Catalan Numbers and Recurrences

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Some notation

We can obtain an integer sequence $a_1, a_2, \ldots, a_n, \ldots$ from a recurrence which gives later terms in terms of previous terms and we begin with certain initial values explicitly. The following makes this more precise.

Proposition. Assume for each $n > k$, $a_n = f(a_1, a_2, \ldots, a_{n-1})$. Assume $a_1, a_2, \ldots, a_k$ are given. Then this uniquely determines a_n for all $n > 0$.

Our recurrences are usually given in terms of the previous k values, namely $a_n = f(a_{n-k}, a_{n-k+1}, \ldots, a_{n-1})$.

Examples

Fibonacci Numbers

They are determined by the recurrence

$$f_n = f_{n-1} + f_{n-2} \qquad\qquad f_1 = f_2 = 1$$

There is an explicit formula for f_n

$$f_n = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Examples

Catalan Numbers

They are detemined by the recurrence

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} \qquad\qquad C_0 = 1$$

There is an explicit formula for C_n

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

We can use recurrences in a variety of ways. The typical way is to obtain a recurrence and then use various results (such as generating functions) to solve the recurrence. You can initially think of recurrences as a kind of induction. In fact, if you miraculously guessed the explicit formula for C_n, you could prove the explicit formula for C_n from the recurrence for C_n using induction on n, but it looks to be a bit of a mess.

A nicer approach in the Combinatorial book by Richard Brualdi, is to consider some object for which both the recurrence and the formula can be seen to hold simultaneously. Then, once proved, you can use this in other circumstances namely when the recurrence holds you have Catalan numbers and when you have Catalan numbers, the recurrence holds. There are other ways to prove this.

Let $g(n)$ denote the number of bracketings of $a_1, a_2, \ldots, a_n$ in that order

$(a_1 \times a_2)$ $\qquad g(2) = 1 = C_1$

$(a_1 \times (a_2 \times a_3)), ((a_1 \times a_2) \times a_3)$ $\qquad g(3) = 2 = C_2$

$(a_1 \times ((a_2 \times a_3) \times a_4)), (a_1 \times (a_2 \times (a_3 \times a_4))),$
$((a_1 \times a_2) \times (a_3 \times a_4)), (((a_1 \times a_2) \times a_3) \times a_4),$
$(((a_1 \times (a_2 \times a_3)) \times a_4)$ $\qquad g(4) = 5 = C_3$

You can check $g(2) = 1$ and can set $g(1) = 1$ (just as we take $C_0 = 1$). These are the Catalan numbers shifted by 1: $g(n) = C_{n-1}$ which is a bit annoying. You can fix this in several ways but we will just proceed with $g(n)$ as given.

It is fairly easy to derive a recurrence for $g(n)$.
First $g(2) = 1$.
Second, consider any bracketing expression exp of $a_1, a_2, \ldots, a_n$ and consider the final multiply, namely

$$exp = (exp_1) \times (exp_2)$$

Each of exp_1, exp_2 will have at least one variable and if exp_1 has k variables, then exp_2 has the remaining $n - k$ variables. Moreover exp_1 will have variables $a_1, a_2, \ldots, a_k$ while exp_2 has $a_{k+1}, a_{k+2}, \ldots, a_n$. The number of choices for exp_1 is $g(k)$ and the number of choices for exp_2 is $g(n-k)$ which yields the recurrence

$$g(n) = \sum_{k=1}^{n-1} g(k)g(n-k); \qquad g(2) = 1$$

Thus $g(n+1) = C_n$ because they satify the same recurrence. Of course we still wish to verify the explicit formula.

It turns out to be easier to count the bracketings where we allow the variables to be in any order

Let $h(n)$ denote the number of bracketings of $a_1, a_2, \ldots, a_n$ (in any order)

Theorem $h(n) = n! \cdot g(n)$

Proof: We note that given a bracketing of $a_1, a_2, \ldots, a_n$ in that order, any of the n! permutations of $a_1, a_2, \ldots, a_n$ can be applied to the expression to obtain a bracketing of $a_1, a_2, \ldots, a_n$ (in any order). ■

Note that our formula above means that if we can compute $h(n)$ then we have computed $g(n)$. Using $g(n+1) = C_n$, we obtain the explicit formula for C_n.

It turns out to be easier to directly compute $h(n)$ using the following recurrence.

Lemma $h(n) = (4n-6)h(n-1)$ for $n \geq 3$. ■

This lemma is quite tricky and it is best to see in terms of computation trees. We can think of any bracketing for $h(n)$ and expression with $n-1$ multiplies sprinkled with the variables.

We can use our Lemma as follows:

$$h(n) = (4n-6)h(n-1) = (4n-6)(4n-10)h(n-2) = \cdots$$
$$\cdots = (4n-6)(4n-10)\cdots 6 \cdot h(2).$$

Thus $h(n) = 2^{n-1}(2n-3)(2n-5)\cdots 3 \cdot 1$ (using $h(2) = 2$)

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Fill in terms top and bottom

$$h(n) = 2^{n-1}\frac{(2n-2)(2n-3)(2n-4)\cdots 1}{(2n-2)(2n-4)\cdot\cdots\cdot 2}$$

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So $g(n) = (n-1)!\binom{2n-2}{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$

To obtain the recurrence for $h(n) = (4n-6)h(n-1)$ we use induction on n. We think of this as $h(n) = (4(n-2))h(n-1) + 2h(n-1)$. Consider an expression *exp* of variables $x_1, x_2, \ldots, x_{n-1}$. We obtain $2h(n-1)$ expresssions of the form $x_n \times exp$ and $exp \times x_n$ which yields the term $2h(n-1)$. All have x_n as one term in the 'last' multiply.
There are $n-2$ choices of '$\times$' in *exp*. For each choice of '$\times$' in *exp* (i.e. $exp = exp_1 \times exp_2$) we have can add x_n and one '$\times$' and obtain four expressions:

$((x_n \times exp_1) \times exp_2)$, $\quad ((exp_1 \times x_n) \times exp_2)$,
$(exp_1 \times (x_n \times exp_1))$, $\quad (exp_1 \times (exp_2 \times x_n))$.

This corresponds to the term $(4(n-2)h(n-1)$.
One also needs to show that this happily reverses removing x_n and one '$\times$' in the reverse of the moves above to obtain the $h(n-1)$ expressions.

Thank you for listening