Some Latin square theorems

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Definition

A latin square of order n is an $n \times n$ array L with entries chosen from $\{1, 2, \ldots, n\}$ with the property, for each $i \in \{1, 2, \ldots, n\}$, that each row contains symbol i and each column contains symbol i. Alternatively symbol i appears in a set of cells in the array L which form a permutation.

Note that a sudoku problem is a partial latin square and the solution to a sudoku puzzle is a 9×9 latin square with some additional constraints.

Completions

We say an $n \times n$ matrix A with some entries specified and the rest blank is a partial latin square if the specified entries $\in \{1, 2, \ldots, n\}$ and no symbol appears twice in any row and no symbol appears twice in any column.

Given an $n \times n$ partial latin square A, one can ask if there is a latin square L where L and A agree on the specified entries. In this case L is called a completion of A.

There are a variety of results including results that promise the existence of completions or indeed unique completions. You can think of a sudoku solution as the unique completion to a given partial latin square when the completion is subject to more conditions that just being a latin square.

Example

$$A = \begin{array}{|c|c|c|c|c|} \hline 1 & & & & \\ \hline & 2 & & & \\ \hline & & 2 & & \\ \hline & & & 2 & \\ \hline & & & & 2 \\ \hline \end{array}$$

This partial latin square A cannot be completed. Moreover if we delete any entry it can be completed. Such a partial latin square is called premature.

Evan's Conjecture

Theorem (Evans, 1960, Smetaniuk 1981) Every partial latin square of order n, with only $n-1$ entries specified, can be completed to a latin square of order n.

Ryser's Theorem

Theorem (Ryser 1951) An $r \times s$ latin rectangle can be completed to a latin square of order n if and only if each symbol in $\{1, 2, \ldots, n\}$ occurs at least $r + s - n$ times.

The proof used ideas of network flows/systems of distinct representatives.

Sudoku

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Sudoku

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Sudoku

Theorem (McGuire et al 2012) The fewest number of filled cells for a sudoku puzzle (namely uniquely completable) is 17.

Transversals

A transversal of a latin square $L = (a_{i,j})$ of order n is a set of n cells $P = (1, \sigma(1)), (2, \sigma(2)), \ldots, (n, \sigma(n))$ where σ is a permutation (σ is a bijection $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$)) such that each entry in $\{1, 2, \ldots, n\}$ appears exactly once in the cells P.

A partial transversal is a transversal with some of the entries missing. The cells P would have fewer than n filled cells and each symbol in $\{1, 2, \ldots, n\}$ appears at most once in cells P.

Conjecture (Ryser) Every odd order Latin square has a transversal and every even order latin square of order n has a partial transversal of at least $n - 1$ cells.

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Theorem (Hatami, Shor 2008) Every Latin square of order n has a partial transversal of at least $n - 11.053(\log(n))^2$ cells.

'*The previous papers proving these results (including one by the second author) not only contained an error, but were sloppily written and quite difficult to understand. We have corrected the error and improved the clarity.*' !

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
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5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
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3	4	5	2	8	6	1	7	9

5	3	4	6	7	8		1	2	9
6	7	2	1	9	5	3	4	8	
1	9	8	3	4	2	5	6	7	
8	5	9	7	6	1	4	2	3	
4	2	6	8	5	3	7	9	1	
7	1	3	9	2	4	8	5	6	
9	6	1	5	3	7	2	8	4	
2	8	7	4	1	9	6	3	5	
3	4	5	2	8	6	1	7	9	
						9			

5	3	4	6	7	8		1	2	9
6	7	2	1	9	5	3		8	4
1	9	8	3	4	2	5	6	7	
8	5	9	7	6	1	4	2	3	
4	2	6	8	5	3	7	9	1	
7	1	3	9	2	4	8	5	6	
9	6	1	5	3	7	2	8	4	
2	8	7	4	1	9	6	3	5	
3	4	5	2	8	6	1	7	9	
						9	4		

5	3	4	6	7	8		1	2	9
6	7	2	1	9	5	3		8	4
1	9	8	3	4	2	5	6		7
	5	9	7	6	1	4	2	3	8
4		6	8	5	3	7	9	1	2
7	1		9	2	4	8	5	6	3
9	6	1		3	7	2	8	4	5
2	8	7	4		9	6	3	5	1
3	4	5	2	8		1	7	9	6
8	2	3	5	1	6	9	4	7	

5	3	4	6	7	8		1	2	9
6	7	2	1	9	5	3		8	4
1	9	8	3	4	2	5	6		7
	5	9	7	6	1	4	2	3	8
4		6	8	5	3	7	9	1	2
7	1		9	2	4	8	5	6	3
9	6	1		3	7	2	8	4	5
2	8	7	4		9	6	3	5	1
3	4	5	2	8		1	7	9	6
8	2	3	5	1	6	9	4	7	

5	3	4	6	7	8	0	1	2	9
6	7	2	1	9	5	3	0	8	4
1	9	8	3	4	2	5	6	0	7
0	5	9	7	6	1	4	2	3	8
4	0	6	8	5	3	7	9	1	2
7	1	0	9	2	4	8	5	6	3
9	6	1	0	3	7	2	8	4	5
2	8	7	4	0	9	6	3	5	1
3	4	5	2	8	0	1	7	9	6
8	2	3	5	1	6	9	4	7	0

5	3	4	6	7	8	0	1	2	9
6	7	2	1	9	5	3	0	8	4
1	9	8	3	4	2	5	6	0	7
0	5	9	7	6	1	4	2	3	8
4	0	6	8	5	3	7	9	1	2
7	1	0	9	2	4	8	5	6	3
9	6	1	0	3	7	2	8	4	5
2	8	7	4	0	9	6	3	5	1
3	4	5	2	8	0	1	7	9	6
8	2	3	5	1	6	9	4	7	0

Transversals

$$\begin{bmatrix} 1 & 2 & \boxed{3} & 4 \\ 3 & 4 & 1 & \boxed{2} \\ \boxed{4} & 3 & 2 & 1 \\ 2 & \boxed{1} & 4 & 3 \end{bmatrix} \text{ has transversal but } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix} \text{ has no transversal}$$

There is an old conjecture of my Ph.D. supervisor Herb Ryser that every odd order latin square has a transversal. This has not been solved!

Problem: Show that the following even order latin square does not have a transversal.

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & & n-1 & n \\ n & 1 & 2 & 3 & \cdots & & n-1 \\ n-1 & n & 1 & 2 & 3 & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ 3 & \cdots & & n-1 & n & 1 & 2 \\ 2 & 3 & \cdots & & n-1 & n & 1 \end{bmatrix}$$

The cells in the array can be labeled (row,col) where the top left entry is in cell $(1,1)$. One idea is to note that any entry 2 is in cell $(i, i+1)$ where when $i = n$, then we reduce $n+1$ mod n from $n+1$ to 1. Thus the 2's are in the cells $(1,2)$ $(2,3)$ $(3,4)$ ... $(n-1, n)$ $(n, 1)$. Similarly the ℓ's are in the cells $(i, i+\ell-1)$ where we reduce mod n.

E.g. 4's are in cells $(1,4)$ $(2,5)$ $(3,6)$ ... $(n-3, n)$, $(n-2, 1)$, $(n-1, 2)$, $(n, 3)$.

Solution

k's are in cells $(a_k, a_k + k - 1)$, indices modulo n
Fact: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.
Assume there is a transversal with k in cell $(a_k, a_k + k - 1)$.

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Assume there is a transversal with k in cell $(a_k, a_k + k - 1)$.

Because it is a transversal:

$$\sum_k a_k = 1 + 2 + \cdots + n \equiv \frac{n(n+1)}{2} \pmod{n} \text{(each row present)}$$

and $$\sum_k a_k + k - 1 = 1 + 2 + \cdots + n \equiv \frac{n(n+1)}{2} \pmod{n} \text{(each col present)}$$

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Because it is a transversal:

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and $$\sum_k a_k + k - 1 = 1 + 2 + \cdots + n \equiv \frac{n(n+1)}{2} (\text{mod } n) \text{(each col present)}$$

We check

$$\sum_k k - 1 = 0 + 1 + \cdots + n - 1 = \frac{n(n-1)}{2} \equiv -\frac{n}{2} (\text{mod } n) \not\equiv 0 (\text{mod } n)$$

This is a contradiction, so we cannot find the a_k's. ■

Thank you for listening!