Rectangles with one integer side

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Introduction

There is a marvellous result about a decomposition of a rectangle into a (finite) number of rectangles. Some call this a tiling. I first heard this in a lecture of Stan Wagon. A paper by Stan Wagon explores some proofs of the result . It is titled 'Fourteen Proofs of a Result about Tiling a Rectangle'. Some use complex integration! I remember this proof some decades later (I occasionaly forget the details).

Theorem 1. Consider a rectangle R with height a and width b. Assume there is a decomposition of R into smaller rectangles for which each smaller rectangle has at least one dimension (height or width) being an integer. Then at least one of a, b is an integer. ■

This theorem seems quite peculiar. It takes a while to see that it is saying something quite interesting. It is not ‘obvious’. Doesn’t really help to draw pictures but you can try.

Assume the (big) rectangle is placed in the integral grid where the rectangle is oriented so the height and width are parallel to the two axes. This will be inherited by the smaller rectangles.
Now imagine the integral grid divided up on the lines $x = i + 1/2$ for all $i \in \mathbf{Z}$ and $y = j + 1/2$ for all $j \in \mathbf{Z}$. Now apply the checkerboard colouring to this grid so that the $1/2 \times 1/2$ squares are coloured either black or white and the square next to origin in the first quadrant is coloured white. For each rectangle (whose sides are parallel to the axes) we can easily compute the area of white and black in the rectangle.

Lemma 2. Assume we have a rectangle R whose sides are parallel to the axes and at least one of the height or the width is an integer. Then the area of white and the area of black in the rectangle are equal.

Proof: Assume the height is an integer. Let us consider a small rectangle R' of the same height as R but which lies between two vertical lines $x = i + 1/2$ for $x = i, i + 1$ for all $i \in \mathbf{Z}$. Then the regions of R' cut by the grid lines $y = j$ for $j \in \mathbf{Z}$ are alternately black and white of sizes $b_1, w_1, b_2, w_2, \ldots, b_k, w_k$. Assume $b_1 \neq 0$ (if not, flip black and white). Assume $b_k \neq 0$ (we are allowing $w_k = 0$ or $w_k \neq 0$). Then $w_1 = b_2 = w_2 = \cdots = w_{k-1} = 1/2$ because of alternating stripes. Since the height is an integer, we have $b_1 + b_k = 1/2$ and $w_k = 0$. Then the black heights are equal to the white heights ($b_1 + b_2 + \cdots + b_k = w_1 + w_2 + \cdots + w_k$) and so the area of white and the area of black in the strip are equal. Our original rectangle can be split it a number of such strips and hence the area of white and the area of black in the entire rectangle are equal.

By transposing the axes (or rotating by 90°), the result follows

Lemma 3 Assume we have a rectangle whose sides are parallel to the axes and whose bottom left corner is placed at the origin If the area of white and the area of black in the rectangle are equal then at least one of the height or the width is an integer.

Proof: Note the need to place the rectangle at the origin. The idea is we can reduce to a rectangle of dimensions $a \times b$ where $a, b < 1$. Say the rectangle is of size $c \times d$ where $c = a + k$ and $d = b + \ell$ where k, ℓ are integers. Then by chopping off the $k \times d$ rectangle (which has an equal amount of black and white since k is an integer) and then chopping of the $a \times \ell$ rectangle (which has an equal amount of black and white since ℓ is an integer) we are left with an $a \times b$ rectangle which has an equal amount of black and white and $a, b < 1$.

We are done if $a = b = 0$. Now let $a = a' + a''$ and $b = b' + b''$ where $a'' = \max\{a' - 1/2, 0\}$ and $b'' = \max\{b' - 1/2, 0\}$. Note that $a' > a''$ and $b' > b''$. This splits the $a \times b$ rectangle into 4 parts, some of which may be empty, where the amount of white is $a' \cdot b' + a'' \cdot b''$ and the amount of black is $a' \cdot b'' + a'' \cdot b'$. Given that $a' > a''$ and $b' > b''$, we readily (I might lose marks for this) conclude that the amount of white exceeds the amount of black, a contradiction. Let us be more careful so that I don't lose marks. We do have $a' \cdot b' > a' \cdot b''$. We are done if $a'' \cdot b'' \geq a'' \cdot b'$ and so may assume the contrary $a'' \cdot b'' < a'' \cdot b'$ so that $a'' > 0$ hence $a' = 1/2$. We can also assume $b'' > 0$ and $b' = 1/2$, since if $b'' = 0$ then the amount of white is $a' \cdot b' + a'' \cdot b'' = a' \cdot b'$ and the amount of black is $a' \cdot b'' + a'' \cdot b' = a'' \cdot b' < a' \cdot b'$. Then by $a < 1$ and $a'' < a'$ (from the definition of a'') , we have more white than black, a contradiction. ■

Now we can prove the main result:

Proof of Theorem 1. We start by considering the rectangle R with bottom left placed at origin and the plane coloured black/white as described above. Then since each subrectangle has either an integer width or integer height, then by Lemma 2 each such rectangle has an equal amount of white and black. Thus the large rectangle R has an equal amount of white and black. We now use Lemma 3 to deduce that one side of R has an integer side. This finishes the proof. ■

Note that there is some care in Lemma 3 required to place rectangle at the origin. Still it is a relatively easy proof to remember.

Thanks for listening.