Directed Towers of Hanoi

Richard Anstee,
UBC, Vancouver

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Introduction

The Tower of Hanoi puzzle is sometimes refered to as the Tower of Brahma puzzle. The puzzle originated with Édouard Lucas in 1883. The original Towers of Hanoi problem considers a problem 3 pegs and with n different sized discs that fit on the pegs. A legal move is to move the top disc from one peg to the top disc on another peg while preserving the property that at no point is a smaller disc under a larger disc. Imagine that we start with all the discs on peg 1. How many moves are required to move the n discs to peg 2? We also consider a cyclic variation of this problem where the pegs are 1,2,3 and we can only move discs from peg 1 to peg 2 or from peg 2 to peg 3 or from peg 3 to peg 1.

Tower of Hanoi, https://en.wikipedia.org/wiki/Tower_of_Hanoi, downloaded Jan 14, 2021

Towers of Hanoi

First consider the original problem. Let $h(n)$ denote the (minimum) number of moves required to move all n discs from one peg to another peg. Note that $h(1) = 1$.

Theorem For all $n \geq 2$, $h(n) = h(n-1) + 1 + h(n-1)$.

Proof: The induction hypothesis would be $H(n) : h(n) = h(n-1) + 1 + h(n-1)$. The base case could be $n = 1$ or perhaps you would prefer $n = 2$. Assume all the discs begin on peg 1. Consider the largest disc. To move the largest disc, all smaller discs must be move to one other peg, say peg 2, leaving one peg empty, in this case peg 3. Certainly this requires $h(n-1)$ moves. Then we move the largest disc with 1 move to peg 3. Then we must move all the smaller discs from peg 2 to peg 3 which will again require at least $h(n-1)$ moves.

We have shown $h(n) \geq h(n-1) + 1 + h(n-1)$. Equality is possible and since $h(n)$ is the minimum number of moves. ■

Using the recurrence with $h(0) = 0$ and $h(1) = 1$
$h(n) = h(n-1) + 1 + h(n-1)$ $h(1) = 1$ we compute
($h(0) = 0$ not needed but consistent)

$h(2) = 3$,

$h(3) = 7$,

$h(4) = 15$,

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We guess that $h(n) = 2^n - 1$

Theorem For all $n \geq 2$, $h(n) = 2^n - 1$.
Proof: The induction hypothesis would be $H(n) : h(n) = 2^n - 1$. The base case would be $n = 1$ (or if your prefer $n = 2$). We apply the recurrence from the previous result to obtain a proof by induction.
The recurrence yields $h(n) = 2h(n-1) + 1$
Substiute for $h(n-1)$ using induction to obtain:
$h(n) = 2(2^{n-1} - 1) + 1 = 2^n - 1$ ∎

Cyclic Towers of Hanoi

I was introduced to this at the Imagine Day orientation session in September 2018.
We have three pegs 1,2,3 and we have n discs, the ith disc of size i. The clockwise order of the pegs is $1 \to 2$, $2 \to 3$ and $3 \to 1$. A legal move is to move the top disc (smallest disc) on one peg onto the next peg in clockwise order where the move is legal namely the moved disc is placed on a larger disc.
WIth these rules, there is always a legal moves since if peg i has a top disc of size t_i then there is no legal move only of $t_1 > t_2$, $t_2 > t_3$ and $t_3 > t_1$ which is a contradiction.

If we try to define h(n) as for the Tower of Hanoi as above, we expect that we need two cases. moving n discs to next peg in clockwise order and moving n discs to the peg two over in clockwise order.
Let $f(n)$ denote the (minimum) number of moves required to move n discs from one peg to the next peg in clockwise order
Let $g(n)$ denote the (minimum) number of moves required to move n discs from one peg to the peg two moves over in clockwise order.

Let $f(n)$ denote the (minimum) number of moves required to move n discs from one peg to the next peg in clockwise order
Let $g(n)$ denote the (minimum) number of moves required to move n discs from one peg to the peg two moves over in clockwise order.
Theorem For all $n \geq 2$, $f(n) = g(n-1) + 1 + g(n-1)$ and $g(n) = g(n-1) + 1 + f(n-1) + 1 + g(n-1)$
Proof: Try formulating the proof as for Towers of Hanoi ∎

Let $f(n)$ denote the (minimum) number of moves required to move n discs from one peg to the next peg in clockwise order
Let $g(n)$ denote the (minimum) number of moves required to move n discs from one peg to the peg two moves over in clockwise order.
We have $f(1) = 1$ and $g(1) = 2$.
We can play with these recurrences:

$$g(n) = 2g(n-1) + f(n-1) + 2$$
$$= 2g(n-1) + 2g(n-2) + 3$$

We will also need $g(2) = 7$ which can be established using the first form of the recurrence $g(2) = 2g(1) + f(1) + 2$.
We compute the sequence $2, 7, 21, 59, 163, \ldots$ for $g(n)$.

We computed the sequence 2,7,21,59,163 for $g(n)$ and can enter it into Sloan's calculator (Online Encyclopedia of Integer Sequences at https://oeis.org) to obtain many references such as $g(n) = (1/(4\sqrt{3}))\left((1+\sqrt{3})^{n+2} - (1-\sqrt{3})^{n+2}\right) - 1$.

Thanks for your attention