathfrak{d} g(x) = 0$, because $g(x) = A_{(n)}$.

Thus it is clear that $\mathfrak{d}\Delta : \Omega_{B/B_{(n)}} \otimes_B h^0(A) \to \Omega_{A/A_{(n)}} \otimes_A h^0(A)$ is a quasi-isomorphism, which is all that we needed to prove. □ check this last part again.

Corollary 5.2 *The two morphisms $A \to A \otimes A[\xi]$ given by $a \mapsto a \otimes 1$ and $a \mapsto 1 \otimes a$ are homotopic. (They are also quasi-isomorphisms.)*

Proof. Both of these morphisms are 2-inverses of the quasi-isomorphism $A \otimes A[\xi] \to A$, constructed in the proposition. Use Corollary 1.21. □

Theorem 5.3 *Let $A \to B$ be a morphism of finite resolving algebras. Then there exists a resolution $A \to B' \to B$, where $A \to B'$ is a finite resolving morphism.*

Proof. See the proof of Corollary 8.3 in Section II of [5]. In our case, the resolution of $A \to B$ is given by

$$
\begin{aligned}
&A \longrightarrow B \otimes A[\xi] \stackrel{B\otimes\sigma}{\longrightarrow} B \\
&a \longmapsto 1 \otimes a
\end{aligned}
$$

where $\sigma : A \otimes A[\xi] \to A$ is any resolution of the diagonal. □

Corollary 5.4 *Let $f : A \to B$ and $g : A \to C$ be morphisms of finite resolving algebras. Then there exists a finite resolving algebra R and a homotopy commutative square*

$$\begin{array}{ccc} R & \longleftarrow & C \\ \uparrow & \nearrow\!\!\!\nearrow & \uparrow{\scriptstyle g} \\ B & \underset{f}{\longleftarrow} & A \end{array} \tag{24}$$

such that, whenever we resolve f as $A \to B' \to B$, then there exists a factorization of (24) as

$$\begin{array}{ccccc} & & \curvearrowleft & & \\ R & \longleftarrow & B' \otimes_A C & \longleftarrow & C \\ \uparrow & \nearrow\!\!\!\nearrow & \uparrow & & \uparrow \\ B & \longleftarrow & B' & \longleftarrow & A \end{array}$$

where $B' \otimes_A C \to R$ is a quasi-isomorphism.

Proof. This is a formal consequence of Theorem 5.3. More explicitly, we can construct R as follows: let A be as above, generated by $x_1, \ldots, x_n$, with $dx_i = f_i(x)$. Choose for every i an element $h_i(y, z, \xi) \in A \otimes A[\xi]$ as in the inductive proof of Proposition 5.1. In particular, $dh_i(y, z, \xi) = f_i(y) - f_i(z)$. Denote the images of the x_i in B by b_i and in C by c_i. Consider the differential graded algebra

$$R = B \otimes C[\xi]$$

with differential defined by $d\xi_i = c_i - b_i + h_i(b, c, \xi)$. Any homotopy between the two morphisms $A \to A \otimes A[\xi]$ (see Corollary 5.2), composed with $A \otimes A[\xi] \to B \otimes C[\xi]$ gives rise to a homotopy commutative diagram

$$\begin{array}{ccc} B \otimes C[\xi] & \longleftarrow & C \\ \uparrow & \nearrow\!\!\!\nearrow & \uparrow{\scriptstyle g} \\ B & \underset{f}{\longleftarrow} & A \end{array}$$

as required. □

Scholum 5.5 *If a, b and c denote the numbers of elements of finite bases for A, B and C, respectively, then there exists a basis for R with $a+b+c$ elements.*

Remark 5.6 The results of this section also apply to perfect resolving algebras and perfect resolutions, in place of finite resolving algebras and finite resolutions. The proofs are easier and require only the use of Lemma 1.33 (iii).

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