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## CLP-1 Differential Calculus

# CLP-1 Differential Calculus 

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## PREFACE

This text is a merger of the CLP Differential Calculus textbook and problembook. It is, at the time that we write this, still a work in progress; some bits and pieces around the edges still need polish. Consequently we recommend to the student that they still consult the textbook webpage for links to the errata - especially if they think there might be a typo or error. We also request that you send us an email at clp@ugrad.math.ubc.ca

Additionally, if you are not a student at UBC and using these texts please send us an email - we'd love to hear from you.

Joel Feldman, Andrew Rechnitzer and Elyse Yeager

To our students.
And to the many generations of scholars who have freely shared all this knowledge with us.

## Acknowledgements

Elyse would like to thank her husband Seçkin Demirbaş for his endless patience, tireless support, and insightful feedback.

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- Rob Beezer and David Farmer for their help converting this book from ETEX to this online PreTeXt format.
- Nick Loewen for designing the cover art, help with figures, colours, spelling and many discussions.
- The many people who have collaborated over the last couple of decades making exams and tests for first year calculus courses at UBC Mathematics. A great many of the exercises in the text come from questions in those tests and exams.

Finally, we'd like to thank those students who reported typos and errors they found in the text. Many of these students did so through our "bug bounty" program which was supported by the Department of Mathematics, Skylight and the Loafe Cafe all at UBC.

## Using THE EXERCISES IN THIS BOOK

Each problem in this book is split into four parts: Question, Hint, Answer, and Solution. As you are working problems, resist the temptation to prematurely peek at the hint or to click through to the answers and solutions in the appendix! It's important to allow yourself to struggle for a time with the material. Even professional mathematicians don't always know right away how to solve a problem. The art is in gathering your thoughts and figuring out a strategy to use what you know to find out what you don't.

If you find yourself at a real impasse, go ahead and look at the linked hint. Think about it for a while, and don't be afraid to read back in the notes to look for a key idea that will help you proceed. If you still can't solve the problem, well, we included the Solutions section for a reason! As you're reading the solutions, try hard to understand why we took the steps we did, instead of memorizing step-by-step how to solve that one particular problem.

If you struggled with a question quite a lot, it's probably a good idea to return to it in a few days. That might have been enough time for you to internalize the necessary ideas, and you might find it easily conquerable. Pat yourself on the back - sometimes math makes you feel good! If you're still having troubles, read over the solution again, with an emphasis on understanding why each step makes sense.

One of the reasons so many students are required to study calculus is the hope that it will improve their problem-solving skills. In this class, you will learn lots of concepts, and be asked to apply them in a variety of situations. Often, this will involve answering one really big problem by breaking it up into manageable chunks, solving those chunks, then putting the pieces back together. When you see a particularly long question, remain calm and look for a way to break it into pieces you can handle.

## - Working with Friends:

Study buddies are fantastic! If you don't already have friends in your class, you can ask your neighbours in lecture to form a group. Often, a question that you might bang your head against for an hour can be easily cleared up by a friend who sees what you've missed. Regular study times make sure you don't procrastinate too much, and friends help you maintain a positive attitude when
you might otherwise succumb to frustration. Struggle in mathematics is desirable, but suffering is not.
When working in a group, make sure you try out problems on your own before coming together to discuss with others. Learning is a process, and getting answers to questions that you haven't considered on your own can rob you of the practice you need to master skills and concepts, and the tenacity you need to develop to become a competent problem-solver.

## - Types of Questions:

Questions outlined by a blue box make up the "RQS" representative question set. This set of questions is intended to cover the most essential ideas in each section. These questions are usually highly typical of what you'd see on an exam, although some of them are atypical but carry an important moral. If you find yourself unconfident with the idea behind one of these, it's probably a good idea to practice similar questions.
This representative question set is our suggestion for a minimal selection of questions to work on. You are highly encouraged to work on more.
In addition to original problems, this book contains problems pulled from quizzes and exams given at UBC for Math 100 and 180 (first-semester calculus) and Math 120 (honours first-semester calculus). These problems are marked by "(*)". The authors would like to acknowledge the contributions of the many people who collaborated to produce these exams over the years.
Finally, the questions are organized into three types: Stage 1, Stage 2 and Stage 3.

- Exercises - Stage 1

The first category is meant to test and improve your understanding of basic underlying concepts. These often do not involve much calculation. They range in difficulty from very basic reviews of definitions to questions that require you to be thoughtful about the concepts covered in the section.

- Exercises - Stage 2

Questions in this category are for practicing skills. It's not enough to understand the philosophical grounding of an idea: you have to be able to apply it in appropriate situations. This takes practice!

- Exercises - Stage 3

The last questions in each section go a little farther than "Exercises Stage 2". Often they will combine more than one idea, incorporate review material, or ask you to apply your understanding of a concept to a new situation.

In exams, as in life, you will encounter questions of varying difficulty. A good skill to practice is recognizing the level of difficulty a problem poses. Exams will have some easy questions, some standard questions, and some harder questions.

## Feedback about The TEXT

This combined edition of the CLP differential calculus text is still undergoing testing and changes. Because of this we request that if you find a problem or error in the text then:

1. Please check the errata list that can be found at the textbook webpage.
2. Is the problem in the online version or the PDF version or both?
3. Note the URL of the online version and the page number in the PDF
4. Send an email to clp@ugrad.math.ubc.ca. Please be sure to include

- a description of the error
- the URL of the page, if found in the online edition
- and if the problem also exists in the PDF, then the page number in the PDF and the compile date on the front page of PDF.


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## THE BASICS

We won't make this section of the text too long - all we really want to do here is to take a short memory-jogging excursion through little bits and pieces you should remember about sets and numbers. The material in this chapter will not be (directly) examined.

## 0.1 ^ Numbers

Before we do anything else, it is very important that we agree on the definitions and names of some important collections of numbers.

- Natural numbers - These are the "whole numbers" $1,2,3, \ldots$ that we learn first at about the same time as we learn the alphabet. We will denote this collection of numbers by the symbol " $\mathbb{N}$ ". The symbol $\mathbb{N}$ is written in a type of bold-face font that we call "black-board bold" (and is definitely not the same symbol as $N$ ). You should become used to writing a few letters in this way since it is typically used to denote collections of important numbers. Unfortunately there is often some confusion as to whether or not zero should be included ${ }^{1}$. In this text the natural numbers does not include zero.

Notice that the set of natural numbers is closed under addition and multiplication. This means that if you take any two natural numbers and add them you get another natural number. Similarly if you take any two natural numbers and multiply them you get another natural number. However the set is not closed

1 This lack of agreement comes from some debate over how "natural" zero is - "how can nothing be something?" It was certainly not used by the ancient Greeks who really first looked at proof and number. If you are a mathematician then generally 0 is not a natural number. If you are a computer scientist then 0 generally is.
under subtraction or division; we need negative numbers and fractions to make collections of numbers closed under subtraction and division.

Two important subsets of natural numbers are:

- Prime numbers - a natural number is prime when the only natural numbers that divide it exactly are 1 and itself. Equivalently it cannot be written as the product of two natural numbers neither of which are 1 . Note that 1 is not a prime number ${ }^{2}$.
- Composite numbers - a natural number is a composite number when it is not prime.

Hence the number 7 is prime, but $6=3 \times 2$ is composite.

- Integers - all positive and negative numbers together with the number zero. We denote the collection of all integers by the symbol " $\mathbb{Z}$ ". Again, note that this is not the same symbol as " $Z$ ", and we must write it in the same black-board bold font. The $\mathbb{Z}$ stands for the German Zahlen meaning numbers ${ }^{3}$. Note that $\mathbb{Z}$ is closed under addition, subtraction and multiplication, but not division.
Two important subsets of integers are:
- Even numbers - an integer is even if it is exactly divisible by 2, or equivalently if it can be written as the product of 2 and another integer. This means that $-14,6$ and 0 are all even.
- Odd numbers - an integer is odd when it is not even. Equivalently it can be written as $2 k+1$ where $k$ is another integer. Thus $11=2 \times 5+1$ and $-7=2 \times(-4)+1$ are both odd.
- Rational numbers - this is all numbers that can be written as the ratio of two integers. That is, any rational number $r$ can be written as $p / q$ where $p, q$ are integers. We denote this collection by $\mathbb{Q}$ standing for quoziente which is Italian for quotient or ratio. Now we finally have a set of numbers which is closed under addition, subtraction, multiplication and division (of course you still need to be careful not to divide by zero).
- Real numbers - generally we think of these numbers as numbers that can be written as decimal expansions and we denote it by $\mathbb{R}$. It is beyond the scope of this text to go into the details of how to give a precise definition of real numbers, and the notion that a real number can be written as a decimal expansion will be sufficient.
It took mathematicians quite a long time to realise that there were numbers that

[^0]could not be written as ratios of integers ${ }^{4}$. The first numbers that were shown to be not-rational are square-roots of prime numbers, like $\sqrt{2}$. Other well known examples are $\pi$ and $e$. Usually the fact that some numbers cannot be represented as ratios of integers is harmless because those numbers can be approximated by rational numbers to any desired precision.

The reason that we can approximate real numbers in this way is the surprising fact that between any two real numbers, one can always find a rational number. So if we are interested in a particular real number we can always find a rational number that is extremely close. Mathematicians refer to this property by saying that $\mathbb{Q}$ is dense in $\mathbb{R}$.

So to summarise

## Definition 0.1.1 Sets of numbers.

This is not really a definition, but you should know these symbols

- $\mathbb{N}=$ the natural numbers,
- $\mathbb{Z}=$ the integers,
- $\mathbb{Q}=$ the rationals, and
- $\mathbb{R}=$ the reals.


### 0.1.0.1 More on Real Numbers

In the preceding paragraphs we have talked about the decimal expansions of real numbers and there is just one more point that we wish to touch on. The decimal expansions of rational numbers are always periodic, that is the expansion eventually starts to repeat itself. For example

$$
\begin{aligned}
& \frac{2}{15}=0.133333333 \ldots \\
& \frac{5}{17}=0.2941176470588235 \\
&
\end{aligned}
$$

4 The existence of such numbers caused mathematicians (particularly the ancient Greeks) all sorts of philosophical problems. They thought that the natural numbers were somehow fundamental and beautiful and "natural". The rational numbers you can get very easily by taking "ratios" - a process that is still somehow quite sensible. There were quite influential philosophers (in Greece at least) called Pythagoreans (disciples of Pythagoras originally) who saw numbers as almost mystical objects explaining all the phenomena in the universe, including beauty - famously they found fractions in musical notes etc and "numbers constitute the entire heavens". They believed that everything could be explained by whole numbers and their ratios. But soon after Pythagoras' theorem was discovered, so were numbers that are not rational. The first proof of the existence of irrational numbers is sometimes attributed to Hippasus in around 400BCE (not really known). It seems that his philosopher "friends" were not very happy about this and essentially exiled him. Some accounts suggest that he was drowned by them.
where we have underlined some of the last example to make the period clearer. On the other hand, irrational numbers, such as $\sqrt{2}$ and $\pi$, have expansions that never repeat.

If we want to think of real numbers as their decimal expansions, then we need those expansions to be unique. That is, we don't want to be able to write down two different expansions, each giving the same real number. Unfortunately there are an infinite set of numbers that do not have unique expansions. Consider the number 1 . We usually just write " 1 ", but as a decimal expansion it is

$$
1.00000000000 \ldots
$$

that is, a single 1 followed by an infinite string of 0 's. Now consider the following number

$$
0.999999999999 \ldots
$$

This second decimal expansions actually represents the same number - the number 1. Let's prove this. First call the real number this represents $q$, then

$$
q=0.99999999999 \ldots
$$

Let's use a little trick to get rid of the long string of trailing 9's. Consider 10q:

$$
\begin{aligned}
q & =0.99999999999 \ldots \\
10 q & =9.99999999999 \ldots
\end{aligned}
$$

If we now subtract one from the other we get

$$
9 q=9.0000000000 \ldots
$$

and so we are left with $q=1.0000000 \ldots$. So both expansions represent the same real number.

Thankfully this sort of thing only happens with rational numbers of a particular form - those whose denominators are products of 2 s and 5 s . For example

$$
\begin{aligned}
\frac{3}{25} & =0.1200000 \cdots=0.119999999 \cdots \\
-\frac{7}{32} & =-0.2187500000 \cdots=-0.2187499999 \cdots \\
\frac{9}{20} & =0.45000000 \cdots=0.4499999 \cdots
\end{aligned}
$$

We can formalise this result in the following theorem (which we haven't proved in general, but it's beyond the scope of the text to do so):

## Theorem 0.1.2

Let $x$ be a real number. Then $x$ must fall into one of the following two categories,

- $x$ has a unique decimal expansion, or
- $x$ is a rational number of the form $\frac{a}{2^{k} 5^{l}}$ where $a \in \mathbb{Z}$ and $k, l$ are non-negative integers.

In the second case, $x$ has exactly two expansions, one that ends in an infinite string of 9 's and the other ending in an infinite string of 0's.

When we do have a choice of two expansions, it is usual to avoid the one that ends in an infinite string of 9's and write the other instead (omitting the infinite trailing string of 0's).

## $0.2 \wedge$ Sets

All of you will have done some basic bits of set-theory in school. Sets, intersection, unions, Venn diagrams etc etc. Set theory now appears so thoroughly throughout mathematics that it is difficult to imagine how Mathematics could have existed without it. It is really quite surprising that set theory is a much newer part of mathematics than calculus. Mathematically rigorous set theory was really only developed in the 19th Century - primarily by Georg Cantor ${ }^{1}$. Mathematicians were using sets before then (of course), however they were doing so without defining things too rigorously and formally.

In mathematics (and elsewhere, including "real life") we are used to dealing with collections of things. For example

- a family is a collection of relatives.
- hockey team is a collection of hockey players.
- shopping list is a collection of items we need to buy.

Generally when we give mathematical definitions we try to make them very formal and rigorous so that they are as clear as possible. We need to do this so that when we

1 An extremely interesting mathematician who is responsible for much of our understanding of infinity. Arguably his most famous results are that there are more real numbers than integers, and that there are an infinite number of different infinities. His work, though now considered to be extremely important, was not accepted by his peers, and he was labelled "a corrupter of youth" for teaching it. For some reason we know that he spent much of his honeymoon talking and doing mathematics with Richard Dedekind.
come across a mathematical object we can decide with complete certainty whether or not it satisfies the definition.

Unfortunately, it is the case that giving a completely rigorous definition of "set" would take up far more of our time than we would really like ${ }^{2}$.

## Definition 0.2.1 A not-so-formal definition of set.

A "set" is a collection of distinct objects. The objects are referred to as "elements" or "members" of the set.

Now - just a moment to describe some conventions. There are many of these in mathematics. These are not firm mathematical rules, but just traditions. It makes it much easier for people reading your work to understand what you are trying to say.

- Use capital letters to denote sets, $A, B, C, X, Y$ etc.
- Use lower case letters to denote elements of the sets $a, b, c, x, y$.

So when you are writing up homework, or just describing what you are doing, then if you stick with these conventions people reading your work (including the person marking your exams) will know - "Oh $A$ is that set they are talking about" and " $a$ is an element of that set.". On the other hand, if you use any old letter or symbol it is correct, but confusing for the reader. Think of it as being a bit like spelling - if you don't spell words correctly people can usually still understand what you mean, but it is much easier if you spell words the same way as everyone else.

We will encounter more of these conventions as we go - another good one is

- The letters $i, j, k, l, m, n$ usually denote integers (like $1,2,3,-5,18, \cdots>$ ).
- The letters $x, y, z, w$ usually denote real numbers (like $1.4323, \pi, \sqrt{2}, 6.0221415 \times$ $10^{23}, \ldots$ and so forth).

So now that we have defined sets, what can we do with them? There is only thing we can ask of a set
"Is this object in the set?"
and the set will answer
"yes" or "no"
For example, if $A$ is the set of even numbers we can ask "Is 4 in $A$ ?" We get back the answer "yes". We write this as

$$
4 \in A
$$

2 The interested reader is invited to google (or whichever search engine you prefer - DuckDuckGo?) "Russell's paradox", "Axiomatic set theory" and "Zermelo-Fraenkel set theory" for a more complete and far more detailed discussion of the basics of sets and why, when you dig into them a little, they are not so basic.

While if we ask "Is 3 in $A$ ?", we get back the answer "no". Mathematically we would write this as

$$
3 \notin A
$$

So this symbol " $\in$ " is mathematical shorthand for "is an element of", while the same symbol with a stroke through it " $\neq$ " is shorthand for "is not an element of".

Notice that both of these statements, though they are written down as short strings of three symbols, are really complete sentences. That is, when we read them out we have

$$
\begin{array}{lll}
" 4 \in A " & \text { is read as } & \text { "Four is an element of } A . " \\
" 3 \notin A " & \text { is read as } & \text { "Three is not an element of } A . "
\end{array}
$$

The mathematical symbols like " + ", " $=$ " and " $\in$ " are shorthand ${ }^{3}$ and mathematical statements like " $4+3=7$ " are complete sentences.

This is an important point - mathematical writing is just like any other sort of writing. It is very easy to put a bunch of symbols or words down on the page, but if we would like it to be easy to read and understand, then we have to work a bit harder. When you write mathematics you should keep in mind that someone else should be able to read it and understand it.

Easy reading is damn hard writing.

- Nathaniel Hawthorne, but possibly also a few others like Richard Sheridan.

We will come across quite a few different sets when doing mathematics. It must be completely clear from the definition how to answer the question "Is this object in the set or not?"

- "Let $A$ be the set of even integers between 1 and 13." - nice and clear.
- "Let $B$ be the set of tall people in this class room." - not clear.

More generally if there are only a small number of elements in the set we just list them all out

- "Let $C=\{1,2,3\} . "$

When we write out the list we put the elements inside braces " $\{\cdot\}$ ". Note that the order we write things in doesn't matter

$$
C=\{1,2,3\}=\{2,1,3\}=\{3,2,1\}
$$

3 Precise definitions aside, by "shorthand" we mean a collection of accepted symbols and abbreviations to allow us to write more quickly and hopefully more clearly. People have been using various systems of shorthand as long as people have been writing. Many of these are used and understood only by the individual, but if you want people to be able to understand what you have written, then you need to use shorthand that is commonly understood.
because the only thing we can ask is "Is this object an element of $C$ ?" We cannot ask more complex questions like "What is the third element of $C$ ?" - we require more sophisticated mathematical objects to ask such questions ${ }^{4}$. Similarly, it doesn't matter how many times we write the same object in the list

$$
C=\{1,1,1,2,3,3,3,3,1,2,1,2,1,3\}=\{1,2,3\}
$$

because all we ask is "Is $1 \in C$ ?". Not "how many times is 1 in $C$ ?".
Now - if the set is a bit bigger then we might write something like this

- $C=\{1,2,3, \ldots, 40\}$ the set of all integers between 1 and 40 (inclusive).
- $A=\{1,4,9,16, \ldots\}$ the set of all perfect squares ${ }^{5}$

The "..." is again shorthand for the missing entries. You have to be careful with this as you can easily confuse the reader

- $B=\{3,5,7, \ldots\}$ - is this all odd primes, or all odd numbers bigger than 1 or ?? What is written is not sufficient for us to have a firm idea of what the writer intended.

Only use this where it is completely clear by context. A few extra words can save the reader (and yourself) a lot of confusion.

Always think about the reader.

## 0.3ム Other Important Sets

We have seen a few important sets above - namely $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$. However, arguably the most important set in mathematics is the empty set.

## Definition 0.3.1 Empty set.

The empty set (or null set or void set) is the set which contains no elements. It is denoted $\varnothing$. For any object $x$, we always have $x \notin \varnothing$; hence $\varnothing=\{ \}$.

Note that it is important to realise that the empty set is not nothing; think of it as an empty bag. Also note that with quite a bit of hard work you can actually define the natural numbers in terms of the empty set. Doing so is very formal and well beyond the scope of this text.

When a set does not contain too many elements it is fine to specify it by listing out its elements. But for infinite sets or even just big sets we can't do this and instead we

4 The interested reader is invited to look at "lists", "multisets", "totally ordered sets" and "partially ordered sets" amongst many other mathematical objects that generalise the basic idea of sets.
5 i.e. integers that can be written as the square of another integer.
have to give the defining rule. For example the set of all perfect square numbers we write as

$$
S=\left\{x \text { s.t. } x=k^{2} \text { where } k \in \mathbb{Z}\right\}
$$

Notice we have used another piece of shorthand here, namely s.t., which stands for "such that" or "so that". We read the above statement as " $S$ is the set of elements $x$ such that $x$ equals $k$-squared where $k$ is an integer". This is the standard way of writing a set defined by a rule, though there are several shorthands for "such that". We shall use two them:

$$
P=\{p \text { s.t. } p \text { is prime }\}=\{p \mid p \text { is prime }\}
$$

Other people also use "." is shorthand for "such that". You should recognise all three of these shorthands.

Example 0.3.2 examples of sets.
Even more examples...

- Let $A=\{2,3,5,7,11,13,17,19\}$ and let

$$
B=\{a \in A \mid a<8\}=\{2,3,5,7\}
$$

the set of elements of $A$ that are strictly less than 8 .

- Even and odd integers

$$
\begin{aligned}
E & =\{n \mid n \text { is an even integer }\} \\
& =\{n \mid n=2 k \text { for some } k \in \mathbb{Z}\} \\
& =\{2 n \mid n \in \mathbb{Z}\},
\end{aligned}
$$

and similarly

$$
\begin{aligned}
O & =\{n \mid n \text { is an odd integer }\} \\
& =\{2 n+1 \mid n \in \mathbb{Z}\} .
\end{aligned}
$$

- Square integers

$$
S=\left\{n^{2} \mid n \in \mathbb{Z}\right\}
$$

The set ${ }^{a} S^{\prime}=\left\{n^{2} \mid n \in \mathbb{N}\right\}$ is not the same as $S$ because $S^{\prime}$ does not contain the number 0 , which is definitely a square integer and 0 is in $S$. We could also write $S=\left\{n^{2} \mid n \in \mathbb{Z}, n \geq 0\right\}$ and $S=\left\{n^{2} \mid n=0,1,2, \ldots\right\}$.
$a \quad$ Notice here we are using another common piece of mathematical short-hand. Very often in mathematics we will be talking or writing about some object, like the set $S$ above, and then we will create a closely related object. Rather than calling this new object by a new symbol (we could have used $T$ or $R$ or...), we instead use the same symbol but with some sort of accent - such as the little single quote mark we added to the symbol $S$ to make $S^{\prime}$ (read " S prime"). The point of this is to let the reader know that this new object is related to the original one, but not the same. You might also see $\dot{S}, \hat{S}, \bar{S}, \tilde{S}$ and others.

The sets $A$ and $B$ in the above example illustrate an important point. Every element in $B$ is an element in $A$, and so we say that $B$ is a subset of $A$

## Definition 0.3.3

Let $A$ and $B$ be sets. We say " $A$ is a subset of $B$ " if every element of $A$ is also an element of $B$. We denote this $A \subseteq B$ (or $B \supseteq A$ ). If $A$ is a subset of $B$ and $A$ and $B$ are not the same, so that there is some element of $B$ that is not in $A$ then we say that $A$ is a proper subset of $B$. We denote this by $A \subset B$ (or $B \supset A$ ).

Two things to note about subsets:

- Let $A$ be a set. It is always the case that $\varnothing \subseteq A$.
- If $A$ is not a subset of $B$ then we write $A \nsubseteq B$. This is the same as saying that there is some element of $A$ that is not in $B$. That is, there is some $a \in A$ such that $a \notin B$.


## Example 0.3.4 subsets.

Let $S=\{1,2\}$. What are all the subsets of $S$ ? Well - each element of $S$ can either be in the subset or not (independent of the other elements of the set). So we have $2 \times 2=4$ possibilities: neither 1 nor 2 is in the subset, 1 is but 2 is not, 2 is but 1 is not, and both 1 and 2 are. That is

$$
\varnothing,\{1\},\{2\},\{1,2\} \subseteq S
$$

This argument can be generalised with a little work to show that a set that contains exactly $n$ elements has exactly $2^{n}$ subsets.

In much of our work with functions later in the text we will need to work with subsets of real numbers, particularly segments of the "real line". A convenient and standard way of representing such subsets is with interval notation.

## Definition 0.3.5 Open and closed intervals of $\mathbb{R}$.

Let $a, b \in \mathbb{R}$ such that $a<b$. We name the subset of all numbers between $a$ and $b$ in different ways depending on whether or not the ends of the interval ( $a$ and $b)$ are elements of the subset.

- The closed interval $[a, b]=\{x \in \mathbb{R}: a \leq x \leq b\}$ - both end points are included.
- The open interval $(a, b)=\{x \in \mathbb{R}: a<x<b\}$ - neither end point is included.

We also define half-open ${ }^{a}$ intervals which contain one end point but not the other:

$$
(a, b]=\{x \in \mathbb{R}: a<x \leq b\} \quad[a, b)=\{x \in \mathbb{R}: a \leq x<b\}
$$

We sometimes also need unbounded intervals

$$
\begin{aligned}
{[a, \infty) } & =\{x \in \mathbb{R}: a \leq x\} & (a, \infty) & =\{x \in \mathbb{R}: a<x\} \\
(-\infty, b] & =\{x \in \mathbb{R}: x \leq b\} & (-\infty, b) & =\{x \in \mathbb{R}: x<b\}
\end{aligned}
$$

These unbounded intervals do not include " $\pm \infty$ ", so that end of the interval is always open ${ }^{b}$.
$a$ Also called "half-closed". The preference for one term over the other may be related to whether a 500 ml glass containing 250 ml of water is half-full or half-empty.
$b$ Infinity is not a real number. As mentioned in an earlier footnote, Cantor proved that there are an infinite number of different infinities and so it is incorrect to think of $\infty$ as being a single number. As such it cannot be an element in an interval of the real line. We suggest that the reader that wants to learn more about how mathematics handles infinity look up transfinite numbers and transfinite arithmetic. Needless to say these topics are beyond the scope of this text.

### 0.3.1 M More on Sets

So we now know how to say that one set is contained within another. We will now define some other operations on sets. Let us also start to be a bit more precise with our definitions and set them out carefully as we get deeper into the text.

## Definition 0.3.6

Let $A$ and $B$ be sets. We define the union of $A$ and $B$, denoted $A \cup B$, to be the set of all elements that are in at least one of $A$ or $B$.

$$
A \cup B=\{x \mid x \in A \text { or } x \in B\}
$$

It is important to realise that we are using the word "or" in a careful mathematical sense. We mean that $x$ belongs to $A$ or $x$ belongs to $B$ or both. Whereas in normal every-day English "or" is often used to be "exclusive or" - $A$ or $B$ but not both ${ }^{1}$.

We also start the definition by announcing "Definition" so that the reader knows "We are about to define something important". We should also make sure that everything is (reasonably) self-contained - we are not assuming the reader already knows $A$ and $B$ are sets.

It is vital that we make our definitions clear otherwise anything we do with the definitions will be very difficult to follow. As writers we must try to be nice to our

1 When you are asked for your dining preferences on a long flight you are usually asked something like "Chicken or beef?" - you get one or the other, but not both. Unless you are way at the back near the toilets in which case you will be presented with which ever meal was less popular. Probably fish.
readers ${ }^{2}$.

## Definition 0.3.7

Let $A$ and $B$ be sets. We define the intersection of $A$ and $B$, denoted $A \cap B$, to be the set of elements that belong to both $A$ and $B$.

$$
A \cap B=\{x \mid x \in A \text { and } x \in B\}
$$

Again note that we are using the word "and" in a careful mathematical sense (which is pretty close to the usual use in English).

Example 0.3.8 Union and intersection.
Let $A=\{1,2,3,4\}, \quad B=\{p: p$ is prime $\}, C=\{5,7,9\}$ and $D=$ \{even positive integers\}. Then

$$
\begin{aligned}
& A \cap B=\{2,3\} \\
& B \cap D=\{2\} \\
& A \cup C=\{1,2,3,4,5,7,9\} \\
& A \cap C=\varnothing
\end{aligned}
$$

In this last case we see that the two sets have no elements in common - they are said to be disjoint.

## 0.4^ Functions

Now that we have reviewed basic ideas about sets we can start doing more interesting things with them - functions.

When we are introduced to functions in mathematics, it is almost always as formulas. We take a number $x$ and do some things to it to get a new number $y$. For example,

$$
y=f(x)=3 x-7
$$

Here, we take a number $x$, multiply it by 3 and then subtract seven to get the result.
This view of functions - a function is a formula - was how mathematicians defined them up until the 19th century. As basic ideas of sets became better defined, people revised ideas surrounding functions. The more modern definition of a function between two sets is that it is a rule which assigns to each element of the first set a unique element of the second set.

Consider the set of days of the week, and the set containing the alphabet
$A=\{$ Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday $\}$

2 If you are finding this text difficult to follow then please complain to us authors and we will do our best to improve it.

$$
B=\{\mathrm{a}, \mathrm{~b}, \mathrm{c}, \mathrm{~d}, \mathrm{e}, \ldots, \mathrm{x}, \mathrm{y}, \mathrm{z}\}
$$

We can define a function $f$ that takes a day (that is, an element of $A$ ) and turns it into the first letter of that day (that is, an element of $B$ ). This is a valid function, though there is no formula. We can draw a picture of the function as


Clearly such pictures will work for small sets, but will get very messy for big ones. When we shift back to talking about functions on real numbers, then we will switch to using graphs of functions on the Cartesian plane.

This example is pretty simple, but this serves to illustrate some important points. If our function gives us a rule for taking elements in $A$ and turning them into elements from $B$ then

- the function must be defined for all elements of $A$ - that is, no matter which element of $A$ we choose, the function must be able to give us an answer. Every function must have this property.
- on the other hand, we don't have to "hit" every element from $B$. In the above example, we miss almost all the letters in $B$. A function that does reach every element of $B$ is said to be "surjective" or "onto".
- a given element of $B$ may be reached by more than one element of $A$. In the above example, the days "Tuesday" and "Thursday" both map to the letter $T$ and similarly the letters $S$ is mapped to by both "Sunday" and "Saturday". A function which does not do this, that is, every element in $A$ maps to a different element in $B$ is called "injective" or "one-to-one" - again we will come back to this later when we discuss inverse function in Section 0.6.

Summarising this more formally, we have

## Definition 0.4.1

Let $A, B$ be non-empty sets. A function $f$ from $A$ to $B$, is a rule or formula that takes elements of $A$ as inputs and returns elements of $B$ as outputs. We write this as

$$
f: A \rightarrow B
$$

and if $f$ takes $a \in A$ as an input and returns $b \in B$ then we write this as $f(a)=b$. Every function must satisfy the following two conditions

- The function must be defined on every possible input from the set $A$. That is, no matter which element $a \in A$ we choose, the function must return an element $b \in B$ so that $f(a)=b$.
- The function is only allowed to return one result for each input ${ }^{a}$. So if we find that $f(a)=b_{1}$ and $f(a)=b_{2}$ then the only way that $f$ can be a function is if $b_{1}$ is exactly the same as $b_{2}$.
$a \quad$ You may have learned this in the context of plotting functions on the Cartesian plane, as "the vertical line test". If the graph intersects a vertical line twice, then the same $x$-value will give two $y$-values and so the graph does not represent a function.

We must include the input and output sets $A$ and $B$ in the definition of the function. This is one of the reasons that we should not think of functions as just formulas. The input and output sets have proper mathematical names, which we give below:

## Definition 0.4.2

Let $f: A \rightarrow B$ be a function. Then

- the set $A$ of inputs to our function is the "domain" of $f$,
- the set $B$ which contains all the results is called the codomain,
- We read " $f(a)=b$ " as " $f$ of $a$ is $b$ ", but sometimes we might say " $f$ maps $a$ to $b$ " or " $b$ is the image of $a$ ".
- The codomain must contain all the possible results of the function, but it might also contain a few other elements. The subset of $B$ that is exactly the outputs of $A$ is called the "range" of $f$. We define it more formally by

$$
\text { range of } \begin{aligned}
f & =\{b \in B \mid \text { there is some } a \in A \text { so that } f(a)=b\} \\
& =\{f(a) \in B \mid a \in A\}
\end{aligned}
$$

The only elements allowed in that set are those elements of $B$ that are the images of elements in $A$.

Example 0.4.3 domains and ranges.
Let us go back to the "days of the week" function example that we worked on above, we can define the domain, codomain and range:

- The domain, $A$, is the set of days of the week.
- The codomain, $B$, is the 26 letters of the alphabet.
- The range is the set $\{F, M, T, S, W\}$ - no other elements of $B$ are images of
inputs from $A$.
亿 $\quad$ Example 0.4.3

Example 0.4.4 more domains and ranges.
A more numerical example - let $g: \mathbb{R} \rightarrow \mathbb{R}$ be defined by the formula $g(x)=x^{2}$. Then

- the domain and codomain are both the set of all real numbers, but
- the range is the set $[0, \infty)$.

Now - let $h:[0, \infty) \rightarrow[0, \infty)$ be defined by the formula $h(x)=\sqrt{x}$. Then

- the domain and codomain are both the set $[0, \infty)$, that is all non-negative real numbers, and
- in this case the range is equal to the codomain, namely $[0, \infty)$.

Example 0.4.5 piece-wise function.
Yet another numerical example.

$$
V:[-1,1] \rightarrow \mathbb{R} \quad \text { defined by } V(t)= \begin{cases}0 & \text { if }-1 \leq t<0 \\ 120 & \text { if } 0 \leq t \leq 1\end{cases}
$$

This is an example of a "piece-wise" function - that is, one that is not defined by a single formula, but instead defined piece-by-piece. This function has domain $[-1,1]$ and its range is $\{0,120\}$. We could interpret this function as measuring the voltage $\uparrow$ across a switch that is flipped on at time $t=0$.

Almost all the functions we look at from here on will be formulas. However it is important to note, that we have to include the domain and codomain when we describe the function. If the domain and codomain are not stated explicitly then we should assume that both are $\mathbb{R}$.

## $0.5 \wedge$ Parsing Formulas

Consider the formula

$$
f(x)=\frac{1+x}{1+2 x-x^{2}}
$$

This is an example of a simple rational function - that is, the ratio of two polynomials. When we start to examine these functions later in the text, it is important that we are able to understand how to evaluate such functions at different values of $x$. For example

$$
f(5)=\frac{1+5}{1+10-25}=\frac{6}{-14}=-\frac{3}{7}
$$

More important, however, is that we understand how we decompose this function into simpler pieces. Since much of your calculus course will involve creating and studying complicated functions by building them up from simple pieces, it is important that you really understand this point.

Now to get there we will take a small excursion into what are called parse-trees. You already implicitly use these when you evaluate the function at a particular value of $x$, but our aim here is to formalise this process a little more.

We can express the steps used to evaluate the above formula as a tree-like diagram ${ }^{1}$. We can decompose this formula as the following tree-like diagram


Figure 0.5.1: A parse tree of the function $\frac{1+x}{1+2 x-x^{2}}$.
Let us explain the pieces here.

- The picture consists of boxes and arrows which are called "nodes" and "edges" respectively.
- There are two types of boxes, those containing numbers and the variable $x$, and those containing arithmetic operations " + "," - ", " $\times$ " and "/".
- If we wish to represent the formula $3+5$, then we can draw this as the following cherry-like configuration

1 Such trees appear in many areas of mathematics and computer science. The reason for the name is that they look rather like trees - starting from their base they grow and branch out towards their many leaves. For some reason, which remains mysterious, they are usually drawn upside down.

which tells us to take the numbers " 3 " and " 5 " and add them together to get 8 .


- By stringing such little "cherries" together we can describe more complicated formulas. For example, if we compute " $(3+5) \times 2$ ", we first compute " $(3+5)$ " and then multiply the result by 2 . The corresponding diagrams are


The tree we drew in Figure 0.5.1 above representing our formula has $x$ in some of the boxes, and so when we want to compute the function at a particular value of $x$ say at $x=5$ - then we replace those " $x$ "s in the tree by that value and then compute back up the tree. See the example below



This is not the only parse tree associated with the formula for $f(x)$; we could also decompose it as


We are able to do this because when we compute the denominator $1+2 x-x^{2}$, we can compute it as

$$
1+2 x-x^{2}=\text { either }(1+2 x)-x^{2} \text { or }=1+\left(2 x-x^{2}\right)
$$

Both ${ }^{2}$ are correct because addition is "associative". Namely

$$
a+b+c=(a+b)+c=a+(b+c) .
$$

Multiplication is also associative:

$$
a \times b \times c=(a \times b) \times c=a \times(b \times c) .
$$

2 We could also use, for example, $1+2 x-x^{2}=\left(1-x^{2}\right)+2 x$.

Example 0.5.2 parsing a formula.
Consider the formula

$$
g(t)=\left(\frac{t+\pi}{t-\pi}\right) \cdot \sin \left(\frac{t+\pi}{2}\right)
$$

This introduces a new idea - we have to evaluate $\frac{t+\pi}{2}$ and then compute the sine of that number. The corresponding tree can be written as


If we want to evaluate this at $t=\pi / 2$ then we get the following...


It is highly unlikely that you will ever need to explicitly construct such a tree for any problem in the remainder of the text. The main point of introducing these objects and working through a few examples is to realise that all the functions that we will examine are constructed from simpler pieces. In particular we have constructed all the above examples from simple "building blocks"

- constants - fixed numbers like $1, \pi$ and so forth
- variables - usually $x$ or $t$, but sometimes other symbols
- standard functions - like trigonometric functions (sine, cosine and tangent), exponentials and logarithms.

These simple building blocks are combined using arithmetic

- addition and subtraction $-a+b$ and $a-b$
- multiplication and division $-a \cdot b$ and $\frac{a}{b}$
- raising to a power $-a^{n}$
- composition - given two functions $f(x)$ and $g(x)$ we form a new function $f(g(x))$ by evaluating $y=g(x)$ and then evaluating $f(y)=f(g(x))$.

During the rest of the course when we learn how to compute limits and derivatives, our computations require us to understand the way we construct functions as we have just described.

That is, in order to compute the derivative ${ }^{3}$ of a function we have to see how to construct the function from these building blocks (i.e. the constants, variables and standard functions) using arithmetic operations. We will then construct the derivative by following these same steps. There will be simple rules for finding the derivatives of the simpler pieces and then rules for putting them together following the arithmetic used to construct the function.

## 0.6 」 Inverse Functions

There is one last thing that we should review before we get into the main material of the course and that is inverse functions. As we have seen above functions are really just rules for taking an input (almost always a number), processing it somehow (usually by a formula) and then returning an output (again, almost always a number).

$$
\text { input number } x \quad \mapsto \quad f \text { does "stuff" to } x \quad \mapsto \quad \text { return number } y
$$

In many situations it will turn out to be very useful if we can undo whatever it is that our function has done. ie

$$
\text { take output } y \quad \mapsto \quad \text { do "stuff" to } y \quad \mapsto \quad \text { return the original } x
$$

When it exists, the function "which undoes" the function $f(x)$ is found by solving $y=f(x)$ for $x$ as a function of $y$ and is called the inverse function of $f$. It turns out that it is not always possible to solve $y=f(x)$ for $x$ as a function of $y$. Even when it is possible, it can be really hard to do ${ }^{1}$.

For example - a particle's position, $s$, at time $t$ is given by the formula $s(t)=7 t$ (sketched below). Given a calculator, and any particular number $t$, you can quickly work out the corresponding positions $s$. However, if you are asked the question "When does the particle reach $s=4$ ?" then to answer it we need to be able to "undo" $s(t)=4$

3 We get to this in Chapter 2 - don't worry about exactly what it is just now.
1 Indeed much of encryption exploits the fact that you can find functions that are very quick to do, but very hard to undo. For example - it is very fast to multiply two large prime numbers together, but very hard to take that result and factor it back into the original two primes. The interested reader should look up trapdoor functions.
to isolate $t$. In this case, because $s(t)$ is always increasing, we can always undo $s(t)$ to get a unique answer:

$$
s(t)=7 t=4 \quad \text { if and only if } \quad t=\frac{4}{7}
$$

However, this question is not always so easy. Consider the sketch of $y=\sin (x)$ below; when is $y=\frac{1}{2}$ ? That is, for which values $x$ is $\sin (x)=\frac{1}{2}$ ? To rephrase it again, at which values of $x$ does the curve $y=\sin x$ (which is sketched in the right half of Figure 0.6.1) cross the horizontal straight line $y=\frac{1}{2}$ (which is also sketched in the same figure)?


Figure 0.6.1

We can see that there are going to be an infinite number of $x$-values that give $y=\sin (x)=\frac{1}{2}$; there is no unique answer.

Recall (from Definition 0.4.1) that for any given input, a function must give a unique output. So if we want to find a function that undoes $s(t)$, then things are good because each $s$-value corresponds to a unique $t$-value. On the other hand, the situation with $y=\sin x$ is problematic - any given $y$-value is mapped to by many different $x$-values. So when we look for an unique answer to the question "When is $\sin x=\frac{1}{2}$ ?" we cannot answer it.

This "uniqueness" condition can be made more precise:

## Definition 0.6.2

A function $f$ is one-to-one (injective) when it never takes the same $y$ value more than once. That is

$$
\text { if } x_{1} \neq x_{2} \text { then } f\left(x_{1}\right) \neq f\left(x_{2}\right)
$$

There is an easy way to test this when you have a plot of the function - the horizontal line test.

## Definition 0.6.3 Horizontal line test.

A function is one-to-one if and only if no horizontal line $y=c$ intersects the graph $y=f(x)$ more than once.
i.e. every horizontal line intersects the graph either zero or one times. Never twice or more. This test tell us that $y=x^{3}$ is one-to-one, but $y=x^{2}$ is not. However note that if we restrict the domain of $y=x^{2}$ to $x \geq 0$ then the horizontal line test is passed. This is one of the reasons we have to be careful to consider the domain of the function.




When a function is one-to-one then it has an inverse function.

## Definition 0.6.4

Let $f$ be a one-to-one function with domain $A$ and range $B$. Then its inverse function is denoted $f^{-1}$ and has domain $B$ and range $A$. It is defined by

$$
f^{-1}(y)=x \quad \text { whenever } \quad f(x)=y
$$

for any $y \in B$.


So if $f$ maps $x$ to $y$, then $f^{-1}$ maps $y$ back to $x$. That is $f^{-1}$ "undoes" $f$. Because of this we have

$$
\begin{array}{ll}
f^{-1}(f(x))=x & \text { for any } x \in A \\
f\left(f^{-1}(y)\right)=y & \text { for any } y \in B
\end{array}
$$

We have to be careful not to confuse $f^{-1}(x)$ with $\frac{1}{f(x)}$. The " -1 " is not an exponent.

## Example 0.6.5 Inverse of $x^{5}+3$.

Let $f(x)=x^{5}+3$ on domain $\mathbb{R}$. To find its inverse we do the following

- Write $y=f(x)$; that is $y=x^{5}+3$.
- Solve for $x$ in terms of $y$ (this is not always easy) $-x^{5}=y-3$, so $x=(y-3)^{1 / 5}$.
- The solution is $f^{-1}(y)=(y-3)^{1 / 5}$.
- Recall that the " $y$ " in $f^{-1}(y)$ is a dummy variable. That is, $f^{-1}(y)=(y-3)^{1 / 5}$ means that if you feed the number $y$ into the function $f^{-1}$ it outputs the number $(y-3)^{1 / 5}$. You may call the input variable anything you like. So if you wish to call the input variable " $x$ " instead of " $y$ " then just replace every $y$ in $f^{-1}(y)$ with an $x$.
- That is $f^{-1}(x)=(x-3)^{1 / 5}$.

Example 0.6.5

Example 0.6.6 Inverse of $\sqrt{x-1}$.
Let $g(x)=\sqrt{x-1}$ on the domain $x \geq 1$. We can find the inverse in the same way:

$$
\begin{aligned}
y & =\sqrt{x-1} \\
y^{2} & =x-1 \\
x & =y^{2}+1=f^{-1}(y) \quad \text { or, writing input variable as " } x \text { ": } \\
f^{-1}(x) & =x^{2}+1 .
\end{aligned}
$$

Let us now turn to finding the inverse of $\sin (x)$ - it is a little more tricky and we have to think carefully about domains.

Example 0.6.7 Inverse of $\sin (x)$.
We have seen (back in Figure 0.6.1) that $\sin (x)$ takes each value $y$ between -1 and +1 for infinitely many different values of $x$ (see the left-hand graph in the figure below). Consequently $\sin (x)$, with domain $-\infty<x<\infty$ does not have an inverse function.



But notice that as $x$ runs from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}, \sin (x)$ increases from -1 to +1 . (See the middle graph in the figure above.) In particular, $\sin (x)$ takes each value $-1 \leq y \leq 1$ for exactly one $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. So if we restrict $\sin x$ to have domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, it does have an inverse function, which is traditionally called arcsine (see Appendix A.9). That is, by definition, for each $-1 \leq y \leq 1, \arcsin (y)$ is the unique $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ obeying $\sin (x)=y$. Equivalently, exchanging the dummy variables x and y throughout the last sentence gives that for each $-1 \leq x \leq 1, \arcsin (x)$ is the unique $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ obeying
$\sin (y)=x$.

It is an easy matter to construct the graph of an inverse function from the graph of the original function. We just need to remember that

$$
Y=f^{-1}(X) \Longleftrightarrow f(Y)=X
$$

which is $y=f(x)$ with $x$ renamed to $Y$ and $y$ renamed to $X$.
Start by drawing the graph of $f$, labelling the $x$ - and $y$-axes and labelling the curve $y=f(x)$.


Now replace each $x$ by $Y$ and each $y$ by $X$ and replace the resulting label $X=f(Y)$ on the curve by the equivalent $Y=f^{-1}(X)$.


Finally we just need to redraw the sketch with the $Y$ axis running vertically (with $Y$ increasing upwards) and the $X$ axis running horizontally (with $X$ increasing to the right). To do so, pretend that the sketch is on a transparency or on a very thin piece of paper that you can see through. Lift the sketch up and flip it over so that the $Y$ axis runs vertically and the $X$ axis runs horizontally. If you want, you can also convert the upper case $X$ into a lower case $x$ and the upper case $Y$ into a lower case $y$.



Another way to say "flip the sketch over so as to exchange the $x$ - and $y$-axes" is "reflect in the line $y=x$ ". In the figure below the blue "horizontal" elliptical disk that is centred on $(a, b)$ has been reflected in the line $y=x$ to give the red "vertical" elliptical disk centred on $(b, a)$.


Example 0.6.8 Sketching inverse of $y=x^{2}$.
As an example, let $f(x)=x^{2}$ with domain $0 \leq x<\infty$.

- When $x=0, f(x)=0^{2}=0$.
- As $x$ increases, $x^{2}$ gets bigger and bigger.
- When $x$ is very large and positive, $x^{2}$ is also very large and positive. (For example, think $x=100$.)

The graph of $y=f(x)=x^{2}$ is the blue curve below. By definition, $Y=f^{-1}(X)$ if $X=f(Y)=Y^{2}$. That is, if $Y=\sqrt{X}$. (Remember that, to be in the domain of $f$, we must have $Y \geq 0$.) So the inverse function of "square" is "square root". The graph of $f^{-1}$ is the red curve below. The red curve is the reflection of the blue curve in the line $y=x$.



So very roughly speaking, "Differential Calculus" is the study of how a function changes as its input changes. The mathematical object we use to describe this is the "derivative" of a function. To properly describe what this thing is we need some machinery; in particular we need to define what we mean by "tangent" and "limit". We'll get back to defining the derivative in Chapter 2.

## 1.1^ Drawing Tangents and a First Limit

### 1.1.1 $\leadsto$ Drawing Tangents and a First Limit

Our motivation for developing "limit" - being the title and subject of this chapter is going to be two related problems of drawing tangent lines and computing velocity.

Now - our treatment of limits is not going to be completely mathematically rigorous, so we won't have too many formal definitions. There will be a few mathematically precise definitions and theorems as we go, but we'll make sure there is plenty of explanation around them.

Let us start with the "tangent line" problem. Of course, we need to define "tangent", but we won't do this formally. Instead let us draw some pictures.



Here we have drawn two very rough sketches of the curve $y=x^{2}$ for $x \geq 0$. These are not very good sketches for a couple of reasons

- The curve in the figure does not pass through $(0,0)$, even though $(0,0)$ lies on $y=x^{2}$.
- The top-right end of the curve doubles back on itself and so fails the vertical line test that all functions must satisfy ${ }^{1}$ - for each $x$-value there is exactly one $y$-value for which $(x, y)$ lies on the curve $y=x^{2}$.

So let's draw those more carefully.


Figure 1.1.1: Sketches of the curve $y=x^{2}$. (left) shows a tangent line, while (right) shows a line that is not a tangent.

These are better. In both cases we have drawn $y=x^{2}$ (carefully) and then picked a point on the curve - call it $P$. Let us zoom in on the "good" example:

1 Take a moment to go back and reread Definition 0.4.1.


Figure 1.1.2: We see that, the more we zoom in on the point $P$, the more the graph of the function (drawn in black) looks like a straight line - that line is the tangent line (drawn in blue).

We see that as we zoom in on the point $P$, the graph of the function looks more and more like a straight line. If we kept on zooming in on $P$ then the graph of the function would be indistinguishable from a straight line. That line is the tangent line (which we have drawn in blue). A little more precisely, the blue line is "the tangent line to the function at $P^{\prime \prime}$. We have to be a little careful, because if we zoom in at a different point, then we will find a different tangent line.

Now let's zoom in on the "bad" example we see that the blue line looks very different from the function; because of this, the blue line is not the tangent line at $P$.


Figure 1.1.3: Zooming in on $P$ we see that the function (drawn in black) looks more and more like a straight line - however it is not the same line as that drawn in blue. Because of this the blue line is not the tangent line.

Here are a couple more examples of tangent lines


Figure 1.1.4: More examples of tangent lines.
The one on the left is very similar to the good example on $y=x^{2}$ that we saw above, while the one on the right is different - it looks a little like the "bad" example, in that it crosses our function the curve at some distant point. Why is the line in Figure 1.1.4(right) a tangent while the line in Figure 1.1.1(right) not a tangent? To see why, we should again zoom in close to the point where we are trying to draw the tangent.


As we saw above in Figure 1.1.3, when we zoom in around our example of "not a tangent line" we see that the straight line looks very different from the curve at the "point of tangency" - i.e. where we are trying to draw the tangent. The line drawn in Figure 1.1.4(right) looks more and more like the function as we zoom in.

This example raises an important point - when we are trying to draw a tangent line, we don't care what the function does a long way from the point; the tangent line to the curve at a particular point $P$, depends only on what the function looks like close to that point $P$.

To illustrate this consider the sketch of the function $y=\sin (x)$ and its tangent line at $(x, y)=(0,0)$ :


As we zoom in, the graph of $\sin (x)$ looks more and more like a straight line - in fact it looks more and more like the line $y=x$. We have also sketched this tangent line. What makes this example a little odd is that the tangent line crosses the function. In the examples above, our tangent lines just "kissed" the curve and did not cross it (or at least did not cross it nearby).

Using this idea of zooming in at a particular point, drawing a tangent line is not too hard. However, finding the equation of the tangent line presents us with a few challenges. Rather than leaping into the general theory, let us do a specific example. Let us find the the equation of the tangent line to the curve $y=x^{2}$ at the point $P$ with coordinates ${ }^{2}(x, y)=(1,1)$.

To find the equation of a line we either need

- the slope of the line and a point on the line, or
- two points on the line, from which we can compute the slope via the formula

$$
m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}
$$

and then write down the equation for the line via a formula such as

$$
y=m \cdot\left(x-x_{1}\right)+y_{1} .
$$

We cannot use the first method because we do not know what the slope of the tangent line should be. To work out the slope we need calculus - so we'll be able to use this method once we get to the next chapter on "differentiation".

It is not immediately obvious how we can use the second method, since we only have one point on the curve, namely $(1,1)$. However we can use it to "sneak up" on the answer. Let's approximate the tangent line, by drawing a line that passes through $(1,1)$ and some nearby point - call it $Q$. Here is our recipe:

- We are given the point $P=(1,1)$ and we are told

Find the tangent line to the curve $y=x^{2}$ that passes through $P=(1,1)$.

2 Note that the coordinates $(x, y)$ is an ordered pair of two numbers $x$ and $y$. Traditionally the first number is called the abscissa while the second is the ordinate, but these terms are a little archaic. It is now much more common to hear people refer to the first number as the $x$-coordinate and the second as $y$-coordinate.

- We don't quite know how to find a line given just 1 point, however we do know how to find a line passing through 2 points. So pick another point on the curves whose coordinates are very close to $P$. Now rather than picking some actual numbers, I am going to write our second point as $Q=\left(1+h,(1+h)^{2}\right)$. That is, a point $Q$ whose $x$-coordinate is equal to that of $P$ plus a little bit - where the little bit is some small number $h$. And since this point lies on the curve $y=x^{2}$, and $Q$ 's x-coordinate is $1+h, Q$ 's y-coordinate must be $(1+h)^{2}$.
If having $h$ as an variable rather than a number bothers you, start by thinking of $h$ as 0.1.
- A picture of the situation will help.

- This line that passes through the curve in two places $P$ and $Q$ is called a "secant line".
- The slope of the line is then

$$
\begin{aligned}
m & =\frac{y_{2}-y_{1}}{x_{2}-x_{1}} \\
& =\frac{(1+h)^{2}-1}{(1+h)-1}=\frac{1+2 h+h^{2}-1}{h}=\frac{2 h+h^{2}}{h}=2+h
\end{aligned}
$$

where we have expanded $(1+h)^{2}=1+2 h+h^{2}$ and then cleaned up a bit.
Now this isn't our tangent line because it passes through 2 nearby points on the curve however it is a reasonable approximation of it. Now we can make that approximation better and so "sneak up" on the tangent line by considering what happens when we move this point $Q$ closer and closer to $P$. i.e. make the number $h$ closer and closer to zero.



First look at the picture. The original choice of $Q$ is on the left, while on the right we have drawn what happens if we choose $h^{\prime}$ to be some number a little smaller than $h$, so that our point $Q$ becomes a new point $Q^{\prime}$ that is a little closer to $P$. The new approximation is better than the first.

So as we make $h$ smaller and smaller, we bring $Q$ closer and closer to $P$, and make our secant line a better and better approximation of the tangent line. We can observe what happens to the slope of the line as we make $h$ smaller by plugging some numbers into our formula $m=2+h$ :

$$
\begin{array}{rr}
h=0.1 & m=2.1 \\
h=0.01 & m=2.01 \\
h=0.001 & m=2.001 .
\end{array}
$$

So again we see that as this difference in $x$ becomes smaller and smaller, the slope appears to be getting closer and closer to 2 . We can write this more mathematically as

$$
\lim _{h \rightarrow 0} \frac{(1+h)^{2}-1}{h}=2
$$

This is read as
The limit, as $h$ approaches 0 , of $\frac{(1+h)^{2}-1}{h}$ is 2 .
This is our first limit! Notice that we can see this a little more clearly with a quick bit of algebra:

$$
\begin{align*}
\frac{(1+h)^{2}-1}{h} & =\frac{\left(1+2 h+h^{2}\right)-1}{h} \\
& =\frac{2 h+h^{2}}{h} \tag{2+h}
\end{align*}
$$

So it is not unreasonable to expect that

$$
\lim _{h \rightarrow 0} \frac{(1+h)^{2}-1}{h}=\lim _{h \rightarrow 0}(2+h)=2 .
$$

Our tangent line can be thought of as the end of this process - namely as we bring $Q$ closer and closer to $P$, the slope of the secant line comes closer and closer to that of the tangent line we want. Since we have worked out what the slope is - that is the limit we saw just above - we now know the slope of the tangent line is 2. Given this, we can work out the equation for the tangent line.

- The equation for the line is $y=m x+c$. We have 2 unknowns $m$ and $c-$ so we need 2 pieces of information to find them.
- Since the line is tangent to $P=(1,1)$ we know the line must pass through $(1,1)$. From the limit we computed above, we also know that the line has slope 2 .
- Since the slope is 2 we know that $m=2$. Thus the equation of the line is $y=2 x+c$.
- We know that the line passes through $(1,1)$, so that $y=2 x+c$ must be 1 when $x=1$. So $1=2 \cdot 1+c$, which forces $c=-1$.

So our tangent line is $y=2 x-1$.

### 1.1.2 $\leftrightarrow$ Exercises

## Exercises - Stage 1

1. On the graph below, draw:
a The tangent line to $y=f(x)$ at $P$,
b the tangent line to $y=f(x)$ at $Q$, and
c the secant line to $y=f(x)$ through $P$ and $Q$.

2. Suppose a curve $y=f(x)$ has tangent line $y=2 x+3$ at the point $x=2$.
a True or False: $f(2)=7$
b True or False: $f(3)=9$
3. Let $L$ be the tangent line to a curve $y=f(x)$ at some point $P$. How many times will $L$ intersect the curve $y=f(x)$ ?

## 1.2^ Another Limit and Computing Velocity

### 1.2.1 Another Limit and Computing Velocity

Computing tangent lines is all very well, but what does this have to do with applications or the "Real World"? Well - at least initially our use of limits (and indeed of calculus) is going to be a little removed from real world applications. However as we go further and learn more about limits and derivatives we will be able to get closer to real problems and their solutions.

So stepping just a little closer to the real world, consider the following problem. You drop a ball from the top of a very very tall building. Let $t$ be elapsed time measured in seconds, and $s(t)$ be the distance the ball has fallen in metres. So $s(0)=0$.

Quick aside: there is quite a bit going on in the statement of this problem. We have described the general picture - tall building, ball, falling - but we have also introduced notation, variables and units. These will be common first steps in applications and are necessary in order to translate a real world problem into mathematics in a clear and consistent way.

Galileo ${ }^{1}$ worked out that $s(t)$ is a quadratic function:

$$
s(t)=4.9 t^{2}
$$

The question that is posed is
How fast is the ball falling after 1 second?
Now before we get to answering this question, we should first be a little more precise. The wording of this question is pretty sloppy for a couple of reasons:

- What we do mean by "after 1 second"? We know the ball will move faster and faster as time passes, so after 1 second it does not fall at one fixed speed.
- As it stands a reasonable answer to the question would be just "really fast". If the person asking the question wants a numerical answer it would be better to ask "At what speed" or "With what velocity".

[^1]We should also be careful using the words "speed" and "velocity" - they are not interchangeable.

- Speed means the distance travelled per unit time and is always a non-negative number. An unmoving object has speed 0 , while a moving object has positive speed.
- Velocity, on the other hand, also specifies the direction of motion. In this text we will almost exclusively deal with objects moving along straight lines. Because of this velocities will be positive or negative numbers indicating which direction the object is moving along the line. We will be more precise about this later ${ }^{2}$.

A better question is
What is the velocity of the ball precisely 1 second after it is dropped?
or even better:
What is the velocity of the ball at the 1 second mark?
This makes it very clear that we want to know what is happening at exactly 1 second after the ball is dropped.

There is something a little subtle going on in this question. In particular, what do we mean by the velocity at $t=1$ ?. Surely if we freeze time at $t=1$ second, then the object is not moving at all? This is definitely not what we mean.

If an object is moving at a constant velocity ${ }^{3}$ in the positive direction, then that velocity is just the distance travelled divided by the time taken. That is

$$
v=\frac{\text { distance moved }}{\text { time taken }}
$$

An object moving at constant velocity that moves 27 metres in 3 seconds has velocity

$$
v=\frac{27 \mathrm{~m}}{3 \mathrm{~s}}=9 \mathrm{~m} / \mathrm{s}
$$

When velocity is constant everything is easy.
However, in our falling object example, the object is being acted on by gravity and its speed is definitely not constant. Instead of asking for $T H E$ velocity, let us examine the "average velocity" of the object over a certain window of time. In this case the formula is very similar

$$
\text { average velocity }=\frac{\text { distance moved }}{\text { time taken }}
$$

But now I want to be more precise, instead write

$$
\text { average velocity }=\frac{\text { difference in distance }}{\text { difference in time }}
$$

2 Getting the sign of velocity wrong is a very common error - you should be careful with it.
3 Newton's first law of motion states that an object in motion moves with constant velocity unless a force acts on it - for example gravity or friction.

Now in spoken English we haven't really changed much - the distance moved is the difference in position, and the time taken is just the difference in time - but the latter is more mathematically precise, and is easy to translate into the following equation

$$
\text { average velocity }=\frac{s\left(t_{2}\right)-s\left(t_{1}\right)}{t_{2}-t_{1}}
$$

This is the formula for the average velocity of our object between time $t_{1}$ and $t_{2}$. The denominator is just the difference between these times and the numerator is the difference in position - i.e. position at time $t_{1}$ is just $s\left(t_{1}\right)$ and position at time $t_{2}$ is just $s\left(t_{2}\right)$.

So what is the average velocity of the falling ball between 1 and 1.1 seconds? All we need to do now is plug some numbers into our formula

$$
\begin{aligned}
\text { average velocity } & =\frac{\text { difference in position }}{\text { difference in time }} \\
& =\frac{s(1.1)-s(1)}{1.1-1} \\
& =\frac{4.9(1.1)^{2}-4.9(1)}{0.1}=\frac{4.9 \times 0.21}{0.1}=10.29 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

And we have our average velocity. However there is something we should notice about this formula and it is easier to see if we sketch a graph of the function $s(t)$



So on the left I have drawn the graph and noted the times $t=1$ and $t=1.1$. The corresponding positions on the axes and the two points on the curve. On the right I have added a few more details. In particular I have noted the differences in position and time, and the line joining the two points. Notice that the slope of this line is

$$
\text { slope }=\frac{\text { change in } y}{\text { change in } x}=\frac{\text { difference in } s}{\text { difference in } t}
$$

which is precisely our expression for the average velocity.
Let us examine what happens to the average velocity as we look over smaller and smaller time-windows.

$$
\begin{array}{rr}
\text { time window } & \text { average velocity } \\
1 \leq t \leq 1.1 & 10.29 \\
1 \leq t \leq 1.01 & 9.849
\end{array}
$$

$$
\begin{array}{rr}
1 \leq t \leq 1.001 & 9.8049 \\
1 \leq t \leq 1.0001 & 9.80049
\end{array}
$$

As we make the time interval smaller and smaller we find that the average velocity is getting closer and closer to 9.8 . We can be a little more precise by finding the average velocity between $t=1$ and $t=1+h$ - this is very similar to what we did for tangent lines.

$$
\begin{aligned}
\text { average velocity } & =\frac{s(1+h)-s(1)}{(1+h)-1} \\
& =\frac{4.9(1+h)^{2}-4.9}{h} \\
& =\frac{9.8 h+4.9 h^{2}}{h} \\
& =9.8+4.9 h
\end{aligned}
$$

Now as we squeeze this window between $t=1$ and $t=1+h$ down towards zero, the average velocity becomes the "instantaneous velocity" - just as the slope of the secant line becomes the slope of the tangent line. This is our second limit

$$
v(1)=\lim _{h \rightarrow 0} \frac{s(1+h)-s(1)}{h}=9.8
$$

More generally we define the instantaneous velocity at time $t=a$ to be the limit

$$
v(a)=\lim _{h \rightarrow 0} \frac{s(a+h)-s(a)}{h}
$$

We read this as
The velocity at time $a$ is equal to the limit as $h$ goes to zero of $\frac{s(a+h)-s(a)}{h}$.
While we have solved the problem stated at the start of this section, it is clear that if we wish to solve similar problems that we will need to understand limits in a more general and systematic way.

### 1.2.2 $\leadsto$ Exercises

## Exercises - Stage 1

1. As they are used in this section, what is the difference between speed and velocity?
2. $\quad$ Speed can never be negative; can it be zero?
3. Suppose you wake up in the morning in your room, then you walk two kilometres to school, walk another two kilometres to lunch, walk four kilometres to a coffee shop to study, then return to your room until the next
morning. In the 24 hours from morning to morning, what was your average velocity? (In CLP-1, we are considering functions of one variable. So, at this stage, think of our whole world as being contained in the $x$-axis.)
4. Suppose you drop an object, and it falls for a few seconds. Which is larger: its speed at the one second mark, or its average speed from the zero second mark to the one second mark?
5. The position of an object at time $t$ is given by $s(t)$. Then its average velocity over the time interval $t=a$ to $t=b$ is given by $\frac{s(b)-s(a)}{b-a}$. Explain why this fraction also gives the slope of the secant line of the curve $y=s(t)$ from the point $t=a$ to the point $t=b$.
6. Below is the graph of the position of an object at time $t$. For what periods of time is the object's velocity positive?


## Exercises - Stage 2

7. Suppose the position of a body at time $t$ (measured in seconds) is given by $s(t)=3 t^{2}+5$.
a What is the average velocity of the object from 3 seconds to 5 seconds?
b What is the velocity of the object at time $t=1$ ?
8. Suppose the position of a body at time $t$ (measured in seconds) is given by $s(t)=\sqrt{t}$.
a What is the average velocity of the object from $t=1$ second to $t=9$ seconds?
b What is the velocity of the object at time $t=1$ ?
c What is the velocity of the object at time $t=9$ ?

## 1.3^ The Limit of a Function

### 1.3.1 $\leadsto$ The Limit of a Function

Before we come to definitions, let us start with a little notation for limits.

## Definition 1.3.1

We will often write

$$
\lim _{x \rightarrow a} f(x)=L
$$

which should be read as
The limit of $f(x)$ as $x$ approaches $a$ is $L$.
The notation is just shorthand - we don't want to have to write out long sentences as we do our mathematics. Whenever you see these symbols you should think of that sentence.

This shorthand also has the benefit of being mathematically precise (we'll see this later), and (almost) independent of the language in which the author is writing. A mathematician who does not speak English can read the above formula and understand exactly what it means.

In mathematics, like most languages, there is usually more than one way of writing things and we can also write the above limit as

$$
f(x) \rightarrow L \text { as } x \rightarrow a
$$

This can also be read as above, but also as

$$
f(x) \text { goes to } L \text { as } x \text { goes to } a
$$

They mean exactly the same thing in mathematics, even though they might be written, read and said a little differently.

To arrive at the definition of limit, we want to start ${ }^{1}$ with a very simple example.

1 Well, we had two limits in the previous sections, so perhaps we really want to "restart" with a very simple example.

Example 1.3.2 A simple limit.
Consider the following function.

$$
f(x)= \begin{cases}2 x & x<3 \\ 9 & x=3 \\ 2 x & x>3\end{cases}
$$

This is an example of a piece-wise function ${ }^{a}$. That is, a function defined in several pieces, rather than as a single formula. We evaluate the function at a particular value of $x$ on a case-by-case basis. Here is a sketch of it


Notice the two circles in the plot. One is open, o and the other is closed •

- A filled circle has quite a precise meaning - a filled circle at $(x, y)$ means that the function takes the value $f(x)=y$.
- An open circle is a little harder - an open circle at $(3,6)$ means that the point $(3,6)$ is not on the graph of $y=f(x)$, i.e. $f(3) \neq 6$. We should only use the open circle where it is absolutely necessary in order to avoid confusion.

This function is quite contrived, but it is a very good example to start working with limits more systematically. Consider what the function does close to $x=3$. We already know what happens exactly at $3-f(x)=9$ - but I want to look at how the function behaves very close to $x=3$. That is, what does the function do as we look at a point $x$ that gets closer and closer to $x=3$.
If we plug in some numbers very close to 3 (but not exactly 3 ) into the function we see the following:

| $x$ | 2.9 | 2.99 | 2.999 | $\circ$ | 3.001 | 3.01 | 3.1 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $f(x)$ | 5.8 | 5.98 | 5.998 | $\circ$ | 6.002 | 6.02 | 6.2 |

So as $x$ moves closer and closer to 3 , without being exactly 3 , we see that the function moves closer and closer to 6 . We can write this as

$$
\lim _{x \rightarrow 3} f(x)=6
$$

That is

The limit as $x$ approaches 3 of $f(x)$ is 6 .
So for $x$ very close to 3 , without being exactly 3 , the function is very close to 6 - which is a long way from the value of the function exactly at $3, f(3)=9$. Note well that the behaviour of the function as $x$ gets very close to 3 does not depend on the value of the function at 3 .
$a$ We saw another piecewise function back in Example 0.4.5.

We now have enough to make an informal definition of a limit, which is actually sufficient for most of what we will do in this text.

Definition 1.3.3 Informal definition of limit.
We write

$$
\lim _{x \rightarrow a} f(x)=L
$$

if the value of the function $f(x)$ is sure to be arbitrarily close to $L$ whenever the value of $x$ is close enough to $a$, without ${ }^{a}$ being exactly $a$.
$a \quad$ You may find the condition "without being exactly $a$ " a little strange, but there is a good reason for it. One very important application of limits, indeed the main reason we teach the topic, is in the definition of derivatives (see Definition 2.2.1 in the next chapter). In that definition we need to compute the limit $\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a}$. In this case the function whose limit is being taken, namely $\frac{f(x)-f(a)}{x-a}$, is not defined at all at $x=a$.

In order to make this definition more mathematically correct, we need to make the idea of "closer and closer" more precise - we do this in Section 1.7. It should be emphasised that the formal definition and the contents of that section are optional material.

For now, let us use the above definition to examine a more substantial example.
Example 1.3.4 $\lim _{x \rightarrow 2} \frac{x-2}{x^{2}+x-6}$.
Let $f(x)=\frac{x-2}{x^{2}+x-6}$ and consider its limit as $x \rightarrow 2$.

- We are really being asked

$$
\lim _{x \rightarrow 2} \frac{x-2}{x^{2}+x-6}=\text { what } ?
$$

- Now if we try to compute $f(2)$ we get $0 / 0$ which is undefined. The function is not defined at that point - this is a good example of why we need limits. We have to sneak up on these places where a function is not defined (or is badly behaved).
- Very important point: the fraction $\frac{0}{0}$ is not $\infty$ and it is not 1 , it is not defined. We cannot ever divide by zero in normal arithmetic and obtain a consistent and mathematically sensible answer. If you learned otherwise in high-school, you should quickly unlearn it.
- Again, we can plug in some numbers close to 2 and see what we find

| $x$ | 1.9 | 1.99 | 1.999 | $\circ$ | 2.001 | 2.01 | 2.1 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $f(x)$ | 0.20408 | 0.20040 | 0.20004 | $\circ$ | 0.19996 | 0.19960 | 0.19608 |

- So it is reasonable to suppose that

$$
\lim _{x \rightarrow 2} \frac{x-2}{x^{2}+x-6}=0.2
$$

Example 1.3.4
The previous two examples are nicely behaved in that the limits we tried to compute actually exist. We now turn to two nastier examples ${ }^{2}$ in which the limits we are interested in do not exist.

Example 1.3.5 A bad example.
Consider the following function $f(x)=\sin (\pi / x)$. Find the limit as $x \rightarrow 0$ of $f(x)$.
We should see something interesting happening close to $x=0$ because $f(x)$ is undefined there. Using your favourite graph-plotting software you can see that the graph looks roughly like


2 Actually, they are good examples, but the functions in them are nastier.

How to explain this? As $x$ gets closer and closer to zero, $\pi / x$ becomes larger and larger (remember what the plot of $y=1 / x$ looks like). So when you take sine of that number, it oscillates faster and faster the closer you get to zero. Since the function does not approach a single number as we bring $x$ closer and closer to zero, the limit does not exist.
We write this as

$$
\lim _{x \rightarrow 0} \sin \left(\frac{\pi}{x}\right) \text { does not exist }
$$

It's not very inventive notation, however it is clear. We frequently abbreviate "does not exist" to "DNE" and rewrite the above as

$$
\lim _{x \rightarrow 0} \sin \left(\frac{\pi}{x}\right)=\mathrm{DNE}
$$



In the following example, the limit we are interested in does not exist. However the way in which things go wrong is quite different from what we just saw.

Example 1.3.6 A non-existent limit.
Consider the function

$$
f(x)= \begin{cases}x & x<2 \\ -1 & x=2 \\ x+3 & x>2\end{cases}
$$

- The plot of this function looks like this

- So let us plug in numbers close to 2 .

| $x$ | 1.9 | 1.99 | 1.999 | $\circ$ | 2.001 | 2.01 | 2.1 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $f(x)$ | 1.9 | 1.99 | 1.999 | $\circ$ | 5.001 | 5.01 | 5.1 |

- This isn't like before. Now when we approach from below, we seem to be getting closer to 2 , but when we approach from above we seem to be getting closer to 5 . Since we are not approaching the same number the limit does not exist.

$$
\lim _{x \rightarrow 2} f(x)=\mathrm{DNE}
$$

Example 1.3.6
While the limit in the previous example does not exist, the example serves to introduce the idea of "one-sided limits". For example, we can say that

As $x$ moves closer and closer to two from below the function approaches 2 . and similarly

As $x$ moves closer and closer to two from above the function approaches 5 .

## Definition 1.3.7 Informal definition of one-sided limits.

We write

$$
\lim _{x \rightarrow a^{-}} f(x)=K
$$

when the value of $f(x)$ gets closer and closer to $K$ when $x<a$ and $x$ moves closer and closer to $a$. Since the $x$-values are always less than $a$, we say that $x$ approaches a from below. This is also often called the left-hand limit since the $x$-values lie to the left of $a$ on a sketch of the graph.
We similarly write

$$
\lim _{x \rightarrow a^{+}} f(x)=L
$$

when the value of $f(x)$ gets closer and closer to $L$ when $x>a$ and $x$ moves closer and closer to $a$. For similar reasons we say that $x$ approaches $a$ from above, and sometimes refer to this as the right-hand limit.

Note - be careful to include the superscript + and - when writing these limits. You might also see the following notations:

$$
\begin{array}{rlrl}
\lim _{x \rightarrow a^{+}} f(x) & =\lim _{x \rightarrow a+} f(x) & =\lim _{x \downarrow a} f(x)=\lim _{x \searrow a} f(x)=L & \text { right-hand limit } \\
\lim _{x \rightarrow a^{-}} f(x) & =\lim _{x \rightarrow a-} f(x)=\lim _{x \uparrow a} f(x)=\lim _{x \nearrow a} f(x)=L & \text { left-hand limit }
\end{array}
$$

but please use with the notation in Definition 1.3.7 above.
Given these two similar notions of limits, when are they the same? The following theorem tell us.

Theorem 1.3.8 Limits and one sided limits.

$$
\lim _{x \rightarrow a} f(x)=L \quad \text { if and only if } \quad \lim _{x \rightarrow a^{-}} f(x)=L \text { and } \lim _{x \rightarrow a^{+}} f(x)=L
$$

Notice that this is really two separate statements because of the "if and only if"

- If the limit of $f(x)$ as $x$ approaches $a$ exists and is equal to $L$, then both the left-hand and right-hand limits exist and are equal to $L$. AND,
- If the left-hand and right-hand limits as $x$ approaches $a$ exist and are equal, then the limit as $x$ approaches $a$ exists and is equal to the one-sided limits.

That is - the limit of $f(x)$ as $x$ approaches $a$ will only exist if it doesn't matter which way we approach $a$ (either from left or right) $A N D$ if we get the same one-sided limits when we approach from left and right, then the limit exists.

We can rephrase the above by writing the contrapositives ${ }^{3}$ of the above statements.

- If either of the left-hand and right-hand limits as $x$ approaches $a$ fail to exist, or if they both exist but are different, then the limit as $x$ approaches $a$ does not exist. AND,
- If the limit as $x$ approaches $a$ does not exist, then the left-hand and right-hand limits are either different or at least one of them does not exist.

Here is another limit example

## Example 1.3.9 Left- and right-handed limits.

Consider the following two functions and compute their limits and one-sided limits as $x$ approaches 1 :


These are a little different from our previous examples, in that we do not have formulas, only the sketch. But we can still compute the limits.

- Function on the left - $f(x)$ :

$$
\lim _{x \rightarrow 1^{-}} f(x)=2 \quad \lim _{x \rightarrow 1^{+}} f(x)=2
$$

so by the previous theorem

$$
\lim _{x \rightarrow 1} f(x)=2
$$

- Function on the right - $g(t)$ :

$$
\lim _{t \rightarrow 1^{-}} g(t)=2
$$

and $\lim _{t \rightarrow 1^{+}} g(t)=-2$
so by the previous theorem

$$
\lim _{t \rightarrow 1} g(t)=\mathrm{DNE}
$$

Example 1.3.9
We have seen 2 ways in which a limit does not exist - in one case the function oscillated wildly, and in the other there was some sort of "jump" in the function, so that the left-hand and right-hand limits were different.

There is a third way that we must also consider. To describe this, consider the following four functions:


None of these functions are defined at $x=a$, nor do the limits as $x$ approaches $a$ exist. However we can say more than just "the limits do not exist".

Notice that the value of function 1 can be made bigger and bigger as we bring $x$ closer and closer to $a$. Similarly the value of the second function can be made arbitrarily large and negative (i.e. make it as big a negative number as we want) by bringing $x$
closer and closer to $a$. Based on this observation we have the following definition.

## Definition 1.3.11

We write

$$
\lim _{x \rightarrow a} f(x)=+\infty
$$

when the value of the function $f(x)$ becomes arbitrarily large and positive as $x$ gets closer and closer to $a$, without being exactly $a$.
Similarly, we write

$$
\lim _{x \rightarrow a} f(x)=-\infty
$$

when the value of the function $f(x)$ becomes arbitrarily large and negative as $x$ gets closer and closer to $a$, without being exactly $a$.

A good examples of the above is

$$
\lim _{x \rightarrow 0} \frac{1}{x^{2}}=+\infty \quad \lim _{x \rightarrow 0}-\frac{1}{x^{2}}=-\infty
$$

IMPORTANT POINT: Please do not think of " $+\infty$ " and " $-\infty$ " in these statements as numbers. You should think of $\lim _{x \rightarrow a} f(x)=+\infty$ and $\lim _{x \rightarrow a} f(x)=-\infty$ as special cases of $\lim _{x \rightarrow a} f(x)=$ DNE. The statement

$$
\lim _{x \rightarrow a} f(x)=+\infty
$$

does not say "the limit of $f(x)$ as $x$ approaches $a$ is positive infinity". It says "the function $f(x)$ becomes arbitrarily large as $x$ approaches $a$ ". These are different statements; remember that $\infty$ is not a number ${ }^{4}$.

Now consider functions 3 and 4 in Figure 1.3.10. Here we can make the value of the function as big and positive as we want (for function 3) or as big and negative as we want (for function 4) but only when $x$ approaches $a$ from one side. With this in mind we can construct similar notation and a similar definition:

## Definition 1.3.12

We write

$$
\lim _{x \rightarrow a^{+}} f(x)=+\infty
$$

when the value of the function $f(x)$ becomes arbitrarily large and positive as $x$

4 One needs to be very careful making statements about infinity. At some point in our lives we get around to asking ourselves "what is the biggest number", and we realise there isn't one. That is, we can go on counting integer after integer, for ever and not stop. Indeed the set of integers is the first infinite thing we really encounter. It is an example of a countably infinite set. The set of real-numbers is actually much bigger and is uncountably infinite. In fact there are an infinite number of different sorts of infinity! Much of the theory of infinite sets was developed by Georg Cantor; we mentioned him back in Section 0.2 and he is well worth googling.
gets closer and closer to $a$ from above (equivalently - from the right), without being exactly $a$.
Similarly, we write

$$
\lim _{x \rightarrow a^{+}} f(x)=-\infty
$$

when the value of the function $f(x)$ becomes arbitrarily large and negative as $x$ gets closer and closer to $a$ from above (equivalently - from the right), without being exactly $a$.
The notation

$$
\lim _{x \rightarrow a^{-}} f(x)=+\infty \quad \lim _{x \rightarrow a^{-}} f(x)=-\infty
$$

has a similar meaning except that limits are approached from below / from the left.

So for function 3 we have

$$
\lim _{x \rightarrow a^{-}} f(x)=+\infty \quad \lim _{x \rightarrow a^{+}} f(x)=\text { some positive number }
$$

and for function 4

$$
\lim _{x \rightarrow a^{-}} f(x)=\text { some positive number } \quad \lim _{x \rightarrow a^{+}} f(x)=-\infty
$$

More examples:
Example 1.3.13 $\lim _{x \rightarrow \pi} \frac{1}{\sin (x)}$.
Consider the function

$$
g(x)=\frac{1}{\sin (x)}
$$

Find the one-sided limits of this function as $x \rightarrow \pi$.
Probably the easiest way to do this is to first plot the graph of $\sin (x)$ and $1 / x$ and then think carefully about the one-sided limits:


- As $x \rightarrow \pi$ from the left, $\sin (x)$ is a small positive number that is getting closer and closer to zero. That is, as $x \rightarrow \pi^{-}$, we have that $\sin (x) \rightarrow 0$ through positive numbers (i.e. from above). Now look at the graph of $1 / x$, and think what happens as we move $x \rightarrow 0^{+}$, the function is positive and becomes larger and larger.
So as $x \rightarrow \pi$ from the left, $\sin (x) \rightarrow 0$ from above, and so $1 / \sin (x) \rightarrow+\infty$.
- By very similar reasoning, as $x \rightarrow \pi$ from the right, $\sin (x)$ is a small negative number that gets closer and closer to zero. So as $x \rightarrow \pi$ from the right, $\sin (x) \rightarrow 0$ through negative numbers (i.e. from below) and so $1 / \sin (x)$ to $-\infty$.

Thus

$$
\lim _{x \rightarrow \pi^{-}} \frac{1}{\sin (x)}=+\infty \quad \lim _{x \rightarrow \pi^{+}} \frac{1}{\sin (x)}=-\infty
$$

Example 1.3.13
Again, we can make Definitions 1.3.11 and 1.3.12 into mathematically precise formal definitions using techniques very similar to those in the optional Section 1.7. This is not strictly necessary for this course.

Up to this point we explored limits by sketching graphs or plugging values into a calculator. This was done to help build intuition, but it is not really the basis of a systematic method for computing limits. We have also avoided more formal approaches ${ }^{5}$ since we do not have time in the course to go into that level of detail and (arguably) we don't need that detail to achieve the aims of the course. Thankfully we can develop a more systematic approach based on the idea of building up complicated limits from simpler ones by examining how limits interact with the basic operations of arithmetic.

### 1.3.2 $\leadsto$ Exercises

## Exercises - Stage 1

1. Given the function shown below, evaluate the following:
a $\lim _{x \rightarrow-2} f(x)$
b $\lim _{x \rightarrow 0} f(x)$
c $\lim _{x \rightarrow 2} f(x)$

5 The formal approaches are typically referred to as "epsilon-delta limits" or "epsilon-delta proofs" since the symbols $\epsilon$ and $\delta$ are traditionally used throughout. Take a peek at Section 1.7 to see.

2. Given the function shown below, evaluate $\lim _{x \rightarrow 0} f(x)$.

3. Given the function shown below, evaluate:
a $\lim _{x \rightarrow-1^{-}} f(x)$
$\mathrm{b} \lim _{x \rightarrow-1^{+}} f(x)$
c $\lim _{x \rightarrow-1} f(x)$
$\mathrm{d} \lim _{x \rightarrow-2^{+}} f(x)$
e $\lim _{x \rightarrow 2^{-}} f(x)$

4. Draw a curve $y=f(x)$ with $\lim _{x \rightarrow 3} f(x)=f(3)=10$.
5. Draw a curve $y=f(x)$ with $\lim _{x \rightarrow 3} f(x)=10$ and $f(3)=0$.
6. Suppose $\lim _{x \rightarrow 3} f(x)=10$. True or false: $f(3)=10$.
7. Suppose $f(3)=10$. True or false: $\lim _{x \rightarrow 3} f(x)=10$.
8. Suppose $f(x)$ is a function defined on all real numbers, and $\lim _{x \rightarrow-2} f(x)=16$. What is $\lim _{x \rightarrow-2^{-}} f(x)$ ?
9. Suppose $f(x)$ is a function defined on all real numbers, and $\lim _{x \rightarrow-2^{-}} f(x)=$ 16. What is $\lim _{x \rightarrow-2} f(x)$ ?

Exercises - Stage 2 In Questions 1.3.2.10 through 1.3.2.17, evaluate the given limits. If you aren't sure where to begin, it's nice to start by drawing the function.
10. $\lim _{t \rightarrow 0} \sin t$
11. $\lim _{x \rightarrow 0^{+}} \log x$
12. $\lim _{y \rightarrow 3} y^{2}$
13. $\lim _{x \rightarrow 0^{-}} \frac{1}{x}$
14. $\lim _{x \rightarrow 0} \frac{1}{x}$
15. $\lim _{x \rightarrow 0} \frac{1}{x^{2}}$
16. $\lim _{x \rightarrow 3} \frac{1}{10}$
17. $\lim _{x \rightarrow 3} f(x)$, where $f(x)=\left\{\begin{array}{ll}\sin x & x \leq 2.9 \\ x^{2} & x>2.9\end{array}\right.$.

## 1.4 • Calculating Limits with Limit Laws

### 1.4.1 $\rightarrow$ Calculating Limits with Limit Laws

Think back to the functions you know and the sorts of things you have been asked to draw, factor and so on. Then they are all constructed from simple pieces, such as

- constants - $c$
- monomials - $x^{n}$
- trigonometric functions - $\sin (x), \cos (x)$ and $\tan (x)$

These are the building blocks from which we construct functions. Soon we will add a few more functions to this list, especially the exponential function and various inverse functions.

We then take these building blocks and piece them together using arithmetic

- addition and subtraction - $f(x)=g(x)+h(x)$ and $f(x)=g(x)-h(x)$
- multiplication - $f(x)=g(x) \cdot h(x)$
- division - $f(x)=\frac{g(x)}{h(x)}$
- substitution - $f(x)=g(h(x))$ - this is also called the composition of $g$ with $h$.

The idea of building up complicated functions from simpler pieces was discussed in Section 0.5.

What we will learn in this section is how to compute the limits of the basic building blocks and then how we can compute limits of sums, products and so forth using "limit laws". This process allows us to compute limits of complicated functions, using very
simple tools and without having to resort to "plugging in numbers" or "closer and closer" or " $\epsilon-\delta$ arguments".

In the examples we saw above, almost all the interesting limits happened at points where the underlying function was badly behaved - where it jumped, was not defined or blew up to infinity. In those cases we had to be careful and think about what was happening. Thankfully most functions we will see do not have too many points at which these sorts of things happen.

For example, polynomials do not have any nasty jumps and are defined everywhere and do not "blow up". If you plot them, they look smooth ${ }^{1}$. Polynomials and limits behave very nicely together, and for any polynomial $P(x)$ and any real number $a$ we have that

$$
\lim _{x \rightarrow a} P(x)=P(a)
$$

That is - to evaluate the limit we just plug in the number. We will build up to this result over the next few pages.

Let us start with the two easiest limits ${ }^{2}$

## Theorem 1.4.1 Easiest limits.

Let $a, c \in \mathbb{R}$. The following two limits hold

$$
\lim _{x \rightarrow a} c=c \quad \text { and } \quad \lim _{x \rightarrow a} x=a
$$

Since we have not seen too many theorems yet, let us examine it carefully piece by piece.

- Let $a, c \in \mathbb{R}$ - just as was the case for definitions, we start a theorem by defining terms and setting the scene. There is not too much scene to set: the symbols $a$ and $c$ are real numbers.
- The following two limits hold - this doesn't really contribute much to the statement of the theorem, it just makes it easier to read.
- $\lim _{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{c}=\mathbf{c}$ - when we take the limit of a constant function (for example think of $c=3$ ), the limit is (unsurprisingly) just that same constant.
- $\lim _{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{x}=\mathbf{a}$ - as we noted above for general polynomials, the limit of the function $f(x)=x$ as $x$ approaches a given point $a$, is just $a$. This says something quite obvious - as $x$ approaches $a, x$ approaches $a$ (if you are not convinced then sketch the graph).

[^2]Armed with only these two limits, we cannot do very much. But combining these limits with some arithmetic we can do quite a lot. For a moment, take a step back from limits for a moment and think about how we construct functions. To make the discussion a little more precise think about how we might construct the function

$$
h(x)=\frac{2 x-3}{x^{2}+5 x-6}
$$

If we want to compute the value of the function at $x=2$, then we would

- compute the numerator at $x=2$
- compute the denominator at $x=2$
- compute the ratio

Now to compute the numerator we

- take $x$ and multiply it by 2
- subtract 3 to the result

While for the denominator

- multiply $x$ by $x$
- multiply $x$ by 5
- add these two numbers and subtract 6

This sequence of operations can be represented pictorially as the tree shown in Figure 1.4.2 below.


Figure 1.4.2

Such trees were discussed in Section 0.5 (now is not a bad time to quickly review that section before proceeding). The point here is that in order to compute the value of the function we just repeatedly add, subtract, multiply and divide constants and $x$.

To compute the limit of the above function at $x=2$ we can do something very similar. From the previous theorem we know how to compute

$$
\lim _{x \rightarrow 2} c=c \quad \text { and } \quad \lim _{x \rightarrow 2} x=2
$$

and the next theorem will tell us how to stitch together these two limits using the arithmetic we used to construct the function.

## Theorem 1.4.3 Arithmetic of limits.

Let $a, c \in \mathbb{R}$, let $f(x)$ and $g(x)$ be defined for all $x$ 's that lie in some interval about $a$ (but $f, g$ need not be defined exactly at $a$ ).

$$
\lim _{x \rightarrow a} f(x)=F \quad \lim _{x \rightarrow a} g(x)=G
$$

exist with $F, G \in \mathbb{R}$. Then the following limits hold

- $\lim _{x \rightarrow a}(f(x)+g(x))=F+G$ - limit of the sum is the sum of the limits.
- $\lim _{x \rightarrow a}(f(x)-g(x))=F-G$ - limit of the difference is the difference of the limits.
- $\lim _{x \rightarrow a} c f(x)=c F$.
- $\lim _{x \rightarrow a}(f(x) \cdot g(x))=F \cdot G$ - limit of the product is the product of limits.
- If $G \neq 0$ then $\lim _{x \rightarrow a} \frac{f(x)}{g(x)}=\frac{F}{G}$ and, in particular, $\lim _{x \rightarrow a} \frac{1}{g(x)}=\frac{1}{G}$.

Note - be careful with this last one - the denominator cannot be zero.

The above theorem shows that limits interact very simply with arithmetic. If you are asked to find the limit of a sum then the answer is just the sum of the limits. Similarly the limit of a product is just the product of the limits.

How do we apply the above theorem to the rational function $h(x)$ we defined above? Here is a warm-up example:

## Example 1.4.4 Using limit laws.

You are given two functions $f, g$ (not explicitly) which have the following limits as $x$ approaches 1:

$$
\lim _{x \rightarrow 1} f(x)=3 \quad \text { and } \quad \lim _{x \rightarrow 1} g(x)=2
$$

Using the above theorem we can compute

$$
\begin{aligned}
\lim _{x \rightarrow 1} 3 f(x) & =3 \times 3=9 \\
\lim _{x \rightarrow 1} 3 f(x)-g(x) & =3 \times 3-2=7 \\
\lim _{x \rightarrow 1} f(x) g(x) & =3 \times 2=6 \\
\lim _{x \rightarrow 1} \frac{f(x)}{f(x)-g(x)} & =\frac{3}{3-2}=3
\end{aligned}
$$

Example 1.4.4
Another simple example
Example 1.4.5 More using limit laws.
Find $\lim _{x \rightarrow 3} 4 x^{2}-1$
We use the arithmetic of limits:

$$
\begin{array}{rlrl}
\lim _{x \rightarrow 3} 4 x^{2}-1 & =\left(\lim _{x \rightarrow 3} 4 x^{2}\right)-\lim _{x \rightarrow 3} 1 & & \text { difference of limits } \\
& =\left(\lim _{x \rightarrow 3} 4 \cdot \lim _{x \rightarrow 3} x^{2}\right)-\lim _{x \rightarrow 3} 1 & & \text { product of limits } \\
& =4 \cdot\left(\lim _{x \rightarrow 3} x^{2}\right)-1 & & \text { limit of constant } \\
& =4 \cdot\left(\lim _{x \rightarrow 3} x\right) \cdot\left(\lim _{x \rightarrow 3} x\right)-1 & & \text { product of limits } \\
& =4 \cdot 3 \cdot 3-1 & & \text { limit of } x \\
& =36-1 & & \\
& =35 &
\end{array}
$$

This is an excruciating level of detail, but when you first use this theorem and try some examples it is a good idea to do things step by step by step until you are comfortable with it.

## Example 1.4.6 Yet more using limit laws.

Yet another limit - compute $\lim _{x \rightarrow 2} \frac{x}{x-1}$.
To apply the arithmetic of limits, we need to examine numerator and denominator separately and make sure the limit of the denominator is non-zero. Numerator first:

$$
\lim _{x \rightarrow 2} x=2 \quad \quad \text { limit of } x
$$

and now the denominator:

$$
\lim _{x \rightarrow 2} x-1=\left(\lim _{x \rightarrow 2} x\right)-\left(\lim _{x \rightarrow 2} 1\right) \quad \text { difference of limits }
$$

$$
=2-1 \quad \text { limit of } x \text { and limit of constant }=1
$$

Since the limit of the denominator is non-zero we can put it back together to get

$$
\begin{aligned}
\lim _{x \rightarrow 2} \frac{x}{x-1} & =\frac{\lim _{x \rightarrow 2} x}{\lim _{x \rightarrow 2}(x-1)} \\
& =\frac{2}{1} \\
& =2
\end{aligned}
$$

In the next example we show that many different things can happen if the limit of the denominator is zero.

## Example 1.4.7 Be careful with limits of ratios.

We must be careful when computing the limit of a ratio - it is the ratio of the limits except when the limit of the denominator is zero. When the limit of the denominator is zero Theorem 1.4.3 does not apply and a few interesting things can happen

- If the limit of the numerator is non-zero then the limit of the ratio does not exist

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)}=D N E \quad \text { when } \lim _{x \rightarrow a} f(x) \neq 0 \text { and } \lim _{x \rightarrow a} g(x)=0
$$

For example, $\lim _{x \rightarrow 0} \frac{1}{x^{2}}=D N E$.

- If the limit of the numerator is zero then the above theorem does not give us enough information to decide whether or not the limit exists. It is possible that
- the limit does not exist, eg. $\lim _{x \rightarrow 0} \frac{x}{x^{2}}=\lim _{x \rightarrow 0} \frac{1}{x}=D N E$
- the limit is $\pm \infty$, eg. $\lim _{x \rightarrow 0} \frac{x^{2}}{x^{4}}=\lim _{x \rightarrow 0} \frac{1}{x^{2}}=+\infty$ or $\lim _{x \rightarrow 0} \frac{-x^{2}}{x^{4}}=\lim _{x \rightarrow 0} \frac{-1}{x^{2}}=-\infty$.
- the limit is zero, eg. $\lim _{x \rightarrow 0} \frac{x^{2}}{x}=0$
- the limit exists and is non-zero, eg. $\lim _{x \rightarrow 0} \frac{x}{x}=1$

Now while the above examples are very simple and a little contrived they serve to illustrate the point we are trying to make - be careful if the limit of the denominator is zero.

We now have enough theory to return to our rational function and compute its limit as $x$ approaches 2 .

Example 1.4.8 More on limits of ratios.
Let $h(x)=\frac{2 x-3}{x^{2}+5 x-6}$ and find its limit as $x$ approaches 2 .
Since this is the limit of a ratio, we compute the limit of the numerator and denominator separately. Numerator first:

$$
\begin{array}{rlr}
\lim _{x \rightarrow 2} 2 x-3 & =\left(\lim _{x \rightarrow 2} 2 x\right)-\left(\lim _{x \rightarrow 2} 3\right) & \text { difference of limits } \\
& =2 \cdot\left(\lim _{x \rightarrow 2} x\right)-3 & \text { product of limits and limit of constant } \\
& =2 \cdot 2-3 & \text { limits of } x \\
& =1 &
\end{array}
$$

Denominator next:

$$
\begin{array}{rlr}
\lim _{x \rightarrow 2} x^{2}+5 x-6 & =\left(\lim _{x \rightarrow 2} x^{2}\right)+\left(\lim _{x \rightarrow 2} 5 x\right)-\left(\lim _{x \rightarrow 2} 6\right) & \\
& =\left(\lim _{x \rightarrow 2} x\right) \cdot\left(\lim _{x \rightarrow 2} x\right)+5 \cdot\left(\lim _{x \rightarrow 2} x\right)-6 & \\
& \quad \text { product of limits and limit of constant } & \\
& =2 \cdot 2+5 \cdot 2-6 & \text { limits of } x \\
& =8 &
\end{array}
$$

Since the limit of the denominator is non-zero, we can obtain our result by taking the ratio of the separate limits.

$$
\lim _{x \rightarrow 2} \frac{2 x-3}{x^{2}+5 x-6}=\frac{\lim _{x \rightarrow 2} 2 x-3}{\lim _{x \rightarrow 2} x^{2}+5 x-6}=\frac{1}{8}
$$

The above works out quite simply. However, if we were to take the limit as $x \rightarrow 1$ then things are a bit harder. The limit of the numerator is:

$$
\lim _{x \rightarrow 1} 2 x-3=2 \cdot 1-3=-1
$$

(we have not listed all the steps). And the limit of the denominator is

$$
\lim _{x \rightarrow 1} x^{2}+5 x-6=1 \cdot 1+5-6=0
$$

Since the limit of the numerator is non-zero, while the limit of the denominator is zero, the limit of the ratio does not exist.

$$
\lim _{x \rightarrow 1} \frac{2 x-3}{x^{2}+5 x-6}=D N E
$$

It is IMPORTANT TO NOTE that it is not correct to write

$$
\lim _{x \rightarrow 1} \frac{2 x-3}{x^{2}+5 x-6}=\frac{-1}{0}=D N E
$$

Because we can only write

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)}=\frac{\lim _{x \rightarrow a} f(x)}{\lim _{x \rightarrow a} g(x)}=\text { something }
$$

when the limit of the denominator is non-zero (see Example 1.4.7 above).
With a little care you can use the arithmetic of limits to obtain the following rules for limits of powers of functions and limits of roots of functions:

## Theorem 1.4.9 More arithmetic of limits - powers and roots.

Let $n$ be a positive integer, let $a \in \mathbb{R}$ and let $f$ be a function so that

$$
\lim _{x \rightarrow a} f(x)=F
$$

for some real number $F$. Then the following holds

$$
\lim _{x \rightarrow a}(f(x))^{n}=\left(\lim _{x \rightarrow a} f(x)\right)^{n}=F^{n}
$$

so that the limit of a power is the power of the limit. Similarly, if

- $n$ is an even number and $F>0$, or
- $n$ is an odd number and $F$ is any real number
then

$$
\lim _{x \rightarrow a}(f(x))^{1 / n}=\left(\lim _{x \rightarrow a} f(x)\right)^{1 / n}=F^{1 / n}
$$

More generally ${ }^{a}$, if $F>0$ and $p$ is any real number,

$$
\lim _{x \rightarrow a}(f(x))^{p}=\left(\lim _{x \rightarrow a} f(x)\right)^{p}=F^{p}
$$

$a \quad$ You may not know the definition of the power $b^{p}$ when $p$ is not a rational number, so here it is. If $b>0$ and $p$ is any real number, then $b^{p}$ is the limit of $b^{r}$ as $r$ approaches $p$ through rational numbers. We won't do so here, but it is possible to prove that the limit exists.

Notice that we have to be careful when taking roots of limits that might be negative numbers. To see why, consider the case $n=2$, the limit

$$
\lim _{x \rightarrow 4} x^{1 / 2}=4^{1 / 2}=2
$$

$$
\lim _{x \rightarrow 4}(-x)^{1 / 2}=(-4)^{1 / 2}=\text { not a real number }
$$

In order to evaluate such limits properly we need to use complex numbers which are beyond the scope of this text.

Also note that the notation $x^{1 / 2}$ refers to the positive square root of $x$. While 2 and $(-2)$ are both square-roots of 4 , the notation $4^{1 / 2}$ means 2 . This is something we must be careful of ${ }^{3}$.

So again - let us do a few examples and carefully note what we are doing.

## Example 1.4.10 $\lim _{x \rightarrow 2}\left(4 x^{2}-3\right)^{1 / 3}$.

$$
\begin{aligned}
\lim _{x \rightarrow 2}\left(4 x^{2}-3\right)^{1 / 3} & =\left(\left(\lim _{x \rightarrow 2} 4 x^{2}\right)-\left(\lim _{x \rightarrow 2} 3\right)\right)^{1 / 3} \\
& =\left(4 \cdot 2^{2}-3\right)^{1 / 3} \\
& =(16-3)^{1 / 3} \\
& =13^{1 / 3}
\end{aligned}
$$

By combining the last few theorems we can make the evaluation of limits of polynomials and rational functions much easier:

## Theorem 1.4.11 Limits of polynomials and rational functions.

Let $a \in \mathbb{R}$, let $P(x)$ be a polynomial and let $R(x)$ be a rational function. Then

$$
\lim _{x \rightarrow a} P(x)=P(a)
$$

and provided $R(x)$ is defined at $x=a$ then

$$
\lim _{x \rightarrow a} R(x)=R(a)
$$

If $R(x)$ is not defined at $x=a$ then we are not able to apply this result.

So the previous examples are now much easier to compute:

$$
\begin{array}{rlrl}
\lim _{x \rightarrow 2} \frac{2 x-3}{x^{2}+5 x-6} & = & \frac{4-3}{4+10-6} & = \\
\lim _{x \rightarrow 2}\left(4 x^{2}-1\right) & = & \frac{1}{8} \\
\lim _{x \rightarrow 2} \frac{x}{x-1} & = & \frac{2}{2-1} & =
\end{array}
$$

3 Like ending sentences in prepositions - "This is something up with which we will not put." This quote is attributed to Churchill though there is some dispute as to whether or not he really said it.

It is clear that limits of polynomials are very easy, while those of rational functions are easy except when the denominator might go to zero. We have seen examples where the resulting limit does not exist, and some where it does. We now work to explain this more systematically. The following example demonstrates that it is sometimes possible to take the limit of a rational function to a point at which the denominator is zero. Indeed we must be able to do exactly this in order to be able to define derivatives in the next chapter.

Example 1.4.12 Numerator and denominator both go to 0 .
Consider the limit

$$
\lim _{x \rightarrow 1} \frac{x^{3}-x^{2}}{x-1} .
$$

If we try to apply the arithmetic of limits then we compute the limits of the numerator and denominator separately

$$
\begin{array}{r}
\lim _{x \rightarrow 1} x^{3}-x^{2}=1-1=0 \\
\lim _{x \rightarrow 1} x-1=1-1=0
\end{array}
$$

Since the denominator is zero, we cannot apply our theorem and we are, for the moment, stuck. However, there is more that we can do here - the hint is that the numerator and denominator both approach zero as $x$ approaches 1 . This means that there might be something we can cancel.
So let us play with the expression a little more before we take the limit:

$$
\frac{x^{3}-x^{2}}{x-1}=\frac{x^{2}(x-1)}{x-1}=x^{2} \quad \text { provided } x \neq 1
$$

So what we really have here is the following function

$$
\frac{x^{3}-x^{2}}{x-1}= \begin{cases}x^{2} & x \neq 1 \\ \text { undefined } & x=1\end{cases}
$$

If we plot the above function the graph looks exactly the same as $y=x^{2}$ except that the function is not defined at $x=1$ (since at $x=1$ both numerator and denominator are zero).


When we compute a limit as $x \rightarrow a$, the value of the function exactly at $x=a$ is irrelevant. We only care what happens to the function as we bring $x$ very close to $a$. So for the above problem we can write

$$
\frac{x^{3}-x^{2}}{x-1}=x^{2} \quad \text { when } x \text { is close to } 1 \text { but not at } x=1
$$

So the limit as $x \rightarrow 1$ of the function is the same as the limit $\lim _{x \rightarrow 1} x^{2}$ since the functions are the same except exactly at $x=1$. By this reasoning we get

$$
\lim _{x \rightarrow 1} \frac{x^{3}-x^{2}}{x-1}=\lim _{x \rightarrow 1} x^{2}=1
$$

The reasoning in the above example can be made more general:

## Theorem 1.4.13

If $f(x)=g(x)$ except when $x=a$ then $\lim _{x \rightarrow a} f(x)=\lim _{x \rightarrow a} g(x)$ provided the limit of $g$ exists.

How do we know when to use this theorem? The big clue is that when we try to compute the limit in a naive way, we end up with $\frac{0}{0}$. We know that $\frac{0}{0}$ does not make sense, but it is an indication that there might be a common factor between numerator and denominator that can be cancelled. In the previous example, this common factor was $(x-1)$.

## Example 1.4.14 Another numerator and denominator both go to 0 limit.

Using this idea compute

$$
\lim _{h \rightarrow 0} \frac{(1+h)^{2}-1}{h}
$$

- First we should check that we cannot just substitute $h=0$ into this - clearly we cannot because the denominator would be 0 .
- But we should also check the numerator to see if we have $\frac{0}{0}$, and we see that the numerator gives us $1-1=0$.
- Thus we have a hint that there is a common factor that we might be able to cancel. So now we look for the common factor and try to cancel it.

$$
\frac{(1+h)^{2}-1}{h}=\frac{1+2 h+h^{2}-1}{h}
$$

expand

$$
\begin{aligned}
& =\frac{2 h+h^{2}}{h}=\frac{h(2+h)}{h} \quad \text { factor and then cancel } \\
& =2+h
\end{aligned}
$$

- Thus we really have that

$$
\frac{(1+h)^{2}-1}{h}= \begin{cases}2+h & h \neq 0 \\ \text { undefined } & h=0\end{cases}
$$

and because of this

$$
\begin{aligned}
\lim _{h \rightarrow 0} \frac{(1+h)^{2}-1}{h} & =\lim _{h \rightarrow 0} 2+h \\
& =2
\end{aligned}
$$

Example 1.4.14
Of course - we have written everything out in great detail here and that is way more than is required for a solution to such a problem. Let us do it again a little more succinctly.

Example 1.4.15 $\lim _{h \rightarrow 0} \frac{(1+h)^{2}-1}{h}$.
Compute the following limit:

$$
\lim _{h \rightarrow 0} \frac{(1+h)^{2}-1}{h}
$$

If we try to use the arithmetic of limits, then we see that the limit of the numerator and the limit of the denominator are both zero. Hence we should try to factor them and cancel any common factor. This gives

$$
\begin{aligned}
\lim _{h \rightarrow 0} \frac{(1+h)^{2}-1}{h} & =\lim _{h \rightarrow 0} \frac{1+2 h+h^{2}-1}{h} \\
& =\lim _{h \rightarrow 0} 2+h \\
& =2
\end{aligned}
$$

Notice that even though we did this example carefully above, we have still written some text in our working explaining what we have done. You should always think about the reader and if in doubt, put in more explanation rather than less. We could make the above example even more terse

Example 1.4.16 Redoing previous example with fewer words.
Compute the following limit:

$$
\lim _{h \rightarrow 0} \frac{(1+h)^{2}-1}{h}
$$

Numerator and denominator both go to zero as $h \rightarrow 0$. So factor and simplify:

$$
\begin{aligned}
\lim _{h \rightarrow 0} \frac{(1+h)^{2}-1}{h} & =\lim _{h \rightarrow 0} \frac{1+2 h+h^{2}-1}{h} \\
& =\lim _{h \rightarrow 0} 2+h=2
\end{aligned}
$$

A slightly harder one now
Example 1.4.17 A harder limit with cancellations.
Compute the limit

$$
\lim _{x \rightarrow 0} \frac{x}{\sqrt{1+x}-1}
$$

If we try to use the arithmetic of limits we get

$$
\begin{aligned}
\lim _{x \rightarrow 0} x & =0 \\
\lim _{x \rightarrow 0} \sqrt{1+x}-1 & =\sqrt{\lim _{x \rightarrow 0} 1+x}-1=1-1=0
\end{aligned}
$$

So doing the naive thing we'd get $0 / 0$. This suggests a common factor that can be cancelled. Since the numerator and denominator are not polynomials we have to try other tricks ${ }^{a}$. We can simplify the denominator $\sqrt{1+x}-1$ a lot, and in particular eliminate the square root, by multiplying it by its conjugate $\sqrt{1+x}+1$.

$$
\begin{array}{rlr}
\frac{x}{\sqrt{1+x}-1} & =\frac{x}{\sqrt{1+x}-1} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} & \text { multiply by } \frac{\text { conjugate }}{\text { conjugate }}=1 \\
& =\frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)} & \text { bring things together } \\
& =\frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x})^{2}-1 \cdot 1} & \text { since }(a-b)(a+b)=a^{2}-b^{2} \\
& =\frac{x(\sqrt{1+x}+1)}{1+x-1} & \\
& =\frac{x(\sqrt{1+x}+1)}{x} & \text { clean up a little } \\
& =\sqrt{1+x}+1 &
\end{array}
$$

So now we have

$$
\begin{aligned}
\lim _{x \rightarrow 0} \frac{x}{\sqrt{1+x}-1} & =\lim _{x \rightarrow 0} \sqrt{1+x}+1 \\
& =\sqrt{1+0}+1=2
\end{aligned}
$$

$a$ While these tricks are useful (and even cute ${ }^{b}$ - this footnote is better in the online edition), Taylor polynomials (see Section 3.4) give us a more systematic way of approaching this problem.

Example 1.4.17
How did we know what to multiply by? Our function was of the form

$$
\frac{a}{\sqrt{b}-c}
$$

so, to eliminate the square root from the denominator, we employ a trick - we multiply by 1 . Of course, multiplying by 1 doesn't do anything. But if you multiply by 1 carefully you can leave the value the same, but change the form of the expression. More precisely

$$
\begin{array}{rlr}
\frac{a}{\sqrt{b}-c} & =\frac{a}{\sqrt{b}-c} \cdot 1 \\
& =\frac{a}{\sqrt{b}-c} \cdot \underbrace{\frac{\sqrt{b}+c}{\sqrt{b}+c}}_{=1} \\
& =\frac{a(\sqrt{b}+c)}{(\sqrt{b}-c)(\sqrt{b}+c)} \quad \text { expand denominator carefully } \\
& =\frac{a(\sqrt{b}+c)}{\sqrt{b} \cdot \sqrt{b}-c \sqrt{b}+c \sqrt{b}-c \cdot c} \quad \text { do some cancellation } \\
& =\frac{a(\sqrt{b}+c)}{b-c^{2}} &
\end{array}
$$

Now the numerator contains roots, but the denominator is just a polynomial.
Before we move on to limits at infinity, there is one more theorem to see. While the scope of its application is quite limited, it can be extremely useful. It is called a sandwich theorem or a squeeze theorem for reasons that will become apparent.

Sometimes one is presented with an unpleasant ugly function such as

$$
f(x)=x^{2} \sin (\pi / x)
$$

It is a fact of life, that not all the functions that are encountered in mathematics will be elegant and simple; this is especially true when the mathematics gets applied to real world problems. One just has to work with what one gets. So how can we compute

$$
\lim _{x \rightarrow 0} x^{2} \sin (\pi / x) ?
$$

Since it is the product of two functions, we might try

$$
\begin{aligned}
\lim _{x \rightarrow 0} x^{2} \sin (\pi / x) & =\left(\lim _{x \rightarrow 0} x^{2}\right) \cdot\left(\lim _{x \rightarrow 0} \sin (\pi / x)\right) \\
& =0 \cdot\left(\lim _{x \rightarrow 0} \sin (\pi / x)\right) \\
& =0
\end{aligned}
$$

But we just cheated - we cannot use the arithmetic of limits theorem here, because the limit

$$
\lim _{x \rightarrow 0} \sin (\pi / x)=D N E
$$

does not exist. Now we did see the function $\sin (\pi / x)$ before (in Example 1.3.5), so you should go back and look at it again. Unfortunately the theorem "the limit of a product is the product of the limits" only holds when the limits you are trying to multiply together actually exist. So we cannot use it.

However, we do see that the function naturally decomposes into the product of two pieces - the functions $x^{2}$ and $\sin (\pi / x)$. We have sketched the two functions in the figure on the left below.


While $x^{2}$ is a very well behaved function and we know quite a lot about it, the function $\sin (\pi / x)$ is quite ugly. One of the few things we can say about it is the following

$$
-1 \leq \sin (\pi / x) \leq 1 \quad \text { provided } x \neq 0
$$

But if we multiply this expression by $x^{2}$ we get (because $x^{2} \geq 0$ )

$$
-x^{2} \leq x^{2} \sin (\pi / x) \leq x^{2} \quad \text { provided } x \neq 0
$$

and we have sketched the result in the figure above (on the right). So the function we are interested in is squeezed or sandwiched between the functions $x^{2}$ and $-x^{2}$.

If we focus in on the picture close to $x=0$ we see that $x$ approaches 0 , the functions $x^{2}$ and $-x^{2}$ both approach 0 . Further, because $x^{2} \sin (\pi / x)$ is sandwiched between them, it seems that it also approaches 0 .

The following theorem tells us that this is indeed the case:

Theorem 1.4.18 Squeeze theorem (or sandwich theorem or pinch theorem).
Let $a \in \mathbb{R}$ and let $f, g, h$ be three functions so that

$$
f(x) \leq g(x) \leq h(x)
$$

for all $x$ in an interval around $a$, except possibly exactly at $x=a$. Then if

$$
\lim _{x \rightarrow a} f(x)=\lim _{x \rightarrow a} h(x)=L
$$

then it is also the case that

$$
\lim _{x \rightarrow a} g(x)=L
$$

Using the above theorem we can compute the limit we want and write it up nicely.

Example 1.4.19 $\lim _{x \rightarrow 0} x^{2} \sin (\pi / x)$.
Compute the limit

$$
\lim _{x \rightarrow 0} x^{2} \sin (\pi / x)
$$

Since $-1 \leq \sin (\theta) \leq 1$ for all real numbers $\theta$, we have

$$
-1 \leq \sin (\pi / x) \leq 1 \quad \text { for all } x \neq 0
$$

Multiplying the above by $x^{2}$ we see that

$$
-x^{2} \leq x^{2} \sin (\pi / x) \leq x^{2} \quad \text { for all } x \neq 0
$$

Since $\lim _{x \rightarrow 0} x^{2}=\lim _{x \rightarrow 0}\left(-x^{2}\right)=0$ by the sandwich (or squeeze or pinch) theorem we have

$$
\lim _{x \rightarrow 0} x^{2} \sin (\pi / x)=0
$$

Notice how we have used "words". We have remarked on this several times already in the text, but we will keep mentioning it. It is okay to use words in your answers to maths problems - and you should do so! These let the reader know what you are doing and help you understand what you are doing.

Example 1.4.20 Another sandwich theorem example.
Let $f(x)$ be a function such that $1 \leq f(x) \leq x^{2}-2 x+2$. What is $\lim _{x \rightarrow 1} f(x)$ ?
We are already supplied with an inequality, so it is likely that it is going to help us.

We should examine the limits of each side to see if they are the same:

$$
\begin{aligned}
\lim _{x \rightarrow 1} 1 & =1 \\
\lim _{x \rightarrow 1} x^{2}-2 x+2 & =1-2+2=1
\end{aligned}
$$

So we see that the function $f(x)$ is trapped between two functions that both approach 1 as $x \rightarrow 1$. Hence by the sandwich / pinch / squeeze theorem, we know that

$$
\lim _{x \rightarrow 1} f(x)=1
$$

To get some intuition as to why the squeeze theorem is true, consider when $x$ is very very close to $a$. In particular, consider when $x$ is sufficiently close to $a$ that we know $h(x)$ is within $10^{-6}$ of $L$ and that $f(x)$ is also within $10^{-6}$ of $L$. That is

$$
|h(x)-L|<10^{-6} \quad \text { and } \quad|f(x)-L|<10^{-6}
$$

This means that

$$
L-10^{-6}<f(x) \leq h(x)<L+10^{-6}
$$

since we know that $f(x) \leq h(x)$.
But now by the hypothesis of the squeeze theorem we know that $f(x) \leq g(x) \leq h(x)$ and so we have

$$
L-10^{-6}<f(x) \leq g(x) \leq h(x)<L+10^{-6}
$$

And thus we know that

$$
L-10^{-6} \leq g(x) \leq L+10^{-6}
$$

That is $g(x)$ is also within $10^{-6}$ of $L$.
In this argument our choice of $10^{-6}$ was arbitrary, so we can really replace $10^{-6}$ with any small number we like. Hence we know that we can force $g(x)$ as close to $L$ as we like, by bringing $x$ sufficiently close to $a$. We give a more formal and rigorous version of this argument at the end of Section 1.9.

### 1.4.2 $\leadsto$ Exercises

## Exercises - Stage 1

1. Suppose $\lim _{x \rightarrow a} f(x)=0$ and $\lim _{x \rightarrow a} g(x)=0$. Which of the following limits can you compute, given this information?
a $\lim _{x \rightarrow a} \frac{f(x)}{2}$
b $\lim _{x \rightarrow a} \frac{2}{f(x)}$
c $\lim _{x \rightarrow a} \frac{f(x)}{g(x)}$
d $\lim _{x \rightarrow a} f(x) g(x)$
2. Give two functions $f(x)$ and $g(x)$ that satisfy $\lim _{x \rightarrow 3} f(x)=\lim _{x \rightarrow 3} g(x)=0$ and $\lim _{x \rightarrow 3} \frac{f(x)}{g(x)}=10$.
3. Give two functions $f(x)$ and $g(x)$ that satisfy $\lim _{x \rightarrow 3} f(x)=\lim _{x \rightarrow 3} g(x)=0$ and $\lim _{x \rightarrow 3} \frac{f(x)}{g(x)}=0$.
4. Give two functions $f(x)$ and $g(x)$ that satisfy $\lim _{x \rightarrow 3} f(x)=\lim _{x \rightarrow 3} g(x)=0$ and $\lim _{x \rightarrow 3} \frac{f(x)}{g(x)}=\infty$.
5. Suppose $\lim _{x \rightarrow a} f(x)=\lim _{x \rightarrow a} g(x)=0$. What are the possible values of $\lim _{x \rightarrow a} \frac{f(x)}{g(x)}$ ?

Exercises - Stage 2 For Questions 1.4.2.6 through 1.4.2.41, evaluate the given limits.
6. $\lim _{t \rightarrow 10} \frac{2(t-10)^{2}}{t}$
7. $\lim _{y \rightarrow 0} \frac{(y+1)(y+2)(y+3)}{\cos y}$
8. $\lim _{x \rightarrow 3}\left(\frac{4 x-2}{x+2}\right)^{4}$
9. *. $\lim _{t \rightarrow-3}\left(\frac{1-t}{\cos (t)}\right)$
10. *. $\lim _{h \rightarrow 0} \frac{(2+h)^{2}-4}{2 h}$
11. *. $\lim _{t \rightarrow-2}\left(\frac{t-5}{t+4}\right)$
12. *. $\lim _{x \rightarrow 1} \sqrt{5 x^{3}+4}$
13. *. $\lim _{t \rightarrow-1}\left(\frac{t-2}{t+3}\right)$
14. *. $\lim _{x \rightarrow 1} \frac{\log (1+x)-x}{x^{2}}$
15. *. $\lim _{x \rightarrow 2}\left(\frac{x-2}{x^{2}-4}\right)$
16. *. $\lim _{x \rightarrow 4} \frac{x^{2}-4 x}{x^{2}-16}$
17. *. $\lim _{x \rightarrow 2} \frac{x^{2}+x-6}{x-2}$
18. *. $\lim _{x \rightarrow-3} \frac{x^{2}-9}{x+3}$
19. $\lim _{t \rightarrow 2} \frac{1}{2} t^{4}-3 t^{3}+t$
20. *. $\lim _{x \rightarrow-1} \frac{\sqrt{x^{2}+8}-3}{x+1}$.
21. *. $\lim _{x \rightarrow 2} \frac{\sqrt{x+7}-\sqrt{11-x}}{2 x-4}$.
22. *. $\lim _{x \rightarrow 1} \frac{\sqrt{x+2}-\sqrt{4-x}}{x-1}$
23. *. $\lim _{x \rightarrow 3} \frac{\sqrt{x-2}-\sqrt{4-x}}{x-3}$.
24. *. $\lim _{t \rightarrow 1} \frac{3 t-3}{2-\sqrt{5-t}}$.
25. $\lim _{x \rightarrow 0}-x^{2} \cos \left(\frac{3}{x}\right)$
26. $\lim _{x \rightarrow 0} \frac{x^{4} \sin \left(\frac{1}{x}\right)+5 x^{2} \cos \left(\frac{1}{x}\right)+2}{(x-2)^{2}}$
27. *. $\lim _{x \rightarrow 0} x \sin ^{2}\left(\frac{1}{x}\right)$
28. $\lim _{w \rightarrow 5} \frac{2 w^{2}-50}{(w-5)(w-1)}$
29. $\lim _{r \rightarrow-5} \frac{r}{r^{2}+10 r+25}$
30. $\lim _{x \rightarrow-1} \sqrt{\frac{x^{3}+x^{2}+x+1}{3 x+3}}$
31. $\lim _{x \rightarrow 0} \frac{x^{2}+2 x+1}{3 x^{5}-5 x^{3}}$
32. $\lim _{t \rightarrow 7} \frac{t^{2} x^{2}+2 t x+1}{t^{2}-14 t+49}$, where $x$ is a positive constant
33. $\lim _{d \rightarrow 0} x^{5}-32 x+15$, where $x$ is a constant
34. $\lim _{x \rightarrow 1}(x-1)^{2} \sin \left[\left(\frac{x^{2}-3 x+2}{x^{2}-2 x+1}\right)^{2}+15\right]$
35. *. Evaluate $\lim _{x \rightarrow 0} x^{1 / 101} \sin \left(x^{-100}\right)$ or explain why this limit does not exist.
36. *. $\lim _{x \rightarrow 2} \frac{x^{2}-4}{x^{2}-2 x}$
37. $\lim _{x \rightarrow 5} \frac{(x-5)^{2}}{x+5}$
38. Evaluate $\lim _{t \rightarrow \frac{1}{2}} \frac{\frac{1}{3 t^{2}}+\frac{1}{t^{2}-1}}{2 t-1}$.
39. Evaluate $\lim _{x \rightarrow 0}\left(3+\frac{|x|}{x}\right)$.
40. Evaluate $\lim _{d \rightarrow-4} \frac{|3 d+12|}{d+4}$
41. Evaluate $\lim _{x \rightarrow 0} \frac{5 x-9}{|x|+2}$.
42. Suppose $\lim _{x \rightarrow-1} f(x)=-1$. Evaluate $\lim _{x \rightarrow-1} \frac{x f(x)+3}{2 f(x)+1}$.
43. *. Find the value of the constant $a$ for which $\lim _{x \rightarrow-2} \frac{x^{2}+a x+3}{x^{2}+x-2}$ exists.
44. Suppose $f(x)=2 x$ and $g(x)=\frac{1}{x}$. Evaluate the following limits.
a $\lim _{x \rightarrow 0} f(x)$
b $\lim _{x \rightarrow 0} g(x)$
c $\lim _{x \rightarrow 0} f(x) g(x)$
d $\lim _{x \rightarrow 0} \frac{f(x)}{g(x)}$
e $\lim _{x \rightarrow 2}[f(x)+g(x)]$
f $\lim _{x \rightarrow 0} \frac{f(x)+1}{g(x+1)}$

## Exercises - Stage 3

45. The curve $y=f(x)$ is shown in the graph below. Sketch the graph of $y=\frac{1}{f(x)}$.

46. The graphs of functions $f(x)$ and $g(x)$ are shown in the graphs below. Use these to sketch the graph of $\frac{f(x)}{g(x)}$.

47. Suppose the position of a white ball, at time $t$, is given by $s(t)$, and the position of a red ball is given by $2 s(t)$. Using the definition from Section 1.2 of the velocity of a particle, and the limit laws from this section, answer the following question: if the white ball has velocity 5 at time $t=1$, what is the velocity of the red ball?
48. Let $f(x)=\frac{1}{x}$ and $g(x)=\frac{-1}{x}$.
a Evaluate $\lim _{x \rightarrow 0} f(x)$ and $\lim _{x \rightarrow 0} g(x)$.
b Evaluate $\lim _{x \rightarrow 0}[f(x)+g(x)]$
c Is it always true that $\lim _{x \rightarrow a}[f(x)+g(x)]=\lim _{x \rightarrow a} f(x)+\lim _{x \rightarrow a} g(x)$ ?
49. Suppose

$$
f(x)= \begin{cases}x^{2}+3 & \text { if } x>0 \\ 0 & \text { if } x=0 \\ x^{2}-3 & \text { if } x<0\end{cases}
$$

a Evaluate $\lim _{x \rightarrow 0^{-}} f(x)$.
b Evaluate $\lim _{x \rightarrow 0^{+}} f(x)$.
c Evaluate $\lim _{x \rightarrow 0} f(x)$.
50. Suppose

$$
f(x)= \begin{cases}\frac{x^{2}+8 x+16}{x^{2}+30 x-4} & \text { if } x>-4 \\ x^{3}+8 x^{2}+16 x & \text { if } x \leq-4\end{cases}
$$

a Evaluate $\lim _{x \rightarrow-4^{-}} f(x)$.
b Evaluate $\lim _{x \rightarrow-4^{+}} f(x)$.
c Evaluate $\lim _{x \rightarrow-4} f(x)$.

### 1.5 4 Limits at Infinity

### 1.5.1 $\leadsto$ Limits at Infinity

Up until this point we have discussed what happens to a function as we move its input $x$ closer and closer to a particular point $a$. For a great many applications of limits we need to understand what happens to a function when its input becomes extremely large - for example what happens to a population at a time far in the future.

The definition of a limit at infinity has a similar flavour to the definition of limits at finite points that we saw above, but the details are a little different. We also need to distinguish between positive and negative infinity. As $x$ becomes very large and positive it moves off towards $+\infty$ but when it becomes very large and negative it moves off towards $-\infty$.

Again we give an informal definition; the full formal definition can be found in (the optional) Section 1.8 near the end of this chapter.

## Definition 1.5.1 Limits at infinity - informal.

We write

$$
\lim _{x \rightarrow \infty} f(x)=L
$$

when the value of the function $f(x)$ gets closer and closer to $L$ as we make $x$ larger and larger and positive.
Similarly we write

$$
\lim _{x \rightarrow-\infty} f(x)=L
$$

when the value of the function $f(x)$ gets closer and closer to $L$ as we make $x$ larger and larger and negative.

Example 1.5.2 Limits to $+\infty$ and $-\infty$.
Consider the two functions depicted below



The dotted horizontal lines indicate the behaviour as $x$ becomes very large. The function on the left has limits as $x \rightarrow \infty$ and as $x \rightarrow-\infty$ since the function "settles down" to a particular value. On the other hand, the function on the right does not have a limit as $x \rightarrow-\infty$ since the function just keeps getting bigger and bigger.

Example 1.5.2
Just as was the case for limits as $x \rightarrow a$ we will start with two very simple building blocks and build other limits from those.

## Theorem 1.5.3

Let $c \in \mathbb{R}$ then the following limits hold

$$
\begin{array}{ll}
\lim _{x \rightarrow \infty} c=c & \lim _{x \rightarrow-\infty} c=c \\
\lim _{x \rightarrow \infty} \frac{1}{x}=0 & \lim _{x \rightarrow-\infty} \frac{1}{x}=0
\end{array}
$$

Again, these limits interact nicely with standard arithmetic:

## Theorem 1.5.4 Arithmetic of limits at infinity.

Let $f(x), g(x)$ be two functions for which the limits

$$
\lim _{x \rightarrow \infty} f(x)=F \quad \lim _{x \rightarrow \infty} g(x)=G
$$

exist. Then the following limits hold

$$
\begin{aligned}
\lim _{x \rightarrow \infty} f(x) \pm g(x) & =F \pm G \\
\lim _{x \rightarrow \infty} f(x) g(x) & =F G \\
\lim _{x \rightarrow \infty} \frac{f(x)}{g(x)} & =\frac{F}{G}
\end{aligned}
$$

and for real numbers $p$

$$
\lim _{x \rightarrow \infty} f(x)^{p}=F^{p} \quad \text { provided } F^{p} \text { and } f(x)^{p} \text { are defined for all } x
$$

The analogous results hold for limits to $-\infty$.

Note that, as was the case in Theorem 1.4.9, we need a little extra care with powers of functions. We must avoid taking square roots of negative numbers, or indeed any even root of a negative number ${ }^{1}$.

Hence we have for all rational $r>0$

$$
\lim _{x \rightarrow \infty} \frac{1}{x^{r}}=0
$$

but we have to be careful with

$$
\lim _{x \rightarrow-\infty} \frac{1}{x^{r}}=0
$$

This is only true if the denominator of $r$ is not an even number ${ }^{2}$.
For example

- $\lim _{x \rightarrow \infty} \frac{1}{x^{1 / 2}}=0$, but $\lim _{x \rightarrow-\infty} \frac{1}{x^{1 / 2}}$ does not exist, because $x^{1 / 2}$ is not defined for $x<0$.
- On the other hand, $x^{4 / 3}$ is defined for negative values of $x$ and $\lim _{x \rightarrow-\infty} \frac{1}{x^{4 / 3}}=0$.

Our first application of limits at infinity will be to examine the behaviour of a rational function for very large $x$. To do this we use a "trick".

Example 1.5.5 $\lim _{x \rightarrow \infty} \frac{x^{2}-3 x+4}{3 x^{2}+8 x+1}$.
Compute the following limit:

$$
\lim _{x \rightarrow \infty} \frac{x^{2}-3 x+4}{3 x^{2}+8 x+1}
$$

As $x$ becomes very large, it is the $x^{2}$ term that will dominate in both the numerator and denominator and the other bits become irrelevant. That is, for very large $x, x^{2}$ is much much larger than $x$ or any constant. So we pull out these dominant parts

$$
\frac{x^{2}-3 x+4}{3 x^{2}+8 x+1}=\frac{x^{2}\left(1-\frac{3}{x}+\frac{4}{x^{2}}\right)}{x^{2}\left(3+\frac{8}{x}+\frac{1}{x^{2}}\right)}
$$

$$
=\frac{1-\frac{3}{x}+\frac{4}{x^{2}}}{3+\frac{8}{x}+\frac{1}{x^{2}}} \quad \text { remove the common factors }
$$

1 To be more precise, there is no real number $x$ so that $x^{\text {even power }}$ is a negative number. Hence we cannot take the even-root of a negative number and express it as a real number. This is precisely what complex numbers allow us to do, but alas there is not space in the course for us to explore them.
2 where we write $r=\frac{p}{q}$ with $p, q$ integers with no common factors. For example, $r=\frac{6}{14}$ should be written as $r=\frac{3}{7}$ when considering this rule.

$$
\begin{array}{rlr}
\lim _{x \rightarrow \infty} \frac{x^{2}-3 x+4}{3 x^{2}+8 x+1} & =\lim _{x \rightarrow \infty} \frac{1-\frac{3}{x}+\frac{4}{x^{2}}}{3+\frac{8}{x}+\frac{1}{x^{2}}} \\
& =\frac{\lim _{x \rightarrow \infty}\left(1-\frac{3}{x}+\frac{4}{x^{2}}\right)}{\lim _{x \rightarrow \infty}\left(3+\frac{8}{x}+\frac{1}{x^{2}}\right)} \\
& =\frac{\lim _{x \rightarrow \infty} 1-\lim _{x \rightarrow \infty} \frac{3}{x}+\lim _{x \rightarrow \infty} \frac{4}{x^{2}}}{\lim _{x \rightarrow \infty} 3+\lim _{x \rightarrow \infty} \frac{8}{x}+\lim _{x \rightarrow \infty} \frac{1}{x^{2}}} \quad \text { arithmetic of limits } \\
& =\frac{1+0+0}{3+0+0}=\frac{1}{3} &
\end{array}
$$

Example 1.5.5
The following one gets a little harder
Example 1.5.6 Be careful of limits involving roots.
Find the limit as $x \rightarrow \infty$ of $\frac{\sqrt{4 x^{2}+1}}{5 x-1}$
We use the same trick - try to work out what is the biggest term in the numerator and denominator and pull it to one side.

- The denominator is dominated by $5 x$.
- The biggest contribution to the numerator comes from the $4 x^{2}$ inside the squareroot. When we pull $x^{2}$ outside the square-root it becomes $x$, so the numerator is dominated by $x \cdot \sqrt{4}=2 x$
- To see this more explicitly rewrite the numerator

$$
\sqrt{4 x^{2}+1}=\sqrt{x^{2}\left(4+1 / x^{2}\right)}=\sqrt{x^{2}} \sqrt{4+1 / x^{2}}=x \sqrt{4+1 / x^{2}} .
$$

- Thus the limit as $x \rightarrow \infty$ is

$$
\begin{aligned}
\lim _{x \rightarrow \infty} \frac{\sqrt{4 x^{2}+1}}{5 x-1} & =\lim _{x \rightarrow \infty} \frac{x \sqrt{4+1 / x^{2}}}{x(5-1 / x)} \\
& =\lim _{x \rightarrow \infty} \frac{\sqrt{4+1 / x^{2}}}{5-1 / x} \\
& =\frac{2}{5}
\end{aligned}
$$

Now let us also think about the limit of the same function, $\frac{\sqrt{4 x^{2}+1}}{5 x-1}$, as $x \rightarrow-\infty$. There is something subtle going on because of the square-root. First consider the
function ${ }^{3}$

$$
h(t)=\sqrt{t^{2}}
$$

Evaluating this at $t=7$ gives

$$
h(7)=\sqrt{7^{2}}=\sqrt{49}=7
$$

We'll get much the same thing for any $t \geq 0$. For any $t \geq 0, h(t)=\sqrt{t^{2}}$ returns exactly $t$. However now consider the function at $t=-3$

$$
h(-3)=\sqrt{(-3)^{2}}=\sqrt{9}=3=-(-3)
$$

that is the function is returning -1 times the input.
This is because when we defined $\sqrt{ }$, we defined it to be the positive square-root.
i.e. the function $\sqrt{t}$ can never return a negative number. So being more careful

$$
h(t)=\sqrt{t^{2}}=|t|
$$

Where the $|t|$ is the absolute value of $t$. You are perhaps used to thinking of absolute value as "remove the minus sign", but this is not quite correct. Let's sketch the function


It is a piecewise function defined by

$$
|x|= \begin{cases}x & x \geq 0 \\ -x & x<0\end{cases}
$$

Hence our function $h(t)$ is really

$$
h(t)=\sqrt{t^{2}}= \begin{cases}t & t \geq 0 \\ -t & t<0\end{cases}
$$

So that when we evaluate $h(-7)$ it is

$$
h(-7)=\sqrt{(-7)^{2}}=\sqrt{49}=7=-(-7)
$$

We are now ready to examine the limit as $x \rightarrow-\infty$ in our previous example. Mostly it is copy and paste from above.


3 Just to change things up let's use $t$ and $h(t)$ instead of the ubiquitous $x$ and $f(x)$.

Example 1.5.7 Be careful of limits involving roots - continued.
Find the limit as $x \rightarrow-\infty$ of $\frac{\sqrt{4 x^{2}+1}}{5 x-1}$
We use the same trick - try to work out what is the biggest term in the numerator and denominator and pull it to one side. Since we are taking the limit as $x \rightarrow-\infty$ we should think of $x$ as a large negative number.

- The denominator is dominated by $5 x$.
- The biggest contribution to the numerator comes from the $4 x^{2}$ inside the squareroot. When we pull the $x^{2}$ outside a square-root it becomes $|x|=-x$ (since we are taking the limit as $x \rightarrow-\infty$ ), so the numerator is dominated by $-x \cdot \sqrt{4}=-2 x$
- To see this more explicitly rewrite the numerator

$$
\begin{aligned}
\sqrt{4 x^{2}+1} & =\sqrt{x^{2}\left(4+1 / x^{2}\right)}=\sqrt{x^{2}} \sqrt{4+1 / x^{2}} \\
& =|x| \sqrt{4+1 / x^{2}} \quad \text { and since } x<0 \text { we have } \\
& =-x \sqrt{4+1 / x^{2}}
\end{aligned}
$$

- Thus the limit as $x \rightarrow-\infty$ is

$$
\begin{aligned}
\lim _{x \rightarrow-\infty} \frac{\sqrt{4 x^{2}+1}}{5 x-1} & =\lim _{x \rightarrow-\infty} \frac{-x \sqrt{4+1 / x^{2}}}{x(5-1 / x)} \\
& =\lim _{x \rightarrow-\infty} \frac{-\sqrt{4+1 / x^{2}}}{5-1 / x} \\
& =-\frac{2}{5}
\end{aligned}
$$

So the limit as $x \rightarrow-\infty$ is almost the same but we gain a minus sign. This $i s$ definitely not the case in general - you have to think about each example separately. Here is a sketch of the function in question.


Example 1.5.8 Do not try to add and subtract infinity.
Compute the following limit:

$$
\lim _{x \rightarrow \infty}\left(x^{7 / 5}-x\right)
$$

In this case we cannot use the arithmetic of limits to write this as

$$
\begin{aligned}
\lim _{x \rightarrow \infty}\left(x^{7 / 5}-x\right) & =\left(\lim _{x \rightarrow \infty} x^{7 / 5}\right)-\left(\lim _{x \rightarrow \infty} x\right) \\
& =\infty-\infty
\end{aligned}
$$

because the limits do not exist. We can only use the limit laws when the limits exist. So we should go back and think some more.
When $x$ is very large, $x^{7 / 5}=x \cdot x^{2 / 5}$ will be much larger than $x$, so the $x^{7 / 5}$ term will dominate the $x$ term. So factor out $x^{7 / 5}$ and rewrite it as

$$
x^{7 / 5}-x=x^{7 / 5}\left(1-\frac{1}{x^{2 / 5}}\right)
$$

Consider what happens to each of the factors as $x \rightarrow \infty$

- For large $x, x^{7 / 5}>x$ (this is actually true for any $x>1$ ). In the limit as $x \rightarrow+\infty$, $x$ becomes arbitrarily large and positive, and $x^{7 / 5}$ must be bigger still, so it follows that

$$
\lim _{x \rightarrow \infty} x^{7 / 5}=+\infty
$$

- On the other hand, $\left(1-x^{-2 / 5}\right)$ becomes closer and closer to 1 - we can use the arithmetic of limits to write this as

$$
\lim _{x \rightarrow \infty}\left(1-x^{-2 / 5}\right)=\lim _{x \rightarrow \infty} 1-\lim _{x \rightarrow \infty} x^{-2 / 5}=1-0=1
$$

So the product of these two factors will be come larger and larger (and positive) as $x$ moves off to infinity. Hence we have

$$
\lim _{x \rightarrow \infty} x^{7 / 5}\left(1-1 / x^{2 / 5}\right)=+\infty
$$

But remember $+\infty$ and $-\infty$ are not numbers; the last equation in the example is shorthand for "the function becomes arbitrarily large".

In the previous section we saw that finite limits and arithmetic interact very nicely (see Theorems 1.4.3 and 1.4.9). This enabled us to compute the limits of more complicated function in terms of simpler ones. When limits of functions go to plus or minus infinity we are quite a bit more restricted in what we can deduce. The next theorem states some results concerning the sum, difference, ratio and product of infinite limits - unfortunately in many cases we cannot make general statements and the results will
depend on the details of the problem at hand.

Theorem 1.5.9 Arithmetic of infinite limits.
Let $a, c, H \in \mathbb{R}$ and let $f, g, h$ be functions defined in an interval around $a$ (but they need not be defined at $x=a$ ), so that

$$
\lim _{x \rightarrow a} f(x)=+\infty \quad \lim _{x \rightarrow a} g(x)=+\infty \quad \quad \lim _{x \rightarrow a} h(x)=H
$$

- $\lim _{x \rightarrow a}(f(x)+g(x))=+\infty$
- $\lim _{x \rightarrow a}(f(x)+h(x))=+\infty$
- $\lim _{x \rightarrow a}(f(x)-g(x))$ undetermined
- $\lim _{x \rightarrow a}(f(x)-h(x))=+\infty$
- $\lim _{x \rightarrow a} c f(x)= \begin{cases}+\infty & c>0 \\ 0 & c=0 \\ -\infty & c<0\end{cases}$
- $\lim _{x \rightarrow a}(f(x) \cdot g(x))=+\infty$.
- $\lim _{x \rightarrow a} f(x) h(x)= \begin{cases}+\infty & H>0 \\ -\infty & H<0 \\ \text { undetermined } & H=0\end{cases}$
- $\lim _{x \rightarrow a} \frac{f(x)}{g(x)}$ undetermined
- $\lim _{x \rightarrow a} \frac{f(x)}{h(x)}= \begin{cases}+\infty & H>0 \\ -\infty & H<0 \\ \text { undetermined } & H=0\end{cases}$
- $\lim _{x \rightarrow a} \frac{h(x)}{f(x)}=0$
- $\lim _{x \rightarrow a} f(x)^{p}= \begin{cases}+\infty & p>0 \\ 0 & p<0 \\ 1 & p=0\end{cases}$

Note that by "undetermined" we mean that the limit may or may not exist, but
cannot be determined from the information given in the theorem. See Example 1.4.7 for an example of what we mean by "undetermined". Additionally consider the following example.

Example 1.5.10 Be careful with the arithmetic of infinite limits.
Consider the following 3 functions:

$$
f(x)=x^{-2}
$$

$$
g(x)=2 x^{-2}
$$

$$
h(x)=x^{-2}-1
$$

Their limits as $x \rightarrow 0$ are:

$$
\lim _{x \rightarrow 0} f(x)=+\infty \quad \lim _{x \rightarrow 0} g(x)=+\infty \quad \quad \lim _{x \rightarrow 0} h(x)=+\infty
$$

Say we want to compute the limit of the difference of two of the above functions as $x \rightarrow 0$. Then the previous theorem cannot help us. This is not because it is too weak, rather it is because the difference of two infinite limits can be, either plus infinity, minus infinity or some finite number depending on the details of the problem. For example,

$$
\begin{aligned}
& \lim _{x \rightarrow 0}(f(x)-g(x))=\lim _{x \rightarrow 0}-x^{-2}=-\infty \\
& \lim _{x \rightarrow 0}(f(x)-h(x))=\lim _{x \rightarrow 0} 1=1 \\
& \lim _{x \rightarrow 0}(g(x)-h(x))=\lim _{x \rightarrow 0} x^{-2}+1=+\infty
\end{aligned}
$$

### 1.5.2 $\leadsto$ Exercises

## Exercises - Stage 1

1. Give a polynomial $f(x)$ with the property that both $\lim _{x \rightarrow \infty} f(x)$ and $\lim _{x \rightarrow-\infty} f(x)$ are (finite) real numbers.
2. Give a polynomial $f(x)$ that satisfies $\lim _{x \rightarrow \infty} f(x) \neq \lim _{x \rightarrow-\infty} f(x)$.

## Exercises - Stage 2

3. Evaluate $\lim _{x \rightarrow \infty} 2^{-x}$
4. Evaluate $\lim _{x \rightarrow \infty} 2^{x}$
5. Evaluate $\lim _{x \rightarrow-\infty} 2^{x}$
6. Evaluate $\lim _{x \rightarrow-\infty} \cos x$
7. Evaluate $\lim _{x \rightarrow \infty} x-3 x^{5}+100 x^{2}$.
8. Evaluate $\lim _{x \rightarrow \infty} \frac{\sqrt{3 x^{8}+7 x^{4}}+10}{x^{4}-2 x^{2}+1}$.
9. *. $\lim _{x \rightarrow \infty}\left[\sqrt{x^{2}+5 x}-\sqrt{x^{2}-x}\right]$
10. *. Evaluate $\lim _{x \rightarrow-\infty} \frac{3 x}{\sqrt{4 x^{2}+x}-2 x}$.
11. *. Evaluate $\lim _{x \rightarrow-\infty} \frac{1-x-x^{2}}{2 x^{2}-7}$.
12. *. Evaluate $\lim _{x \rightarrow \infty}\left(\sqrt{x^{2}+x}-x\right)$
13. *. Evaluate $\lim _{x \rightarrow+\infty} \frac{5 x^{2}-3 x+1}{3 x^{2}+x+7}$.
14. *. Evaluate $\lim _{x \rightarrow+\infty} \frac{\sqrt{4 x+2}}{3 x+4}$.
15. *. Evaluate $\lim _{x \rightarrow+\infty} \frac{4 x^{3}+x}{7 x^{3}+x^{2}-2}$.
16. Evaluate $\lim _{x \rightarrow-\infty} \frac{\sqrt[3]{x^{2}+x}-\sqrt[4]{x^{4}+5}}{x+1}$
17. *. Evaluate $\lim _{x \rightarrow+\infty} \frac{5 x^{2}+10}{3 x^{3}+2 x^{2}+x}$.
18. Evaluate $\lim _{x \rightarrow-\infty} \frac{x+1}{\sqrt{x^{2}}}$.
19. Evaluate $\lim _{x \rightarrow \infty} \frac{x+1}{\sqrt{x^{2}}}$
20. *. Find the limit $\lim _{x \rightarrow-\infty} \sin \left(\frac{\pi}{2} \frac{|x|}{x}\right)+\frac{1}{x}$.
21. *. Evaluate $\lim _{x \rightarrow-\infty} \frac{3 x+5}{\sqrt{x^{2}+5}-x}$.
22. *. Evaluate $\lim _{x \rightarrow-\infty} \frac{5 x+7}{\sqrt{4 x^{2}+15}-x}$
23. Evaluate $\lim _{x \rightarrow-\infty} \frac{3 x^{7}+x^{5}-15}{4 x^{2}+32 x}$.
24. *. Evaluate $\lim _{n \rightarrow \infty}\left(\sqrt{n^{2}+5 n}-n\right)$.
25. Evaluate $\lim _{a \rightarrow 0^{+}} \frac{a^{2}-\frac{1}{a}}{a-1}$.
26. Evaluate $\lim _{x \rightarrow 3} \frac{2 x+8}{\frac{1}{x-3}+\frac{1}{x^{2}-9}}$.

## Exercises - Stage 3

27. Give a rational function $f(x)$ with the properties that $\lim _{x \rightarrow \infty} f(x) \neq$ $\lim _{x \rightarrow-\infty} f(x)$, and both limits are (finite) real numbers.
28. Suppose the concentration of a substance in your body $t$ hours after injection is given by some formula $c(t)$, and $\lim _{t \rightarrow \infty} c(t) \neq 0$. What kind of substance might have been injected?

## 1.6」 Continuity

### 1.6.1 $\leadsto$ Continuity

We have seen that computing the limits some functions - polynomials and rational functions - is very easy because

$$
\lim _{x \rightarrow a} f(x)=f(a)
$$

That is, the the limit as $x$ approaches $a$ is just $f(a)$. Roughly speaking, the reason we can compute the limit this way is that these functions do not have any abrupt jumps near $a$.

Many other functions have this property, $\sin (x)$ for example. A function with this property is called "continuous" and there is a precise mathematical definition for it. If you do not recall interval notation, then now is a good time to take a quick look back at Definition 0.3.5.

## Definition 1.6.1

A function $f(x)$ is continuous at $a$ if

$$
\lim _{x \rightarrow a} f(x)=f(a)
$$

If a function is not continuous at $a$ then it is said to be discontinuous at $a$.
When we write that $f$ is continuous without specifying a point, then typically this means that $f$ is continuous at $a$ for all $a \in \mathbb{R}$.
When we write that $f(x)$ is continuous on the open interval $(a, b)$ then the function is continuous at every point $c$ satisfying $a<c<b$.

So if a function is continuous at $x=a$ we immediately know that

- $f(a)$ exists
- $\lim _{x \rightarrow a^{-}}$exists and is equal to $f(a)$, and
- $\lim _{x \rightarrow a^{+}}$exists and is equal to $f(a)$.


### 1.6.2 ( Quick Aside - One-sided Continuity

Notice in the above definition of continuity on an interval $(a, b)$ we have carefully avoided saying anything about whether or not the function is continuous at the endpoints of the interval - i.e. is $f(x)$ continuous at $x=a$ or $x=b$. This is because talking of continuity at the endpoints of an interval can be a little delicate.

In many situations we will be given a function $f(x)$ defined on a closed interval $[a, b]$. For example, we might have:

$$
f(x)=\frac{x+1}{x+2} \quad \text { for } x \in[0,1]
$$

For any $0 \leq x \leq 1$ we know the value of $f(x)$. However for $x<0$ or $x>1$ we know nothing about the function - indeed it has not been defined.

So now, consider what it means for $f(x)$ to be continuous at $x=0$. We need to have

$$
\lim _{x \rightarrow 0} f(x)=f(0)
$$

however this implies that the one-sided limits

$$
\lim _{x \rightarrow 0^{+}} f(x)=f(0) \quad \text { and } \quad \lim _{x \rightarrow 0^{-}} f(x)=f(0)
$$

Now the first of these one-sided limits involves examining the behaviour of $f(x)$ for $x>0$. Since this involves looking at points for which $f(x)$ is defined, this is something we can do. On the other hand the second one-sided limit requires us to understand the behaviour of $f(x)$ for $x<0$. This we cannot do because the function hasn't been defined for $x<0$.

One way around this problem is to generalise the idea of continuity to one-sided continuity, just as we generalised limits to get one-sided limits.

## Definition 1.6.2

A function $f(x)$ is continuous from the right at $a$ if

$$
\lim _{x \rightarrow a^{+}} f(x)=f(a)
$$

Similarly a function $f(x)$ is continuous from the left at $a$ if

$$
\lim _{x \rightarrow a^{-}} f(x)=f(a)
$$

Using the definition of one-sided continuity we can now define what it means for a function to be continuous on a closed interval.

## Definition 1.6.3

A function $f(x)$ is continuous on the closed interval $[a, b]$ when

- $f(x)$ is continuous on $(a, b)$,
- $f(x)$ is continuous from the right at $a$, and
- $f(x)$ is continuous from the left at $b$.

Note that the last two condition are equivalent to

$$
\lim _{x \rightarrow a^{+}} f(x)=f(a) \quad \text { and } \quad \lim _{x \rightarrow b^{-}} f(x)=f(b)
$$

### 1.6.3 Back to the Main Text

We already know from our work above that polynomials are continuous, and that rational functions are continuous at all points in their domains - i.e. where their denominators are non-zero. As we did for limits, we will see that continuity interacts "nicely" with arithmetic. This will allow us to construct complicated continuous functions from simpler continuous building blocks (like polynomials).

But first, a few examples...
Example 1.6.4 Simple continuous and discontinuous functions.
Consider the functions drawn below




These are

$$
\begin{aligned}
& f(x)= \begin{cases}x & x<1 \\
x+2 & x \geq 1\end{cases} \\
& g(x)= \begin{cases}1 / x^{2} & x \neq 0 \\
0 & x=0\end{cases} \\
& h(x)= \begin{cases}\frac{x^{3}-x^{2}}{x-1} & x \neq 1 \\
0 & x=1\end{cases}
\end{aligned}
$$

Determine where they are continuous and discontinuous:

- When $x<1$ then $f(x)$ is a straight line (and so a polynomial) and so it is continuous at every point $x<1$. Similarly when $x>1$ the function is a straight line and so it is continuous at every point $x>1$. The only point which might be a discontinuity is at $x=1$. We see that the one sided limits are different. Hence the limit at $x=1$ does not exist and so the function is discontinuous at $x=1$.
But note that that $f(x)$ is continuous from one side - which?
- The middle case is much like the previous one. When $x \neq 0$ the $g(x)$ is a rational function and so is continuous everywhere on its domain (which is all reals except $x=0$ ). Thus the only point where $g(x)$ might be discontinuous is at $x=0$. We see that neither of the one-sided limits exist at $x=0$, so the limit does not exist at $x=0$. Hence the function is discontinuous at $x=0$.
- We have seen the function $h(x)$ before. By the same reasoning as above, we know it is continuous except at $x=1$ which we must check separately.
By definition of $h(x), h(1)=0$. We must compare this to the limit as $x \rightarrow 1$. We did this before.

$$
\frac{x^{3}-x^{2}}{x-1}=\frac{x^{2}(x-1)}{x-1}=x^{2}
$$

So $\lim _{x \rightarrow 1} \frac{x^{3}-x^{2}}{x-1}=\lim _{x \rightarrow 1} x^{2}=1 \neq h(1)$. Hence $h$ is discontinuous at $x=1$.

This example illustrates different sorts of discontinuities:

- The function $f(x)$ has a "jump discontinuity" because the function "jumps" from one finite value on the left to another value on the right.
- The second function, $g(x)$, has an "infinite discontinuity" since $\lim f(x)=+\infty$.
- The third function, $h(x)$, has a "removable discontinuity" because we could make the function continuous at that point by redefining the function at that point. i.e. setting $h(1)=1$. That is

$$
\text { new function } h(x)= \begin{cases}\frac{x^{3}-x^{2}}{x-1} & x \neq 1 \\ 1 & x=1\end{cases}
$$

Showing a function is continuous can be a pain, but just as the limit laws help us compute complicated limits in terms of simpler limits, we can use them to show that complicated functions are continuous by breaking them into simpler pieces.

## Theorem 1.6.5 Arithmetic of continuity.

Let $a, c \in \mathbb{R}$ and let $f(x)$ and $g(x)$ be functions that are continuous at $a$. Then the following functions are also continuous at $x=a$ :

- $f(x)+g(x)$ and $f(x)-g(x)$,
- $c f(x)$ and $f(x) g(x)$, and
- $\frac{f(x)}{g(x)}$ provided $g(a) \neq 0$.

Above we stated that polynomials and rational functions are continuous (being careful about domains of rational functions - we must avoid the denominators being zero) without making it a formal statement. This is easily fixed...

## Lemma 1.6.6

Let $c \in \mathbb{R}$. The functions

$$
f(x)=x \quad g(x)=c
$$

are continuous everywhere on the real line

This isn't quite the result we wanted (that's a couple of lines below) but it is a small result that we can combine with the arithmetic of limits to get the result we want. Such small helpful results are called "lemmas" and they will arise more as we go along.

Now since we can obtain any polynomial and any rational function by carefully adding, subtracting, multiplying and dividing the functions $f(x)=x$ and $g(x)=c$, the
above lemma combines with the "arithmetic of continuity" theorem to give us the result we want:

## Theorem 1.6.7 Continuity of polynomials and rational functions.

Every polynomial is continuous everywhere. Similarly every rational function is continuous except where its denominator is zero (i.e. on all its domain).

With some more work this result can be extended to wider families of functions:

## Theorem 1.6.8

The following functions are continuous everywhere in their domains

- polynomials, rational functions
- roots and powers
- trig functions and their inverses
- exponential and the logarithm

We haven't encountered inverse trigonometric functions, nor exponential functions or logarithms, but we will see them in the next chapter. For the moment, just file the information away.

Using a combination of the above results you can show that many complicated functions are continuous except at a few points (usually where a denominator is equal to zero).

## Example 1.6.9 Continuity of $\frac{\sin (x)}{2+\cos (x)}$.

Where is the function $f(x)=\frac{\sin (x)}{2+\cos (x)}$ continuous?
We just break things down into pieces and then put them back together keeping track of where things might go wrong.

- The function is a ratio of two pieces - so check if the numerator is continuous, the denominator is continuous, and if the denominator might be zero.
- The numerator is $\sin (x)$ which is "continuous on its domain" according to one of the above theorems. Its domain is all real numbers ${ }^{a}$, so it is continuous everywhere. No problems here.
- The denominator is the sum of 2 and $\cos (x)$. Since 2 is a constant it is continuous everywhere. Similarly (we just checked things for the previous point) we know that $\cos (x)$ is continuous everywhere. Hence the denominator is continuous.
- So we just need to check if the denominator is zero. One of the facts that we should know ${ }^{b}$ is that

$$
-1 \leq \cos (x) \leq 1
$$

and so by adding 2 we get

$$
1 \leq 2+\cos (x) \leq 3
$$

Thus no matter what value of $x, 2+\cos (x) \geq 1$ and so cannot be zero.

- So the numerator is continuous, the denominator is continuous and nowhere zero, so the function is continuous everywhere.

If the function were changed to $\frac{\sin (x)}{x^{2}-5 x+6}$ much of the same reasoning can be used. Being a little terse we could answer with:

- Numerator and denominator are continuous.
- Since $x^{2}-5 x+6=(x-2)(x-3)$ the denominator is zero when $x=2,3$.
- So the function is continuous everywhere except possibly at $x=2,3$. In order to verify that the function really is discontinuous at those points, it suffices to verify that the numerator is non-zero at $x=2,3$. Indeed we know that $\sin (x)$ is zero only when $x=n \pi$ (for any integer $n$ ). Hence $\sin (2), \sin (3) \neq 0$. Thus the numerator is non-zero, while the denominator is zero and hence $x=2,3$ really are points of discontinuity.

Note that this example raises a subtle point about checking continuity when numerator and denominator are simultaneously zero. There are quite a few possible outcomes in this case and we need more sophisticated tools to adequately analyse the behaviour of functions near such points. We will return to this question later in the text after we have developed Taylor expansions (see Section 3.4).
$a \quad$ Remember that sin and cos are defined on all real numbers, so $\tan (x)=\sin (x) / \cos (x)$ is continuous everywhere except where $\cos (x)=0$. This happens when $x=\frac{\pi}{2}+n \pi$ for any integer $n$. If you cannot remember where $\tan (x)$ "blows up" or $\sin (x)=0$ or $\cos (x)=0$ then you should definitely revise trigonometric functions. Come to think of it - just revise them anyway.
$b$ If you do not know this fact then you should revise trigonometric functions. See the previous footnote.


So we know what happens when we add subtract multiply and divide, what about when we compose functions? Well - limits and compositions work nicely when things are continuous.

Theorem 1.6.10 Compositions and continuity.
If $f$ is continuous at $b$ and $\lim _{x \rightarrow a} g(x)=b$ then $\lim _{x \rightarrow a} f(g(x))=f(b)$. I.e.

$$
\lim _{x \rightarrow a} f(g(x))=f\left(\lim _{x \rightarrow a} g(x)\right)
$$

Hence if $g$ is continuous at $a$ and $f$ is continuous at $g(a)$ then the composite function $(f \circ g)(x)=f(g(x))$ is continuous at $a$.

So when we compose two continuous functions we get a new continuous function. We can put this to use

Example 1.6.11 Continuity of composed functions.
Where are the following functions continuous?

$$
\begin{aligned}
& f(x)=\sin \left(x^{2}+\cos (x)\right) \\
& g(x)=\sqrt{\sin (x)}
\end{aligned}
$$

Our first step should be to break the functions down into pieces and study them. When we put them back together we should be careful of dividing by zero, or falling outside the domain.

- The function $f(x)$ is the composition of $\sin (x)$ with $x^{2}+\cos (x)$.
- These pieces, $\sin (x), x^{2}, \cos (x)$ are continuous everywhere.
- So the sum $x^{2}+\cos (x)$ is continuous everywhere
- And hence the composition of $\sin (x)$ and $x^{2}+\cos (x)$ is continuous everywhere.

The second function is a little trickier.

- The function $g(x)$ is the composition of $\sqrt{x}$ with $\sin (x)$.
- $\sqrt{x}$ is continuous on its domain $x \geq 0$.
- $\sin (x)$ is continuous everywhere, but it is negative in many places.
- In order for $g(x)$ to be defined and continuous we must restrict $x$ so that $\sin (x) \geq$ 0 .
- Recall the graph of $\sin (x)$ :


Hence $\sin (x) \geq 0$ when $x \in[0, \pi]$ or $x \in[2 \pi, 3 \pi]$ or $x \in[-2 \pi,-\pi]$ or.... To be more precise $\sin (x)$ is positive when $x \in[2 n \pi,(2 n+1) \pi]$ for any integer $n$.

- Hence $g(x)$ is continuous when $x \in[2 n \pi,(2 n+1) \pi]$ for any integer $n$.

Example 1.6.11
Continuous functions are very nice (mathematically speaking). Functions from the "real world" tend to be continuous (though not always). The key aspect that makes them nice is the fact that they don't jump about.

The absence of such jumps leads to the following theorem which, while it can be quite confusing on first glance, actually says something very natural - obvious even. It says, roughly speaking, that, as you draw the graph $y=f(x)$ starting at $x=a$ and ending at $x=b, y$ changes continuously from $y=f(a)$ to $y=f(b)$, with no jumps, and consequently $y$ must take every value between $f(a)$ and $f(b)$ at least once. We'll start by just giving the precise statement and then we'll explain it in detail.

Theorem 1.6.12 Intermediate value theorem (IVT).
Let $a<b$ and let $f$ be a function that is continuous at all points $a \leq x \leq b$. If $Y$ is any number between $f(a)$ and $f(b)$ then there exists some number $c \in[a, b]$ so that $f(c)=Y$.

Like the $\epsilon-\delta$ definition of limits ${ }^{1}$, we should break this theorem down into pieces. Before we do that, keep the following pictures in mind.


Now the break-down

1 The interested student is invited to take a look at the optional Section 1.7

- Let $a<b$ and let $f$ be a function that is continuous at all points $a \leq x \leq$ $b$. - This is setting the scene. We have $a, b$ with $a<b$ (we can safely assume these to be real numbers). Our function must be continuous at all points between $a$ and $b$.
- if $Y$ is any number between $f(a)$ and $f(b)$ - Now we need another number $Y$ and the only restriction on it is that it lies between $f(a)$ and $f(b)$. That is, if $f(a) \leq f(b)$ then $f(a) \leq Y \leq f(b)$. Or if $f(a) \geq f(b)$ then $f(a) \geq Y \geq f(b)$. So notice that $Y$ could be equal to $f(a)$ or $f(b)$ - if we wanted to avoid that possibility, then we would normally explicitly say $Y \neq f(a), f(b)$ or we would write that $Y$ is strictly between $f(a)$ and $f(b)$.
- there exists some number $c \in[a, b]$ so that $f(c)=Y$ - so if we satisfy all of the above conditions, then there has to be some real number $c$ lying between $a$ and $b$ so that when we evaluate $f(c)$ it is $Y$.

So that breaks down the proof statement by statement, but what does it actually mean?

- Draw any continuous function you like between $a$ and $b$ - it must be continuous.
- The function takes the value $f(a)$ at $x=a$ and $f(b)$ at $x=b$ - see the left-hand figure above.
- Now we can pick any $Y$ that lies between $f(a)$ and $f(b)$ - see the middle figure above. The IVT ${ }^{2}$ tells us that there must be some $x$-value that when plugged into the function gives us $Y$. That is, there is some $c$ between $a$ and $b$ so that $f(c)=Y$. We can also interpret this graphically; the IVT tells us that the horizontal straight line $y=Y$ must intersect the graph $y=f(x)$ at some point $(c, Y)$ with $a \leq c \leq b$.
- Notice that the IVT does not tell us how many such $c$-values there are, just that there is at least one of them. See the right-hand figure above. For that particular choice of $Y$ there are three different $c$ values so that $f\left(c_{1}\right)=f\left(c_{2}\right)=f\left(c_{3}\right)=Y$.

This theorem says that if $f(x)$ is a continuous function on all of the interval $a \leq x \leq b$ then as $x$ moves from $a$ to $b, f(x)$ takes every value between $f(a)$ and $f(b)$ at least once. To put this slightly differently, if $f$ were to avoid a value between $f(a)$ and $f(b)$ then $f$ cannot be continuous on $[a, b]$.

It is not hard to convince yourself that the continuity of $f$ is crucial to the IVT. Without it one can quickly construct examples of functions that contradict the theorem. See the figure below for a few non-continuous examples:

2 Often with big important useful theorems like this one, writing out the full name again and again becomes tedious, so we abbreviate it. Such abbreviations are okay provided the reader knows this is what you are doing, so the first time you use an abbreviation you should let the reader know. Much like we are doing here in this footnote: : "IVT" stands for "intermediate value theorem", which is Theorem 1.6.12.



In the left-hand example we see that a discontinuous function can "jump" over the $Y$-value we have chosen, so there is no $x$-value that makes $f(x)=Y$. The right-hand example demonstrates why we need to be be careful with the ends of the interval. In particular, a function must be continuous over the whole interval $[a, b]$ including the end-points of the interval. If we only required the function to be continuous on $(a, b)$ (so strictly between $a$ and $b$ ) then the function could "jump" over the $Y$-value at $a$ or $b$.

If you are still confused then here is a "real-world" example

## Example 1.6.13 The IVT in the "real world".

You are climbing the Grouse-grind ${ }^{a}$ with a friend - call him Bob. Bob was eager and started at 9 am . Bob, while very eager, is also very clumsy; he sprained his ankle somewhere along the path and has stopped moving at 9:21am and is just sitting ${ }^{b}$ enjoying the view. You get there late and start climbing at 10 am and being quite fit you get to the top at 11am. The IVT implies that at some time between 10am and 11am you meet up with Bob.
You can translate this situation into the form of the IVT as follows. Let $t$ be time and let $a=10 \mathrm{am}$ and $b=11 \mathrm{am}$. Let $g(t)$ be your distance along the trail. Hence ${ }^{c}$ $g(a)=0$ and $g(b)=2.9 \mathrm{~km}$. Since you are a mortal, your position along the trail is a continuous function - no helicopters or teleportation or... We have no idea where Bob is sitting, except that he is somewhere between $g(a)$ and $g(b)$, call this point $Y$. The IVT guarantees that there is some time $c$ between $a$ and $b$ (so between 10am and 11 am ) with $g(c)=Y$ (and your position will be the same as Bob's).

$a$ If you don't know it then google it.
$b$ Hopefully he remembered to carry something warm.
$c \quad$ It's amazing what facts you can find on Wikipedia.

Aside from finding Bob sitting by the side of the trail, one of the most important applications of the IVT is determining where a function is zero. For quadratics we know (or should know) that

$$
a x^{2}+b x+c=0 \quad \text { when } x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}
$$

While the Babylonians could (mostly, but not quite) do the above, the corresponding formula for solving a cubic is uglier and that for a quartic is uglier still. One of the most
famous results in mathematics demonstrates that no such formula exists for quintics or higher degree polynomials ${ }^{3}$.

So even for polynomials we cannot, in general, write down explicit formulae for their zeros and have to make do with numerical approximations - i.e. write down the root as a decimal expansion to whatever precision we desire. For more complicated functions we have no choice - there is no reason that the zeros should be expressible as nice neat little formulas. At the same time, finding the zeros of a function:

$$
f(x)=0
$$

or solving equations of the form ${ }^{4}$

$$
g(x)=h(x)
$$

can be a crucial step in many mathematical proofs and applications.
For this reason there is a considerable body of mathematics which focuses just on finding the zeros of functions. The IVT provides a very simple way to "locate" the zeros of a function. In particular, if we know a continuous function is negative at a point $x=a$ and positive at another point $x=b$, then there must (by the IVT) be a point $x=c$ between $a$ and $b$ where $f(c)=0$.


Consider the leftmost of the above figures. It depicts a continuous function that is negative at $x=a$ and positive at $x=b$. So choose $Y=0$ and apply the IVT - there must be some $c$ with $a \leq c \leq b$ so that $f(c)=Y=0$. While this doesn't tell us $c$ exactly, it does give us bounds on the possible positions of at least one zero - there must be at least one c obeying $a \leq c \leq b$.

See middle figure. To get better bounds we could test a point half-way between $a$ and $b$. So set $a^{\prime}=\frac{a+b}{2}$. In this example we see that $f\left(a^{\prime}\right)$ is negative. Applying the IVT again tells us there is some $c$ between $a^{\prime}$ and $b$ so that $f(c)=0$. Again - we don't have $c$ exactly, but we have halved the range of values it could take.

Look at the rightmost figure and do it again - test the point half-way between $a^{\prime}$ and $b$. In this example we see that $f\left(b^{\prime}\right)$ is positive. Applying the IVT tells us that

3 The similar (but uglier) formula for solving cubics took until the 15th century and the work of del Ferro and Cardano (and Cardano's student Ferrari). A similar (but even uglier) formula for quartics was also found by Ferrari. The extremely famous Abel-Ruffini Theorem (nearly by Ruffini in the late 18th century and completely by Abel in early 19th century) demonstrates that a similar formula for the zeros of a quintic does not exist. Note that the theorem does not say that quintics do not have zeros; rather it says that the zeros cannot in general be expressed using a finite combination of addition, multiplication, division, powers and roots. The interested student should also look up Évariste Galois and his contributions to this area.
4 In fact both of these are the same because we can write $f(x)=g(x)-h(x)$ and then the zeros of $f(x)$ are exactly when $g(x)=h(x)$.
there is some $c$ between $a^{\prime}$ and $b^{\prime}$ so that $f(c)=0$. This new range is a quarter of the length of the original. If we keep doing this process the range will halve each time until we know that the zero is inside some tiny range of possible values. This process is called the bisection method.

Consider the following zero-finding example
Example 1.6.14 Show that $f(x)=x-1+\sin (\pi x / 2)$ has a zero.
Show that the function $f(x)=x-1+\sin (\pi x / 2)$ has a zero in $0 \leq x \leq 1$.
This question has been set up nicely to lead us towards using the IVT; we are already given a nice interval on which to look. In general we might have to test a few points and experiment a bit with a calculator before we can start narrowing down a range.
Let us start by testing the endpoints of the interval we are given

$$
\begin{aligned}
& f(0)=0-1+\sin (0)=-1<0 \\
& f(1)=1-1+\sin (\pi / 2)=1>0
\end{aligned}
$$

So we know a point where $f$ is positive and one where it is negative. So by the IVT there is a point in between where it is zero.
$B U T$ in order to apply the IVT we have to show that the function is continuous, and we cannot simply write
it is continuous
We need to explain to the reader why it is continuous. That is - we have to prove it. So to write up our answer we can put something like the following - keeping in mind we need to tell the reader what we are doing so they can follow along easily.

- We will use the IVT to prove that there is a zero in $[0,1]$.
- First we must show that the function is continuous.
- Since $x-1$ is a polynomial it is continuous everywhere.
- The function $\sin (\pi x / 2)$ is a trigonometric function and is also continuous everywhere.
- The sum of two continuous functions is also continuous, so $f(x)$ is continuous everywhere.
- Let $a=0, b=1$, then

$$
\begin{aligned}
& f(0)=0-1+\sin (0)=-1<0 \\
& f(1)=1-1+\sin (\pi / 2)=1>0
\end{aligned}
$$

- The function is negative at $x=0$ and positive at $x=1$. Since the function is continuous we know there is a point $c \in[0,1]$ so that $f(c)=0$.
Notice that though we have not used full sentences in our explanation here, we are still using words. Your mathematics, unless it is very straight-forward computation, should contain words as well as symbols.
$\uparrow$ The zero of this function is actually located at about $x=0.4053883559$.

The bisection method is really just the idea that we can keep repeating the above reasoning (with a calculator handy). Each iteration will tell us the location of the zero more precisely. The following example illustrates this.

Example 1.6.15 Using the bisection method.
Use the bisection method to find a zero of

$$
f(x)=x-1+\sin (\pi x / 2)
$$

that lies between 0 and 1 .
So we start with the two points we worked out above:

- $a=0, b=1$ and

$$
\begin{aligned}
& f(0)=-1 \\
& f(1)=1
\end{aligned}
$$

- Test the point in the middle $x=\frac{0+1}{2}=0.5$

$$
f(0.5)=0.2071067813>0
$$

- So our new interval will be $[0,0.5]$ since the function is negative at $x=0$ and positive at $x=0.5$

Repeat

- $a=0, b=0.5$ where $f(0)<0$ and $f(0.5)>0$.
- Test the point in the middle $x=\frac{0+0.5}{2}=0.25$

$$
f(0.25)=-0.3673165675<0
$$

- So our new interval will be $[0.25,0.5]$ since the function is negative at $x=0.25$ and positive at $x=0.5$

Repeat

- $a=0.25, b=0.5$ where $f(0.25)<0$ and $f(0.5)>0$.
- Test the point in the middle $x=\frac{0.25+0.5}{2}=0.375$

$$
f(0.375)=-0.0694297669<0
$$

- So our new interval will be $[0.375,0.5]$ since the function is negative at $x=0.375$ and positive at $x=0.5$

Below is an illustration of what we have observed so far together with a plot of the actual function.


And one final iteration:

- $a=0.375, b=0.5$ where $f(0.375)<0$ and $f(0.5)>0$.
- Test the point in the middle $x=\frac{0.375+0.5}{2}=0.4375$

$$
f(0.4375)=0.0718932843>0
$$

- So our new interval will be $[0.375,0.4375]$ since the function is negative at $x=$ 0.375 and positive at $x=0.4375$

So without much work we know the location of a zero inside a range of length $0.0625=$ $2^{-4}$. Each iteration will halve the length of the range and we keep going until we reach the precision we need, though it is much easier to program a computer to do it.

Example 1.6.15

### 1.6.4 $円$ Exercises

## Exercises - Stage 1

1. Give an example of a function (you can write a formula, or sketch a graph) that has infinitely many infinite discontinuities.
2. When I was born, I was less than one meter tall. Now, I am more than one meter tall. What is the conclusion of the Intermediate Value Theorem about my height?
3. Give an example (by sketch or formula) of a function $f(x)$, defined on the interval $[0,2]$, with $f(0)=0, f(2)=2$, and $f(x)$ never equal to 1 . Why does this not contradict the Intermediate Value Theorem?
4. Is the following a valid statement?

Suppose $f$ is a continuous function over $[10,20], f(10)=13$, and $f(20)=-13$. Then $f$ has a zero between $x=10$ and $x=20$.
5. Is the following a valid statement?

Suppose $f$ is a continuous function over $[10,20], f(10)=13$, and $f(20)=-13$. Then $f(15)=0$.
6. Is the following a valid statement?

Suppose $f$ is a function over $[10,20], f(10)=13$, and $f(20)=$ -13 , and $f$ takes on every value between -13 and 13. Then $f$ is continuous.
7. Suppose $f(t)$ is continuous at $t=5$. True or false: $t=5$ is in the domain of $f(t)$.
8. Suppose $\lim _{t \rightarrow 5} f(t)=17$, and suppose $f(t)$ is continuous at $t=5$. True or false: $f(5)=17$.
9. Suppose $\lim _{t \rightarrow 5} f(t)=17$. True or false: $f(5)=17$.
10. Suppose $f(x)$ and $g(x)$ are continuous at $x=0$, and let $h(x)=\frac{x f(x)}{g^{2}(x)+1}$. What is $\lim _{x \rightarrow 0^{+}} h(x)$ ?

## Exercises - Stage 2

11. Find a constant $k$ so that the function

$$
a(x)= \begin{cases}x \sin \left(\frac{1}{x}\right) & \text { when } x \neq 0 \\ k & \text { when } x=0\end{cases}
$$

is continuous at $x=0$.
12. Use the Intermediate Value Theorem to show that the function $f(x)=x^{3}+$ $x^{2}+x+1$ takes on the value 12345 at least once in its domain.
13. *. Describe all points for which the function is continuous: $f(x)=\frac{1}{x^{2}-1}$.
14. *. Describe all points for which this function is continuous: $f(x)=$ $\frac{1}{\sqrt{x^{2}-1}}$.
15. *. Describe all points for which this function is continuous: $\frac{1}{\sqrt{1+\cos (x)}}$.
16. *. Describe all points for which this function is continuous: $f(x)=\frac{1}{\sin x}$.
17. *. Find all values of $c$ such that the following function is continuous at $x=c$ :

$$
f(x)=\left\{\begin{array}{cll}
8-c x & \text { if } & x \leq c \\
x^{2} & \text { if } & x>c
\end{array}\right.
$$

Use the definition of continuity to justify your answer.
18. *. Find all values of $c$ such that the following function is continuous everywhere:

$$
f(x)= \begin{cases}x^{2}+c & x \geq 0 \\ \cos c x & x<0\end{cases}
$$

Use the definition of continuity to justify your answer.
19. *. Find all values of $c$ such that the following function is continuous:

$$
f(x)= \begin{cases}x^{2}-4 & \text { if } x<c \\ 3 x & \text { if } x \geq c\end{cases}
$$

Use the definition of continuity to justify your answer.
20. *. Find all values of $c$ such that the following function is continuous:

$$
f(x)=\left\{\begin{array}{ccc}
6-c x & \text { if } & x \leq 2 c \\
x^{2} & \text { if } & x>2 c
\end{array}\right.
$$

Use the definition of continuity to justify your answer.

## Exercises - Stage 3

21. Show that there exists at least one real number $x$ satisfying $\sin x=x-1$
22. *. Show that there exists at least one real number $c$ such that $3^{c}=c^{2}$.
23. *. Show that there exists at least one real number $c$ such that $2 \tan (c)=c+1$.
24. *. Show that there exists at least one real number c such that $\sqrt{\cos (\pi c)}=$ $\sin (2 \pi c)+\frac{1}{2}$.
25. *. Show that there exists at least one real number $c$ such that $\frac{1}{(\cos \pi c)^{2}}=c+\frac{3}{2}$.
26. Use the intermediate value theorem to find an interval of length one containing a root of $f(x)=x^{7}-15 x^{6}+9 x^{2}-18 x+15$.
27. Use the intermediate value theorem to give a decimal approximation of $\sqrt[3]{7}$ that is correct to at least two decimal places. You may use a calculator, but only to add, subtract, multiply, and divide.
28. Suppose $f(x)$ and $g(x)$ are functions that are continuous over the interval $[a, b]$, with $f(a) \leq g(a)$ and $g(b) \leq f(b)$. Show that there exists some $c \in[a, b]$ with $f(c)=g(c)$.

## 1.7^ (Optional) - Making the Informal a Little More Formal

As we noted above, the definition of limits that we have been working with was quite informal and not mathematically rigorous. In this (optional) section we will work to understand the rigorous definition of limits.

Here is the formal definition - we will work through it all very slowly and carefully afterwards, so do not panic.

## Definition 1.7.1

Let $a \in \mathbb{R}$ and let $f(x)$ be a function defined everywhere in a neighbourhood of $a$, except possibly at $a$. We say that
the limit as $x$ approaches $a$ of $f(x)$ is $L$
or equivalently
as $x$ approaches $a, f(x)$ approaches $L$
and write

$$
\lim _{x \rightarrow a} f(x)=L
$$

if and only if for every $\epsilon>0$ there exists $\delta>0$ so that

$$
|f(x)-L|<\epsilon \text { whenever } 0<|x-a|<\delta
$$

Note that an equivalent way of writing this very last statement is

$$
\text { if } 0<|x-a|<\delta \text { then }|f(x)-L|<\epsilon
$$

This is quite a lot to take in, so let us break it down into pieces.

## Definition 1.7.2 The typical 3 pieces of a definition.

Usually a definition can be broken down into three pieces.

- Scene setting - define symbols and any restrictions on the objects that we are talking about.
- Naming - state the name and any notation for the property or object that the definition is about.
- Properties and restrictions - this is the heart of the definition where we explain to the reader what it is that the object (in our case a function) has to do in order to satisfy the definition.

Let us go back to the definition and look at each of these pieces in turn.

- Setting things up - The first sentence of the definition is really just setting up the picture. It is telling us what the definition is about and sorting out a few technical details.
- Let $a \in \mathbb{R}$ - This simply tells us that the symbol " $a$ " is a real number ${ }^{1}$.
- Let $f(x)$ be a function - This is just setting the scene so that we understand all of the terms and symbols.
- defined everywhere in a neighbourhood of a, except possibly at a This is just a technical requirement; we need our function to be defined in a little region ${ }^{2}$ around $a$. The function doesn't have to be defined everywhere, but it must be defined for all $x$-values a little less than $a$ and a little more than $a$. The definition does not care about what the function does outside this little window, nor does it care what happens exactly at $a$.
- Names, phrases and notation - The next part of the definition is simply naming the property we are discussing and tells us how to write it down. i.e. we are talking about "limits" and we write them down using the symbols indicated.

1 The symbol " $\in$ " is read as "is an element of" - it is definitely not the same as $e$ or $\epsilon$ or $\varepsilon$. If you do not recognise " $\mathbb{R}$ " or understand the difference between $\mathbb{R}$ and $R$, then please go back and read Chapter 0 carefully.
2 The term "neighbourhood of $a$ " means a small open interval around $a$ - for example ( $a-0.01, a+$ 0.01 ). Typically we don't really care how big this little interval is.

- The heart of things - we explain this at length below, but for now we will give a quick explanation. Work on these two points. They are hard.
- for all $\epsilon>0$ there exists $\delta>0$ - It is important we read this in order. It means that we can pick any positive number $\epsilon$ we want and there will always be another positive number $\delta$ that is going to make what ever follows be true.
- if $0<|x-a|<\delta$ then $|f(x)-L|<\epsilon$ - From the previous point we have our two numbers - any $\epsilon>0$ then based on that choice of $\epsilon$ we have a positive number $\delta$. The current statement says that whenever we have chosen $x$ so that it is very close to $a$, then $f(x)$ has to be very close to $L$. How close it "very close"? Well $0<|x-a|<\delta$ means that $x$ has to be within a distance $\delta$ of $a$ (but not exactly $a$ ) and similarly $|f(x)-L|<\epsilon$ means that $f(x)$ has to be within a distance $\epsilon$ of $L$.

That is the definition broken up into pieces which hopefully now make more sense, but what does it actually mean? Consider a function we saw earlier

$$
f(x)= \begin{cases}2 x & x \neq 3 \\ 9 & x=3\end{cases}
$$

and sketch it again:


We know (from our earlier work) that $\lim _{x \rightarrow 3} f(x)=6$, so zoom in around $(x, y)=$ $(3,6)$. To make this look more like our definition, we have $a=3$ and $L=6$.

- Pick some small number $\epsilon>0$ and highlight the horizontal strip of all points $(x, y)$ for which $|y-L|<\epsilon$. This means all the $y$-values have to satisfy $L-\epsilon<y<L+\epsilon$.
- You can see that the graph of the function passes through this strip for some $x$-values close to $a$. What we need to be able to do is to pick a vertical strip of $x$-values around $a$ so that the function lies inside the horizontal strip.
- That is, we must find a small number $\delta>0$ so that for any $x$-value inside the vertical strip $a-\delta<x<a+\delta$, except exactly at $x=a$, the value of the function lies inside the horizontal strip, namely $L-\epsilon<y=f(x)<L+\epsilon$.
- We see (pictorially) that we can do this. If we were to choose a smaller value of $\epsilon$ making the horizontal strip narrower, it is clear that we can choose the vertical strip to be narrower. Indeed, it doesn't matter how small we make the horizontal strip, we will always be able to construct the second vertical strip.

The above is a pictorial argument, but we can quite easily make it into a mathematical one. We want to show the limit is 6 . That means for any $\epsilon$ we need to find a $\delta$ so that when

$$
3-\delta<x<3+\delta \text { with } x \neq 3 \quad \text { we have } \quad 6-\epsilon<f(x)<6+\epsilon
$$

Now we note that when $x \neq 3$, we have $f(x)=2 x$ and so

$$
6-\epsilon<f(x)<6+\epsilon \quad \text { implies that } \quad 6-\epsilon<2 x<6+\epsilon
$$

this nearly specifies a range of $x$ values, we just need to divide by 2

$$
3-\epsilon / 2<x<3+\epsilon / 2
$$

Hence if we choose $\delta=\epsilon / 2$ then we get the desired inequality

$$
3-\delta<x<3+\delta
$$

i.e. - no matter what $\epsilon>0$ is chosen, if we put $\delta=\epsilon / 2$ then when $3-\delta<x<3+\delta$ with $x \neq 3$ we will have $6-\epsilon<f(x)<6+\epsilon$. This is exactly what we need to satisfy the definition of "limit" above.

The above work gives us the argument we need, but it still needs to be written up properly. We do this below.

## Example 1.7.3 Formal limit of a simple function.

Find the limit as $x \rightarrow 3$ of the following function

$$
f(x)= \begin{cases}2 x & x \neq 3 \\ 9 & x=3\end{cases}
$$

Proof. We will show that the limit is equal to 6 . Let $\epsilon>0$ and $\delta=\epsilon / 2$. It remains to show that $|f(x)-6|<\epsilon$ whenever $|x-3|<\delta$.
So assume that $|x-3|<\delta$, and so

$$
\begin{array}{rr}
3-\delta<x<3+\delta & \text { multiply both sides by } 2 \\
6-2 \delta<2 x<6+2 \delta &
\end{array}
$$

Recall that $f(x)=2 x$ and that since $\delta=\epsilon / 2$

$$
6-\epsilon<f(x)<6+\epsilon
$$

We can conclude that $|f(x)-6|<\epsilon$ as required.

Because of the $\epsilon$ and $\delta$ in the definition of limits, we need to have $\epsilon$ and $\delta$ in the proof. While $\epsilon$ and $\delta$ are just symbols playing particular roles, and could be replaced with other symbols, this style of proof is usually called $\epsilon-\delta$ proof.

In the above example everything works, but it can be very instructive to see what happens in an example that doesn't work.

Example 1.7.4 Formal limit where limit does not exist.
Look again at the function

$$
f(x)= \begin{cases}x & x<2 \\ -1 & x=2 \\ x+3 & x>2\end{cases}
$$

and let us see why, according to the definition of the limit, that $\lim _{x \rightarrow 2} f(x) \neq 2$. Again, start by sketching a picture and zooming in around $(x, y)=(2,2)$ :


Try to proceed through the same steps as before:

- Pick some small number $\epsilon>0$ and highlight a horizontal strip that contains all $y$-values with $|y-L|<\epsilon$. This means all the $y$-values have to satisfy $L-\epsilon<y<$ $L+\epsilon$.
- You can see that the graph of the function passes through this strip for some $x$ values close to $a$. To the left of $a$, we can always find some $x$-values that make the function sit inside the horizontal- $\epsilon$-strip. However, unlike the previous example, there is a problem to the right of $a$. Even for $x$-values just a little larger than $a$, the value of $f(x)$ lies well outside the horizontal- $\epsilon$-strip.
- So given this choice of $\epsilon$, we can find a $\delta>0$ so that for $x$ inside the vertical strip $a-\delta<x<a$, the value of the function sits inside the horizontal- $\epsilon$-strip.
- Unfortunately, there is no way to choose a $\delta>0$ so that for $x$ inside the vertical strip $a<x<a+\delta$ (with $x \neq a$ ) the value of the function sits inside the horizontal-epsilon-strip.
- So it is impossible to choose $\delta$ so that for $x$ inside the vertical strip $a-\delta<x<a+\delta$ the value of the function sits inside the horizontal strip $L-\epsilon<y=f(x)<L+\epsilon$.
- Thus the limit of $f(x)$ as $x \rightarrow 2$ is not 2 .

Example 1.7.4
Doing things formally with $\epsilon$ 's and $\delta$ 's is quite painful for general functions. It is far better to make use of the arithmetic of limits (Theorem 1.4.3) and some basic building blocks (like those in Theorem 1.4.1). Thankfully for most of the problems we deal with in calculus (at this level at least) can be approached in exactly this way.

This does leave the problem of proving the arithmetic of limits and the limits of the basic building blocks. The proof of the Theorem 1.4.3 is quite involved and we leave it to the very end of this Chapter. Before we do that we will prove Theorem 1.4.1 by a formal $\epsilon-\delta$ proof. Then in the next section we will look at the formal definition of limits at infinity and prove Theorem 1.5.3. The proof of the Theorem 1.5.9, the arithmetic of infinite limits, is very similar to that of Theorem 1.4.3 and so we do not give it.

So let us now prove Theorem 1.4.1 in which we stated two simple limits:

$$
\lim _{x \rightarrow a} c=c \quad \text { and } \quad \lim _{x \rightarrow a} x=a .
$$

Here is the formal $\epsilon-\delta$ proof:

## Proof. Proof of Theorem 1.4.1

Since there are two limits to prove, we do each in turn. Let $a, c$ be real numbers.

- Let $\epsilon>0$ and set $f(x)=c$. Choose $\delta=1$, then for any $x$ satisfying $|x-a|<\delta$ (or indeed any real number $x$ at all) we have $|f(x)-c|=0<\epsilon$. Hence $\lim _{x \rightarrow a} c=c$ as required.
- Let $\epsilon>0$ and set $f(x)=x$. Choose $\delta=\epsilon$, then for any $x$ satisfying $|x-a|<\delta$ we have

$$
\begin{aligned}
a-\delta<x & <a+\delta \text { but } f(x)=x \text { and } \delta=\epsilon \\
a-\epsilon<f(x) & <a+\epsilon
\end{aligned}
$$

Thus we have $|f(x)-a|<\epsilon$. Hence $\lim _{x \rightarrow a} x=a$ as required.
This completes the proof.

## 1.8^ (Optional) - Making Infinite Limits a Little More Formal

For those of you who made it through the formal $\epsilon-\delta$ definition of limits we give the formal definition of limits involving infinity:

## Definition 1.8.1 Limits involving infinity - formal.

a Let $f$ be a function defined on the whole real line. We say that the limit as $x$ approaches $\infty$ of $f(x)$ is $L$
or equivalently

$$
f(x) \text { converges to } L \text { as } x \text { goes to } \infty
$$

and write

$$
\lim _{x \rightarrow \infty} f(x)=L
$$

if and only if for every $\epsilon>0$ there exists $M \in \mathbb{R}$ so that $|f(x)-L|<\epsilon$ whenever $x>M$.

Similarly we write

$$
\lim _{x \rightarrow-\infty} f(x)=K
$$

if and only if for every $\epsilon>0$ there exists $N \in \mathbb{R}$ so that $|f(x)-K|<\epsilon$ whenever $x<N$.
b Let $a$ be a real number and $f(x)$ be a function defined for all $x \neq a$. We write

$$
\lim _{x \rightarrow a} f(x)=\infty
$$

if and only if for every $P>0$ there exists $\delta>0$ so that $f(x)>P$ whenever $0<|x-a|<\delta$.
c Let $f$ be a function defined on the whole real line. We write

$$
\lim _{x \rightarrow \infty} f(x)=\infty
$$

if and only if for every $P>0$ there exists $M>0$ so that $f(x)>P$ whenever $x>M$.

Note that we can loosen the above requirements on the domain of definition of $f$ for example, in part (a) all we actually require is that $f(x)$ be defined for all $x$ larger
than some value. It would be sufficient to require "there is some $x_{0} \in \mathbb{R}$ so that $f$ is defined for all $x>x_{0}$ ". Also note that there are obvious variations of parts (b) and (c) with $\infty$ replaced by $-\infty$.

For completeness let's prove Theorem 1.5.3 using this form definition. The layout of the proof will be very similar to our proof of Theorem 1.4.1.

## Proof. Proof of Theorem 1.5.3.

There are four limits to prove in total and we do each in turn. Let $c \in \mathbb{R}$.

- Let $\epsilon>0$ and set $f(x)=c$. Choose $M=0$, then for any $x$ satisfying $x>M$ (or indeed any real number $x$ at all) we have $|f(x)-c|=0<\epsilon$. Hence $\lim _{x \rightarrow \infty} c=c$ as required.
- The proof that $\lim _{x \rightarrow-\infty} c=c$ is nearly identical. Again, let $\epsilon>0$ and set $f(x)=c$. Choose $N=0$, then for any $x$ satisfying $x<N$ we have $\mid f(x)-$ $c \mid=0<\epsilon$. Hence $\lim _{x \rightarrow-\infty} c=c$ as required.
- Let $\epsilon>0$ and set $f(x)=x$. Choose $M=\frac{1}{\epsilon}$. Then when $x>M$ we have

$$
\begin{array}{ll}
0<M<x & \text { divide through by } x M \text { to get } \\
0<\frac{1}{x}<\frac{1}{M}=\epsilon &
\end{array}
$$

Since $x>0, \frac{1}{x}=\left|\frac{1}{x}\right|=\left|\frac{1}{x}-0\right|<\epsilon$ as required.

- Again, the proof in the limit to $-\infty$ is similar but we have to be careful of signs. Let $\epsilon>0$ and set $f(x)=x$. Choose $N=-\frac{1}{\epsilon}$. Then when $x<N$ we have

$$
\begin{array}{ll}
0>N>x & \quad \text { divide through by } x N \text { to get } \\
0>\frac{1}{x}>\frac{1}{N}=-\epsilon &
\end{array}
$$

Notice that by assumption both $x, N<0$, so $x N>0$. Now since $x<0$, $\frac{1}{x}=-\left|\frac{1}{x}\right|=\left|\frac{1}{x}-0\right|<\epsilon$ as required.

This completes the proof.

## 1.9^ (Optional) - Proving the Arithmetic of Limits

Perhaps the most useful theorem of this chapter is Theorem 1.4.3 which shows how limits interact with arithmetic. In this (optional) section we will prove both the arithmetic of limits Theorem 1.4.3 and the Squeeze Theorem 1.4.18. Before we get to the proofs
it is very helpful to prove three technical lemmas that we'll need. The first is a very general result about absolute values of numbers:

Lemma 1.9.1 The triangle inequality.
For any $x, y \in \mathbb{R}$

$$
|x+y| \leq|x|+|y|
$$

Proof. Notice that for any real number $x$, we always have $-x, x \leq|x|$ and either $|x|=x$ or $|x|=-x$. So now let $x, y \in \mathbb{R}$. Then we must have either

$$
|x+y|=x+y \quad \leq|x|+|y|
$$

or

$$
|x+y|=-x-y \quad \leq|x|+|y|
$$

In both cases we end up with $|x+y| \leq|x|+|y|$.

The second lemma is more specialised. It proves that if we have a function $f(x) \rightarrow F$ as $x \rightarrow a$ then there must be a small window around $x=a$ where the function $f(x)$ must only take values not far from $F$. In particular it tells us that $|f(x)|$ cannot be bigger than $|F|+1$ when $x$ is very close to $a$.

## Lemma 1.9.2

Let $a \in \mathbb{R}$ and let $f$ be a function so that $\lim _{x \rightarrow a} f(x)=F$. Then there exists a $\delta>0$ so that if $0<|x-a|<\delta$ then we also have $|f(x)| \leq|F|+1$.

The proof is mostly just manipulating the $\epsilon-\delta$ definition of a limit with $\epsilon=1$.
Proof. Let $\epsilon=1$. Then since $f(x) \rightarrow F$ as $x \rightarrow a$, there exists $\delta>0$ so that when $0<|x-a|<\delta$, we also have $|f(x)-F| \leq \epsilon=1$. So now assume $0<|x-a|<\delta$. Then

$$
\begin{aligned}
-\epsilon & \leq f(x)-F \leq \epsilon & \text { rearrange a little } \\
-\epsilon+F & \leq f(x) \leq \epsilon+F &
\end{aligned}
$$

Now $\epsilon+F \leq \epsilon+|F|$ and $-\epsilon+F \geq-\epsilon-|F|$, so

$$
-\epsilon-|F| \leq f(x) \leq \epsilon+|F|
$$

Hence we have $|f(x)| \leq \epsilon+|F|=|F|+1$.

Finally our third technical lemma gives us a bound in the other direction; it tells us that when $x$ is close to $a$, the value of $|f(x)|$ cannot be much smaller than $|F|$.

## Lemma 1.9.3

Let $a \in \mathbb{R}$ and $F \neq 0$ and let $f$ be a function so that $\lim _{x \rightarrow a} f(x)=F$. Then there exists $\delta>0$ so that when $0<|x-a|<\delta$, we have $|f(x)|>\frac{|F|}{2}$.

Proof. Set $\epsilon=\frac{|F|}{2}>0$. Since $f(x) \rightarrow F$, we know there exists a $\delta>0$ so that when $0<|x-a|<\delta$ we have $|f(x)-F|<\epsilon$. So now assume $0<|x-a|<\delta$ so that $|f(x)-F|<\epsilon=\frac{|F|}{2}$. Then

$$
\begin{array}{rlr}
|F| & =|F-f(x)+f(x)| & \text { sneaky trick } \\
& \leq|f(x)-F|+|f(x)| & \text { but }|f(x)-F|<\epsilon \\
& <\epsilon+|f(x)| &
\end{array}
$$

Hence $|f(x)|>|F|-\epsilon=\frac{|F|}{2}$ as required.

Now we are in a position to prove Theorem 1.4.3. The proof has more steps than the previous $\epsilon-\delta$ proofs we have seen. This is mostly because we do not have specific functions $f(x)$ and $g(x)$ and instead must play with them in the abstract - and make good use of the formal definition of limits.

We will break the proof into three pieces. The minimum that is required is to prove that

$$
\begin{aligned}
\lim _{x \rightarrow a}(f(x)+g(x)) & =F+G \\
\lim _{x \rightarrow a} f(x) \cdot g(x) & =F \cdot G \\
\lim _{x \rightarrow a} \frac{1}{g(x)} & =\frac{1}{G} \quad \text { if } G \neq 0 .
\end{aligned}
$$

From these three we can prove that

$$
\begin{aligned}
\lim _{x \rightarrow a} f(x) \cdot c & =F \cdot c \\
\lim _{x \rightarrow a}(f(x)-g(x)) & =F-G \\
\lim _{x \rightarrow a} \frac{f(x)}{g(x)} & =\frac{F}{G} \quad \text { if } G \neq 0 .
\end{aligned}
$$

The first follows by setting $g(x)=c$ and using $\lim f(x) \cdot g(x)$. The second follows by setting $c=-1$, putting $h(x)=(-1) \cdot g(x)$ and then applying both $\lim f(x) \cdot g(x)$ and
$\lim f(x)+g(x)$. The third follows by setting $h(x)=\frac{1}{g(x)}$ and then using $\lim f(x) \cdot h(x)$.
Starting with addition, in order to satisfy the definition of limit, we are going to have to show that

$$
|(f(x)+g(x))-(F+G)| \text { is small }
$$

when we know that $|f(x)-F|,|g(x)-G|$ are small. To do this we use the triangle inequality above showing that

$$
|(f(x)+g(x))-(F+G)|=|(f(x)-F)+(g(x)-G)| \leq|f(x)-F|+|g(x)-G|
$$

This is the key technical piece of the proof. So if we want the LHS of the above to be size $\epsilon$, we need to make sure that each term on the RHS is of size $\frac{\epsilon}{2}$. The rest of the proof is setting up facts based on the definition of limits and then rearranging facts to reach the conclusion.

## Proof. Proof of Theorem 1.4.3-limit of a sum.

Let $a \in \mathbb{R}$ and assume that

$$
\lim _{x \rightarrow a} f(x)=F \quad \text { and } \quad \lim _{x \rightarrow a} g(x)=G
$$

We wish to show that

$$
\lim _{x \rightarrow a} f(x)+g(x)=F+G
$$

Let $\epsilon>0$ - we have to find a $\delta>0$ so that when $|x-a|<\delta$ we have $\mid(f(x)+$ $g(x))-(F+G) \mid<\epsilon$.
Let $\epsilon>0$ and set $\epsilon_{1}=\epsilon_{2}=\frac{\epsilon}{2}$. By the definition of limits, because $f(x) \rightarrow F$ there exists some $\delta_{1}>0$ so that whenever $|x-a|<\delta_{1}$, we also have $|f(x)-F|<\epsilon_{1}$. Similarly there exists $\delta_{2}>0$ so that if $|x-a|<\delta_{2}$, then we must have $|g(x)-G|<$ $\epsilon_{2}$. So now choose $\delta=\min \left\{\delta_{1}, \delta_{2}\right\}$ and assume $|x-a|<\delta$. Then we must have that $|x-a|<\delta_{1}, \delta_{2}$ and so we also have

$$
|f(x)-F|<\epsilon_{1} \quad|g(x)-G|<\epsilon_{2}
$$

Now consider $|(f(x)+g(x))-(F+G)|$ and rearrange the terms:

$$
|(f(x)+g(x))-(F+G)|=|(f(x)-F)+(g(x)-G)|
$$

now apply triangle inequality

$$
\begin{aligned}
& \leq|f(x)-F|+|g(x)-G| \quad \text { use facts from above } \\
& <\epsilon_{1}+\epsilon_{2} \\
& =\epsilon
\end{aligned}
$$

Hence we have shown that for any $\epsilon>0$ there exists some $\delta>0$ so that when $|x-a|<\delta$ we also have $|(f(x)+g(x))-(F+G)|<\epsilon$. Which is exactly the formal definition of the limit we needed to prove.

Let us do similarly for the limit of a product. Some of the details of the proof are very similar, but there is a little technical trick in the middle to make it work. In particular we need to show that

$$
|f(x) \cdot g(x)-F \cdot G| \text { is small }
$$

when we know that $|f(x)-F|$ and $|g(x)-G|$ are both small. Notice that

$$
\begin{aligned}
f(x) \cdot g(x)-F \cdot G & =f(x) \cdot g(x)-F \cdot G+\underbrace{f(x) \cdot G-f(x) \cdot G}_{=0} \\
& =f(x) \cdot g(x)-f(x) \cdot G+f(x) \cdot G-F \cdot G \\
& =f(x) \cdot(g(x)-G)+(f(x)-F) \cdot G
\end{aligned}
$$

So if we know $|f(x)-F|$ is small and $|g(x)-G|$ is small then we are done - except that we also need to know that $f(x)$ doesn't become really large near $a$ - this is exactly why we needed to prove Lemma 1.9.2.

As was the case in the previous proof, we want the LHS to be of size at most $\epsilon$, so we want, for example, the two terms on the RHS to be of size at most $\frac{\epsilon}{2}$. This means

- we need $|G| \cdot|f(x)-F|$ to be of size at most $\frac{\epsilon}{2}$, and
- we need $|g(x)-G|$ to be of size at most $\frac{\epsilon}{2(|F|+1)}$ since we know that $|f(x)| \leq|F|+1$ when $x$ is close to $a$.

Armed with these tricks we turn to the proofs.

## Proof. Proof of Theorem 1.4.3-limit of a product.

Let $a \in \mathbb{R}$ and assume that

$$
\lim _{x \rightarrow a} f(x)=F \quad \text { and } \quad \lim _{x \rightarrow a} g(x)=G
$$

We wish to show that

$$
\lim _{x \rightarrow a} f(x) \cdot g(x)=F \cdot G
$$

Let $\epsilon>0$. Set $\epsilon_{1}=\frac{\epsilon}{2(|G|+1)}$ (the extra +1 in the denominator is just there to make sure that $\epsilon_{1}$ is well-defined even if $G=0$ ), and $\epsilon_{2}=\frac{\epsilon}{2(|F|+1)}$. From this we establish the existence of $\delta_{1}, \delta_{2}, \delta_{3}$ which we need below.

- By assumption $f(x) \rightarrow F$ so there exists $\delta_{1}>0$ so that whenever $|x-a|<\delta_{1}$, we also have $|f(x)-F|<\epsilon_{1}$.
- Similarly because $g(x) \rightarrow G$, there exists $\delta_{2}>0$ so that whenever $|x-a|<$ $\delta_{2}$, we also have $|g(x)-G|<\epsilon_{2}$.
- By Lemma 1.9.2 there exists $\delta_{3}>0$ so that whenever $|x-a|<\delta_{3}$, we also have $|f(x)| \leq|F|+1$.
Let $\delta=\min \left\{\delta_{1}, \delta_{2}, \delta_{3}\right\}$, assume $|x-a|<\delta$ and consider $|f(x) \cdot g(x)-F \cdot G|$. Rearrange the terms as we did above:

$$
|f(x) \cdot g(x)-F \cdot G|=|f(x) \cdot(g(x)-G)+(f(x)-F) \cdot G|
$$

$$
\leq|f(x)| \cdot|g(x)-G|+|G| \cdot|f(x)-F|
$$

By our three dot-points above we know that $|f(x)-F|<\epsilon_{1}$ and $|g(x)-G|<\epsilon_{2}$ and $|f(x)| \leq|F|+1$, so we have

$$
|f(x) \cdot g(x)-F \cdot G|<|f(x)| \cdot \epsilon_{2}+|G| \cdot \epsilon_{1}
$$

sub in $\epsilon_{1}, \epsilon_{2}$ and bound on $f(x)$

$$
\begin{aligned}
& <(|F|+1) \cdot \frac{\epsilon}{2(|F|+1)}+|G| \cdot \frac{\epsilon}{2(|G|+1)} \\
& \leq \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon
\end{aligned}
$$

Thus we have shown that for any $\epsilon>0$ there exists $\delta>0$ so that when $|x-a|<\delta$ we also have $|f(x) \cdot g(x)-F \cdot G|<\epsilon$. Hence $f(x) \cdot g(x) \rightarrow F \cdot G$.

Finally we can prove the limit of a reciprocal. Notice that

$$
\frac{1}{g(x)}-\frac{1}{G}=\frac{G-g(x)}{g(x) \cdot G}
$$

We need to show the LHS is of size at most $\epsilon$ when $x$ is close enough to $a$, so if $G-g(x)$ is small we are done - except if $g(x)$ or $G$ are close to zero. By assumption (go back and read Theorem 1.4.3) we have $G \neq 0$, and we know from Lemma 1.9.3 that $|g(x)|$ cannot be smaller than $\frac{|G|}{2}$. Together these imply that the denominator on the RHS cannot be zero and indeed must be of magnitude at least $\frac{|G|^{2}}{2}$. Thus we need $|G-g(x)|$ to be of size at most $\epsilon \cdot \frac{|G|^{2}}{2}$.

## Proof. Proof of Theorem 1.4.3-limit of a reciprocal.

Let $\epsilon>0$ and set $\epsilon_{1}=\epsilon|G|^{2} \cdot \frac{1}{2}$. We now use this and Lemma 1.9.3 to establish the existence of $\delta_{1}, \delta_{2}$.

- Since $g(x) \rightarrow G$ we know that there exists $\delta_{1}>0$ so that when $|x-a|<\delta_{1}$ we also have $|g(x)-G|<\epsilon_{1}$.
- By Lemma 1.9.3 there exists $\delta_{2}$ so that when $|x-a|<\delta_{2}$ we also have $|g(x)|>\frac{|G|}{2}$. Equivalently, when $|x-a|<\delta_{2}$ we also have $\left|\frac{G}{2 g(x)}\right|<1$.
Set $\delta=\min \left\{\delta_{1}, \delta_{2}\right\}$ and assume $|x-a|<\delta$. Then

$$
\begin{array}{rlr}
\left|\frac{1}{g(x)}-\frac{1}{G}\right| & =\left|\frac{G-g(x)}{g(x) \cdot G}\right| & \\
& =|g(x)-G| \cdot \frac{1}{|G| \cdot|g(x)|} & \text { by assumption } \\
& <\frac{\epsilon_{1}}{|G| \cdot|g(x)|} & \text { sub in } \epsilon_{1}
\end{array}
$$

$$
\begin{aligned}
& =\epsilon \cdot \frac{|G|}{2|g(x)|} \quad \text { since }\left|\frac{G}{2 g(x)}\right|<1 \\
& <\epsilon
\end{aligned}
$$

Thus we have shown that for any $\epsilon>0$ there exists $\delta>0$ so that when $|x-a|<\delta$ we also have $\left|\frac{1}{g(x)}-\frac{1}{G}\right|<\epsilon$. Hence $\frac{1}{g(x)} \rightarrow \frac{1}{G}$.

We can also now prove the Squeeze / sandwich / pinch theorem.
Proof. Proof of Theorem 1.4.18-Squeeze / sandwich / pinch.
In the squeeze theorem, we are given three functions $f(x), g(x)$ and $h(x)$ and are told that

$$
f(x) \leq g(x) \leq h(x) \quad \text { and } \quad \lim _{x \rightarrow a} f(x)=\lim _{x \rightarrow a} h(x)=L
$$

and we must conclude from this that $\lim _{x \rightarrow a} g(x)=L$ too. That is, we are given some fixed, but unspecified, $\epsilon>0$ and it is up to us to find a $\delta>0$ with the property that $|g(x)-L|<\epsilon$ whenever $|x-a|<\delta$. Now because we have been told that $f$ and $h$ both converge to $L$, there exist $\delta_{1}>0$ and $\delta_{2}>0$ such that

- $|f(x)-L|<\epsilon$, i.e. $L-\epsilon<f(x)<L+\epsilon$, whenever $|x-a|<\delta_{1}$, and
- $|h(x)-L|<\epsilon$, i.e. $L-\epsilon<h(x)<L+\epsilon$, whenever $|x-a|<\delta_{2}$

So set $\delta=\min \left\{\delta_{1}, \delta_{2}\right\}$ and assume $|x-a|<\delta$. Then both $L-\epsilon<f(x)<L+\epsilon$ and $L-\epsilon<h(x)<L+\epsilon$ so that

$$
\begin{array}{cr}
L-\epsilon<f(x) \leq g(x) \leq h(x)<L+\epsilon & \text { which implies that } \\
L-\epsilon<g(x)<L+\epsilon & \text { which in turn gives us } \\
|g(x)-L|<\epsilon &
\end{array}
$$

as desired.


Calculus is built on two operations - differentiation, which is used to analyse instantaneous rate of change, and integration, which is used to analyse areas. Understanding differentiation and using it to compute derivatives of functions is one of the main aims of this course.

We had a glimpse of derivatives in the previous chapter on limits - in particular Sections 1.1 and 1.2 on tangents and velocities introduced derivatives in disguise. One of the main reasons that we teach limits is to understand derivatives. Fortunately, as we shall see, while one does need to understand limits in order to correctly understand derivatives, one does not need the full machinery of limits in order to compute and work with derivatives. The other main part of calculus, integration, we (mostly) leave until a later course.

The derivative finds many applications in many different areas of the sciences. Indeed the reason that calculus is taken by so many university students is so that they may then use the ideas both in subsequent mathematics courses and in other fields. In almost any field in which you study quantitative data you can find calculus lurking somewhere nearby.

Its development ${ }^{1}$ came about over a very long time, starting with the ancient Greek geometers. Indian, Persian and Arab mathematicians made significant contributions from around the $6^{\text {th }}$ century. But modern calculus really starts with Newton and Leibniz in the $17^{\text {th }}$ century who developed independently based on ideas of others including Descartes. Newton applied his work to many physical problems (including orbits of moons and planets) but didn't publish his work. When Leibniz subsequently published his "calculus", Newton accused him of plagiarism - this caused a huge rift between British and continental-European mathematicians which wasn't closed for another century.

[^3]
## DERIVATIVES

## 2.1」 Revisiting Tangent Lines

### 2.1.1 Revisiting Tangent Lines

By way of motivation for the definition of the derivative, we return to the discussion of tangent lines that we started in the previous chapter on limits. We consider, in Examples 2.1.2 and 2.1.5, below, the problem of finding the slope of the tangent line to a curve at a point. But let us start by recalling, in Example 2.1.1, what is meant by the slope of a straight line.

Example 2.1.1 What is slope.
In this example, we recall what is meant by the slope of the straight line

$$
y=\frac{1}{2} x+\frac{3}{2}
$$

- We claim that if, as we walk along this straight line, our $x$-coordinate changes by an amount $\Delta x$, then our $y$-coordinate changes by exactly $\Delta y=\frac{1}{2} \Delta x$.
- For example, in the figure on the left below, we move from the point

$$
\left(x_{0}, y_{0}\right)=\left(1,2=\frac{1}{2} \times 1+\frac{3}{2}\right)
$$

on the line to the point

$$
\left(x_{1}, y_{1}\right)=\left(5,4=\frac{1}{2} \times 5+\frac{3}{2}\right)
$$

on the line. In this move our $x$-coordinate changes by

$$
\Delta x=5-1=4
$$

and our $y$-coordinate changes by

$$
\Delta y=4-2=2
$$

which is indeed $\frac{1}{2} \times 4=\frac{1}{2} \Delta x$, as claimed.



- In general, when we move from the point

$$
\left(x_{0}, y_{0}\right)=\left(x_{0}, \frac{1}{2} x_{0}+\frac{3}{2}\right)
$$

on the line to the point

$$
\left(x_{1}, y_{1}\right)=\left(x_{1}, \frac{1}{2} x_{1}+\frac{3}{2}\right)
$$

on the line, our $x$-coordinate changes by

$$
\Delta x=x_{1}-x_{0}
$$

and our $y$-coordinate changes by

$$
\begin{aligned}
\Delta y & =y_{1}-y_{0} \\
& =\left[\frac{1}{2} x_{1}+\frac{3}{2}\right]-\left[\frac{1}{2} x_{0}+\frac{3}{2}\right] \\
& =\frac{1}{2}\left(x_{1}-x_{0}\right)
\end{aligned}
$$

which is indeed $\frac{1}{2} \Delta x$, as claimed.

- So, for the straight line $y=\frac{1}{2} x+\frac{3}{2}$, the ratio $\frac{\Delta y}{\Delta x}=\frac{y_{1}-y_{0}}{x_{1}-x_{0}}$ always takes the value $\frac{1}{2}$, regardless of the choice of initial point $\left(x_{0}, y_{0}\right)$ and final point $\left(x_{1}, y_{1}\right)$. This constant ratio is the slope of the line $y=\frac{1}{2} x+\frac{3}{2}$.

Example 2.1.1
Straight lines are special in that for each straight line, there is a fixed number $m$, called the slope of the straight line, with the property that if you take any two different points, $\left(x_{0}, y_{0}\right)$ and $\left(x_{1}, y_{1}\right)$, on the line, the ratio $\frac{\Delta y}{\Delta x}=\frac{y_{1}-y_{0}}{x_{1}-x_{0}}$, which is called the rate of change of $y$ per unit rate of change ${ }^{1}$ of $x$, always takes the value $m$. This is the property that distinguishes lines from other curves.

Other curves do not have this property. In the next two examples we illustrate this point with the parabola $y=x^{2}$. Recall that we studied this example back in Section 1.1. In Example 2.1.2 we find the slope of the tangent line to $y=x^{2}$ at a particular point. We generalise this in Example 2.1.5, to show that we can define "the slope of the curve $y=x^{2 \prime \prime}$ at an arbitrary point $x=x_{0}$ by considering $\frac{\Delta y}{\Delta x}=\frac{y_{1}-y_{0}}{x_{1}-x_{0}}$ with $\left(x_{1}, y_{1}\right)$ very close to $\left(x_{0}, y_{0}\right)$.

Example 2.1.2 Slope of secants of $y=x^{2}$.
In this example, let us fix $\left(x_{0}, y_{0}\right)$ to be the point $(2,4)$ on the parabola $y=x^{2}$. Now let $\left(x_{1}, y_{1}\right)=\left(x_{1}, x_{1}^{2}\right)$ be some other point on the parabola; that is, a point with $x_{1} \neq x_{0}$.

1 In the "real world" the phrase "rate of change" usually refers to rate of change per unit time. In science it used more generally.

- Draw the straight line through $\left(x_{0}, y_{0}\right)$ and $\left(x_{1}, y_{1}\right)$ - this is a secant line and we saw these in Chapter 1 when we discussed tangent lines ${ }^{a}$.
- The following table gives the slope, $\frac{y_{1}-y_{0}}{x_{1}-x_{0}}$, of the secant line through $\left(x_{0}, y_{0}\right)=$ $(2,4)$ and $\left(x_{1}, y_{1}\right)$, for various different choices of $\left(x_{1}, y_{1}=x_{1}^{2}\right)$.

| $x_{1}$ | 1 | 1.5 | 1.9 | 1.99 | 1.999 | $\circ$ | 2.001 | 2.01 | 2.1 | 2.5 | 3 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $y_{1}=x_{1}^{2}$ | 1 | 2.25 | 3.61 | 3.9601 | 3.9960 | $\circ$ | 4.0040 | 4.0401 | 4.41 | 6.25 | 9 |
| $\frac{y_{1}-y_{0}}{x_{1}-x_{0}}$ | 3 | 3.5 | 3.9 | 3.99 | 3.999 | $\circ$ | 4.001 | 4.01 | 4.1 | 4.5 | 5 |

- So now we have a big table of numbers - what do we do with them? Well, there are messages we can take away from this table.
- Different choices of $x_{1}$ give different values for the slope, $\frac{y_{1}-y_{0}}{x_{1}-x_{0}}$, of the secant through $\left(x_{0}, y_{0}\right)$ and $\left(x_{1}, y_{1}\right)$. This is illustrated in Figure 2.1.3 below - the slope of the secant through $\left(x_{0}, y_{0}\right)$ and $\left(x_{1}, y_{1}\right)$ is different from the slope of the secant through $\left(x_{0}, y_{0}\right)$ and $\left(x_{1}^{\prime}, y_{1}^{\prime}\right)$.


Figure 2.1.3: For a curvy curve, different secants have different slopes.

If the parabola were a straight line this would not be the case - the secant through any two different points on a line is always identical to the line itself and so always has exactly the same slope as the line itself, as is illustrated in Figure 2.1.4 below - the (yellow) secant through $\left(x_{0}, y_{0}\right)$ and $\left(x_{1}, y_{1}\right)$ lies exactly on top of the (red) line $y=\frac{1}{2} x+\frac{3}{2}$.


Figure 2.1.4: For a straight line, all secants have the same slope.

- Now look at the columns of the table closer to the middle. As $x_{1}$ gets closer and closer to $x_{0}=2$, the slope, $\frac{y_{1}-y_{0}}{x_{1}-x_{0}}$, of the secant through $\left(x_{0}, y_{0}\right)$ and $\left(x_{1}, y_{1}\right)$ appears to get closer and closer to the value 4.
$a \quad$ If you do not remember this, then please revisit the first couple of sections of Chapter 1.
$\qquad$
Example 2.1.4

Example 2.1.5 More on secants of $y=x^{2}$.
It is very easy to generalise what is happening in Example 2.1.2.

- Fix any point $\left(x_{0}, y_{0}\right)$ on the parabola $y=x^{2}$. If $\left(x_{1}, y_{1}\right)$ is any other point on the parabola $y=x^{2}$, then $y_{1}=x_{1}^{2}$ and the slope of the secant through $\left(x_{0}, y_{0}\right)$ and $\left(x_{1}, y_{1}\right)$ is

$$
\begin{aligned}
\text { slope } & =\frac{y_{1}-y_{0}}{x_{1}-x_{0}}=\frac{x_{1}^{2}-x_{0}^{2}}{x_{1}-x_{0}} & & \text { since } y=x^{2} \\
& =\frac{\left(x_{1}-x_{0}\right)\left(x_{1}+x_{0}\right)}{x_{1}-x_{0}} & & \text { remember } a^{2}-b^{2}=(a-b)(a+b) \\
& =x_{1}+x_{0} & &
\end{aligned}
$$

You should check the values given in the table of Example 2.1.2 above to convince yourself that the slope $\frac{y_{1}-y_{0}}{x_{1}-x_{0}}$ of the secant line really is $x_{0}+x_{1}=2+x_{1}$ (since we set $x_{0}=2$ ).

- Now as we move $x_{1}$ closer and closer to $x_{0}$, the slope should move closer and closer to $2 x_{0}$. Indeed if we compute the limit carefully - we now have the technology
to do this - we see that in the limit as $x_{1} \rightarrow x_{0}$ the slope becomes $2 x_{0}$. That is

$$
\begin{aligned}
\lim _{x_{1} \rightarrow x_{0}} \frac{y_{1}-y_{0}}{x_{1}-x_{0}} & =\lim _{x_{1} \rightarrow x_{0}}\left(x_{1}+x_{0}\right) \quad \text { by the work we did just above } \\
& =2 x_{0}
\end{aligned}
$$

Taking this limit gives us our first derivative. Of course we haven't yet given the definition of a derivative, so we perhaps wouldn't recognise it yet. We rectify this in the next section.


Figure 2.1.6: Secants approaching a tangent line

- So it is reasonable to say "as $x_{1}$ approaches $x_{0}$, the secant through $\left(x_{0}, y_{0}\right)$ and $\left(x_{1}, y_{1}\right)$ approaches the tangent line to the parabola $y=x^{2}$ at $\left(x_{0}, y_{0}\right)$ ". This is what we did back in Section 1.1.
The figure above shows four different secants through $\left(x_{0}, y_{0}\right)$ for the curve $y=x^{2}$. The four hollow circles are four different choices of $\left(x_{1}, y_{1}\right)$. As $\left(x_{1}, y_{1}\right)$ approaches $\left(x_{0}, y_{0}\right)$, the corresponding secant does indeed approach the tangent to $y=x^{2}$ at $\left(x_{0}, y_{0}\right)$, which is the heavy (red) straight line in the figure.
Using limits we determined the slope of the tangent line to $y=x^{2}$ at $x_{0}$ to be $2 x_{0}$. Often we will be a little sloppy with our language and instead say "the slope of the parabola $y=x^{2}$ at $\left(x_{0}, y_{0}\right)$ is $2 x_{0}$ " - where we really mean the slope of the line tangent to the parabola at $x_{0}$.

Example 2.1.6

### 2.1.2 $\leadsto$ Exercises

## Exercises - Stage 1

1. Shown below is the graph $y=f(x)$. If we choose a point $Q$ on the graph to the left of the $y$-axis, is the slope of the secant line through $P$ and $Q$ positive or negative? If we choose a point $Q$ on the graph to the right of the $y$-axis, is the slope of the secant line through $P$ and $Q$ positive or negative?

2. Shown below is the graph $y=f(x)$.
a If we want the slope of the secant line through $P$ and $Q$ to increase, should we slide $Q$ closer to $P$, or further away?
b Which is larger, the slope of the tangent line at $P$, or the slope of the secant line through $P$ and $Q$ ?

3. Group the functions below into collections whose secant lines from $x=-2$ to $x=2$ all have the same slopes.


## Exercises - Stage 2

4. Give your best approximation of the slope of the tangent line to the graph below at the point $x=5$.

5. On the graph below, sketch the tangent line to $y=f(x)$ at $P$. Then, find two points $Q$ and $R$ on the graph so that the secant line through $Q$ and $R$ has the same slope as the tangent line at $P$.

6. Mark the points where the curve shown below has a tangent line with slope 0.

(Later on, we'll learn how these points tell us a lot about the shape of a graph.)

### 2.2 Definition of the Derivative

We now define the "derivative" explicitly, based on the limiting slope ideas of the previous section. Then we see how to compute some simple derivatives.

Let us now generalise what we did in the last section so as to find "the slope of the
curve $y=f(x)$ at $\left(x_{0}, y_{0}\right)$ " for any smooth enough ${ }^{1}$ function $f(x)$.
As before, let $\left(x_{0}, y_{0}\right)$ be any point on the curve $y=f(x)$. So we must have $y_{0}=f\left(x_{0}\right)$. Now let $\left(x_{1}, y_{1}\right)$ be any other point on the same curve. So $y_{1}=f\left(x_{1}\right)$ and $x_{1} \neq x_{0}$. Think of $\left(x_{1}, y_{1}\right)$ as being pretty close to $\left(x_{0}, y_{0}\right)$ so that the difference

$$
\Delta x=x_{1}-x_{0}
$$

in $x$-coordinates is pretty small. In terms of this $\Delta x$ we have

$$
x_{1}=x_{0}+\Delta x \quad \text { and } \quad y_{1}=f\left(x_{0}+\Delta x\right)
$$

We can construct a secant line through $\left(x_{0}, y_{0}\right)$ and $\left(x_{1}, y_{1}\right)$ just as we did for the parabola above. It has slope

$$
\frac{y_{1}-y_{0}}{x_{1}-x_{0}}=\frac{f\left(x_{0}+\Delta x\right)-f\left(x_{0}\right)}{\Delta x}
$$

If $f(x)$ is reasonably smooth ${ }^{2}$, then as $x_{1}$ approaches $x_{0}$, i.e. as $\Delta x$ approaches 0 , we would expect the secant through $\left(x_{0}, y_{0}\right)$ and $\left(x_{1}, y_{1}\right)$ to approach the tangent line to the curve $y=f(x)$ at $\left(x_{0}, y_{0}\right)$, just as happened in Figure 2.1.6. And more importantly, the slope of the secant through $\left(x_{0}, y_{0}\right)$ and $\left(x_{1}, y_{1}\right)$ should approach the slope of the tangent line to the curve $y=f(x)$ at $\left(x_{0}, y_{0}\right)$.

Thus we would expect ${ }^{3}$ the slope of the tangent line to the curve $y=f(x)$ at $\left(x_{0}, y_{0}\right)$ to be

$$
\lim _{\Delta x \rightarrow 0} \frac{f\left(x_{0}+\Delta x\right)-f\left(x_{0}\right)}{\Delta x}
$$

When we talk of the "slope of the curve" at a point, what we really mean is the slope of the tangent line to the curve at that point. So "the slope of the curve $y=f(x)$ at $\left(x_{0}, y_{0}\right)$ " is also the limit ${ }^{4}$ expressed in the above equation. The derivative of $f(x)$ at $x=x_{0}$ is also defined to be this limit. Which leads ${ }^{5}$ us to the most important definition in this text:

## Definition 2.2.1 Derivative at a point.

Let $a \in \mathbb{R}$ and let $f(x)$ be defined on an open interval ${ }^{a}$ that contains $a$.

- The derivative of $f(x)$ at $x=a$ is denoted $f^{\prime}(a)$ and is defined by

$$
f^{\prime}(a)=\lim _{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}
$$

if the limit exists.

1 The idea of "smooth enough" can be made quite precise. Indeed the word "smooth" has a very precise meaning in mathematics, which we won't cover here. For now think of "smooth" as meaning roughly just "smooth".
2 Again the term "reasonably smooth" can be made more precise.
3 Indeed, we don't have to expect - it is!
4 This is of course under the assumption that the limit exists - we will talk more about that below.
5 We will rename " $x_{0}$ " to " $a$ " and " $\Delta x$ " to " $h$ ".

- When the above limit exists, the function $f(x)$ is said to be differentiable at $x=a$. When the limit does not exist, the function $f(x)$ is said to be not differentiable at $x=a$.
- We can equivalently define the derivative $f^{\prime}(a)$ by the limit

$$
f^{\prime}(a)=\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a} .
$$

To see that these two definitions are the same, we set $x=a+h$ and then the limit as $h$ goes to 0 is equivalent to the limit as $x$ goes to $a$.
$a$ Recall, from Definition 0.3.5, that the open interval $(c, d)$ is just the set of all real numbers obeying $c<x<d$.

Lets now compute the derivatives of some very simple functions. This is our first step towards building up a toolbox for computing derivatives of complicated functions - this process will very much parallel what we did in Chapter 1 with limits. The two simplest functions we know are $f(x)=c$ and $g(x)=x$.

Example 2.2.2 Derivative of $f(x)=c$.
Let $a, c \in \mathbb{R}$ be a constants. Compute the derivative of the constant function $f(x)=c$ at $x=a$.
We compute the desired derivative by just substituting the function of interest into the formal definition of the derivative.

$$
\begin{aligned}
f^{\prime}(a) & =\lim _{h \rightarrow 0} \frac{f(a+h)-f(a)}{h} & & \text { (the definition) } \\
& =\lim _{h \rightarrow 0} \frac{c-c}{h} & & \text { (substituted in the function) } \\
& =\lim _{h \rightarrow 0} 0 & & \text { (simplified things) } \\
& =0 & &
\end{aligned}
$$

That was easy! What about the next most complicated function - arguably it's this one:

Example 2.2.3 Derivative of $g(x)=x$.
Let $a \in \mathbb{R}$ and compute the derivative of $g(x)=x$ at $x=a$.
Again, we compute the derivative of $g$ by just substituting the function of interest into
the formal definition of the derivative and then evaluating the resulting limit.

$$
\begin{aligned}
g^{\prime}(a) & =\lim _{h \rightarrow 0} \frac{g(a+h)-g(a)}{h} & & \text { (the definition) } \\
& =\lim _{h \rightarrow 0} \frac{(a+h)-a}{h} & & \text { (substituted in the function) } \\
& =\lim _{h \rightarrow 0} \frac{h}{h} & & \text { (simplified things) } \\
& =\lim _{h \rightarrow 0} 1 & & \text { (simplified a bit more) } \\
& =1 & &
\end{aligned}
$$

That was a little harder than the first example, but still quite straight forward start with the definition and apply what we know about limits.

Thanks to these two examples, we have our first theorem about derivatives:

## Theorem 2.2.4 Easiest derivatives.

Let $a, c \in \mathbb{R}$ and let $f(x)=c$ be the constant function and $g(x)=x$. Then

$$
f^{\prime}(a)=0
$$

and

$$
g^{\prime}(a)=1
$$

To ratchet up the difficulty a little bit more, let us redo the example we have already done a few times $f(x)=x^{2}$. To make it a little more interesting let's change the names of the function and the variable so that it is not exactly the same as Examples 2.1.2 and 2.1.5.

Example 2.2.5 Derivative of $h(t)=t^{2}$.
Compute the derivative of

$$
h(t)=t^{2} \quad \text { at } t=a
$$

- This function isn't quite like the ones we saw earlier - it's a function of $t$ rather than $x$. Recall that a function is a rule which assigns to each input value an output value. So far, we have usually called the input value $x$. But this " $x$ " is just a dummy variable representing a generic input value. There is nothing wrong with calling a generic input value $t$ instead. Indeed, from time to time you will see
functions that are not written as formulas involving $x$, but instead are written as formulas in $t$ (for example representing time - see Section 1.2), or $z$ (for example representing height), or other symbols.
- So let us write the definition of the derivative

$$
f^{\prime}(a)=\lim _{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}
$$

and then translate it to the function names and variables at hand:

$$
h^{\prime}(a)=\lim _{h \rightarrow 0} \frac{h(a+h)-h(a)}{h}
$$

- But there is a problem - " $h$ " plays two roles here - it is both the function name and the small quantity that is going to zero in our limit. It is extremely dangerous to have a symbol represent two different things in a single computation. We need to change one of them. So let's rename the small quantity that is going to zero in our limit from " $h$ " to " $\Delta t$ ":

$$
h^{\prime}(a)=\lim _{\Delta t \rightarrow 0} \frac{h(a+\Delta t)-h(a)}{\Delta t}
$$

- Now we are ready to begin. Substituting in what the function $h$ is,

$$
\begin{aligned}
h^{\prime}(a) & =\lim _{\Delta t \rightarrow 0} \frac{(a+\Delta t)^{2}-a^{2}}{\Delta t} \\
& =\lim _{\Delta t \rightarrow 0} \frac{a^{2}+2 a \Delta t+\Delta t^{2}-a^{2}}{\Delta t} \quad\left(\text { just squared out }(a+\Delta t)^{2}\right) \\
& =\lim _{\Delta t \rightarrow 0} \frac{2 a \Delta t+\Delta t^{2}}{\Delta t} \\
& =\lim _{\Delta t \rightarrow 0}(2 a+\Delta t) \\
& =2 a
\end{aligned}
$$

- You should go back check that this is what we got in Example 2.1.5 - just some names have been changed.


### 2.2.1 An Important Point (and Some Notation)

Notice here that the answer we get depends on our choice of $a$ - if we want to know the derivative at $a=3$ we can just substitute $a=3$ into our answer $2 a$ to get the slope is 6. If we want to know at $a=1$ (like at the end of Section 1.1) we substitute $a=1$ and
get the slope is 2 . The important thing here is that we can move from the derivative being computed at a specific point to the derivative being a function itself - input any value of $a$ and it returns the slope of the tangent line to the curve at the point $x=a$, $y=h(a)$. The variable $a$ is a dummy variable. We can rename $a$ to anything we want, like $x$, for example. So we can replace every $a$ in

$$
h^{\prime}(a)=2 a \quad \text { by } x, \text { giving } \quad h^{\prime}(x)=2 x
$$

where all we have done is replaced the symbol $a$ by the symbol $x$.
We can do this more generally and tweak the derivative at a specific point $a$ to obtain the derivative as a function of $x$. We replace

$$
f^{\prime}(a)=\lim _{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}
$$

with

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
$$

which gives us the following definition

## Definition 2.2.6 Derivative as a function.

Let $f(x)$ be a function.

- The derivative of $f(x)$ with respect to $x$ is

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
$$

provided the limit exists.

- If the derivative $f^{\prime}(x)$ exists for all $x \in(a, b)$ we say that $f$ is differentiable on $(a, b)$.
- Note that we will sometimes be a little sloppy with our discussions and simply write " $f$ is differentiable" to mean " $f$ is differentiable on an interval we are interested in" or " $f$ is differentiable everywhere".

Notice that we are no longer thinking of tangent lines, rather this is an operation we can do on a function. For example:

Example 2.2.7 The derivative of $f(x)=\frac{1}{x}$.
Let $f(x)=\frac{1}{x}$ and compute its derivative with respect to $x$ - think carefully about where the derivative exists.

- Our first step is to write down the definition of the derivative - at this stage, we
know of no other strategy for computing derivatives.

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \quad \text { (the definition) }
$$

- And now we substitute in the function and compute the limit.

$$
\begin{aligned}
f^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} & & \text { (the definition) } \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{1}{x+h}-\frac{1}{x}\right] & & \text { (substituted in the function) } \\
& =\lim _{h \rightarrow 0} \frac{1}{h} \frac{x-(x+h)}{x(x+h)} & & \text { (wrote over a common denominator) } \\
& =\lim _{h \rightarrow 0} \frac{1}{h} \frac{-h}{x(x+h)} & & \text { (started cleanup) } \\
& =\lim _{h \rightarrow 0} \frac{-1}{x(x+h)} & & \\
& =-\frac{1}{x^{2}} & &
\end{aligned}
$$

- Notice that the original function $f(x)=\frac{1}{x}$ was not defined at $x=0$ and the derivative is also not defined at $x=0$. This does happen more generally - if $f(x)$ is not defined at a particular point $x=a$, then the derivative will not exist at that point either.

Example 2.2.7
So we now have two slightly different ideas of derivatives:

- The derivative $f^{\prime}(a)$ at a specific point $x=a$, being the slope of the tangent line to the curve at $x=a$, and
- The derivative as a function, $f^{\prime}(x)$ as defined in Definition 2.2.6.

Of course, if we have $f^{\prime}(x)$ then we can always recover the derivative at a specific point by substituting $x=a$.

As we noted at the beginning of the chapter, the derivative was discovered independently by Newton and Leibniz in the late $17^{\text {th }}$ century. Because their discoveries were independent, Newton and Leibniz did not have exactly the same notation. Stemming from this, and from the many different contexts in which derivatives are used, there are quite a few alternate notations for the derivative:

## Definition 2.2.8

The following notations are all used for "the derivative of $f(x)$ with respect to $x$ "

$$
f^{\prime}(x) \quad \frac{\mathrm{d} f}{\mathrm{~d} x} \quad \frac{\mathrm{~d}}{\mathrm{~d} x} f(x) \quad \dot{f}(x) \quad D f(x) \quad D_{x} f(x)
$$

while the following notations are all used for "the derivative of $f(x)$ at $x=a$ "

$$
\left.f^{\prime}(a) \quad \frac{\mathrm{d} f}{\mathrm{~d} x}(a) \quad \frac{\mathrm{d}}{\mathrm{~d} x} f(x)\right|_{x=a} \quad \dot{f}(a) \quad D f(a) \quad D_{x} f(a)
$$

Some things to note about these notations:

- We will generally use the first three, but you should recognise them all. The notation $f^{\prime}(a)$ is due to Lagrange, while the notation $\frac{\mathrm{d} f}{\mathrm{~d} x}(a)$ is due to Leibniz. They are both very useful. Neither can be considered "better".
- Leibniz notation writes the derivative as a "fraction" - however it is definitely not a fraction and should not be thought of in that way. It is just shorthand, which is read as "the derivative of $f$ with respect to $x$ ".
- You read $f^{\prime}(x)$ as " $f$-prime of $x$ ", and $\frac{\mathrm{d} f}{\mathrm{~d} x}$ as "dee $-f-$ dee $-x$ ", and $\frac{\mathrm{d}}{\mathrm{d} x} f(x)$ as "dee-by-dee- $x$ of $f$ ".
- Similarly you read $\frac{\mathrm{d} f}{\mathrm{~d} x}(a)$ as "dee- $f$-dee- $x$ at $a$ ", and $\left.\frac{\mathrm{d}}{\mathrm{d} x} f(x)\right|_{x=a}$ as "dee-by-dee- $x$ of $f$ of $x$ at $x$ equals $a$ ".
- The notation $\dot{f}$ is due to Newton. In physics, it is common to use $\dot{f}(t)$ to denote the derivative of $f$ with respect to time.


### 2.2.2 Back to Computing Some Derivatives

At this point we could try to start working out how derivatives interact with arithmetic and make an "Arithmetic of derivatives" theorem just like the one we saw for limits (Theorem 1.4.3). We will get there shortly, but before that it is important that we become more comfortable with computing derivatives using limits and then understanding what the derivative actually means. So - more examples.

Example 2.2.9 $\frac{\mathrm{d}}{\mathrm{d} x} \sqrt{x}$.
Compute the derivative, $f^{\prime}(a)$, of the function $f(x)=\sqrt{x}$ at the point $x=a$ for any $a>0$.

- So again we start with the definition of derivative and go from there:

$$
f^{\prime}(a)=\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a}=\lim _{x \rightarrow a} \frac{\sqrt{x}-\sqrt{a}}{x-a}
$$

- As $x$ tends to $a$, the numerator and denominator both tend to zero. But $\frac{0}{0}$ is not defined. So to get a well defined limit we need to exhibit a cancellation between
the numerator and denominator - just as we saw in Examples 1.4.12 and 1.4.17. Now there are two equivalent ways to proceed from here, both based on a similar "trick".
- For the first, review Example 1.4.17, which concerned taking a limit involving square-roots, and recall that we used "multiplication by the conjugate" there:

$$
\begin{aligned}
& \frac{\sqrt{x}-\sqrt{a}}{x-a} \\
& =\frac{\sqrt{x}-\sqrt{a}}{x-a} \times \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} \quad\left(\text { multiplication by } 1=\frac{\text { conjugate }}{\text { conjugate }}\right) \\
& =\frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{(x-a)(\sqrt{x}+\sqrt{a})} \\
& =\frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})} \\
& =\frac{1}{\sqrt{x}+\sqrt{a}}
\end{aligned}
$$

- Alternatively, we can arrive at $\frac{\sqrt{x}-\sqrt{a}}{x-a}=\frac{1}{\sqrt{x}+\sqrt{a}}$ by using almost the same trick to factor the denominator. Just set $A=\sqrt{x}$ and $B=\sqrt{a}$ in $A^{2}-B^{2}=(A-B)(A+B)$ to get

$$
x-a=(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})
$$

and then substitute this little fact into our expression

$$
\begin{aligned}
\frac{\sqrt{x}-\sqrt{a}}{x-a} & =\frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})} \quad \text { (now cancel common factors) } \\
& =\frac{1}{(\sqrt{x}+\sqrt{a})}
\end{aligned}
$$

- Once we know that $\frac{\sqrt{x}-\sqrt{a}}{x-a}=\frac{1}{\sqrt{x}+\sqrt{a}}$, we can take the limit we need:

$$
\begin{aligned}
f^{\prime}(a) & =\lim _{x \rightarrow a} \frac{\sqrt{x}-\sqrt{a}}{x-a} \\
& =\lim _{x \rightarrow a} \frac{1}{\sqrt{x}+\sqrt{a}} \\
& =\frac{1}{2 \sqrt{a}}
\end{aligned}
$$

- We should think about the domain of $f^{\prime}$ here - that is, for which values of $a$ is $f^{\prime}(a)$ defined? The original function $f(x)$ was defined for all $x \geq 0$, however the derivative $f^{\prime}(a)=\frac{1}{2 \sqrt{a}}$ is undefined at $a=0$.
If we draw a careful picture of $\sqrt{x}$ around $x=0$ we can see why this has to be the case. The figure below shows three different tangent lines to the graph of
$y=f(x)=\sqrt{x}$. As the point of tangency moves closer and closer to the origin, the tangent line gets steeper and steeper. The slope of the tangent line at $(a, \sqrt{a})$ blows up as $a \rightarrow 0$.



Example 2.2.10 $\frac{\mathrm{d}}{\mathrm{d} x}\{|x|\}$.
Compute the derivative, $f^{\prime}(a)$, of the function $f(x)=|x|$ at the point $x=a$.

- We should start this example by recalling the definition of $|x|$ (we saw this back in Example 1.5.6):

$$
|x|= \begin{cases}-x & \text { if } x<0 \\ 0 & \text { if } x=0 \\ x & \text { if } x>0\end{cases}
$$

It is definitely not just "chop off the minus sign".


- This breaks our computation of the derivative into 3 cases depending on whether $x$ is positive, negative or zero.
- Assume $x>0$. Then

$$
\begin{aligned}
\frac{\mathrm{d} f}{\mathrm{~d} x} & =\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{|x+h|-|x|}{h}
\end{aligned}
$$

Since $x>0$ and we are interested in the behaviour of this function as $h \rightarrow 0$ we can assume $h$ is much smaller than $x$. This means $x+h>0$ and so $|x+h|=x+h$.

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{x+h-x}{h} \\
& =\lim _{h \rightarrow 0} \frac{h}{h}=1
\end{aligned}
$$

as expected

- Assume $x<0$. Then

$$
\begin{aligned}
\frac{\mathrm{d} f}{\mathrm{~d} x} & =\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{|x+h|-|x|}{h}
\end{aligned}
$$

Since $x<0$ and we are interested in the behaviour of this function as $h \rightarrow 0$ we can assume $h$ is much smaller than $x$. This means $x+h<0$ and so $|x+h|=-(x+h)$.

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{-(x+h)-(-x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{-h}{h}=-1
\end{aligned}
$$

- When $x=0$ we have

$$
\begin{aligned}
f^{\prime}(0) & =\lim _{h \rightarrow 0} \frac{f(0+h)-f(0)}{h} \\
& =\lim _{h \rightarrow 0} \frac{|0+h|-|0|}{h} \\
& =\lim _{h \rightarrow 0} \frac{|h|}{h}
\end{aligned}
$$

To proceed we need to know if $h>0$ or $h<0$, so we must use one-sided limits. The limit from above is:

$$
\begin{array}{rlr}
\lim _{h \rightarrow 0^{+}} \frac{|h|}{h} & =\lim _{h \rightarrow 0^{+}} \frac{h}{h} \quad \text { since } h>0,|h|=h \\
& =1 &
\end{array}
$$

Whereas, the limit from below is:

$$
\begin{array}{rlr}
\lim _{h \rightarrow 0^{-}} \frac{|h|}{h} & =\lim _{h \rightarrow 0^{-}} \frac{-h}{h} \quad \text { since } h<0,|h|=-h \\
& =-1
\end{array}
$$

Since the one-sided limits differ, the limit as $h \rightarrow 0$ does not exist. And thus the derivative does not exist as $x=0$.

In summary:

$$
\frac{\mathrm{d}}{\mathrm{~d} x}|x|= \begin{cases}-1 & \text { if } x<0 \\ D N E & \text { if } x=0 \\ 1 & \text { if } x>0\end{cases}
$$



### 2.2.3 $円$ Where is the Derivative Undefined?

According to Definition 2.2.1, the derivative $f^{\prime}(a)$ exists precisely when the limit $\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a}$ exists. That limit is also the slope of the tangent line to the curve $y=f(x)$ at $x=a$. That limit does not exist when the curve $y=f(x)$ does not have a tangent line at $x=a$ or when the curve does have a tangent line, but the tangent line has infinite slope. We have already seen some examples of this.

- In Example 2.2.7, we considered the function $f(x)=\frac{1}{x}$. This function "blows up" (i.e. becomes infinite) at $x=0$. It does not have a tangent line at $x=0$ and its derivative does not exist at $x=0$.
- In Example 2.2.10, we considered the function $f(x)=|x|$. This function does not have a tangent line at $x=0$, because there is a sharp corner in the graph of $y=|x|$ at $x=0$. (Look at the graph in Example 2.2.10.) So the derivative of $f(x)=|x|$ does not exist at $x=0$.

Here are a few more examples.

Example 2.2.11 Derivative at a discontinuity.
Visually, the function

$$
H(x)= \begin{cases}0 & \text { if } x \leq 0 \\ 1 & \text { if } x>0\end{cases}
$$


does not have a tangent line at $(0,0)$. Not surprisingly, when $a=0$ and $h$ tends to 0 with $h>0$,

$$
\frac{H(a+h)-H(a)}{h}=\frac{H(h)-H(0)}{h}=\frac{1}{h}
$$

blows up. The same sort of computation shows that $f^{\prime}(a)$ cannot possibly exist whenever the function $f$ is not continuous at $a$. We will formalize, and prove, this statement $\uparrow$ in Theorem 2.2.14, below.

Example 2.2.12 $\frac{\mathrm{d}}{\mathrm{d} x} x^{1 / 3}$.
Visually, it looks like the function $f(x)=x^{1 / 3}$, sketched below, (this might be a good point to recall that cube roots of negative numbers are negative - for example, since $(-1)^{3}=-1$, the cube root of -1 is -1 ,

has the $y$-axis as its tangent line at $(0,0)$. So we would expect that $f^{\prime}(0)$ does not exist. Let's check. With $a=0$,

$$
\begin{aligned}
f^{\prime}(a) & =\lim _{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}=\lim _{h \rightarrow 0} \frac{f(h)-f(0)}{h}=\lim _{h \rightarrow 0} \frac{h^{1 / 3}}{h} \\
& =\lim _{h \rightarrow 0} \frac{1}{h^{2 / 3}}=D N E
\end{aligned}
$$

as expected.

Example $2.2 .13 \frac{\mathrm{~d}}{\mathrm{~d} x} \sqrt{|x|}$.
We have already considered the derivative of the function $\sqrt{x}$ in Example 2.2.9. We'll now look at the function $f(x)=\sqrt{|x|}$. Recall, from Example 2.2.10, the definition of $|x|$.
When $x>0$, we have $|x|=x$ and $f(x)$ is identical to $\sqrt{x}$. When $x<0$, we have $|x|=-x$ and $f(x)=\sqrt{-x}$. So to graph $y=\sqrt{|x|}$ when $x<0$, you just have to graph $y=\sqrt{x}$ for $x>0$ and then send $x \rightarrow-x$ - i.e. reflect the graph in the $y$-axis. Here is the graph.


The pointy thing at the origin is called a cusp. The graph of $y=f(x)$ does not have a tangent line at $(0,0)$ and, correspondingly, $f^{\prime}(0)$ does not exist because

$$
\lim _{h \rightarrow 0^{+}} \frac{f(h)-f(0)}{h}=\lim _{h \rightarrow 0^{+}} \frac{\sqrt{|h|}}{h}=\lim _{h \rightarrow 0^{+}} \frac{1}{\sqrt{h}}=D N E
$$

## Theorem 2.2.14

If the function $f(x)$ is differentiable at $x=a$, then $f(x)$ is also continuous at $x=a$.

Proof. The function $f(x)$ is continuous at $x=a$ if and only if the limit of

$$
f(a+h)-f(a)=\frac{f(a+h)-f(a)}{h} h
$$

as $h \rightarrow 0$ exists and is zero. But if $f(x)$ is differentiable at $x=a$, then, as $h \rightarrow 0$, the first factor, $\frac{f(a+h)-f(a)}{h}$ converges to $f^{\prime}(a)$ and the second factor, $h$, converges to zero. So the product provision of our arithmetic of limits Theorem 1.4.3 implies that the product $\frac{f(a+h)-f(a)}{h} h$ converges to $f^{\prime}(a) \cdot 0=0$ too.

Notice that while this theorem is useful as stated, it is (arguably) more often applied in its contrapositive ${ }^{6}$ form:

6 If you have forgotten what the contrapositive is, then quickly reread Footnote 1.3.5 in Section 1.3.

## Theorem 2.2.15 The contrapositive of Theorem 2.2.14.

If $f(x)$ is not continuous at $x=a$ then it is not differentiable at $x=a$.

As the above examples illustrate, this statement does not tell us what happens if $f$ is continuous at $x=a$ - we have to think!

### 2.2.4 $\leadsto$ Exercises

## Exercises - Stage 1

1. The function $f(x)$ is shown. Select all options below that describe its derivative, $\frac{\mathrm{d} f}{\mathrm{~d} x}$ :

- (a) constant
- (b) increasing
- (c) decreasing
- (d) always positive
- (e) always negative


2. The function $f(x)$ is shown. Select all options below that describe its derivative, $\frac{\mathrm{d} f}{\mathrm{~d} x}$ :

- (a) constant
- (b) increasing
- (c) decreasing
- (d) always positive
- (e) always negative


3. The function $f(x)$ is shown. Select all options below that describe its derivative, $\frac{\mathrm{d} f}{\mathrm{~d} x}$ :

- (a) constant
- (b) increasing
- (c) decreasing
- (d) always positive
- (e) always negative


4. *. State, in terms of a limit, what it means for $f(x)=x^{3}$ to be differentiable at $x=0$.
5. For which values of $x$ does $f^{\prime}(x)$ not exist?

6. Suppose $f(x)$ is a function defined at $x=a$ with

$$
\lim _{h \rightarrow 0^{-}} \frac{f(a+h)-f(a)}{h}=\lim _{h \rightarrow 0^{+}} \frac{f(a+h)-f(a)}{h}=1 .
$$

True or false: $f^{\prime}(a)=1$.
7. Suppose $f(x)$ is a function defined at $x=a$ with

$$
\lim _{x \rightarrow a^{-}} f^{\prime}(x)=\lim _{x \rightarrow a^{+}} f^{\prime}(x)=1
$$

True or false: $f^{\prime}(a)=1$.
8. Suppose $s(t)$ is a function, with $t$ measured in seconds, and $s$ measured in metres. What are the units of $s^{\prime}(t)$ ?

## Exercises - Stage 2

9. Use the definition of the derivative to find the equation of the tangent line to the curve $y(x)=x^{3}+5$ at the point $(1,6)$.
10. Use the definition of the derivative to find the derivative of $f(x)=\frac{1}{x}$.
11. *. Let $f(x)=x|x|$. Using the definition of the derivative, show that $f(x)$ is differentiable at $x=0$.
12. *. Use the definition of the derivative to compute the derivative of the function $f(x)=\frac{2}{x+1}$.
13. *. Use the definition of the derivative to compute the derivative of the function $f(x)=\frac{1}{x^{2}+3}$.
14. Use the definition of the derivative to find the slope of the tangent line to the curve $f(x)=x \log _{10}(2 x+10)$ at the point $x=0$.
15. *. Compute the derivative of $f(x)=\frac{1}{x^{2}}$ directly from the definition.
16. *. Find the values of the constants $a$ and $b$ for which

$$
f(x)= \begin{cases}x^{2} & x \leq 2 \\ a x+b & x>2\end{cases}
$$

is differentiable everywhere.
Remark: In the text, you have already learned the derivatives of $x^{2}$ and $a x+b$. In this question, you are only asked to find the values of $a$ and $b$-not to justify how you got them-so you don't have to use the definition of the derivative. However, on an exam, you might be asked to justify your answer, in which case you would show how to differentiate the two branches of $f(x)$ using the definition of a derivative.
17. *. Use the definition of the derivative to compute $f^{\prime}(x)$ if $f(x)=\sqrt{1+x}$. Where does $f^{\prime}(x)$ exist?

## Exercises - Stage 3

18. Use the definition of the derivative to find the velocity of an object whose position is given by the function $s(t)=t^{4}-t^{2}$.
19. *. Determine whether the derivative of following function exists at $x=0$.

$$
f(x)= \begin{cases}x \cos x & \text { if } x \geq 0 \\ \sqrt{x^{2}+x^{4}} & \text { if } x<0\end{cases}
$$

You must justify your answer using the definition of a derivative.
20. *. Determine whether the derivative of the following function exists at $x=0$

$$
f(x)= \begin{cases}x \cos x & \text { if } x \leq 0 \\ \sqrt{1+x}-1 & \text { if } x>0\end{cases}
$$

You must justify your answer using the definition of a derivative.
21. *. Determine whether the derivative of the following function exists at $x=0$

$$
f(x)= \begin{cases}x^{3}-7 x^{2} & \text { if } x \leq 0 \\ x^{3} \cos \left(\frac{1}{x}\right) & \text { if } x>0\end{cases}
$$

You must justify your answer using the definition of a derivative.
22. *. Determine whether the derivative of the following function exists at $x=1$

$$
f(x)= \begin{cases}4 x^{2}-8 x+4 & \text { if } x \leq 1 \\ (x-1)^{2} \sin \left(\frac{1}{x-1}\right) & \text { if } x>1\end{cases}
$$

You must justify your answer using the definition of a derivative.
23. . Sketch a function $f(x)$ with $f^{\prime}(0)=-1$ that takes the following values:

| $\mathbf{x}$ | -1 | $-\frac{1}{2}$ | $-\frac{1}{4}$ | $-\frac{1}{8}$ | 0 | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathbf{f}(\mathbf{x})$ | -1 | $-\frac{1}{2}$ | $-\frac{1}{4}$ | $-\frac{1}{8}$ | 0 | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 |

Remark: you can't always guess the behaviour of a function from its points, even if the points seem to be making a clear pattern.
24. Let $p(x)=f(x)+g(x)$, for some functions $f$ and $g$ whose derivatives exist. Use limit laws and the definition of a derivative to show that $p^{\prime}(x)=f^{\prime}(x)+g^{\prime}(x)$. Remark: this is called the sum rule, and we'll learn more about it in Lemma 2.4.1.
25. Let $f(x)=2 x, g(x)=x$, and $p(x)=f(x) \cdot g(x)$.
a Find $f^{\prime}(x)$ and $g^{\prime}(x)$.
b Find $p^{\prime}(x)$.
c Is $p^{\prime}(x)=f^{\prime}(x) \cdot g^{\prime}(x)$ ?
In Theorem 2.4.3, you'll learn a rule for calculating the derivative of a product of two functions.
26. *. There are two distinct straight lines that pass through the point $(1,-3)$ and are tangent to the curve $y=x^{2}$. Find equations for these two lines. Remark: the point $(1,-3)$ does not lie on the curve $y=x^{2}$.
27. *. For which values of $a$ is the function

$$
f(x)= \begin{cases}0 & x \leq 0 \\ x^{a} \sin \frac{1}{x} & x>0\end{cases}
$$

differentiable at 0 ?

## 2.3ム Interpretations of the Derivative

In the previous sections we defined the derivative as the slope of a tangent line, using a particular limit. This allows us to compute "the slope of a curve" ${ }^{1}$ and provides us with one interpretation of the derivative. However, the main importance of derivatives does not come from this application. Instead, (arguably) it comes from the interpretation of the derivative as the instantaneous rate of change of a quantity.

### 2.3.1 $\leadsto$ Instantaneous Rate of Change

In fact we have already (secretly) used a derivative to compute an instantaneous rate of change in Section 1.2. For your convenience we'll review that computation here, in Example 2.3.1, and then generalise it.

Example 2.3.1 Velocity as a derivative.
You drop a ball from a tall building. After $t$ seconds the ball has fallen a distance of $s(t)=4.9 t^{2}$ metres. What is the velocity of the ball one second after it is dropped?

- In the time interval from $t=1$ to $t=1+h$ the ball travels a distance

$$
s(1+h)-s(1)=4.9(1+h)^{2}-4.9(1)^{2}=4.9\left[2 h+h^{2}\right]
$$

- So the average velocity over this time interval is

$$
\begin{aligned}
& \text { average velocity from } t=1 \text { to } t=1+h \\
& =\frac{\text { distance travelled from } t=1 \text { to } t=1+h}{\text { length of time from } t=1 \text { to } t=1+h} \\
& =\frac{s(1+h)-s(1)}{h} \\
& =\frac{4.9\left[2 h+h^{2}\right]}{h} \\
& =4.9[2+h]
\end{aligned}
$$

- The instantaneous velocity at time $t=1$ is then defined to be the limit

$$
\text { instantaneous velocity at time } t=1
$$

1 Again - recall that we are being a little sloppy with this term — we really mean "The slope of the tangent line to the curve".

$$
\begin{aligned}
& =\lim _{h \rightarrow 0}[\text { average velocity from } t=1 \text { to } t=1+h] \\
& =\lim _{h \rightarrow 0} \frac{s(1+h)-s(1)}{h}=s^{\prime}(1) \\
& =\lim _{h \rightarrow 0} 4.9[2+h] \\
& =9.8 \mathrm{~m} / \mathrm{sec}
\end{aligned}
$$

- We conclude that the instantaneous velocity at time $t=1$, which is the instantaneous rate of change of distance per unit time at time $t=1$, is the derivative $s^{\prime}(1)=9.8 \mathrm{~m} / \mathrm{sec}$.

Example 2.3.1
Now suppose, more generally, that you are taking a walk and that as you walk, you are continuously measuring some quantity, like temperature, and that the measurement at time $t$ is $f(t)$. Then the

$$
\begin{aligned}
& \text { average rate of change of } f(t) \text { from } t=a \text { to } t=a+h \\
& \quad=\frac{\text { change in } f(t) \text { from } t=a \text { to } t=a+h}{\text { length of time from } t=a \text { to } t=a+h} \\
& =\frac{f(a+h)-f(a)}{h}
\end{aligned}
$$

so the

$$
\begin{aligned}
& \text { instantaneous rate of change of } f(t) \text { at } t=a \\
& \quad=\lim _{h \rightarrow 0}[\text { average rate of change of } f(t) \text { from } t=a \text { to } t=a+h] \\
& \quad=\lim _{h \rightarrow 0} \frac{f(a+h)-f(a)}{h} \\
& =f^{\prime}(a)
\end{aligned}
$$

In particular, if you are walking along the $x$-axis and your $x$-coordinate at time $t$ is $x(t)$, then $x^{\prime}(a)$ is the instantaneous rate of change (per unit time) of your $x$-coordinate at time $t=a$, which is your velocity at time $a$. If $v(t)$ is your velocity at time $t$, then $v^{\prime}(a)$ is the instantaneous rate of change of your velocity at time $a$. This is called your acceleration at time $a$.

You might expect that if the instantaneous rate of change of a function at time $c$ is strictly positive, then, in some sense, the function is increasing at $t=c$. You would be right. Indeed, if $f^{\prime}(c)>0$, then, by definition, the limit of $\frac{f(t)-f(c)}{t-c}$ as $t$ approaches $c$ is strictly bigger than zero. So

- for all $t>c$ that are sufficiently close ${ }^{2}$ to $c$

$$
\begin{aligned}
\frac{f(t)-f(c)}{t-c}>0 & \Longrightarrow f(t)-f(c)>0 \quad(\text { since } t-c>0) \\
& \Longrightarrow f(t)>f(c)
\end{aligned}
$$

- for all $t<c$ that are sufficiently close to $c$

$$
\begin{aligned}
\frac{f(t)-f(c)}{t-c}>0 & \Longrightarrow f(t)-f(c)<0 \quad(\text { since } t<c) \\
& \Longrightarrow f(t)<f(c)
\end{aligned}
$$

Consequently we say that " $f(t)$ is increasing at $t=c$ ". If we wish to emphasise that the inequalities above are the strict inequalities $>$ and $<$, as opposed to $\geq$ and $\leq$, we will say that " $f(t)$ is strictly increasing at $t=c$ ".

### 2.3.2 $\leadsto$ Slope

Suppose that $y=f(x)$ is the equation of a curve in the $x y$-plane. That is, $f(x)$ is the $y$-coordinate of the point on the curve whose $x$-coordinate is $x$. Then, as we have already seen,
[the slope of the secant through $(a, f(a))$ and $(a+h, f(a+h))]=\frac{f(a+h)-f(a)}{h}$
This is shown in Figure 2.3.2 below.


Figure 2.3.2

2 This is typical mathematician speak - it allows us to be completely correct, without being terribly precise. In this context, sufficiently close means The following need not be true for all $t$ bigger than $c$, but there must exist some $b>c$ so that the following is true for all $c<t<b$. Typically we do not know what $b$ is. And typically it does not matter what the exact value of $b$ is. All that matters is that $b$ exists and is strictly bigger than $c$.

In order to create the tangent line (as we have done a few times now) we squeeze $h \rightarrow 0$. As we do this, the secant through $(a, f(a))$ and $(a+h, f(a+h))$ approaches ${ }^{3}$ the tangent line to $y=f(x)$ at $x=a$. Since the secant becomes the tangent line in this limit, the slope of the secant becomes the slope of the tangent and

$$
\text { [the slope of the tangent line to } \begin{aligned}
y=f(x) \text { at } x=a] & =\lim _{h \rightarrow 0} \frac{f(a+h)-f(a)}{h} \\
& =f^{\prime}(a) .
\end{aligned}
$$

Let us go a little further and work out a general formula for the equation of the tangent line to $y=f(x)$ at $x=a$. We know that the tangent line

- has slope $f^{\prime}(a)$ and
- passes through the point $(a, f(a))$.

There are a couple of different ways to construct the equation of the tangent line from this information. One is to observe, as in Figure 2.3.3, that if $(x, y)$ is any other point on the tangent line then the line segment from $(a, f(a))$ to $(x, y)$ is part of the tangent line and so also has slope $f^{\prime}(a)$. That is,

$$
\frac{y-f(a)}{x-a}=[\text { the slope of the tangent line }]=f^{\prime}(a)
$$

Cross multiplying gives us the equation of the tangent line:

$$
y-f(a)=f^{\prime}(a)(x-a) \quad \text { or } \quad y=f(a)+f^{\prime}(a)(x-a)
$$



Figure 2.3.3: A line segment of a tangent line
A second way to derive the same equation of the same tangent line is to recall that the general equation for a line, with finite slope, is $y=m x+b$, where $m$ is the slope and $b$ is the $y$-intercept. We already know the slope - so $m=f^{\prime}(a)$. To work out $b$ we

3 We are of course assuming that the curve is smooth enough to have a tangent line at $a$.
use the other piece of information - $(a, f(a))$ is on the line. So $(x, y)=(a, f(a))$ must solve $y=f^{\prime}(a) x+b$. That is,

$$
f(a)=f^{\prime}(a) \cdot a+b \quad \text { and so } \quad b=f(a)-a f^{\prime}(a)
$$

Hence our equation is, once again,

$$
\begin{array}{ll}
y=f^{\prime}(a) \cdot x+\left(f(a)-a f^{\prime}(a)\right) \\
y=f(a)+f^{\prime}(a)(x-a) & \text { or, after rearranging a little, }
\end{array}
$$

This is a very useful formula, so perhaps we should make it a theorem.

## Theorem 2.3.4 Tangent line.

The tangent line to the curve $y=f(x)$ at $x=a$ is given by the equation

$$
y=f(a)+f^{\prime}(a)(x-a)
$$

provided the derivative $f^{\prime}(a)$ exists.

The caveat at the end of the above theorem is necessary - there are certainly cases in which the derivative does not exist and so we do need to be careful.

Example 2.3.5 A tangent line to the curve $y=\sqrt{x}$.
Find the tangent line to the curve $y=\sqrt{x}$ at $x=4$.
Rather than redoing everything from scratch, we can, and for efficiency, should, use Theorem 2.3.4. To write this up properly, we must ensure that we tell the reader what we are doing. So something like the following:

- By Theorem 2.3.4, the tangent line to the curve $y=f(x)$ at $x=a$ is given by

$$
y=f(a)+f^{\prime}(a)(x-a)
$$

provided $f^{\prime}(a)$ exists.

- In Example 2.2.9, we found that, for any $a>0$, the derivative of $\sqrt{x}$ at $x=a$ is

$$
f^{\prime}(a)=\frac{1}{2 \sqrt{a}}
$$

- In the current example, $a=4$ and we have

$$
\begin{aligned}
f(a) & =f(4)=\left.\sqrt{x}\right|_{x=4}=\sqrt{4}=2 \\
\text { and } \quad f^{\prime}(a) & =f^{\prime}(4)=\left.\frac{1}{2 \sqrt{a}}\right|_{a=4}=\frac{1}{2 \sqrt{4}}=\frac{1}{4}
\end{aligned}
$$

- So the equation of the tangent line to $y=\sqrt{x}$ at $x=4$ is

$$
y=2+\frac{1}{4}(x-4) \quad \text { or } \quad y=\frac{x}{4}+1
$$

We don't have to write it up using dot-points as above; we have used them here to help delineate each step in the process of computing the tangent line.

Example 2.3.5

### 2.3.3 Exercises

## Exercises - Stage 2

1. Suppose $h(t)$ gives the height at time $t$ of the water at a dam, where the units of $t$ are hours and the units of $h$ are meters.
a What is the physical interpretation of the slope of the secant line through the points $(0, h(0))$ and $(24, h(24))$ ?
b What is the physical interpretation of the slope of the tangent line to the curve $y=h(t)$ at the point $(0, h(0))$ ?
2. Suppose $p(t)$ is a function that gives the profit generated by selling $t$ widgets. What is the practical interpretation of $p^{\prime}(t)$ ?
3. $\quad T(d)$ gives the temperature of water at a particular location $d$ metres below the surface. What is the physical interpretation of $T^{\prime}(d)$ ? Would you expect the magnitude of $T^{\prime}(d)$ to be larger when $d$ is near 0 , or when $d$ is very large?
4. $\quad C(w)$ gives the calories in $w$ grams of a particular dish. What does $C^{\prime}(w)$ describe?
5. The velocity of a moving object at time $t$ is given by $v(t)$. What is $v^{\prime}(t)$ ?
6. The function $T(j)$ gives the temperature in degrees Celsius of a cup of water after $j$ joules of heat have been added. What is $T^{\prime}(j)$ ?
7. A population of bacteria, left for a fixed amount of time at temperature $T$, grows to $P(T)$ individuals. Interpret $P^{\prime}(T)$.

## Exercises - Stage 3

8. You hammer a small nail into a wooden wagon wheel. $R(t)$ gives the number of rotations the nail has undergone $t$ seconds after the wagon started to roll. Give an equation for how quickly the nail is rotating, measured in degrees per second.
9. A population of bacteria, left for a fixed amount of time at temperature $T$, grows to $P(T)$ individuals. There is one ideal temperature where the bacteria population grows largest, and the closer the sample is to that temperature, the larger the population is (unless the temperature is so extreme that it causes all the bacteria to die by freezing or boiling). How will $P^{\prime}(T)$ tell you whether you are colder or hotter than the ideal temperature?

## 2.4^ Arithmetic of Derivatives - a Differentiation Toolbox

### 2.4.1 Arithmetic of Derivatives - a Differentiation Toolbox

So far, we have evaluated derivatives only by applying Definition 2.2.1 to the function at hand and then computing the required limits directly. It is quite obvious that as the function being differentiated becomes even a little complicated, this procedure quickly becomes extremely unwieldy. It is many orders of magnitude more efficient to have access to

- a list of derivatives of some simple functions and
- a collection of rules for breaking down complicated derivative computations into sequences of simple derivative computations.

This is precisely what we did to compute limits. We started with limits of simple functions and then used "arithmetic of limits" to computed limits of complicated functions.

We have already started building our list of derivatives of simple functions. We have shown, in Examples 2.2.2, 2.2.3, 2.2.5 and 2.2.9, that

$$
\frac{\mathrm{d}}{\mathrm{~d} x} 1=0 \quad \frac{\mathrm{~d}}{\mathrm{~d} x} x=1 \quad \frac{\mathrm{~d}}{\mathrm{~d} x} x^{2}=2 x \quad \frac{\mathrm{~d}}{\mathrm{~d} x} \sqrt{x}=\frac{1}{2 \sqrt{x}}
$$

We'll expand this list later.
We now start building a collection of tools that help reduce the problem of computing the derivative of a complicated function to that of computing the derivatives of a number of simple functions. In this section we give three derivative "rules" as three separate theorems. We'll give the proofs of these theorems in the next section and examples of how they are used in the following section.

As was the case for limits, derivatives interact very cleanly with addition, subtraction and multiplication by a constant. The following result actually follows very directly from the first three points of Theorem 1.4.3.

Lemma 2.4.1 Derivative of sum and difference.
Let $f(x), g(x)$ be differentiable functions and let $c \in \mathbb{R}$ be a constant. Then

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{f(x)+g(x)\} & =f^{\prime}(x)+g^{\prime}(x) \\
\frac{\mathrm{d}}{\mathrm{~d} x}\{f(x)-g(x)\} & =f^{\prime}(x)-g^{\prime}(x) \\
\frac{\mathrm{d}}{\mathrm{~d} x}\{c f(x)\} & =c f^{\prime}(x)
\end{aligned}
$$

That is, the derivative of the sum is the sum of the derivatives, and so forth.

Following this we can combine the three statements in this lemma into a single rule which captures the "linearity of differentiation".

## Theorem 2.4.2 Linearity of differentiation.

Again, let $f(x), g(x)$ be differentiable functions, let $\alpha, \beta \in \mathbb{R}$ be constants and define the "linear combination"

$$
S(x)=\alpha f(x)+\beta g(x)
$$

Then the derivative of $S(x)$ at $x=a$ exists and is

$$
\frac{\mathrm{d} S}{\mathrm{~d} x}=S^{\prime}(x)=\alpha f^{\prime}(x)+\beta g^{\prime}(x)
$$

Note that we can recover the three rules in the previous lemma by setting $\alpha=$ $\beta=1$ or $\alpha=1, \beta=-1$ or $\alpha=c, \beta=0$.

Unfortunately, the derivative does not act quite as simply on products or quotients. The rules for computing derivatives of products and quotients get their own names and theorems:

Theorem 2.4.3 The product rule.
Let $f(x), g(x)$ be differentiable functions, then the derivative of the product $f(x) g(x)$ exists and is given by

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\{f(x) g(x)\}=f^{\prime}(x) g(x)+f(x) g^{\prime}(x)
$$

Before we proceed to the derivative of the ratio of two functions, it is worth noting a
special case of the product rule when $g(x)=f(x)$. In fact, since this is a useful special case, let us call it a corollary ${ }^{1}$ :

## Corollary 2.4.4 Derivative of a square.

Let $f(x)$ be a differentiable function, then the derivative of its square is:

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{f(x)^{2}\right\}=2 f(x) f^{\prime}(x)
$$

With a little work this can be generalised to other powers - but that is best done once we understand how to compute the derivative of the composition of two functions. That requires the chain rule (see Theorem 2.9.2 below). But before we get to that, we need to see how to take the derivative of a quotient of two functions.

## Theorem 2.4.5 The quotient rule.

Let $f(x), g(x)$ be differentiable functions. Then the derivative of their quotient is

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{f(x)}{g(x)}\right\}=\frac{f^{\prime}(x) g(x)-f(x) g^{\prime}(x)}{g(x)^{2}} .
$$

This derivative exists except at points where $g(x)=0$.

There is a useful special case of this theorem which we obtain by setting $f(x)=1$. In that case, the quotient rule tells us how to compute the derivative of the reciprocal of a function.

Corollary 2.4.6 Derivative of a reciprocal.
Let $g(x)$ be a differentiable function. Then the derivative of the reciprocal of $g$ is given by

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{1}{g(x)}\right\}=-\frac{g^{\prime}(x)}{g(x)^{2}}
$$

and exists except at those points where $g(x)=0$.

So we have covered, sums, differences, products and quotients. This allows us to compute derivatives of many different functions - including polynomials and rational functions. However we are still missing trigonometric functions (for example), and a

1 Recall that a corollary is an important result that follows from one or more theorems - typically without too much extra work - as is the case here.
rule for computing derivatives of compositions. These will follow in the near future, but there are a couple of things to do before that - understand where the above theorems come from, and practice using them.

### 2.4.2 Exercises

## Exercises - Stage 1

1. True or false: $\frac{\mathrm{d}}{\mathrm{d} x}\{f(x)+g(x)\}=f^{\prime}(x)+g^{\prime}(x)$ when $f$ and $g$ are differentiable functions.
2. True or false: $\frac{\mathrm{d}}{\mathrm{d} x}\{f(x) g(x)\}=f^{\prime}(x) g^{\prime}(x)$ when $f$ and $g$ are differentiable functions.
3. True or false: $\frac{\mathrm{d}}{\mathrm{d} x}\left\{\frac{f(x)}{g(x)}\right\}=\frac{f^{\prime}(x)}{g(x)}-\frac{f(x) g^{\prime}(x)}{g^{2}(x)}$ when $f$ and $g$ are differentiable functions.
4. Let $f$ be a differentiable function. Use at least three different rules to differentiate $g(x)=3 f(x)$, and verify that they all give the same answer.

## Exercises - Stage 2

5. Differentiate $f(x)=3 x^{2}+4 x^{1 / 2}$ for $x>0$.
6. Use the product rule to differentiate $f(x)=(2 x+5)(8 \sqrt{x}-9 x)$.
7. *. Find the equation of the tangent line to the graph of $y=x^{3}$ at $x=\frac{1}{2}$.
8. *. A particle moves along the $x$-axis so that its position at time $t$ is given by $x=t^{3}-4 t^{2}+1$.
a At $t=2$, what is the particle's speed?
b At $t=2$, in what direction is the particle moving?
c At $t=2$, is the particle's speed increasing or decreasing?
9. *. Calculate and simplify the derivative of $\frac{2 x-1}{2 x+1}$
10. What is the slope of the graph $y=\left(\frac{3 x+1}{3 x-2}\right)^{2}$ when $x=1$ ?
11. Find the equation of the tangent line to the curve $f(x)=\frac{1}{\sqrt{x}+1}$ at the point $\left(1, \frac{1}{2}\right)$.

## Exercises - Stage 3

12. A town is founded in the year 2000. After $t$ years, it has had $b(t)$ births and $d(t)$ deaths. Nobody enters or leaves the town except by birth or death (whoa). Give an expression for the rate the population of the town is growing.
13. *. Find all points on the curve $y=3 x^{2}$ where the tangent line passes through $(2,9)$.
14. *. Evaluate $\lim _{y \rightarrow 0}\left(\frac{\sqrt{100180+y}-\sqrt{100180}}{y}\right)$ by interpreting the limit as a derivative.
15. A rectangle is growing. At time $t=0$, it is a square with side length 1 metre. Its width increases at a constant rate of 2 metres per second, and its length increases at a constant rate of 5 metres per second. How fast is its area increasing at time $t>0$ ?
16. Let $f(x)=x^{2} g(x)$ for some differentiable function $g(x)$. What is $f^{\prime}(0)$ ?
17. Verify that differentiating $f(x)=\frac{g(x)}{h(x)}$ using the quotient rule gives the same answer as differentiating $f(x)=\frac{g(x)}{k(x)} \cdot \frac{k(x)}{h(x)}$ using the product rule and the quotient rule.

### 2.54 Proofs of the Arithmetic of Derivatives

The theorems of the previous section are not too difficult to prove from the definition of the derivative (which we know) and the arithmetic of limits (which we also know). In this section we show how to construct these rules.

Throughout this section we will use our two functions $f(x)$ and $g(x)$. Since the theorems we are going to prove all express derivatives of linear combinations, products and quotients in terms of $f, g$ and their derivatives, it is helpful to recall the definitions
of the derivatives of $f$ and $g$ :

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \quad \text { and } \quad g^{\prime}(x)=\lim _{h \rightarrow 0} \frac{g(x+h)-g(x)}{h} .
$$

Our proofs, roughly speaking, involve doing algebraic manipulations to uncover the expressions that look like the above.

### 2.5.1 Proof of the Linearity of Differentiation (Theorem 2.4.2)

Recall that in Theorem 2.4.2 we defined $S(x)=\alpha f(x)+\beta g(x)$, where $\alpha, \beta \in \mathbb{R}$ are constants. We wish to compute $S^{\prime}(x)$, so we start with the definition:

$$
S^{\prime}(x)=\lim _{h \rightarrow 0} \frac{S(x+h)-S(x)}{h}
$$

Let us concentrate on the numerator of the expression inside the limit and then come back to the full limit in a moment. Substitute in the definition of $S(x)$ :

$$
\begin{aligned}
S(x+h)-S(x) & =[\alpha f(x+h)+\beta g(x+h)]-[\alpha f(x)+\beta g(x)] \quad \text { collect terms } \\
& =\alpha[f(x+h)-f(x)]+\beta[g(x+h)-g(x)]
\end{aligned}
$$

Now it is easy to see the structures we need - namely, we almost have the expressions for the derivatives $f^{\prime}(x)$ and $g^{\prime}(x)$. Indeed, all we need to do is divide by $h$ and take the limit. So let's finish things off.

$$
\begin{array}{rlrl}
S^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{S(x+h)-S(x)}{h} & & \text { from above } \\
& =\lim _{h \rightarrow 0} \frac{\alpha[f(x+h)-f(x)]+\beta[g(x+h)-g(x)]}{h} & & \\
& =\lim _{h \rightarrow 0}\left[\alpha \frac{f(x+h)-f(x)}{h}+\beta \frac{g(x+h)-g(x)}{h}\right] & & \text { limit laws } \\
& =\alpha \lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}+\beta \lim _{h \rightarrow 0} \frac{g(x+h)-g(x)}{h} & \\
& =\alpha f^{\prime}(x)+\beta g^{\prime}(x) & &
\end{array}
$$

as required.

### 2.5.2 $\leadsto$ Proof of the Product Rule (Theorem 2.4.3)

After the warm-up above, we will just jump straight in. Let $P(x)=f(x) g(x)$, the product of our two functions. The derivative of the product is given by

$$
P^{\prime}(x)=\lim _{h \rightarrow 0} \frac{P(x+h)-P(x)}{h}
$$

Again we will focus on the numerator inside the limit and massage it into the form we need. To simplify these manipulations, define

$$
F(h)=\frac{f(x+h)-f(x)}{h} \quad \text { and } \quad G(h)=\frac{g(x+h)-g(x)}{h}
$$

Then we can write

$$
f(x+h)=f(x)+h F(h) \quad \text { and } \quad g(x+h)=g(x)+h G(h)
$$

We can also write

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} F(h) \quad \text { and } \quad g^{\prime}(x)=\lim _{h \rightarrow 0} G(h)
$$

So back to that numerator:

$$
\begin{array}{lr}
P(x+h)-P(x)=f(x+h) \cdot g(x+h)-f(x) \cdot g(x) & \text { substitute } \\
=[f(x)+h F(h)][g(x)+h G(h)]-f(x) \cdot g(x) & \text { expand } \\
=f(x) g(x)+f(x) \cdot h G(h)+h F(h) \cdot g(x)+h^{2} F(h) \cdot G(h)-f(x) \cdot g(x) \\
=f(x) \cdot h G(h)+h F(h) \cdot g(x)+h^{2} F(h) \cdot G(h)
\end{array}
$$

Armed with this we return to the definition of the derivative:

$$
\begin{aligned}
P^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{P(x+h)-P(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{f(x) \cdot h G(h)+h F(h) \cdot g(x)+h^{2} F(h) \cdot G(h)}{h} \\
& =\left(\lim _{h \rightarrow 0} \frac{f(x) \cdot h G(h)}{h}\right)+\left(\lim _{h \rightarrow 0} \frac{h F(h) \cdot g(x)}{h}\right)+\left(\lim _{h \rightarrow 0} \frac{h^{2} F(h) \cdot G(h)}{h}\right) \\
& =\left(\lim _{h \rightarrow 0} f(x) \cdot G(h)\right)+\left(\lim _{h \rightarrow 0} F(h) \cdot g(x)\right)+\left(\lim _{h \rightarrow 0} h F(h) \cdot G(h)\right)
\end{aligned}
$$

Now since $f(x)$ and $g(x)$ do not change as we send $h$ to zero, we can pull them outside. We can also write the third term as the product of 3 limits:

$$
\begin{aligned}
& =\left(f(x) \lim _{h \rightarrow 0} G(h)\right)+\left(g(x) \lim _{h \rightarrow 0} F(h)\right)+\left(\lim _{h \rightarrow 0} h\right) \cdot\left(\lim _{h \rightarrow 0} F(h)\right) \cdot\left(\lim _{h \rightarrow 0} G(h)\right) \\
& =f(x) \cdot g^{\prime}(x)+g(x) \cdot f^{\prime}(x)+0 \cdot f^{\prime}(x) \cdot g^{\prime}(x) \\
& =f(x) \cdot g^{\prime}(x)+g(x) \cdot f^{\prime}(x) .
\end{aligned}
$$

And so we recover the product rule.

### 2.5.3 (Optional) — Proof of the Quotient Rule (Theorem 2.4.5)

We now give the proof of the quotient rule in two steps ${ }^{1}$. We assume throughout that $g(x) \neq 0$ and that $f(x)$ and $g(x)$ are differentiable, meaning that the limits defining $f^{\prime}(x), g^{\prime}(x)$ exist.

1 We thank Serban Raianu for suggesting this approach.

- In the first step, we prove the quotient rule under the assumption that $f(x) / g(x)$ is differentiable.
- In the second step, we prove that $1 / g(x)$ differentiable. Once we know that $1 / g(x)$ is differentiable, the product rule implies that $f(x) / g(x)$ is differentiable.

Step 1: the proof of the quotient rule assumng that $\frac{f(x)}{g(x)}$ is differentiable. Write $Q(x)=\frac{f(x)}{g(x)}$. Then $f(x)=g(x) Q(x)$ so that $f^{\prime}(x)=g^{\prime}(x) Q(x)+g(x) Q^{\prime}(x)$, by the product rule, and

$$
\begin{aligned}
Q^{\prime}(x) & =\frac{f^{\prime}(x)-g^{\prime}(x) Q(x)}{g(x)}=\frac{f^{\prime}(x)-g^{\prime}(x) \frac{f(x)}{g(x)}}{g(x)} \\
& =\frac{f^{\prime}(x) g(x)-f(x) g^{\prime}(x)}{g(x)^{2}}
\end{aligned}
$$

Step 2: the proof that $1 / g(x)$ is differentiable. By definition

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x} \frac{1}{g(x)} & =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{1}{g(x+h)}-\frac{1}{g(x)}\right]=\lim _{h \rightarrow 0} \frac{g(x)-g(x+h)}{h g(x) g(x+h)} \\
& =-\lim _{h \rightarrow 0} \frac{1}{g(x)} \frac{1}{g(x+h)} \frac{g(x+h)-g(x)}{h} \\
& =-\frac{1}{g(x)} \lim _{h \rightarrow 0} \frac{1}{g(x+h)} \lim _{h \rightarrow 0} \frac{g(x+h)-g(x)}{h} \\
& =-\frac{1}{g(x)^{2}} g^{\prime}(x)
\end{aligned}
$$

## 2.6ム Using the Arithmetic of Derivatives - Examples

### 2.6.1 $\leadsto$ Using the Arithmetic of Derivatives - Examples

In this section we illustrate the computation of derivatives using the arithmetic of derivatives - Theorems 2.4.2, 2.4.3 and 2.4.5. To make it clear which rules we are using during the examples we will note which theorem we are using:
$\bullet$ LIN to stand for "linearity" $\quad \frac{\mathrm{d}}{\mathrm{d} x}\{\alpha f(x)+\beta g(x)\}=\alpha f^{\prime}(x)+\beta g^{\prime}(x) \quad$ Theorem 2.4.2
$\bullet$-PR to stand for "product rule" $\quad \frac{d}{\mathrm{~d} x}\{f(x) g(x)\}=f^{\prime}(x) g(x)+f(x) g^{\prime}(x) \quad$ Theorem 2.4.3
$\bullet$ QR to stand for "quotient rule" $\frac{\mathrm{d}}{\mathrm{d} x}\left\{\frac{f(x)}{g(x)}\right\}=\frac{f^{\prime}(x) g(x)-f(x) g^{\prime}(x)}{g(x)^{2}} \quad$ Theorem 2.4.5
We'll start with a really easy example.

Example 2.6.1 $\frac{\mathrm{d}}{\mathrm{d} x}\{4 x+7\}$.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{4 x+7\} & =4 \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\{x\}+7 \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\{1\} \quad \text { LIN } \\
& =4 \cdot 1+7 \cdot 0=4
\end{aligned}
$$

where we have used LIN with $f(x)=x, g(x)=1, \alpha=4, \beta=7$.

## Example 2.6.2 $\frac{\mathrm{d}}{\mathrm{d} x}\{x(4 x+7)\}$.

Continuing on from the previous example, we can use the product rule and the previous result to compute

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{x(4 x+7)\} & =x \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\{4 x+7\}+(4 x+7) \frac{\mathrm{d}}{\mathrm{~d} x}\{x\} \\
& =x \cdot 4+(4 x+7) \cdot 1 \\
& =8 x+7
\end{aligned}
$$

where we have used the product rule PR with $f(x)=x$ and $g(x)=4 x+7$.

## Example 2.6.3 $\frac{\mathrm{d}}{\mathrm{d} x}\left\{\frac{x}{4 x+7}\right\}$.

In the same vein as the previous example, we can use the quotient rule to compute

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{x}{4 x+7}\right\} & =\frac{(4 x+7) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\{x\}-x \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\{4 x+7\}}{(4 x+7)^{2}} \quad \mathrm{QR} \\
& =\frac{(4 x+7) \cdot 1-x \cdot 4}{(4 x+7)^{2}} \\
& =\frac{7}{(4 x+7)^{2}}
\end{aligned}
$$

$\uparrow$ where we have used the quotient rule QR with $f(x)=x$ and $g(x)=4 x+7$.

Now for a messier example.
Example 2.6.4 Some examples should be messy.
Differentiate

$$
f(x)=\frac{x}{2 x+\frac{1}{3 x+1}}
$$

This problem looks nasty. But it isn't so hard if we just build it up a bit at a time.

- First, $f(x)$ is the ratio of

$$
f_{1}(x)=x \quad \text { and } \quad f_{2}(x)=2 x+\frac{1}{3 x+1}
$$

If we can find the derivatives of $f_{1}(x)$ and $f_{2}(x)$, we will be able to get the derivative of $f(x)$ just by applying the quotient rule. The derivative, $f_{1}^{\prime}(x)=1$, of $f_{1}(x)$ is easy, so let's work on $f_{2}(x)$.

- The function $f_{2}(x)$ is the linear combination

$$
f_{2}(x)=2 f_{3}(x)+f_{4}(x) \quad \text { with } \quad f_{3}(x)=x \quad \text { and } \quad f_{4}(x)=\frac{1}{3 x+1}
$$

If we can find the derivatives of $f_{3}(x)$ and $f_{4}(x)$, we will be able to get the derivative of $f_{2}(x)$ just by applying linearity (Theorem 2.4.2). The derivative, $f_{3}^{\prime}(x)=1$, of $f_{3}(x)$ is easy. So let's work of $f_{4}(x)$.

- The function $f_{4}(x)$ is the ratio

$$
f_{4}(x)=\frac{1}{f_{5}(x)} \quad \text { with } \quad f_{5}(x)=3 x+1
$$

If we can find the derivative of $f_{5}(x)$, we will be able to get the derivative of $f_{4}(x)$ just by applying the special case the quotient rule (Corollary 2.4.6). The derivative of $f_{5}(x)$ is easy.

- So we have completed breaking down $f(x)$ into easy pieces. It is now just a matter of reversing the break down steps, putting everything back together, starting with the easy pieces and working up to $f(x)$. Here goes.

$$
\begin{array}{lll}
f_{5}(x)=3 x+1 & & \text { so } \frac{\mathrm{d}}{\mathrm{~d} x} f_{5}(x)=3 \frac{\mathrm{~d}}{\mathrm{~d} x} x+\frac{\mathrm{d}}{\mathrm{~d} x} 1=3 \cdot 1+0=3 \\
f_{4}(x)=\frac{1}{f_{5}(x)} & \text { so } \frac{\mathrm{d}}{\mathrm{~d} x} f_{4}(x)=-\frac{f_{5}^{\prime}(x)}{f_{5}(x)^{2}}=-\frac{3}{(3 x+1)^{2}} & \text { QR } \\
f_{2}(x)=2 f_{3}(x)+f_{4}(x) & \text { LIN } \\
& \text { so } \frac{\mathrm{d}}{\mathrm{~d} x} f_{2}(x)=2 f_{3}^{\prime}(x)+f_{4}^{\prime}(x)=2-\frac{3}{(3 x+1)^{2}} & \text { LIN } \\
f(x)=\frac{f_{1}(x)}{f_{2}(x)} & \text { so } \frac{\mathrm{d}}{\mathrm{~d} x} f(x)=\frac{f_{1}^{\prime}(x) f_{2}(x)-f_{1}(x) f_{2}^{\prime}(x)}{f_{2}(x)^{2}}
\end{array}
$$

$$
=\frac{1\left[2 x+\frac{1}{3 x+1}\right]-x\left[2-\frac{3}{(3 x+1)^{2}}\right]}{\left[2 x+\frac{1}{3 x+1}\right]^{2}}
$$

Oof!

- We now have an answer. But we really should clean it up, not only to make it easier to read, but also because invariably such computations are just small steps inside much larger computations. Any future computations involving this expression will be a lot easier and less error prone if we clean it up now. Cancelling the $2 x$ and the $-2 x$ in

$$
\begin{aligned}
1\left[2 x+\frac{1}{3 x+1}\right]-x\left[2-\frac{3}{(3 x+1)^{2}}\right] & =2 x+\frac{1}{3 x+1}-2 x+\frac{3 x}{(3 x+1)^{2}} \\
& =\frac{1}{3 x+1}+\frac{3 x}{(3 x+1)^{2}}
\end{aligned}
$$

and multiplying both the numerator and denominator by $(3 x+1)^{2}$ gives

$$
\begin{aligned}
f^{\prime}(x) & =\frac{\frac{1}{3 x+1}+\frac{3 x}{(3 x+1)^{2}}}{\left[2 x+\frac{1}{3 x+1}\right]^{2}} \frac{(3 x+1)^{2}}{(3 x+1)^{2}} \\
& =\frac{(3 x+1)+3 x}{[2 x(3 x+1)+1]^{2}} \\
& =\frac{6 x+1}{\left[6 x^{2}+2 x+1\right]^{2}} .
\end{aligned}
$$

While the linearity theorem (Theorem 2.4.2) is stated for a linear combination of two functions, it is not difficult to extend it to linear combinations of three or more functions as the following example shows.

Example 2.6.5 Linearity of the derivative of three or more functions.
We'll start by generalising linearity to three functions.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{a F(x)+b G(x)+c H(x)\}= & \frac{\mathrm{d}}{\mathrm{~d} x}\{a \cdot[F(x)]+1 \cdot[b G(x)+c H(x)]\} \\
= & a F^{\prime}(x)+\frac{\mathrm{d}}{\mathrm{~d} x}\{b G(x)+c H(x)\} \\
& \quad \text { by LIN with } \alpha=a, f(x)=F(x), \beta=1, \\
& \quad \text { and } g(x)=b G(x)+c H(x) \\
= & a F^{\prime}(x)+b G^{\prime}(x)+c H^{\prime}(x) \\
& \quad \text { by LIN with } \alpha=b, f(x)=G(x), \beta=c, \\
& \quad \text { and } g(x)=H(x)
\end{aligned}
$$

This gives us linearity for three terms, namely (just replacing upper case names by lower case names)

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\{a f(x)+b g(x)+c h(x)\}=a f^{\prime}(x)+b g^{\prime}(x)+c h^{\prime}(x)
$$

Just by repeating the above argument many times, we may generalise to linearity for $n$ terms, for any natural number $n$ :

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} x}\left\{a_{1} f_{1}(x)+a_{2} f_{2}(x)+\cdots+a_{n} f_{n}(x)\right\} \\
& \quad=a_{1} f_{1}^{\prime}(x)+a_{2} f_{2}^{\prime}(x)+\cdots+a_{n} f_{n}^{\prime}(x)
\end{aligned}
$$

Example 2.6.5
Similarly, while the product rule is stated for the product of two functions, it is not difficult to extend it to the product of three or more functions as the following example shows.

Example 2.6.6 Extending the product rule to more than two factors.
Once again, we'll start by generalising the product rule to three factors.

$$
\begin{array}{r}
\frac{\mathrm{d}}{\mathrm{~d} x}\{F(x) G(x) H(x)\}=F^{\prime}(x) G(x) H(x)+F(x) \frac{\mathrm{d}}{\mathrm{~d} x}\{G(x) H(x)\} \\
\text { by PR with } f(x)=F(x) \text { and } g(x)=G(x) H(x) \\
=F^{\prime}(x) G(x) H(x)+F(x)\left\{G^{\prime}(x) H(x)+G(x) H^{\prime}(x)\right\}
\end{array}
$$

by PR with $f(x)=G(x)$ and $g(x)=H(x)$
This gives us a product rule for three factors, namely (just replacing upper case names by lower case names)

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\{f(x) g(x) h(x)\}=f^{\prime}(x) g(x) h(x)+f(x) g^{\prime}(x) h(x)+f(x) g(x) h^{\prime}(x)
$$

Observe that when we differentiate a product of three factors, the answer is a sum of three terms and in each term the derivative acts on exactly one of the original factors. Just by repeating the above argument many times, we may generalise the product rule to give the derivative of a product of $n$ factors, for any natural number $n$ :

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{f_{1}(x) f_{2}(x) \cdots f_{n}(x)\right\}= & f_{1}^{\prime}(x) f_{2}(x) \cdots f_{n}(x) \\
& +f_{1}(x) f_{2}^{\prime}(x) \cdots f_{n}(x) \\
\vdots & \\
& +f_{1}(x) f_{2}(x) \cdots f_{n}^{\prime}(x)
\end{aligned}
$$

We can also write the above as

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{f_{1}(x) f_{2}(x) \cdots f_{n}(x)\right\}
$$

$$
=\left[\frac{f_{1}^{\prime}(x)}{f_{1}(x)}+\frac{f_{2}^{\prime}(x)}{f_{2}(x)}+\cdots+\frac{f_{n}^{\prime}(x)}{f_{n}(x)}\right] \cdot f_{1}(x) f_{2}(x) \cdots f_{n}(x)
$$

When we differentiate a product of $n$ factors, the answer is a sum of $n$ terms and in each term the derivative acts on exactly one of the original factors. In the first term, the derivative acts on the first of the original factors. In the second term, the derivative acts on the second of the original factors. And so on.
If we make $f_{1}(x)=f_{2}(x)=\cdots=f_{n}(x)=f(x)$ then each of the $n$ terms on the right hand side of the above equation is the product of $f^{\prime}(x)$ and exactly $n-1 f(x)$ 's, and so is exactly $f(x)^{n-1} f^{\prime}(x)$. So we get the following useful result

$$
\frac{\mathrm{d}}{\mathrm{~d} x} f(x)^{n}=n \cdot f(x)^{n-1} \cdot f^{\prime}(x)
$$

Example 2.6.6
This last result is quite useful, so let us write it as a lemma for future reference.

## Lemma 2.6.7

Let $n$ be a natural number and $f$ be a differentiable function. Then

$$
\frac{\mathrm{d}}{\mathrm{~d} x} f(x)^{n}=n \cdot f(x)^{n-1} \cdot f^{\prime}(x)
$$

This immediately gives us another useful result.
Example 2.6.8 Derivative of $x^{n}$.
We can now compute the derivative of $x^{n}$ for any natural number $n$. Start with Lemma 2.6.7 and substitute $f(x)=x$ and $f^{\prime}(x)=1$ :

$$
\frac{\mathrm{d}}{\mathrm{~d} x} x^{n}=n \cdot x^{n-1} \cdot 1=n x^{n-1}
$$

Again - this is a result we will come back to quite a few times in the future, so we should make sure we can refer to it easily. However, at present this statement only holds when $n$ is a positive integer. With a little more work we can extend this to compute $x^{q}$ where $q$ is any positive rational number and then any rational number at all (positive or negative). So let us hold off for a little longer. Instead we can make it a lemma, since it will be an ingredient in quite a few of the examples following below and in constructing the final corollary.

Lemma 2.6.9 Derivative of $x^{n}$.
Let $n$ be a positive integer then

$$
\frac{\mathrm{d}}{\mathrm{~d} x} x^{n}=n x^{n-1}
$$

Back to more examples.
Example 2.6.10 Derivative of a polynomial.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{2 x^{3}+4 x^{5}\right\}= & 2 \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{x^{3}\right\}+4 \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{x^{5}\right\} \\
& \quad \text { by LIN with } \alpha=2, f(x)=x^{3}, \beta=4 \text { and } g(x)=x^{5} \\
= & 2\left\{3 x^{2}\right\}+4\left\{5 x^{4}\right\}
\end{aligned}
$$

by Lemma 2.6.9, once with $n=3$, and once with $n=5$

$$
=6 x^{2}+20 x^{4}
$$

Example 2.6.11 Derivative of product of polynomials.
In this example we'll compute $\frac{\mathrm{d}}{\mathrm{d} x}\left\{(3 x+9)\left(x^{2}+4 x^{3}\right)\right\}$ in two different ways. For the first, we'll start with the product rule.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{(3 x & \left.+9)\left(x^{2}+4 x^{3}\right)\right\} \\
& =\left\{\frac{\mathrm{d}}{\mathrm{~d} x}(3 x+9)\right\}\left(x^{2}+4 x^{3}\right)+(3 x+9) \frac{\mathrm{d}}{\mathrm{~d} x}\left\{x^{2}+4 x^{3}\right\} \\
& =\{3 \times 1+9 \times 0\}\left(x^{2}+4 x^{3}\right)+(3 x+9)\left\{2 x+4\left(3 x^{2}\right)\right\} \\
& =3\left(x^{2}+4 x^{3}\right)+(3 x+9)\left(2 x+12 x^{2}\right) \\
& =3 x^{2}+12 x^{3}+\left(6 x^{2}+18 x+36 x^{3}+108 x^{2}\right) \\
& =18 x+117 x^{2}+48 x^{3}
\end{aligned}
$$

For the second, we expand the product first and then differentiate.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{(3 x+9)\left(x^{2}+4 x^{3}\right)\right\} & =\frac{\mathrm{d}}{\mathrm{~d} x}\left\{9 x^{2}+39 x^{3}+12 x^{4}\right\} \\
& =9(2 x)+39\left(3 x^{2}\right)+12\left(4 x^{3}\right) \\
& =18 x+117 x^{2}+48 x^{3}
\end{aligned}
$$

Example 2.6.12 Derivative of a rational function.

$$
\left.\begin{array}{rl}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{4 x^{3}-7 x}{4 x^{2}+1}\right\}= & \frac{\left(12 x^{2}-7\right)\left(4 x^{2}+1\right)-\left(4 x^{3}-7 x\right)(8 x)}{\left(4 x^{2}+1\right)^{2}} \\
& \quad \text { by QR with } f(x)=4 x^{3}-7 x, f^{\prime}(x)=12 x^{2}-7, \\
\quad \text { and } g(x)=4 x^{2}+1, g^{\prime}(x)=8 x
\end{array}\right] \begin{aligned}
& \left(48 x^{4}-16 x^{2}-7\right)-\left(32 x^{4}-56 x^{2}\right) \\
& =
\end{aligned}
$$

Example 2.6.13 Derivative of a cube-root.
In this example, we'll use a little trickery to find the derivative of $\sqrt[3]{x}$. The trickery consists of observing that, by the definition of the cube root,

$$
x=(\sqrt[3]{x})^{3}
$$

Since both sides of the expression are the same, they must have the same derivatives:

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\{x\}=\frac{\mathrm{d}}{\mathrm{~d} x}(\sqrt[3]{x})^{3}
$$

We already know by Theorem 2.2.4 that

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\{x\}=1
$$

and that, by Lemma 2.6.7 with $n=3$ and $f(x)=\sqrt[3]{x}$,

$$
\frac{\mathrm{d}}{\mathrm{~d} x}(\sqrt[3]{x})^{3}=3(\sqrt[3]{x})^{2} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\{\sqrt[3]{x}\}=3 x^{2 / 3} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\{\sqrt[3]{x}\}
$$

Since we know that $\frac{\mathrm{d}}{\mathrm{d} x}\{x\}=\frac{\mathrm{d}}{\mathrm{d} x}(\sqrt[3]{x})^{3}$, we must have

$$
1=3 x^{2 / 3} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\{\sqrt[3]{x}\}
$$

which we can rearrange to give the result we need

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\{\sqrt[3]{x}\}=\frac{1}{3} x^{-2 / 3}
$$

Example 2.6.14 Derivative of a positive rational power of $x$.
In this example, we'll use the same trickery as in Example 2.6.13 to find the derivative $x^{p / q}$ for any two natural numbers $p$ and $q$. By definition of the $q^{\text {th }}$ root,

$$
x^{p}=\left(x^{p / q}\right)^{q} .
$$

That is, $x^{p}$ and $\left(x^{p / q}\right)^{q}$ are the same function, and so have the same derivative. So we differentiate both of them. We already know that, by Lemma 2.6 .9 with $n=p$,

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{x^{p}\right\}=p x^{p-1}
$$

and that, by Lemma 2.6 .7 with $n=q$ and $f(x)=x^{p / q}$,

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\left(x^{p / q}\right)^{q}\right\}=q\left(x^{p / q}\right)^{q-1} \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{x^{p / q}\right\}
$$

Remember that $\left(x^{a}\right)^{b}=x^{(a \cdot b)}$. Now these two derivatives must be the same. So

$$
p x^{p-1}=q \cdot x^{(p q-p) / q} \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{x^{p / q}\right\}
$$

and, rearranging things,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{x^{p / q}\right\} & =\frac{p}{q} x^{p-1-(p q-p) / q} \\
& =\frac{p}{q} x^{(p q-q-p q+p) / q} \\
& =\frac{p}{q} x^{\frac{p}{q}-1}
\end{aligned}
$$

So finally

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{x^{p / q}\right\}=\frac{p}{q} x^{\frac{p}{q}-1}
$$

Notice that this has the same form as Lemma 2.6.9, above, except with $n=\frac{p}{q}$ allowed to be any positive rational number, not just a positive integer.

Example 2.6.15 Derivative of $x^{-m}$.
In this example we'll use the quotient rule to find the derivative of $x^{-m}$, for any natural number $m$.
By the special case of the quotient rule (Corollary 2.4.6) with $g(x)=x^{m}$ and $g^{\prime}(x)=$ $m x^{m-1}$

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{x^{-m}\right\}=\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{1}{x^{m}}\right\}=-\frac{m x^{m-1}}{\left(x^{m}\right)^{2}}=-m x^{-m-1}
$$

Again, notice that this has the same form as Lemma 2.6.9, above, except with $n=-m$ being a negative integer.

Example 2.6.15

Example 2.6.16 Derivative of a negative rational power of $x$.
In this example we'll use the quotient rule to find the derivative of $x^{-p / q}$, for any pair of natural numbers $p$ and $q$. By the special case the quotient rule (Corollary 2.4.6) with $g(x)=x^{\frac{p}{q}}$ and $g^{\prime}(x)=\frac{p}{q} x^{\frac{p}{q}-1}$,

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{x^{-\frac{p}{q}}\right\}=\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{1}{x^{\frac{p}{q}}}\right\}=-\frac{\frac{p}{q} x^{\frac{p}{q}-1}}{\left(x^{\frac{p}{q}}\right)^{2}}=-\frac{p}{q} x^{-\frac{p}{q}-1}
$$

Note that we have found, in Examples 2.2.2, 2.6.14 and 2.6.16, the derivative of $x^{a}$ for any rational number $a$, whether 0 , positive, negative, integer or fractional. In all cases, the answer is

## Corollary 2.6.17 Derivative of $x^{a}$.

Let $a$ be a rational number, then

$$
\frac{\mathrm{d}}{\mathrm{~d} x} x^{a}=a x^{a-1}
$$

We shall show, in Example 2.10.5, that the formula $\frac{\mathrm{d}}{\mathrm{d} x} x^{a}=a x^{a-1}$ in fact applies for all real numbers $a$, not just rational numbers.

Back in Example 2.2.9 we computed the derivative of $\sqrt{x}$ from the definition of the derivative. The above corollary (correctly) gives

$$
\frac{\mathrm{d}}{\mathrm{~d} x} x^{\frac{1}{2}}=\frac{1}{2} x^{-\frac{1}{2}}
$$

but with far less work.
Here's an (optional) messy example.
Example 2.6.18 Optional messy example.
Find the derivative of

$$
f(x)=\frac{(\sqrt{x}-1)(2-x)\left(1-x^{2}\right)}{\sqrt{x}(3+2 x)}
$$

- As we seen before, the best strategy for dealing with nasty expressions is to break
them up into easy pieces. We can think of $f(x)$ as the five-fold product

$$
f(x)=f_{1}(x) \cdot f_{2}(x) \cdot f_{3}(x) \cdot \frac{1}{f_{4}(x)} \cdot \frac{1}{f_{5}(x)}
$$

with

$$
\begin{array}{lll}
f_{1}(x)=\sqrt{x}-1 & f_{2}(x)=2-x & f_{3}(x)=1-x^{2} \\
f_{4}(x)=\sqrt{x} & f_{5}(x)=3+2 x &
\end{array}
$$

- By now, the derivatives of the $f_{j}$ 's should be easy to find:

$$
\begin{array}{lll}
f_{1}^{\prime}(x)=\frac{1}{2 \sqrt{x}} & f_{2}^{\prime}(x)=-1 & f_{3}^{\prime}(x)=-2 x \\
f_{4}^{\prime}(x)=\frac{1}{2 \sqrt{x}} & f_{5}^{\prime}(x)=2 &
\end{array}
$$

- Now, to get the derivative $f(x)$ we use the $n$-fold product rule which was developed in Example 2.6.6, together with the special case of the quotient rule (Corollary 2.4.6).

$$
\begin{aligned}
& f^{\prime}(x)= f_{1}^{\prime} f_{2} f_{3} \frac{1}{f_{4}} \frac{1}{f_{5}}+f_{1} f_{2}^{\prime} f_{3} \frac{1}{f_{4}} \frac{1}{f_{5}}+f_{1} f_{2} f_{3}^{\prime} \frac{1}{f_{4}} \frac{1}{f_{5}}-f_{1} f_{2} f_{3} \frac{f_{4}^{\prime}}{f_{4}^{2}} \frac{1}{f_{5}} \\
&-f_{1} f_{2} f_{3} \frac{1}{f_{4}} \frac{f_{5}^{\prime}}{f_{5}^{2}} \\
&= {\left[\frac{f_{1}^{\prime}}{f_{1}}+\frac{f_{2}^{\prime}}{f_{2}}+\frac{f_{3}^{\prime}}{f_{3}}-\frac{f_{4}^{\prime}}{f_{4}}-\frac{f_{5}^{\prime}}{f_{5}}\right] f_{1} f_{2} f_{3} \frac{1}{f_{4}} \frac{1}{f_{5}} } \\
&=\left[\frac{1}{2 \sqrt{x}(\sqrt{x}-1)}-\frac{1}{2-x}-\frac{2 x}{1-x^{2}}-\frac{1}{2 x}-\frac{2}{3+2 x}\right] \\
& \frac{(\sqrt{x}-1)(2-x)\left(1-x^{2}\right)}{\sqrt{x}(3+2 x)}
\end{aligned}
$$

The trick that we used in going from the first line to the second line, namely multiplying term number $j$ by $\frac{f_{j}(x)}{f_{j}(x)}$ is often useful in simplifying the derivative of a product of many factors ${ }^{a}$.

[^4]
### 2.6.2 $円$ Exercises

## Exercises - Stage 1

1. Spot and correct the error(s) in the following calculation.

$$
\begin{aligned}
f(x) & =\frac{2 x}{x+1} \\
f^{\prime}(x) & =\frac{2(x+1)+2 x}{(x+1)^{2}} \\
& =\frac{2(x+1)}{(x+1)^{2}} \\
& =\frac{2}{x+1}
\end{aligned}
$$

2. True or false: $\frac{d}{d x}\left\{2^{x}\right\}=x 2^{x-1}$.

## Exercises - Stage 2

3. Differentiate $f(x)=\frac{2}{3} x^{6}+5 x^{4}+12 x^{2}+9$ and factor the result.
4. Differentiate $s(t)=3 t^{4}+5 t^{3}-\frac{1}{t}$.
5. Differentiate $x(y)=\left(2 y+\frac{1}{y}\right) \cdot y^{3}$.
6. Differentiate $T(x)=\frac{\sqrt{x}+1}{x^{2}+3}$.
7. *. Compute the derivative of $\left(\frac{7 x+2}{x^{2}+3}\right)$.
8. What is $f^{\prime}(0)$, when $f(x)=\left(3 x^{3}+4 x^{2}+x+1\right)(2 x+5)$ ?
9. Differentiate $f(x)=\frac{3 x^{3}+1}{x^{2}+5 x}$.
10. *. Compute the derivative of $\left(\frac{3 x^{2}+5}{2-x}\right)$
11. *. Compute the derivative of $\left(\frac{2-x^{2}}{3 x^{2}+5}\right)$.
12. *. Compute the derivative of $\left(\frac{2 x^{3}+1}{x+2}\right)$.
13. *. For what values of $x$ does the derivative of $\frac{\sqrt{x}}{1-x^{2}}$ exist? Explain your answer.
14. Differentiate $f(x)=(3 \sqrt[5]{x}+15 \sqrt[3]{x}+8)\left(3 x^{2}+8 x-5\right)$.
15. Differentiate $f(x)=\frac{\left(x^{2}+5 x+1\right)(\sqrt{x}+\sqrt[3]{x})}{x}$.
16. Find all $x$-values where $f(x)=\frac{1}{\frac{1}{5} x^{5}+x^{4}-\frac{5}{3} x^{3}}$ has a horizontal tangent line.

## Exercises - Stage 3

17. *. Find an equation of a line that is tangent to both of the curves $y=x^{2}$ and $y=x^{2}-2 x+2$ (at different points).
18. $[1998 \mathrm{H}]$ Find all lines that are tangent to both of the curves $y=x^{2}$ and $y=-x^{2}+2 x-5$. Illustrate your answer with a sketch.
19. *. Evaluate $\lim _{x \rightarrow 2}\left(\frac{x^{2015}-2^{2015}}{x-2}\right)$.

## 2.7』 Derivatives of Exponential Functions

Now that we understand how derivatives interact with products and quotients, we are able to compute derivatives of

- polynomials,
- rational functions, and
- powers and roots of rational functions.

Notice that all of the above come from knowing ${ }^{1}$ the derivative of $x^{n}$ and applying linearity of derivatives and the product rule.

There is still one more "rule" that we need to complete our toolbox and that is the chain rule. However before we get there, we will add a few functions to our list of things we can differentiate ${ }^{2}$. The first of these is the exponential function.

Let $a>0$ and set $f(x)=a^{x}$ - this is what is known as an exponential function. Let's see what happens when we try to compute the derivative of this function just

1 Differentiating powers and roots of functions is actually quite a bit easier once one knows the chain rule - which we will discuss soon.
2 One reason we add these functions is that they interact very nicely with the derivative. Another reason is that they turn up in many "real world" examples.
using the definition of the derivative.

$$
\begin{aligned}
\frac{\mathrm{d} f}{\mathrm{~d} x} & =\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}=\lim _{h \rightarrow 0} \frac{a^{x+h}-a^{x}}{h} \\
& =\lim _{h \rightarrow 0} a^{x} \cdot \frac{a^{h}-1}{h}=a^{x} \cdot \lim _{h \rightarrow 0} \frac{a^{h}-1}{h}
\end{aligned}
$$

Unfortunately we cannot complete this computation because we cannot evaluate the last limit directly. For the moment, let us assume this limit exists and name it

$$
C(a)=\lim _{h \rightarrow 0} \frac{a^{h}-1}{h}
$$

It depends only on $a$ and is completely independent of $x$. Using this notation (which we will quickly improve upon below), our desired derivative is now

$$
\frac{\mathrm{d}}{\mathrm{~d} x} a^{x}=C(a) \cdot a^{x} .
$$

Thus the derivative of $a^{x}$ is $a^{x}$ multiplied by some constant - i.e. the function $a^{x}$ is nearly unchanged by differentiating. If we can tune $a$ so that $C(a)=1$ then the derivative would just be the original function! This turns out to be very useful.

To try finding an $a$ that obeys $C(a)=1$, let us investigate how $C(a)$ changes with $a$. Unfortunately (though this fact is not at all obvious) there is no way to write $C(a)$ as a finite combination of any of the functions we have examined so far ${ }^{3}$. To get started, we'll try to guess $C(a)$, for a few values of $a$, by plugging in some small values of $h$.

Example 2.7.1 Estimates of $C(a)$.
Let $a=1$ then $C(1)=\lim _{h \rightarrow 0} \frac{1^{h}-1}{h}=0$. This is not surprising since $1^{x}=1$ is constant, and so its derivative must be zero everywhere. Let $a=2$ then $C(2)=\lim _{h \rightarrow 0} \frac{2^{h}-1}{h}$. Setting $h$ to smaller and smaller numbers gives

| $h$ | 0.1 | 0.01 | 0.001 | 0.0001 | 0.00001 | 0.000001 | 0.0000001 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\frac{2^{h}-1}{h}$ | 0.7177 | 0.6956 | 0.6934 | 0.6932 | 0.6931 | 0.6931 | 0.6931 |

Similarly when $a=3$ we get

| $h$ | 0.1 | 0.01 | 0.001 | 0.0001 | 0.00001 | 0.000001 | 0.0000001 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\frac{3^{h}-1}{h}$ | 1.1612 | 1.1047 | 1.0992 | 1.0987 | 1.0986 | 1.0986 | 1.0986 |

3 To a bit more be precise, we say that a number $q$ is algebraic if we can write $q$ as the zero of a polynomial with integer coefficients. When $a$ is any positive algebraic number other $1, C(a)$ is not algebraic. A number that is not algebraic is called transcendental. The best known example of a transcendental number is $\pi$ (which follows from the Lindemann-Weierstrass Theorem - way beyond the scope of this course).
and $a=10$

| $h$ | 0.1 | 0.01 | 0.001 | 0.0001 | 0.00001 | 0.000001 | 0.0000001 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\frac{10^{h}-1}{h}$ | 2.5893 | 2.3293 | 2.3052 | 2.3028 | 2.3026 | 2.3026 | 2.3026 |

From this example it appears that $C(a)$ increases as we increase $a$, and that $C(a)=1$ for some value of $a$ between 2 and 3 .

Example 2.7.1
We can learn a lot more about $C(a)$, and, in particular, confirm the guesses that we made in the last example, by making use of logarithms - this would be a good time for you to review them.

### 2.7.1 $~ W h i r l w i n d ~ R e v i e w ~ o f ~ L o g a r i t h m s ~$

Before you read much further into this little review on logarithms, you should first go back and take a look at the review of inverse functions in Section 0.6.

### 2.7.1.1 Logarithmic Functions

We are about to define the "logarithm with base $q$ ". In principle, $q$ is allowed to be any strictly positive real number, except $q=1$. However we shall restrict our attention to $q>1$, because, in practice, the only $q$ 's that are ever used are $e$ (a number that we shall define in the next few pages), 10 and, if you are a computer scientist, 2. So, fix any $q>1$ (if you like, pretend that $q=10$ ). The function $f(x)=q^{x}$

- increases as $x$ increases (for example if $x^{\prime}>x$, then $10^{x^{\prime}}=10^{x} \cdot 10^{x^{\prime}-x}>10^{x}$ since $10^{x^{\prime}-x}>1$ )
- obeys $\lim _{x \rightarrow-\infty} q^{x}=0$ (for example $10^{-1000}$ is really small) and
- obeys $\lim _{x \rightarrow+\infty} q^{x}=+\infty$ (for example $10^{+1000}$ is really big).

Consequently, for any $0<Y<\infty$, the horizontal straight line $y=Y$ crosses the graph of $y=f(x)=q^{x}$ at exactly one point, as illustrated in the figure below.


The $x$-coordinate of that intersection point, denoted $X$ in the figure, is $\log _{q}(Y)$. So $\log _{q}(Y)$ is the power to which you have to raise $q$ to get $Y$. It is the inverse function of $f(x)=q^{x}$. Of course we are free to rename the dummy variables $X$ and $Y$. If, for example, we wish to graph our logarithm function, it is natural to rename $Y \rightarrow x$ and $X \rightarrow y$, giving

## Definition 2.7.2

Let $q>1$. Then the logarithm with base $q$ is defined ${ }^{a}$ by

$$
y=\log _{q}(x) \Leftrightarrow x=q^{y}
$$

$a \quad$ We can also define logarithms with base $0<r<1$ but doing so is not necessary. To see this, set $q=1 / r>1$. Then it is reasonable to define $\log _{r}(x)=-\log _{q}(x)$ since

$$
r^{\log _{r}(x)}=\left(\frac{1}{q}\right)^{\log _{r}(x)}=\left(\frac{1}{q}\right)^{-\log _{q}(x)}=q^{\log _{q}(x)}=x
$$

as required.
Obviously the power to which we have to raise $q$ to get $q^{x}$ is $x$, so we have both

$$
\log _{q}\left(q^{x}\right)=x \quad q^{\log _{q}(x)}=x
$$

From the exponential properties

$$
\begin{aligned}
q^{\log _{q}(x y)} & =x y & & =q^{\log _{q}(x)} q^{\log _{q}(y)}=q^{\log _{q}(x)+\log _{q}(y)} \\
q^{\log _{q}(x / y)} & =x / y & & =q^{\log _{q}(x)} / q^{\log _{q}(y)}=q^{\log _{q}(x)-\log _{q}(y)} \\
q^{\log _{q}\left(x^{r}\right)} & =x^{r} & & =\left(q^{\log _{q}(x)}\right)^{r}=q^{r \log _{q}(x)}
\end{aligned}
$$

we have

$$
\begin{aligned}
\log _{q}(x y) & =\log _{q}(x)+\log _{q}(y) \\
\log _{q}(x / y) & =\log _{q}(x)-\log _{q}(y) \\
\log _{q}\left(x^{r}\right) & =r \log _{q}(x)
\end{aligned}
$$

Can we convert from logarithms in one base to logarithms in another? For example, if our calculator computes logarithms base 10 for us (which it very likely does), can we also use it to compute a logarithm base $q$ ? Yes, using

$$
\log _{q}(x)=\frac{\log _{10} x}{\log _{10} q}
$$

How did we get this? Well, let's start with a number $x$ and suppose that we want to compute

$$
y=\log _{q} x
$$

We can rearrange this by exponentiating both sides

$$
q^{y}=q^{\log _{q} x}=x
$$

Now take log base 10 of both sides

$$
\log _{10} q^{y}=\log _{10} x
$$

But recall that $\log _{q}\left(x^{r}\right)=r \log _{q}(x)$, so

$$
\begin{aligned}
y \log _{10} q & =\log _{10} x \\
y & =\frac{\log _{10} x}{\log _{10} q}
\end{aligned}
$$

### 2.7.2 $\leadsto$ Back to that Limit

Recall that we are trying to choose $a$ so that

$$
\lim _{h \rightarrow 0} \frac{a^{h}-1}{h}=C(a)=1 .
$$

We can estimate the correct value of $a$ by using our numerical estimate of $C(10)$ above. The way to do this is to first rewrite $C(a)$ in terms of logarithms.

$$
a=10^{\log _{10} a} \quad \text { and so } \quad a^{h}=10^{h \log _{10} a} .
$$

Using this we rewrite $C(a)$ as

$$
C(a)=\lim _{h \rightarrow 0} \frac{1}{h}\left(10^{h \log _{10} a}-1\right)
$$

Now set $H=h \log _{10}(a)$, and notice that as $h \rightarrow 0$ we also have $H \rightarrow 0$

$$
=\lim _{H \rightarrow 0} \frac{\log _{10} a}{H}\left(10^{H}-1\right)
$$

$$
\begin{aligned}
& =\log _{10} a \cdot \lim _{H \rightarrow 0} \frac{10^{H}-1}{H} \\
& =\log _{10} a \cdot C(10)
\end{aligned}
$$

Below is a sketch of $C(a)$ against $a$.


Figure 2.7.3
Remember that we are trying to find an $a$ with $C(a)=1$. We can do so by recognising that $C(a)=C(10)\left(\log _{10} a\right)$ has the following properties.

- When $a=1, \log _{10}(a)=\log _{10} 1=0$ so that $C(a)=C(10) \log _{10}(a)=0$. Of course, we should have expected this, because when $a=1$ we have $a^{x}=1^{x}=1$ which is just the constant function and $\frac{\mathrm{d}}{\mathrm{d} x} 1=0$.
- $\log _{10} a$ increases as $a$ increases, and hence $C(a)=C(10) \log _{10} a$ increases as $a$ increases.
- $\log _{10} a$ tends to $+\infty$ as $a \rightarrow \infty$, and hence $C(a)$ tends to $+\infty$ as $a \rightarrow \infty$.

Hence the graph of $C(a)$ passes through $(1,0)$, is always increasing as $a$ increases and goes off to $+\infty$ as $a$ goes off to $+\infty$. See Figure 2.7.3. Consequently ${ }^{4}$ there is exactly one value of $a$ for which $C(a)=1$.

The value of $a$ for which $C(a)=1$ is given the name $e$. It is called Euler's constant ${ }^{5}$. In Example 2.7.1, we estimated $C(10) \approx 2.3026$. So if we assume $C(a)=1$ then the above equation becomes

$$
2.3026 \cdot \log _{10} a \approx 1
$$

4 We are applying the Intermediate Value Theorem here, but we have neglected to verify the hypothesis that $\log _{10}(a)$ is a continuous function. Please forgive us - we could do this if we really had to, but it would make a big mess without adding much understanding, if we were to do so here in the text. Better to just trust us on this.
5 Unfortunately there is another Euler's constant, $\gamma$, which is more properly called the Eu-ler-Mascheroni constant. Anyway like many mathematical discoveries, $e$ was first found by someone else - Napier used the constant $e$ in order to compute logarithms but only implicitly. Bernoulli was probably the first to approximate it when examining continuous compound interest. It first appeared explicitly in work of Leibniz, though he denoted it $b$. It was Euler, though, who established the notation we now use and who showed how important the constant is to mathematics.

$$
\begin{aligned}
\log _{10} a & \approx \frac{1}{2.3026} \approx 0.4343 \\
a & \approx 10^{0.4343} \approx 2.7813
\end{aligned}
$$

This gives us the estimate $a \approx 2.7813$ which is not too bad. In fact ${ }^{6}$

Equation 2.7.4 Euler's constant.

$$
\begin{aligned}
e & =2.7182818284590452354 \ldots \\
& =1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots
\end{aligned}
$$

We will be able to explain this last formula once we develop Taylor polynomials later in the course.

To summarise

## Theorem 2.7.5

The constant $e$ is the unique real number that satisfies

$$
\lim _{h \rightarrow 0} \frac{e^{h}-1}{h}=1
$$

Further,

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left(e^{x}\right)=e^{x}
$$

We plot $e^{x}$ in the graph below


6 Recall $n$ factorial, written $n$ ! is the product $n \times(n-1) \times(n-2) \times \cdots \times 2 \times 1$.

And just a reminder of some of its ${ }^{7}$ properties...

1. $e^{0}=1$
2. $e^{x+y}=e^{x} e^{y}$
3. $e^{-x}=\frac{1}{e^{x}}$
4. $\left(e^{x}\right)^{y}=e^{x y}$
5. $\lim _{x \rightarrow \infty} e^{x}=\infty, \lim _{x \rightarrow-\infty} e^{x}=0$

Now consider again the problem of differentiating $a^{x}$. We saw above that

$$
\begin{gathered}
\frac{\mathrm{d}}{\mathrm{~d} x} a^{x}=C(a) \cdot a^{x} \quad \text { and } \quad C(a)=C(10) \cdot \log _{10} a \\
\text { which gives } \frac{\mathrm{d}}{\mathrm{~d} x} a^{x}=C(10) \cdot \log _{10} a \cdot a^{x}
\end{gathered}
$$

We can eliminate the $C(10)$ term with a little care. Since we know that $\frac{\mathrm{d}}{\mathrm{d} x} e^{x}=e^{x}$, we have $C(e)=1$. This allows us to express

$$
\begin{aligned}
1=C(e) & =C(10) \cdot \log _{10} e \quad \text { and so } \\
C(10) & =\frac{1}{\log _{10} e}
\end{aligned}
$$

Putting things back together gives

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x} a^{x} & =\frac{\log _{10} a}{\log _{10} e} \cdot a^{x} \\
& =\log _{e} a \cdot a^{x} .
\end{aligned}
$$

There is more than one way to get to this result. For example, let $f(x)=a^{x}$, then

$$
\begin{aligned}
\log _{e} f(x) & =x \log _{e} a \\
f(x) & =e^{x \log _{e} a}
\end{aligned}
$$

So if we write $g(x)=e^{x}$ then we are really attempting to differentiate the function

$$
\frac{\mathrm{d} f}{\mathrm{~d} x}=\frac{\mathrm{d}}{\mathrm{~d} x} g\left(x \cdot \log _{e} a\right)
$$

In order to compute this derivative we need to know how to differentiate

$$
\frac{\mathrm{d}}{\mathrm{~d} x} g(q x)
$$

where $q$ is a constant. We'll hold off on learning this for the moment until we have introduced the chain rule (see Section 2.9 and in particular Corollary 2.9.9). Similarly

7 The function $e^{x}$ is of course the special case of the function $a^{x}$ with $a=e$. So it inherits all the usual algebraic properties of $a^{x}$.
we'd like to know how to differentiate logarithms - again this has to wait until we have learned the chain rule.

Notice that the derivatives

$$
\frac{\mathrm{d}}{\mathrm{~d} x} x^{n}=n x^{n-1} \quad \text { and } \quad \frac{\mathrm{d}}{\mathrm{~d} x} e^{x}=e^{x}
$$

are either nearly unchanged or actually unchanged by differentiating. It turns out that some of the trigonometric functions also have this property of being "nearly unchanged" by differentiation. That brings us to the next section.

### 2.7.3 $円$ Exercises

## Exercises - Stage 1

1. Match the curves in the graph to the following functions:
(a) $y=\left(\frac{1}{2}\right)^{x}$
(b) $y=1^{x}$
(c) $y=2^{x}$
(d) $y=2^{-x}$
(e) $y=3^{x}$

2. The graph below shows an exponential function $f(x)=a^{x}$ and its derivative $f^{\prime}(x)$. Choose all the options that describe the constant $a$.
(a) $a<0$
(b) $a>0$
(c) $a<1$
(d) $a>1$
(e) $a<e$
(f) $a>e$

3. True or false: $\frac{\mathrm{d}}{\mathrm{d} x}\left\{e^{x}\right\}=x e^{x-1}$
4. A population of bacteria is described by $P(t)=100 e^{0.2 t}$, for $0 \leq t \leq 10$. Over this time period, is the population increasing or decreasing?
We will learn more about the uses of exponential functions to describe realworld phenomena in Section 3.3.

## Exercises - Stage 2

5. Find the derivative of $f(x)=\frac{e^{x}}{2 x}$.
6. Differentiate $f(x)=e^{2 x}$.
7. Differentiate $f(x)=e^{a+x}$, where $a$ is a constant.
8. For which values of $x$ is the function $f(x)=x e^{x}$ increasing?
9. Differentiate $e^{-x}$.
10. Differentiate $f(x)=\left(e^{x}+1\right)\left(e^{x}-1\right)$.
11. A particle's position is given by

$$
s(t)=t^{2} e^{t}
$$

When is the particle moving in the negative direction?

## Exercises - Stage 3

12. Let $g(x)=f(x) e^{x}$, for a differentiable function $f(x)$. Give a simplified formula for $g^{\prime}(x)$.
Functions of the form $g(x)$ are relatively common. If you remember this formula, you can save yourself some time when you need to differentiate
them. We will explore this more in Question 2.14.2.19, Section 2.14.
13. Which of the following functions describe a straight line?
(a) $y=e^{3 \log x}+1$
(b) $2 y+5=e^{3+\log x}$
(c) $y=e^{2 x}+4$
(d) $y=e^{\log x} 3^{e}+\log 2$
14. *. Find constants $a, b$ so that the following function is differentiable:

$$
f(x)= \begin{cases}a x^{2}+b & x \leq 1 \\ e^{x} & x>1\end{cases}
$$

## 2.8^ Derivatives of Trigonometric Functions

We are now going to compute the derivatives of the various trigonometric functions, $\sin x, \cos x$ and so on. The computations are more involved than the others that we have done so far and will take several steps. Fortunately, the final answers will be very simple.

Observe that we only need to work out the derivatives of $\sin x$ and $\cos x$, since the other trigonometric functions are really just quotients of these two functions. Recall:

$$
\tan x=\frac{\sin x}{\cos x} \quad \cot x=\frac{\cos x}{\sin x} \quad \csc x=\frac{1}{\sin x} \quad \sec x=\frac{1}{\cos x}
$$

The first steps towards computing the derivatives of $\sin x, \cos x$ is to find their derivatives at $x=0$. The derivatives at general points $x$ will follow quickly from these, using trig identities. It is important to note that we must measure angles in radians ${ }^{1}$, rather than degrees, in what follows. Indeed - unless explicitly stated otherwise, any number that is put into a trigonometric function is measured in radians.

### 2.8.1 These Proofs are Optional, the Results are Not.

While we expect you to read and follow these proofs, we do not expect you to be able to reproduce them. You will be required to know the results, in particular Theorem 2.8.5 below.

1 In science, radians is the standard unit for measuring angles. While you may be more familiar with degrees, radians should be used in any computation involving calculus. Using degrees will cause errors. Thankfully it is easy to translate between these two measures since $360^{\circ}=2 \pi$ radians. See Appendix B.2.1.
2.8.2 $\leadsto$ Step 1: $\left.\frac{d}{d x}\{\sin \mathrm{x}\}\right|_{\mathrm{x}=0}$

By definition, the derivative of $\sin x$ evaluated at $x=0$ is

$$
\left.\frac{\mathrm{d}}{\mathrm{~d} x}\{\sin x\}\right|_{x=0}=\lim _{h \rightarrow 0} \frac{\sin h-\sin 0}{h}=\lim _{h \rightarrow 0} \frac{\sin h}{h}
$$

We will prove this limit by use of the squeeze theorem (Theorem 1.4.18). To get there we will first need to do some geometry. But first we will build some intuition.

The figure below contains part of a circle of radius 1. Recall that an arc of length $h$ on such a circle subtends an angle of $h$ radians at the centre of the circle. So the darkened arc in the figure has length $h$ and the darkened vertical line in the figure has length $\sin h$. We must determine what happens to the ratio of the lengths of the darkened vertical line and darkened arc as $h$ tends to zero.



Here is a magnified version of the part of the above figure that contains the darkened arc and vertical line.


This particular figure has been drawn with $h=.4$ radians. Here are three more
such blow ups. In each successive figure, the value of $h$ is smaller. To make the figures clearer, the degree of magnification was increased each time $h$ was decreased.




As we make $h$ smaller and smaller and look at the figure with ever increasing magnification, the arc of length $h$ and vertical line of length $\sin h$ look more and more alike. We would guess from this that

$$
\lim _{h \rightarrow 0} \frac{\sin h}{h}=1
$$

The following tables of values

| $h$ | $\sin h$ | $\frac{\sin h}{h}$ |
| :--- | :--- | :--- |
| 0.4 | .3894 | .9735 |
| 0.2 | .1987 | .9934 |
| 0.1 | .09983 | .9983 |
| 0.05 | .049979 | .99958 |
| 0.01 | .00999983 | .999983 |
| 0.001 | .0099999983 | .9999983 |


| h | $\sin h$ | $\frac{\sin h}{h}$ |
| :--- | :--- | :--- |
| -0.4 | -.3894 | .9735 |
| -0.2 | -.1987 | .9934 |
| -0.1 | -.09983 | .9983 |
| -0.05 | -.049979 | .99958 |
| -0.01 | -.00999983 | .999983 |
| -0.001 | -.0099999983 | .9999983 |

suggests the same guess. Here is an argument that shows that the guess really is correct.

### 2.8.3 $\leadsto$ Proof that $\lim _{h \rightarrow 0} \frac{\sin h}{h}=1$



The circle in the figure above has radius 1. Hence

$$
\begin{array}{rlrl}
|O P|=\mid & |O R|=1 & |P S|=\sin h \\
|O S|=\cos h & |Q R|=\tan h
\end{array}
$$

Now we can use a few geometric facts about this figure to establish both an upper bound and a lower bound on $\frac{\sin h}{h}$ with both the upper and lower bounds tending to 1 as $h$ tends to 0 . So the squeeze theorem will tell us that $\frac{\sin h}{h}$ also tends to 1 as $h$ tends to 0 .

- The triangle $O P R$ has base 1 and height $\sin h$, and hence

$$
\text { area of } \triangle O P R=\frac{1}{2} \times 1 \times \sin h=\frac{\sin h}{2} \text {. }
$$

- The triangle $O Q R$ has base 1 and height $\tan h$, and hence

$$
\text { area of } \triangle O Q R=\frac{1}{2} \times 1 \times \tan h=\frac{\tan h}{2} \text {. }
$$

- The "piece of pie" $O P R$ cut out of the circle is the fraction $\frac{h}{2 \pi}$ of the whole circle (since the angle at the corner of the piece of pie is $h$ radians and the angle for the whole circle is $2 \pi$ radians). Since the circle has radius 1 we have

$$
\text { area of pie } O P R=\frac{h}{2 \pi} \cdot(\text { area of circle })=\frac{h}{2 \pi} \pi \cdot 1^{2}=\frac{h}{2}
$$

Now the triangle $O P R$ is contained inside the piece of pie $O P R$. and so the area of the triangle is smaller than the area of the piece of pie. Similarly, the piece of pie $O P R$ is contained inside the triangle $O Q R$. Thus we have
area of triangle $O P R \leq$ area of pie $O P R \leq$ area of triangle $O Q R$

Substituting in the areas we worked out gives

$$
\frac{\sin h}{2} \leq \frac{h}{2} \leq \frac{\tan h}{2}
$$

which cleans up to give

$$
\sin h \leq h \leq \frac{\sin h}{\cos h}
$$

We rewrite these two inequalities so that $\frac{\sin h}{h}$ appears in both.

- Since $\sin h \leq h$, we have that $\frac{\sin h}{h} \leq 1$.
- Since $h \leq \frac{\sin h}{\cos h}$ we have that $\cos h \leq \frac{\sin h}{h}$.

Thus we arrive at the "squeezable" inequality

$$
\cos h \leq \frac{\sin h}{h} \leq 1
$$

We know ${ }^{2}$ that

$$
\lim _{h \rightarrow 0} \cos h=1
$$

Since $\frac{\sin h}{h}$ is sandwiched between $\cos h$ and 1, we can apply the squeeze theorem for limits (Theorem 1.4.18) to deduce the following lemma:

Lemma 2.8.1

$$
\lim _{h \rightarrow 0} \frac{\sin h}{h}=1
$$

Since this argument took a bit of work, perhaps we should remind ourselves why we needed it in the first place. We were computing

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{~d} x}\{\sin x\}\right|_{x=0} & =\lim _{h \rightarrow 0} \frac{\sin h-\sin 0}{h} \\
& =\lim _{h \rightarrow 0} \frac{\sin h}{h} \quad \text { (This is why!) } \\
& =1
\end{aligned}
$$

This concludes Step 1. We now know that $\left.\frac{\mathrm{d}}{\mathrm{d} x} \sin x\right|_{x=0}=1$. The remaining steps are easier.


2 Again, refresh your memory by looking up Appendix A.5.

### 2.8.4 Step 2: $\left.\frac{\mathrm{d}}{\mathrm{dx}}\{\cos \mathrm{x}\}\right|_{\mathrm{x}=0}$

By definition, the derivative of $\cos x$ evaluated at $x=0$ is

$$
\lim _{h \rightarrow 0} \frac{\cos h-\cos 0}{h}=\lim _{h \rightarrow 0} \frac{\cos h-1}{h}
$$

Fortunately we don't have to wade through geometry like we did for the previous step. Instead we can recycle our work and massage the above limit to rewrite it in terms of expressions involving $\frac{\sin h}{h}$. Thanks to Lemma 2.8.1 the work is then easy.

We'll show you two ways to proceed - one uses a method similar to "multiplying by the conjugate" that we have already used a few times (see Example 1.4.17 and 2.2.9 ), while the other uses a nice trick involving the double-angle formula ${ }^{3}$.

### 2.8.4.1 Method 1 - Multiply by the "Conjugate"

Start by multiplying the expression inside the limit by 1 , written as $\frac{\cos h+1}{\cos h+1}$ :

$$
\begin{array}{rlr}
\frac{\cos h-1}{h} & =\frac{\cos h-1}{h} \cdot \frac{\cos h+1}{\cos h+1} & \\
& =\frac{\cos ^{2} h-1}{h(1+\cos h)} & \\
& =-\frac{\sin ^{2} h}{h(1+\cos h)} & \\
& =-\frac{\sin h}{h} \cdot \frac{\sin h}{1+\cos h} &
\end{array}
$$

Now we can take the limit as $h \rightarrow 0$ via Lemma 2.8.1.

$$
\begin{aligned}
\lim _{h \rightarrow 0} \frac{\cos h-1}{h} & =\lim _{h \rightarrow 0}\left(\frac{-\sin h}{h} \cdot \frac{\sin h}{1+\cos h}\right) \\
& =-\lim _{h \rightarrow 0}\left(\frac{\sin h}{h}\right) \cdot \lim _{h \rightarrow 0}\left(\frac{\sin h}{1+\cos h}\right) \\
& =-1 \cdot \frac{0}{2} \\
& =0
\end{aligned}
$$

3 See Appendix A. 14 if you have forgotten. You should also recall that $\sin ^{2} \theta+\cos ^{2} \theta=1$. Sorry for nagging.

### 2.8.4.2 Method 2 - via the Double Angle Formula

The other way involves the double angle formula ${ }^{4}$,

$$
\cos 2 \theta=1-2 \sin ^{2}(\theta) \quad \text { or } \quad \cos 2 \theta-1=-2 \sin ^{2}(\theta)
$$

Setting $\theta=h / 2$, we have

$$
\frac{\cos h-1}{h}=\frac{-2\left(\sin \frac{h}{2}\right)^{2}}{h}
$$

Now this begins to look like $\frac{\sin h \text { ? }}{h}$, except that inside the $\sin (\cdot)$ we have $h / 2$. So, setting $\theta=h / 2$,

$$
\begin{aligned}
\frac{\cos h-1}{h} & =-\frac{\sin ^{2} \theta}{\theta}=-\theta \cdot \frac{\sin ^{2} \theta}{\theta^{2}} \\
& =-\theta \cdot \frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{\theta}
\end{aligned}
$$

When we take the limit as $h \rightarrow 0$, we are also taking the limit as $\theta=h / 2 \rightarrow 0$, and so

$$
\begin{aligned}
\lim _{h \rightarrow 0} \frac{\cos h-1}{h} & =\lim _{\theta \rightarrow 0}\left[-\theta \cdot \frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{\theta}\right] \\
& =\lim _{\theta \rightarrow 0}[-\theta] \cdot \lim _{\theta \rightarrow 0}\left[\frac{\sin \theta}{\theta}\right] \cdot \lim _{\theta \rightarrow 0}\left[\frac{\sin \theta}{\theta}\right] \\
& =0 \cdot 1 \cdot 1 \\
& =0
\end{aligned}
$$

where we have used the fact that $\lim _{h \rightarrow 0} \frac{\sin h}{h}=1$ and that the limit of a product is the product of limits (i.e. Lemma 2.8.1 and Theorem 1.4.3).
Thus we have now produced two proofs of the following lemma:

Lemma 2.8.2

$$
\lim _{h \rightarrow 0} \frac{\cos h-1}{h}=0
$$

Again, there has been a bit of work to get to here, so we should remind ourselves why we needed it. We were computing

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{~d} x}\{\cos x\}\right|_{x=0} & =\lim _{h \rightarrow 0} \frac{\cos h-\cos 0}{h} \\
& =\lim _{h \rightarrow 0} \frac{\cos h-1}{h} \\
& =0
\end{aligned}
$$

Armed with these results we can now build up the derivatives of sine and cosine.

4 We hope you looked this up in in Appendix A.14. Nag.

### 2.8.5 Step 3: $\frac{\mathrm{d}}{\mathrm{d} x}\{\sin x\}$ and $\frac{\mathrm{d}}{\mathrm{d} x}\{\cos x\}$ for General $x$

To proceed to the general derivatives of $\sin x$ and $\cos x$ we are going to use the above two results and a couple of trig identities. Remember the addition formulae ${ }^{5}$

$$
\begin{aligned}
\sin (a+b) & =\sin (a) \cos (b)+\cos (a) \sin (b) \\
\cos (a+b) & =\cos (a) \cos (b)-\sin (a) \sin (b)
\end{aligned}
$$

To compute the derivative of $\sin (x)$ we just start from the definition of the derivative:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x} \sin x & =\lim _{h \rightarrow 0} \frac{\sin (x+h)-\sin x}{h} \\
& =\lim _{h \rightarrow 0} \frac{\sin x \cos h+\cos x \sin h-\sin x}{h} \\
& =\lim _{h \rightarrow 0}\left[\sin x \frac{\cos h-1}{h}+\cos x \frac{\sin h-0}{h}\right] \\
& =\sin x \lim _{h \rightarrow 0} \frac{\cos h-1}{h}+\cos x \underbrace{\lim _{3} \frac{\sin h-0}{h}}_{h \rightarrow 0} \\
& =\sin x \underbrace{\left[\frac{\mathrm{~d}}{\mathrm{~d} x} \cos x\right]_{x=0}}_{=0}+\cos x \underbrace{\left[\frac{\mathrm{~d}}{\mathrm{~d} x} \sin x\right]_{x=0}}_{=1} \\
& =\cos x
\end{aligned}
$$

The computation of the derivative of $\cos x$ is very similar.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x} \cos x & =\lim _{h \rightarrow 0} \frac{\cos (x+h)-\cos x}{h} \\
& =\lim _{h \rightarrow 0} \frac{\cos x \cos h-\sin x \sin h-\cos x}{h} \\
& =\lim _{h \rightarrow 0}\left[\cos x \frac{\cos h-1}{h}-\sin x \frac{\sin h-0}{h}\right] \\
& =\cos x \lim _{h \rightarrow 0} \frac{\cos h-1}{h}-\sin x \lim _{h \rightarrow 0} \frac{\sin h-0}{h} \\
& =\cos x \underbrace{\left.\left[\frac{\mathrm{~d}}{\mathrm{~d} x} \cos x\right]\right]_{x=0}}_{=0}-\sin x \underbrace{\left[\frac{\mathrm{~d}}{\mathrm{~d} x} \sin x\right]_{x=0}}_{=1} \\
& =-\sin x
\end{aligned}
$$

We have now found the derivatives of both $\sin x$ and $\cos x$, provided $x$ is measured in radians.

[^5]
## Lemma 2.8.3

$$
\frac{\mathrm{d}}{\mathrm{~d} x} \sin x=\cos x \quad \frac{\mathrm{~d}}{\mathrm{~d} x} \cos x=-\sin x
$$

The above formulas hold provided $x$ is measured in radians.

These formulae are pretty easy to remember - applying $\frac{\mathrm{d}}{\mathrm{d} x}$ to $\sin x$ and $\cos x$ just exchanges $\sin x$ and $\cos x$, except for the minus sign ${ }^{6}$ in the derivative of $\cos x$.

Remark 2.8.4 Optional - Another derivation of $\frac{\mathrm{d}}{\mathrm{d} x} \cos x=-\sin x$. We remark that, once one knows that $\frac{\mathrm{d}}{\mathrm{d} x} \sin x=\cos x$, it is easy to use it and the trig identity $\cos (x)=\sin \left(\frac{\pi}{2}-x\right)$ to derive $\frac{\mathrm{d}}{\mathrm{d} x} \cos x=-\sin x$. Here is how ${ }^{a}$.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x} \cos x & =\lim _{h \rightarrow 0} \frac{\cos (x+h)-\cos x}{h}=\lim _{h \rightarrow 0} \frac{\sin \left(\frac{\pi}{2}-x-h\right)-\sin \left(\frac{\pi}{2}-x\right)}{h} \\
& =-\lim _{h^{\prime} \rightarrow 0} \frac{\sin \left(x^{\prime}+h^{\prime}\right)-\sin \left(x^{\prime}\right)}{h^{\prime}} \quad \text { with } x^{\prime}=\frac{\pi}{2}-x, h^{\prime}=-h \\
& =-\left.\frac{\mathrm{d}}{\mathrm{~d} x^{\prime}} \sin x^{\prime}\right|_{x^{\prime}=\frac{\pi}{2}-x}=-\cos \left(\frac{\pi}{2}-x\right) \\
& =-\sin x
\end{aligned}
$$

$a \quad$ We thank Serban Raianu for suggesting that we include this.

Note that if $x$ is measured in degrees, then the formulas of Lemma 2.8.3 are wrong. There are similar formulas, but we need the chain rule to build them - that is the subject of the next section. But first we should find the derivatives of the other trig functions.

### 2.8.6 $\sim$ Step 4: the Remaining Trigonometric Functions

It is now an easy matter to get the derivatives of the remaining trigonometric functions using basic trig identities and the quotient rule. Remember ${ }^{7}$ that

$$
\begin{array}{ll}
\tan x=\frac{\sin x}{\cos x} & \cot x=\frac{\cos x}{\sin x}=\frac{1}{\tan x} \\
\csc x=\frac{1}{\sin x} & \sec x=\frac{1}{\cos x}
\end{array}
$$

6 There is a bad pun somewhere in here about sine errors and sign errors.
7 You really should. If you do not then take a quick look at the appropriate appendix.

So, by the quotient rule,

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} x} \tan x=\frac{\mathrm{d}}{\mathrm{~d} x} \frac{\sin x}{\cos x}=\frac{\overbrace{\left(\frac{\mathrm{d}}{\mathrm{~d} x} \sin x\right)}^{\cos x} \cos x-\sin x \overbrace{\left(\frac{\mathrm{~d}}{\mathrm{~d} x} \cos x\right)}^{-\sin x}}{\cos ^{2} x}=\sec ^{2} x \\
& \frac{\mathrm{~d}}{\mathrm{~d} x} \csc x=\frac{\mathrm{d}}{\mathrm{~d} x} \frac{1}{\sin x}=-\frac{\overbrace{\left(\frac{\mathrm{d}}{\mathrm{~d} x} \sin x\right)}^{\cos x}}{\sin ^{2} x} \quad=-\csc x \cot x \\
& \frac{\mathrm{~d}}{\mathrm{~d} x} \sec x=\frac{\mathrm{d}}{\mathrm{~d} x} \frac{1}{\cos x}=-\frac{\overbrace{\left(\frac{\mathrm{d}}{\mathrm{~d} x} \cos x\right)}^{\sin x}}{\cos ^{2} x} \quad=\sec x \tan x \\
& \frac{\mathrm{~d}}{\mathrm{~d} x} \cot x=\frac{\mathrm{d}}{\mathrm{~d} x} \frac{\cos x}{\sin x}=\frac{\overbrace{\left(\frac{\mathrm{d}}{\mathrm{~d} x} \cos x\right)}^{-\sin x} \sin x-\cos x \overbrace{\left(\frac{\mathrm{~d}}{\mathrm{~d} x} \sin x\right)}^{\cos x}}{\sin ^{2} x}=-\csc ^{2} x
\end{aligned}
$$

### 2.8.7 $\rightarrow$ Summary

To summarise all this work, we can write this up as a theorem:

Theorem 2.8.5 Derivatives of trigonometric functions.
The derivatives of $\sin x$ and $\cos x$ are

$$
\frac{\mathrm{d}}{\mathrm{~d} x} \sin x=\cos x \quad \frac{\mathrm{~d}}{\mathrm{~d} x} \cos x=-\sin x
$$

Consequently the derivatives of the other trigonometric functions are

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x} \tan x & =\sec ^{2} x & \frac{\mathrm{~d}}{\mathrm{~d} x} \cot x & =-\csc ^{2} x \\
\frac{\mathrm{~d}}{\mathrm{~d} x} \csc x & =-\csc x \cot x & \frac{\mathrm{~d}}{\mathrm{~d} x} \sec x & =\sec x \tan x
\end{aligned}
$$

Of these 6 derivatives you should really memorise those of sine, cosine and tangent. We certainly expect you to be able to work out those of cotangent, cosecant and secant.

### 2.8.8 Exercises

## Exercises - Stage 1

1. Graph sine and cosine on the same axes, from $x=-2 \pi$ to $x=2 \pi$. Mark the points where $\sin x$ has a horizontal tangent. What do these points correspond to, on the graph of cosine?
2. Graph sine and cosine on the same axes, from $x=-2 \pi$ to $x=2 \pi$. Mark the points where $\sin x$ has a tangent line of maximum (positive) slope. What do these points correspond to, on the graph of cosine?

## Exercises - Stage 2

3. Differentiate $f(x)=\sin x+\cos x+\tan x$.
4. For which values of $x$ does the function $f(x)=\sin x+\cos x$ have a horizontal tangent?
5. Differentiate $f(x)=\sin ^{2} x+\cos ^{2} x$.
6. Differentiate $f(x)=2 \sin x \cos x$.
7. Differentiate $f(x)=e^{x} \cot x$.
8. Differentiate $f(x)=\frac{2 \sin x+3 \tan x}{\cos x+\tan x}$
9. Differentiate $f(x)=\frac{5 \sec x+1}{e^{x}}$.
10. Differentiate $f(x)=\left(e^{x}+\cot x\right)\left(5 x^{6}-\csc x\right)$.
11. Differentiate $f(\theta)=\sin \left(\frac{\pi}{2}-\theta\right)$.
12. Differentiate $f(x)=\sin (-x)+\cos (-x)$.
13. Differentiate $s(\theta)=\frac{\cos \theta+\sin \theta}{\cos \theta-\sin \theta}$.
14. *. Find the values of the constants $a$ and $b$ for which

$$
f(x)= \begin{cases}\cos (x) & x \leq 0 \\ a x+b & x>0\end{cases}
$$

is differentiable everywhere.
15. *. Find the equation of the line tangent to the graph of $y=\cos (x)+2 x$ at $x=\frac{\pi}{2}$.

## Exercises - Stage 3

16. *. Evaluate $\lim _{x \rightarrow 2015}\left(\frac{\cos (x)-\cos (2015)}{x-2015}\right)$.
17. *. Evaluate $\lim _{x \rightarrow \pi / 3}\left(\frac{\cos (x)-1 / 2}{x-\pi / 3}\right)$.
18. *. Evaluate $\lim _{x \rightarrow \pi}\left(\frac{\sin (x)}{x-\pi}\right)$.
19. Show how you can use the quotient rule to find the derivative of tangent, if you already know the derivatives of sine and cosine.
20. *. The derivative of the function

$$
f(x)= \begin{cases}a x+b & \text { for } x<0 \\ \frac{6 \cos x}{2+\sin x+\cos x} & \text { for } x \geq 0\end{cases}
$$

exists for all $x$. Determine the values of the constants $a$ and $b$.
21. *. For which values of $x$ does the derivative of $f(x)=\tan x$ exist?
22. *. For what values of $x$ does the derivative of $\frac{10 \sin (x)}{x^{2}+x-6}$ exist? Explain your answer.
23. *. For what values of $x$ does the derivative of $\frac{x^{2}+6 x+5}{\sin (x)}$ exist? Explain your answer.
24. *. Find the equation of the line tangent to the graph of $y=\tan (x)$ at $x=\frac{\pi}{4}$.
25. *. Find the equation of the line tangent to the graph of $y=\sin (x)+\cos (x)+e^{x}$ at $x=0$.
26. For which values of $x$ does the function $f(x)=e^{x} \sin x$ have a horizontal tangent line?
27. Let

$$
f(x)=\left\{\begin{array}{cc}
\frac{\sin x}{x} & , x \neq 0 \\
1 & , x=0
\end{array}\right.
$$

Find $f^{\prime}(0)$, or show that it does not exist.
28. *. Differentiate the function

$$
h(x)=\sin (|x|)
$$

and give the domain where the derivative exists.
29. *. For the function

$$
f(x)= \begin{cases}0 & x \leq 0 \\ \frac{\sin (x)}{\sqrt{x}} & x>0\end{cases}
$$

which of the following statements is correct?
i $f$ is undefined at $x=0$.
ii $f$ is neither continuous nor differentiable at $x=0$.
iii $f$ is continuous but not differentiable at $x=0$.
iv $f$ is differentiable but not continuous at $x=0$.
$\mathrm{v} f$ is both continuous and differentiable at $x=0$.
30. *. Evaluate $\lim _{x \rightarrow 0} \frac{\sin x^{27}+2 x^{5} e^{x^{99}}}{\sin ^{5} x}$.

## 2.9^ One More Tool - the Chain Rule

We have built up most of the tools that we need to express derivatives of complicated functions in terms of derivatives of simpler known functions. We started by learning how to evaluate

- derivatives of sums, products and quotients
- derivatives of constants and monomials

These tools allow us to compute derivatives of polynomials and rational functions. In the previous sections, we added exponential and trigonometric functions to our list. The final tool we add is called the chain rule. It tells us how to take the derivative of a composition of two functions. That is if we know $f(x)$ and $g(x)$ and their derivatives, then the chain rule tells us the derivative of $f(g(x))$.

Before we get to the statement of the rule, let us look at an example showing how such a composition might arise (in the "real-world").

## Example 2.9.1 Walking towards a campfire.

You are out in the woods after a long day of mathematics and are walking towards your camp fire on a beautiful still night. The heat from the fire means that the air temperature depends on your position. Let your position at time $t$ be $x(t)$. The temperature of the air at position $x$ is $f(x)$. What instantaneous rate of change of temperature do you feel at time $t$ ?


- Because your position at time $t$ is $x=x(t)$, the temperature you feel at time $t$ is $F(t)=f(x(t))$.
- The instantaneous rate of change of temperature that you feel is $F^{\prime}(t)$. We have a complicated function, $F(t)$, constructed by composing two simpler functions, $x(t)$ and $f(x)$.
- We wish to compute the derivative, $F^{\prime}(t)=\frac{\mathrm{d}}{\mathrm{d} t} f(x(t))$, of the complicated function $F(t)$ in terms of the derivatives, $x^{\prime}(t)$ and $f^{\prime}(x)$, of the two simple functions. This is exactly what the chain rule does.



### 2.9.1 $\leadsto$ Statement of the Chain Rule

## Theorem 2.9.2 The chain rule - version 1.

Let $a \in \mathbb{R}$ and let $g(x)$ be a function that is differentiable at $x=a$. Now let $f(u)$ be a function that is differentiable at $u=g(a)$. Then the function $F(x)=f(g(x))$ is differentiable at $x=a$ and

$$
F^{\prime}(a)=f^{\prime}(g(a)) g^{\prime}(a)
$$

Here, as was the case earlier in this chapter, we have been very careful to give the point at which the derivative is evaluated a special name (i.e. a). But of course this evaluation point can really be any point (where the derivative is defined). So it is very common to just call the evaluation point " $x$ " rather than give it a special name like " $a$ ", like this:

## Theorem 2.9.3 The chain rule - version 2.

Let $f$ and $g$ be differentiable functions then

$$
\frac{\mathrm{d}}{\mathrm{~d} x} f(g(x))=f^{\prime}(g(x)) \cdot g^{\prime}(x)
$$

Notice that when we form the composition $f(g(x))$ there is an "outside" function (namely $f(x)$ ) and an "inside" function (namely $g(x)$ ). The chain rule tells us that when we differentiate a composition that we have to differentiate the outside and then multiply by the derivative of the inside.

$$
\frac{\mathrm{d}}{\mathrm{~d} x} f(g(x))=\underbrace{f^{\prime}(g(x))}_{\text {diff outside }} \cdot \underbrace{g^{\prime}(x)}_{\text {diff inside }}
$$

Here is another statement of the chain rule which makes this idea more explicit.

## Theorem 2.9.4 The chain rule - version 3.

Let $y=f(u)$ and $u=g(x)$ be differentiable functions, then

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=\frac{\mathrm{d} y}{\mathrm{~d} u} \cdot \frac{\mathrm{~d} u}{\mathrm{~d} x}
$$

This particular form is easy to remember because it looks like we can just "cancel" the $\mathrm{d} u$ between the two terms.

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=\frac{\mathrm{d} y}{\mathrm{~d} u} \cdot \frac{\mathrm{~d} u}{\mathrm{~d} x}
$$

Of course, $\mathrm{d} u$ is not, by itself, a number or variable ${ }^{1}$ that can be cancelled. But this is still a good memory aid.

The hardest part about applying the chain rule is recognising when the function you are trying to differentiate is really the composition of two simpler functions. This takes a little practice. We can warm up with a couple of simple examples.

Example 2.9.5 Derivative of a power of $\sin x$.
Let $f(u)=u^{5}$ and $g(x)=\sin (x)$. Then set $F(x)=f(g(x))=(\sin (x))^{5}$. To find the derivative of $F(x)$ we can simply apply the chain rule - the pieces of the composition have been laid out for us. Here they are.

$$
f(u)=u^{5} \quad f^{\prime}(u)=5 u^{4}
$$

1 In this context $\mathrm{d} u$ is called a differential. There are ways to understand and manipulate these in calculus but they are beyond the scope of this course.

$$
g(x)=\sin (x) \quad g^{\prime}(x)=\cos x
$$

We now just put them together as the chain rule tells us

$$
\begin{array}{rlr}
\frac{\mathrm{d} F}{\mathrm{~d} x} & =f^{\prime}(g(x)) \cdot g^{\prime}(x) & \\
& =5(g(x))^{4} \cdot \cos (x) & \text { since } f^{\prime}(u)=5 u^{4} \\
& =5(\sin (x))^{4} \cdot \cos (x) &
\end{array}
$$

Notice that it is quite easy to extend this to any power. Set $f(u)=u^{n}$. Then follow the same steps and we arrive at

$$
F(x)=(\sin (x))^{n} \quad \quad F^{\prime}(x)=n(\sin (x))^{n-1} \cos (x)
$$

亿 Example 2.9.5
This example shows one of the ways that the chain rule appears very frequently when we need to differentiate the power of some simpler function. More generally we have the following.

Example 2.9.6 Derivative of a power of a function.
Let $f(u)=u^{n}$ and let $g(x)$ be any differentiable function. Set $F(x)=f(g(x))=g(x)^{n}$. Then

$$
\frac{\mathrm{d} F}{\mathrm{~d} x}=\frac{\mathrm{d}}{\mathrm{~d} x}\left(g(x)^{n}\right)=n g(x)^{n-1} \cdot g^{\prime}(x)
$$

This is precisely the result in Example 2.6.6 and Lemma 2.6.7.

Example 2.9.7 Derivative of $\cos (3 x-2)$.
Let $f(u)=\cos (u)$ and $g(x)=3 x-2$. Find the derivative of

$$
F(x)=f(g(x))=\cos (3 x-2) .
$$

Again we should approach this by first writing down $f$ and $g$ and their derivatives and then putting everything together as the chain rule tells us.

$$
\begin{array}{ll}
f(u)=\cos (u) & f^{\prime}(u)=-\sin (u) \\
g(x)=3 x-2 & g^{\prime}(x)=3
\end{array}
$$

So the chain rule says

$$
\begin{aligned}
F^{\prime}(x) & =f^{\prime}(g(x)) \cdot g^{\prime}(x) \\
& =-\sin (g(x)) \cdot 3 \\
& =-3 \sin (3 x-2)
\end{aligned}
$$

This example shows a second way that the chain rule appears very frequently when we need to differentiate some function of $a x+b$. More generally we have the following.

Example 2.9.8 Derivative of $f(a x+b)$.
Let $a, b \in \mathbb{R}$ and let $f(x)$ be a differentiable function. Set $g(x)=a x+b$. Then

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x} f(a x+b) & =\frac{\mathrm{d}}{\mathrm{~d} x} f(g(x)) \\
& =f^{\prime}(g(x)) \cdot g^{\prime}(x) \\
& =f^{\prime}(a x+b) \cdot a
\end{aligned}
$$

So the derivative of $f(a x+b)$ with respect to $x$ is just $a f^{\prime}(a x+b)$.

The above is a very useful result that follows from the chain rule, so let's make it a corollary to highlight it.

## Corollary 2.9.9

Let $a, b \in \mathbb{R}$ and let $f(x)$ be a differentiable function, then

$$
\frac{\mathrm{d}}{\mathrm{~d} x} f(a x+b)=a f^{\prime}(a x+b) .
$$

Example 2.9.10 2.9.1 continued.
Let us now go back to our motivating campfire example. There we had

$$
\begin{aligned}
f(x) & =\text { temperature at position } x \\
x(t) & =\text { position at time } t \\
F(t) & =f(x(t))=\text { temperature at time } t
\end{aligned}
$$

The chain rule gave

$$
F^{\prime}(t)=f^{\prime}(x(t)) \cdot x^{\prime}(t)
$$

Notice that the units of measurement on both sides of the equation agree - as indeed they must. To see this, let us assume that $t$ is measured in seconds, that $x(t)$ is measured in metres and that $f(x)$ is measured in degrees. Because of this $F(x(t))$ must also be measured in degrees (since it is a temperature).
What about the derivatives? These are rates of change. So

- $F^{\prime}(t)$ has units $\frac{\text { degrees }}{\text { second }}$,
- $f^{\prime}(x)$ has units $\frac{\text { degrees }}{\text { metre }}$, and
- $x^{\prime}(t)$ has units $\frac{\text { metre }}{\text { second }}$.

Hence the product

$$
f^{\prime}(x(t)) \cdot x^{\prime}(t) \text { has units }=\frac{\text { degrees }}{\text { metre }} \cdot \frac{\text { metre }}{\text { second }}=\frac{\text { degrees }}{\text { second }} .
$$

has the same units as $F^{\prime}(t)$. So the units on both sides of the equation agree. Checking that the units on both sides of an equation agree is a good check of consistency, but of course it does not prove that both sides are in fact the same.

Example 2.9.10

### 2.9.2 (Optional) - Derivation of the Chain Rule

First, let's review what our goal is. We have been given a function $g(x)$, that is differentiable at some point $x=a$, and another function $f(u)$, that is differentiable at the point $u=b=g(a)$. We have defined the composite function $F(x)=f(g(x))$ and we wish to show that

$$
F^{\prime}(a)=f^{\prime}(g(a)) \cdot g^{\prime}(a)
$$

Before we can compute $F^{\prime}(a)$, we need to set up some ground work, and in particular the definitions of our given derivatives:

$$
f^{\prime}(b)=\lim _{H \rightarrow 0} \frac{f(b+H)-f(b)}{H} \quad \text { and } \quad g^{\prime}(a)=\lim _{h \rightarrow 0} \frac{g(a+h)-g(a)}{h} .
$$

We are going to use similar manipulation tricks as we did back in the proofs of the arithmetic of derivatives in Section 2.5. Unfortunately, we have already used up the symbols " $F$ " and " $H$ ", so we are going to make use the Greek letters $\gamma, \varphi$.

As was the case in our derivation of the product rule it is convenient to introduce a couple of new functions. Set

$$
\varphi(H)=\frac{f(b+H)-f(b)}{H}
$$

Then we have

$$
\lim _{H \rightarrow 0} \varphi(H)=f^{\prime}(b)=f^{\prime}(g(a)) \quad \text { since } b=g(a)
$$

and we can also write (with a little juggling)

$$
f(b+H)=f(b)+H \varphi(H)
$$

Similarly set

$$
\gamma(h)=\frac{g(a+h)-g(a)}{h}
$$

which gives us

$$
\lim _{h \rightarrow 0} \gamma(h)=g^{\prime}(a) \quad \text { and } \quad g(a+h)=g(a)+h \gamma(h)
$$

Now we can start computing

$$
\begin{aligned}
F^{\prime}(a) & =\lim _{h \rightarrow 0} \frac{F(a+h)-F(a)}{h} \\
& =\lim _{h \rightarrow 0} \frac{f(g(a+h))-f(g(a))}{h}
\end{aligned}
$$

We know that $g(a)=b$ and $g(a+h)=g(a)+h \gamma(h))$, so

$$
\begin{aligned}
F^{\prime}(a) & =\lim _{h \rightarrow 0} \frac{f(g(a)+h \gamma(h))-f(g(a))}{h} \\
& =\lim _{h \rightarrow 0} \frac{f(b+h \gamma(h))-f(b)}{h}
\end{aligned}
$$

Now for the sneaky bit. We can turn $f(b+h \gamma(h))$ into $f(b+H)$ by setting

$$
H=h \gamma(h)
$$

Now notice that as $h \rightarrow 0$ we have

$$
\begin{aligned}
\lim _{h \rightarrow 0} H & =\lim _{h \rightarrow 0} h \cdot \gamma(h) \\
& =\lim _{h \rightarrow 0} h \cdot \lim _{h \rightarrow 0} \gamma(h) \\
& =0 \cdot g^{\prime}(a)=0
\end{aligned}
$$

So as $h \rightarrow 0$ we also have $H \rightarrow 0$.
We now have

$$
\begin{array}{rlr}
F^{\prime}(a) & =\lim _{h \rightarrow 0} \frac{f(b+H)-f(b)}{h} \\
& =\lim _{h \rightarrow 0} \underbrace{\frac{f(b+H)-f(b)}{H}}_{=\varphi(H)} \cdot \underbrace{\frac{H}{h}}_{=\gamma(h)} & \text { if } H=h \gamma(h) \neq 0 \\
& =\lim _{h \rightarrow 0}(\varphi(H) \cdot \gamma(h)) & \\
& =\lim _{h \rightarrow 0} \varphi(H) \cdot \lim _{h \rightarrow 0} \gamma(h) & \text { since } H \rightarrow 0 \text { as } h \rightarrow 0 \\
& =\lim _{H \rightarrow 0} \varphi(H) \cdot \lim _{h \rightarrow 0} \gamma(h) & =f^{\prime}(b) \cdot g^{\prime}(a)
\end{array}
$$

This is exactly the RHS of the chain rule. It is possible to have $H=0$ in the second line above. But that possibility is easy to deal with:

- If $g^{\prime}(a) \neq 0$, then, since $\lim _{h \rightarrow 0} \gamma(h)=g^{\prime}(a), H=h \gamma(h)$ cannot be 0 for small nonzero $h$. Technically, there is an $h_{0}>0$ such that $H=h \gamma(h) \neq 0$ for all $0<|h|<h_{0}$. In taking the limit $h \rightarrow 0$, above, we need only consider $0<|h|<h_{0}$ and so, in this case, the above computation is completely correct.
- If $g^{\prime}(a)=0$, the above computation is still fine provided we exclude all $h$ 's for which $H=h \gamma(h) \neq 0$. When $g^{\prime}(a)=0$, the right hand side, $f^{\prime}(g(a)) \cdot g^{\prime}(a)$, of the chain rule is 0 . So the above computation gives

$$
\lim _{\substack{h \rightarrow 0 \\ \gamma(h) \neq 0}} \frac{f(b+H)-f(b)}{h}=f^{\prime}(g(a)) \cdot g^{\prime}(a)=0
$$

On the other hand, when $H=0$, we have $f(b+H)-f(b)=0$. So

$$
\lim _{\substack{h \rightarrow 0 \\ \gamma(h)=0}} \frac{f(b+H)-f(b)}{h}=0
$$

too. That's all we need.

### 2.9.3 Chain Rule Examples

We'll now use the chain rule to compute some more derivatives.
Example 2.9.11 $\frac{\mathrm{d}}{\mathrm{d} x}(1+3 x)^{75}$.
Find $\frac{\mathrm{d}}{\mathrm{d} x}(1+3 x)^{75}$.
This is a concrete version of Example 2.9.8. We are to find the derivative of a function that is built up by first computing $1+3 x$ and then taking the $75^{\text {th }}$ power of the result. So we set

$$
\begin{array}{rlr}
f(u) & =u^{75} & f^{\prime}(u)=75 u^{74} \\
g(x) & =1+3 x & g^{\prime}(x)=3 \\
F(x) & =f(g(x))=g(x)^{75}=(1+3 x)^{75} &
\end{array}
$$

By the chain rule

$$
\begin{aligned}
F^{\prime}(x) & =f^{\prime}(g(x)) g^{\prime}(x)=75 g(x)^{74} g^{\prime}(x)=75(1+3 x)^{74} \cdot 3 \\
& =225(1+3 x)^{74}
\end{aligned}
$$

Example 2.9.12 $\frac{\mathrm{d}}{\mathrm{d} x} \sin \left(x^{2}\right)$.
Find $\frac{\mathrm{d}}{\mathrm{d} x} \sin \left(x^{2}\right)$.
In this example we are to compute the derivative of sin with a (slightly) complicated argument. So we apply the chain rule with $f$ being sin and $g(x)$ being the complicated argument. That is, we set

$$
\begin{aligned}
f(u) & =\sin u & f^{\prime}(u)=\cos u \\
g(x) & =x^{2} & g^{\prime}(x)=2 x \\
F(x) & =f(g(x))=\sin (g(x))=\sin \left(x^{2}\right) &
\end{aligned}
$$

By the chain rule

$$
\begin{aligned}
F^{\prime}(x) & =f^{\prime}(g(x)) g^{\prime}(x)=\cos (g(x)) g^{\prime}(x)=\cos \left(x^{2}\right) \cdot 2 x \\
& =2 x \cos \left(x^{2}\right)
\end{aligned}
$$

Example $2.9 .13 \frac{\mathrm{~d}}{\mathrm{~d} x} \sqrt[3]{\sin \left(x^{2}\right)}$
Find $\frac{\mathrm{d}}{\mathrm{d} x} \sqrt[3]{\sin \left(x^{2}\right)}$.
In this example we are to compute the derivative of the cube root of a (moderately) complicated argument, namely $\sin \left(x^{2}\right)$. So we apply the chain rule with $f$ being "cube root" and $g(x)$ being the complicated argument. That is, we set

$$
\begin{array}{ll}
f(u)=\sqrt[3]{u}=u^{\frac{1}{3}} & f^{\prime}(u)=\frac{1}{3} u^{-\frac{2}{3}} \\
g(x)=\sin \left(x^{2}\right) & g^{\prime}(x)=2 x \cos \left(x^{2}\right) \\
F(x)=f(g(x))=\sqrt[3]{g(x)}=\sqrt[3]{\sin \left(x^{2}\right)} &
\end{array}
$$

In computing $g^{\prime}(x)$ here, we have already used the chain rule once (in Example 2.9.12). By the chain rule

$$
\begin{aligned}
F^{\prime}(x) & =f^{\prime}(g(x)) y^{\prime}(x)=\frac{1}{3} g(x)^{-\frac{2}{3}} \cdot 2 x \cos \left(x^{2}\right) \\
& =\frac{2 x}{3} \frac{\cos \left(x^{2}\right)}{\left[\sin \left(x^{2}\right)\right]^{\frac{2}{3}}}
\end{aligned}
$$

Example 2.9.14 Derivative of a double-composition.
Find the derivative of $\frac{\mathrm{d}}{\mathrm{d} x} f(g(h(x)))$.
This is very similar to the previous example. Let us set $F(x)=f(g(h(x)))$ with $u=g(h(x))$ then the chain rule tells us

$$
\begin{aligned}
\frac{\mathrm{d} F}{\mathrm{~d} x} & =\frac{\mathrm{d} f}{\mathrm{~d} u} \cdot \frac{\mathrm{~d} u}{\mathrm{~d} x} \\
& =f^{\prime}(g(h(x))) \cdot \frac{\mathrm{d}}{\mathrm{~d} x} g(h(x))
\end{aligned}
$$

We now just apply the chain rule again

$$
=f^{\prime}(g(h(x))) \cdot g^{\prime}(h(x)) \cdot h^{\prime}(x)
$$

Indeed it is not too hard to generalise further (in the manner of Example 2.6.6 to find the derivative of the composition of 4 or more functions (though things start to become tedious to write down):

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x} f_{1}\left(f_{2}\left(f_{3}\left(f_{4}(x)\right)\right)\right) & =f_{1}^{\prime}\left(f_{2}\left(f_{3}\left(f_{4}(x)\right)\right)\right) \cdot \frac{\mathrm{d}}{\mathrm{~d} x} f_{2}\left(f_{3}\left(f_{4}(x)\right)\right) \\
& =f_{1}^{\prime}\left(f_{2}\left(f_{3}\left(f_{4}(x)\right)\right)\right) \cdot f_{2}^{\prime}\left(f_{3}\left(f_{4}(x)\right)\right) \cdot \frac{\mathrm{d}}{\mathrm{~d} x} f_{3}\left(f_{4}(x)\right) \\
& =f_{1}^{\prime}\left(f_{2}\left(f_{3}\left(f_{4}(x)\right)\right)\right) \cdot f_{2}^{\prime}\left(f_{3}\left(f_{4}(x)\right)\right) \cdot f_{3}^{\prime}\left(f_{4}(x)\right) \cdot f_{4}^{\prime}(x)
\end{aligned}
$$

Example 2.9.15 Finding the quotient rule from the chain rule.
We can also use the chain rule to recover Corollary 2.4.6 and from there we can use the product rule to recover the quotient rule.
We want to differentiate $F(x)=\frac{1}{g(x)}$ so set $f(u)=\frac{1}{u}$ and $u=g(x)$. Then the chain rule tells us

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{1}{g(x)}\right\}=\frac{\mathrm{d} F}{\mathrm{~d} x} & =\frac{\mathrm{d} f}{\mathrm{~d} u} \cdot \frac{\mathrm{~d} u}{\mathrm{~d} x} \\
& =\frac{-1}{u^{2}} \cdot g^{\prime}(x) \\
& =-\frac{g^{\prime}(x)}{g(x)^{2}} .
\end{aligned}
$$

Once we know this, a quick application of the product rule will give us the quotient rule.

$$
\begin{array}{ll}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{f(x)}{g(x)}\right\}=\frac{\mathrm{d}}{\mathrm{~d} x}\left\{f(x) \cdot \frac{1}{g(x)}\right\} & \text { use PR } \\
=f^{\prime}(x) \cdot \frac{1}{g(x)}+f(x) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{1}{g(x)}\right\} & \text { use the result from above }
\end{array}
$$

$$
\begin{array}{ll}
=f^{\prime}(x) \cdot \frac{1}{g(x)}-f(x) \cdot \frac{g^{\prime}(x)}{g(x)^{2}} & \text { place over a common denominator } \\
=\frac{f^{\prime}(x) \cdot g(x)-f(x) \cdot g^{\prime}(x)}{g(x)^{2}} &
\end{array}
$$

which is exactly the quotient rule.

Example 2.9.16 A nice messy example.
Compute the following derivative:

$$
\frac{\mathrm{d}}{\mathrm{~d} x} \cos \left(\frac{x^{5} \sqrt{3+x^{6}}}{\left(4+x^{2}\right)^{3}}\right)
$$

This time we are to compute the derivative of cos with a really complicated argument.

- So, to start, we apply the chain rule with $g(x)=\frac{x^{5} \sqrt{3+x^{6}}}{\left(4+x^{2}\right)^{3}}$ being the really complicated argument and $f$ being cos. That is, $f(u)=\cos (u)$. Since $f^{\prime}(u)=-\sin (u)$, the chain rule gives

$$
\frac{\mathrm{d}}{\mathrm{~d} x} \cos \left(\frac{x^{5} \sqrt{3+x^{6}}}{\left(4+x^{2}\right)^{3}}\right)=-\sin \left(\frac{x^{5} \sqrt{3+x^{6}}}{\left(4+x^{2}\right)^{3}}\right) \frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{x^{5} \sqrt{3+x^{6}}}{\left(4+x^{2}\right)^{3}}\right\}
$$

- This reduced our problem to that of computing the derivative of the really complicated argument $\frac{x^{5} \sqrt{3+x^{6}}}{\left(4+x^{2}\right)^{3}}$. We can think of the argument as being built up out of three pieces, namely $x^{5}$, multiplied by $\sqrt{3+x^{6}}$, divided by $\left(4+x^{2}\right)^{3}$, or, equivalently, multiplied by $\left(4+x^{2}\right)^{-3}$. So we may rewrite $\frac{x^{5} \sqrt{3+x^{6}}}{\left(4+x^{2}\right)^{3}}$ as $x^{5}\left(3+x^{6}\right)^{\frac{1}{2}}\left(4+x^{2}\right)^{-3}$, and then apply the product rule to reduce the problem to that of computing the derivatives of the three pieces.
- Here goes (recall Example 2.6.6):

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left[x^{5}\left(3+x^{6}\right)^{\frac{1}{2}}\left(4+x^{2}\right)^{-3}\right]= & \frac{\mathrm{d}}{\mathrm{~d} x}\left[x^{5}\right] \cdot\left(3+x^{6}\right)^{\frac{1}{2}} \cdot\left(4+x^{2}\right)^{-3} \\
& +x^{5} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\left[\left(3+x^{6}\right)^{\frac{1}{2}}\right] \cdot\left(4+x^{2}\right)^{-3} \\
& +x^{5} \cdot\left(3+x^{6}\right)^{\frac{1}{2}} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\left[\left(4+x^{2}\right)^{-3}\right]
\end{aligned}
$$

This has reduced our problem to computing the derivatives of $x^{5}$, which is easy, and of $\left(3+x^{6}\right)^{\frac{1}{2}}$ and $\left(4+x^{2}\right)^{-3}$, both of which can be done by the chain rule.

Doing so,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left[x^{5}\left(3+x^{6}\right)^{\frac{1}{2}}\left(4+x^{2}\right)^{-3}\right]= & \overbrace{\frac{\mathrm{d}}{\mathrm{~d} x}\left[x^{5}\right]}^{5 x^{4}} \cdot\left(3+x^{6}\right)^{\frac{1}{2}} \cdot\left(4+x^{2}\right)^{-3} \\
& +x^{5} \cdot \overbrace{\frac{\mathrm{~d}}{\mathrm{~d} x}\left[\left(3+x^{6}\right)^{\frac{1}{2}}\right]}^{\frac{1}{2}\left(3+x^{6}\right)^{-\frac{1}{2} \cdot 6 x^{5}}} \cdot\left(4+x^{2}\right)^{-3} \\
& +x^{5} \cdot\left(3+x^{6}\right)^{\frac{1}{2}} \cdot \overbrace{\frac{\mathrm{~d}}{\mathrm{~d} x}\left[\left(4+x^{2}\right)^{-3}\right]}^{-3\left(4+x^{2}\right)^{-4} \cdot 2 x}
\end{aligned}
$$

- Now we can clean things up in a sneaky way by observing
- differentiating $x^{5}$, to get $5 x^{4}$, is the same as multiplying $x^{5}$ by $\frac{5}{x}$, and
- differentiating $\left(3+x^{6}\right)^{\frac{1}{2}}$ to get $\frac{1}{2}\left(3+x^{6}\right)^{-\frac{1}{2}} \cdot 6 x^{5}$ is the same as multiplying $\left(3+x^{6}\right)^{\frac{1}{2}}$ by $\frac{3 x^{5}}{3+x^{6}}$, and
- differentiating $\left(4+x^{2}\right)^{-3}$ to get $-3\left(4+x^{2}\right)^{-4} \cdot 2 x$ is the same as multiplying $\left(4+x^{2}\right)^{-3}$ by $-\frac{6 x}{4+x^{2}}$.

Using these sneaky tricks we can write our solution quite neatly:

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} x} \cos \left(\frac{x^{5} \sqrt{3+x^{6}}}{\left(4+x^{2}\right)^{3}}\right) \\
& =-\sin \left(\frac{x^{5} \sqrt{3+x^{6}}}{\left(4+x^{2}\right)^{3}}\right) \frac{x^{5} \sqrt{3+x^{6}}}{\left(4+x^{2}\right)^{3}}\left\{\frac{5}{x}+\frac{3 x^{5}}{3+x^{6}}-\frac{6 x}{4+x^{2}}\right\}
\end{aligned}
$$

- This method of cleaning up the derivative of a messy product is actually something more systematic in disguise - namely logarithmic differentiation. We will come to this later.

Example 2.9.16

### 2.9.4 $\leadsto$ Exercises

## Exercises - Stage 1

1. Suppose the amount of kelp in a harbour depends on the number of urchins. Urchins eat kelp: when there are more urchins, there is less kelp, and when there are fewer urchins, there is more kelp. Suppose further that the
number of urchins in the harbour depends on the number of otters, who find urchins extremely tasty: the more otters there are, the fewer urchins there are.
Let $O, U$, and $K$ be the populations of otters, urchins, and kelp, respectively.
a Is $\frac{\mathrm{d} K}{\mathrm{~d} U}$ positive or negative?
b Is $\frac{d U}{d O}$ positive or negative?
c Is $\frac{\mathrm{d} K}{\mathrm{~d} O}$ positive or negative?
Remark: An urchin barren is an area where unchecked sea urchin grazing has decimated the kelp population, which in turn causes the other species that shelter in the kelp forests to leave. Introducing otters to urchin barrens is one intervention to increase biodiversity. A short video with a more complex view of otters and urchins in Canadian waters is available on YouTube: https://youtu.be/ASJ82wyHisE
2. $\quad$ Suppose $A, B, C, D$ and $E$ are functions describing an interrelated system, with the following signs: $\frac{\mathrm{d} A}{\mathrm{~d} B}>0, \frac{\mathrm{~d} B}{\mathrm{~d} C}>0, \frac{\mathrm{~d} C}{\mathrm{~d} D}<0$, and $\frac{\mathrm{d} D}{\mathrm{~d} E}>0$. Is $\frac{\mathrm{d} A}{\mathrm{~d} E}$ positive or negative?

## Exercises - Stage 2

3. Evaluate the derivative of $f(x)=\cos (5 x+3)$.
4. Evaluate the derivative of $f(x)=\left(x^{2}+2\right)^{5}$.
5. Evaluate the derivative of $T(k)=\left(4 k^{4}+2 k^{2}+1\right)^{17}$.
6. Evaluate the derivative of $f(x)=\sqrt{\frac{x^{2}+1}{x^{2}-1}}$.
7. Evaluate the derivative of $f(x)=e^{\cos \left(x^{2}\right)}$.
8. *. Evaluate $f^{\prime}(2)$ if $f(x)=g(x / h(x)), h(2)=2, h^{\prime}(2)=3, g^{\prime}(1)=4$.
9. *. Find the derivative of $e^{x \cos (x)}$.
10. *. Evaluate $f^{\prime}(x)$ if $f(x)=e^{x^{2}+\cos x}$.
11. *. Evaluate $f^{\prime}(x)$ if $f(x)=\sqrt{\frac{x-1}{x+2}}$.
12. *. Differentiate the function

$$
f(x)=\frac{1}{x^{2}}+\sqrt{x^{2}-1}
$$

and give the domain where the derivative exists.
13. *. Evaluate the derivative of $f(x)=\frac{\sin 5 x}{1+x^{2}}$
14. Evaluate the derivative of $f(x)=\sec \left(e^{2 x+7}\right)$.
15. Find the tangent line to the curve $y=\left(\tan ^{2} x+1\right)\left(\cos ^{2} x\right)$ at the point $x=\frac{\pi}{4}$.
16. The position of a particle at time $t$ is given by $s(t)=e^{t^{3}-7 t^{2}+8 t}$. For which values of $t$ is the velocity of the particle zero?
17. What is the slope of the tangent line to the curve $y=\tan \left(e^{x^{2}}\right)$ at the point $x=1$ ?
18. *. Differentiate $y=e^{4 x} \tan x$. You do not need to simplify your answer.
19. *. Evaluate the derivative of the following function at $x=1: f(x)=\frac{x^{3}}{1+e^{3 x}}$.
20. *. Differentiate $e^{\sin ^{2}(x)}$.
21. *. Compute the derivative of $y=\sin \left(e^{5 x}\right)$
22. *. Find the derivative of $e^{\cos \left(x^{2}\right)}$.
23. *. Compute the derivative of $y=\cos \left(x^{2}+\sqrt{x^{2}+1}\right)$
24. *. Evaluate the derivative.

$$
y=\left(1+x^{2}\right) \cos ^{2} x
$$

25. *. Evaluate the derivative.

$$
y=\frac{e^{3 x}}{1+x^{2}}
$$

26. *. Find $g^{\prime}(2)$ if $g(x)=x^{3} h\left(x^{2}\right)$, where $h(4)=2$ and $h^{\prime}(4)=-2$.
27. *. At what points $(x, y)$ does the curve $y=x e^{-\left(x^{2}-1\right) / 2}$ have a horizontal tangent?
28. A particle starts moving at time $t=1$, and its position thereafter is given by

$$
s(t)=\sin \left(\frac{1}{t}\right)
$$

When is the particle moving in the negative direction?
29. Compute the derivative of $f(x)=\frac{e^{x}}{\cos ^{3}(5 x-7)}$.
30. *. Evaluate $\frac{\mathrm{d}}{\mathrm{d} x}\left\{x e^{2 x} \cos 4 x\right\}$.

## Exercises - Stage 3

31. A particle moves along the Cartesian plane from time $t=-\pi / 2$ to time $t=$ $\pi / 2$. The $x$-coordinate of the particle at time $t$ is given by $x=\cos t$, and the $y$-coordinate is given by $y=\sin t$, so the particle traces a curve in the plane. When does the tangent line to that curve have slope -1 ?
32. *. Show that, for all $x>0, e^{x+x^{2}}>1+x$.
33. We know that $\sin (2 x)=2 \sin x \cos x$. What other trig identity can you derive from this, using differentiation?
34. Evaluate the derivative of $f(x)=\sqrt[3]{\frac{e^{\csc x^{2}}}{\sqrt{x^{3}-9} \tan x}}$. You do not have to simplify your answer.
35. Suppose a particle is moving in the Cartesian plane over time. For any real number $t \geq 0$, the coordinate of the particle at time $t$ is given by $\left(\sin t, \cos ^{2} t\right)$.
a Sketch a graph of the curve traced by the particle in the plane by plotting points, and describe how the particle moves along it over time.
b What is the slope of the curve traced by the particle at time $t=\frac{10 \pi}{3}$ ?

### 2.10』 The Natural Logarithm

The chain rule opens the way to understanding derivatives of more complicated function. Not only compositions of known functions as we have seen the examples of the previous section, but also functions which are defined implicitly.

Consider the logarithm base $e-\log _{e}(x)$ is the power that $e$ must be raised to to give $x$. That is, $\log _{e}(x)$ is defined by

$$
e^{\log _{e} x}=x
$$

i.e. - it is the inverse of the exponential function with base $e$. Since this choice of base works so cleanly and easily with respect to differentiation, this base turns out to be (arguably) the most natural choice for the base of the logarithm. And as we saw in
our whirlwind review of logarithms in Section 2.7, it is easy to use logarithms of one base to compute logarithms with another base:

$$
\log _{q} x=\frac{\log _{e} x}{\log _{e} q}
$$

So we are (relatively) free to choose a base which is convenient for our purposes.
The logarithm with base $e$, is called the "natural logarithm". The "naturalness" of logarithms base $e$ is exactly that this choice of base works very nicely in calculus (and so wider mathematics) in ways that other bases do not ${ }^{1}$. There are several different "standard" notations for the logarithm base $e$;

$$
\log _{e} x=\log x=\ln x
$$

We recommend that you be able to recognise all of these.
In this text we will write the natural logarithm as "log" with no base. The reason for this choice is that base $e$ is the standard choice of base for logarithms in mathematics ${ }^{2}$

The natural logarithm inherits many properties of general logarithms ${ }^{3}$. So, for all $x, y>0$ the following hold:

- $e^{\log x}=x$,
- for any real number $X, \log \left(e^{X}\right)=X$,
- for any $a>1, \log _{a} x=\frac{\log x}{\log a}$ and $\log x=\frac{\log _{a} x}{\log _{a} e}$
- $\log 1=0, \log e=1$
- $\log (x y)=\log x+\log y$
- $\log \left(\frac{x}{y}\right)=\log x-\log y, \log \left(\frac{1}{y}\right)=-\log y$
- $\log \left(x^{X}\right)=X \log x$
- $\lim _{x \rightarrow \infty} \log x=\infty, \lim _{x \rightarrow 0} \log x=-\infty$

And finally we should remember that $\log x$ has domain (i.e. is defined for) $x>0$ and range (i.e. takes all values in) $-\infty<x<\infty$.

1 The interested reader should head to Wikipedia and look up the natural logarithm.
2 In other disciplines other bases are natural; in computer science, since numbers are stored in binary it makes sense to use the binary logarithm - i.e. base 2. While in some sciences and finance, it makes sense to use the decimal logarithm - i.e. base 10.
3 Again take a quick look at the whirlwind review of logarithms in Section 2.7.


To compute the derivative of $\log x$ we could attempt to start with the limit definition of the derivative

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x} \log x & =\lim _{h \rightarrow 0} \frac{\log (x+h)-\log (x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{\log ((x+h) / x)}{h} \\
& =\text { um... }
\end{aligned}
$$

This doesn't look good. But all is not lost - we have the chain rule, and we know that the logarithm satisfies the equation:

$$
x=e^{\log x}
$$

Since both sides of the equation are the same function, both sides of the equation have the same derivative. i.e. we are using ${ }^{4}$

$$
\text { if } f(x)=g(x) \text { for all } x, \text { then } f^{\prime}(x)=g^{\prime}(x)
$$

So now differentiate both sides:

$$
\frac{\mathrm{d}}{\mathrm{~d} x} x=\frac{\mathrm{d}}{\mathrm{~d} x} e^{\log x}
$$

The left-hand side is easy, and the right-hand side we can process using the chain rule with $f(u)=e^{u}$ and $u=\log x$.

$$
\begin{aligned}
1 & =\frac{\mathrm{d} f}{\mathrm{~d} u} \cdot \frac{\mathrm{~d} u}{\mathrm{~d} x} \\
& =e^{u} \cdot \underbrace{\frac{\mathrm{~d}}{\mathrm{~d} x} \log x}_{\text {what we want to compute }}
\end{aligned}
$$



4 Notice that just because the derivatives are the same, doesn't mean the original functions are the same. Both $f(x)=x^{2}$ and $g(x)=x^{2}+3$ have derivative $f^{\prime}(x)=g^{\prime}(x)=2 x$, but $f(x) \neq g(x)$.

Recall that $e^{u}=e^{\log x}=x$, so

$$
1=x \cdot \underbrace{\frac{\mathrm{~d}}{\mathrm{~d} x} \log x}_{\text {now what? }}
$$

We can now just rearrange this equation to make the thing we want the subject:

$$
\frac{\mathrm{d}}{\mathrm{~d} x} \log x=\frac{1}{x}
$$

Thus we have proved:

## Theorem 2.10.1

$$
\frac{\mathrm{d}}{\mathrm{~d} x} \log x=\frac{1}{x}
$$

where $\log x$ is the logarithm base $e$.

Example 2.10.2 The derivative of $\log 3 x$.
Let $f(x)=\log 3 x$. Find $f^{\prime}(x)$.
There are two ways to approach this - we can simplify then differentiate, or differentiate and then simplify. Neither is difficult.

- Simplify and then differentiate:

$$
\begin{array}{rlr}
f(x) & =\log 3 x & \log \text { of a product } \\
& =\log 3+\log x & \\
f^{\prime}(x) & =\frac{\mathrm{d}}{\mathrm{~d} x} \log 3+\frac{\mathrm{d}}{\mathrm{~d} x} \log x & \\
& =\frac{1}{x}
\end{array}
$$

- Differentiation and then simplify:

$$
\begin{aligned}
f^{\prime}(x) & =\frac{\mathrm{d}}{\mathrm{~d} x} \log (3 x) \quad \text { chain rule } \\
& =\frac{1}{3 x} \cdot 3 \\
& =\frac{1}{x}
\end{aligned}
$$

Example 2.10.3 The derivative of $\log c x$.
Notice that we can extend the previous example for any positive constant - not just
3. Let $c>0$ be a constant, then

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x} \log c x & =\frac{\mathrm{d}}{\mathrm{~d} x}(\log c+\log x) \\
& =\frac{1}{x}
\end{aligned}
$$

Example 2.10.4 The derivative of $\log |x|$.
We can push this further still. Let $g(x)=\log |x|$, then ${ }^{a}$

- If $x>0,|x|=x$ and so

$$
g^{\prime}(x)=\frac{\mathrm{d}}{\mathrm{~d} x} \log x=\frac{1}{x}
$$

- If $x<0$ then $|x|=-x$. If $|h|$ is strictly smaller than $|x|$, then we also have that $x+h<0$ and $|x+h|=-(x+h)=|x|-h$. Write $X=|x|$ and $H=-h$. Then, by the definition of the derivative,

$$
\begin{aligned}
g^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{\log |x+h|-\log |x|}{h}=\lim _{h \rightarrow 0} \frac{\log (|x|-h)-\log |x|}{h} \\
& =\lim _{H \rightarrow 0} \frac{\log (X+H)-\log X}{-H}=-\lim _{H \rightarrow 0} \frac{\log (X+H)-\log X}{H} \\
& =-\frac{\mathrm{d}}{\mathrm{~d} X} \log X=-\frac{1}{X}=-\frac{1}{|x|} \\
& =\frac{1}{x}
\end{aligned}
$$

- Since $\log 0$ is undefined, $g^{\prime}(0)$ does not exist.

Putting this together gives:

$$
\frac{\mathrm{d}}{\mathrm{~d} x} \log |x|=\frac{1}{x}
$$

[^6]Example 2.10.5 The derivative of $x^{a}$.
Just after Corollary 2.6.17, we said that we would, in the future, find the derivative of $x^{a}$ for all real numbers. The future is here. Let $x>0$ and $a$ be any real number. Exponentiating both sides of $\log \left(x^{a}\right)=a \log x$ gives us $x^{a}=e^{a \log x}$ and then

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x} x^{a} & =\frac{\mathrm{d}}{\mathrm{~d} x} e^{a \log x}=e^{a \log x} \frac{\mathrm{~d}}{\mathrm{~d} x}(a \log x) \quad \text { by the chain rule } \\
& =\frac{a}{x} e^{a \log x}=\frac{a}{x} x^{a} \\
& =a x^{a-1}
\end{aligned}
$$

$\uparrow$ as expected.

We can extend Theorem 2.10.1 to compute the derivative of logarithms of other bases in a straightforward way. Since for any positive $a \neq 1$ :

$$
\begin{aligned}
\log _{a} x & =\frac{\log x}{\log a}=\frac{1}{\log a} \cdot \log x \\
\frac{\mathrm{~d}}{\mathrm{~d} x} \log _{a} x & =\frac{1}{\log a} \cdot \frac{1}{x}
\end{aligned}
$$

### 2.10.1 $\rightarrow$ Back to $\frac{d}{d x} \mathbf{a}^{\mathrm{x}}$

We can also now finally get around to computing the derivative of $a^{x}$ (which we started to do back in Section 2.7).

$$
\begin{aligned}
f(x) & =a^{x} & \text { take } \log \text { of both sides } \\
\log f(x) & =x \log a & \text { exponentiate both sides base } e \\
f(x) & =e^{x \log a} & \text { chain rule } \\
f^{\prime}(x) & =e^{x \log a} \cdot \log a & \\
& =a^{x} \cdot \log a &
\end{aligned}
$$

Notice that we could have also done the following:

$$
\begin{aligned}
f(x) & =a^{x} & \text { take log of both sides } \\
\log f(x) & =x \log a & \text { differentiate both sides } \\
\frac{\mathrm{d}}{\mathrm{~d} x}(\log f(x)) & =\log a &
\end{aligned}
$$

We then process the left-hand side using the chain rule

$$
f^{\prime}(x) \cdot \frac{1}{f(x)}=\log a
$$

$$
f^{\prime}(x)=f(x) \cdot \log a=a^{x} \cdot \log a
$$

We will see $\frac{\mathrm{d}}{\mathrm{d} x} \log f(x)$ more below in the subsection on "logarithmic differentiation".
To summarise the results above:

## Corollary 2.10.6

$$
\begin{array}{rlr}
\frac{\mathrm{d}}{\mathrm{~d} x} a^{x} & =\log a \cdot a^{x} & \text { for any } a>0 \\
\frac{\mathrm{~d}}{\mathrm{~d} x} \log _{a} x & =\frac{1}{x \cdot \log a} & \text { for any } a>0, a \neq 1
\end{array}
$$

where $\log x$ is the natural logarithm.

Recall that we need the caveat $a \neq 1$ because the logarithm base 1 is not well defined. This is because $1^{x}=1$ for any $x$. We do not need a similar caveat for the derivative of the exponential because we know (recall Example 2.7.1)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x} 1^{x} & =\frac{\mathrm{d}}{\mathrm{~d} x} 1=0 \quad \text { while the above corollary tells us } \\
& =\log 1 \cdot 1^{x}=0 \cdot 1=0 .
\end{aligned}
$$

### 2.10.2 $\leadsto$ Logarithmic Differentiation

We want to go back to some previous slightly messy examples (Examples 2.6.6 and 2.6.18) and now show you how they can be done more easily.

Example 2.10.7 Derivative of a triple product.
Consider again the derivative of the product of 3 functions:

$$
P(x)=F(x) \cdot G(x) \cdot H(x)
$$

Start by taking the logarithm of both sides:

$$
\begin{aligned}
\log P(x) & =\log (F(x) \cdot G(x) \cdot H(x)) \\
& =\log F(x)+\log G(x)+\log H(x)
\end{aligned}
$$

Notice that the product of functions on the right-hand side has become a sum of functions. Differentiating sums is much easier than differentiating products. So when we differentiate we have

$$
\frac{\mathrm{d}}{\mathrm{~d} x} \log P(x)=\frac{\mathrm{d}}{\mathrm{~d} x} \log F(x)+\frac{\mathrm{d}}{\mathrm{~d} x} \log G(x)+\frac{\mathrm{d}}{\mathrm{~d} x} \log H(x)
$$

A quick application of the chain rule shows that $\frac{\mathrm{d}}{\mathrm{d} x} \log f(x)=f^{\prime}(x) / f(x)$ :

$$
\frac{P^{\prime}(x)}{P(x)}=\frac{F^{\prime}(x)}{F(x)}+\frac{G^{\prime}(x)}{G(x)}+\frac{H^{\prime}(x)}{H(x)}
$$

Multiply through by $P(x)=F(x) G(x) H(x)$ :

$$
\begin{aligned}
P^{\prime}(x) & =\left(\frac{F^{\prime}(x)}{F(x)}+\frac{G^{\prime}(x)}{G(x)}+\frac{H^{\prime}(x)}{H(x)}\right) \cdot F(x) G(x) H(x) \\
& =F^{\prime}(x) G(x) H(x)+F(x) G^{\prime}(x) H(x)+F(x) G(x) H^{\prime}(x)
\end{aligned}
$$

which is what found in Example 2.6.6 by repeated application of the product rule. The above generalises quite easily to more than 3 functions.

Example 2.10.7
This same trick of "take a logarithm and then differentiate" - or logarithmic differentiation - will work any time you have a product (or ratio) of functions.

Example 2.10.8 Derivative of a messy product.
Lets use logarithmic differentiation on the function from Example 2.6.18:

$$
f(x)=\frac{(\sqrt{x}-1)(2-x)\left(1-x^{2}\right)}{\sqrt{x}(3+2 x)}
$$

Beware however, that we may only take the logarithm of positive numbers, and this $f(x)$ is often negative. For example, if $1<x<2$, the factor $\left(1-x^{2}\right)$ in the definition of $f(x)$ is negative while all of the other factors are positive, so that $f(x)<0$. None-the-less, we can use logarithmic differentiation to find $f^{\prime}(x)$, by exploiting the observation that $\frac{\mathrm{d}}{\mathrm{d} x} \log |f(x)|=\frac{f^{\prime}(x)}{f(x)}$. (To see this, use the chain rule and Example 2.10.4.) So we take the logarithm of $|f(x)|$ and expand.

$$
\begin{aligned}
\log |f(x)| & =\log \frac{|\sqrt{x}-1||2-x|\left|1-x^{2}\right|}{\sqrt{x}|3+2 x|} \\
& =\log |\sqrt{x}-1|+\log |2-x|+\log \left|1-x^{2}\right|-\underbrace{\log (\sqrt{x})}_{=\frac{1}{2} \log x}-\log |3+2 x|
\end{aligned}
$$

Now we can essentially just differentiate term-by-term:

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} x} \log |f(x)|=\frac{\mathrm{d}}{\mathrm{~d} x}\left(\log |\sqrt{x}-1|+\log |2-x|+\log \left|1-x^{2}\right|\right. \\
& \left.-\frac{1}{2} \log |x|-\log |3+2 x|\right) \\
& \frac{f^{\prime}(x)}{f(x)}=\frac{1 /(2 \sqrt{x})}{\sqrt{x}-1}+\frac{-1}{2-x}+\frac{-2 x}{1-x^{2}}-\frac{1}{2 x}-\frac{2}{3+2 x}
\end{aligned}
$$

$$
\begin{aligned}
f^{\prime}(x) & =f(x) \cdot\left(\frac{1}{2 \sqrt{x}(\sqrt{x}-1)}-\frac{1}{2-x}-\frac{2 x}{1-x^{2}}-\frac{1}{2 x}-\frac{2}{3+2 x}\right) \\
= & \frac{(\sqrt{x}-1)(2-x)\left(1-x^{2}\right)}{\sqrt{x}(3+2 x)} \\
& \quad\left(\frac{1}{2 \sqrt{x}(\sqrt{x}-1)}-\frac{1}{2-x}-\frac{2 x}{1-x^{2}}-\frac{1}{2 x}-\frac{2}{3+2 x}\right)
\end{aligned}
$$

just as we found previously.
Example 2.10.8

### 2.10.3 $\leadsto$ Exercises

Reminder: in these notes, we use $\log x$ to mean $\log _{e} x$, which is also commonly written elsewhere as $\ln x$.

## Exercises - Stage 1

1. The volume in decibels $(\mathrm{dB})$ of a sound is given by the formula:

$$
V(P)=10 \log _{10}\left(\frac{P}{S}\right)
$$

where $P$ is the intensity of the sound and $S$ is the intensity of a standard baseline sound. (That is: $S$ is some constant.)
How much noise will ten speakers make, if each speaker produces 3 dB of noise? What about one hundred speakers?
2. An investment of $\$ 1000$ with an interest rate of $5 \%$ per year grows to

$$
A(t)=1000 e^{t / 20}
$$

dollars after $t$ years. When will the investment double?
3. Which of the following expressions, if any, is equivalent to $\log \left(\cos ^{2} x\right)$ ?
(a) $2 \log (\cos x)$
(b) $2 \log |\cos x|$
(c) $\log ^{2}(\cos x)$
(d) $\left.\quad \log \left(\cos x^{2}\right)\right)$

## Exercises - Stage 2

4. Differentiate $f(x)=\log (10 x)$.
5. Differentiate $f(x)=\log \left(x^{2}\right)$.
6. Differentiate $f(x)=\log \left(x^{2}+x\right)$.
7. $\quad$ Differentiate $f(x)=\log _{10} x$.
8. *. Find the derivative of $y=\frac{\log x}{x^{3}}$.
9. Evaluate $\frac{\mathrm{d}}{\mathrm{d} \theta} \log (\sec \theta)$.
10. Differentiate the function $f(x)=e^{\cos (\log x)}$.
11. *. Evaluate the derivative. You do not need to simplify your answer.

$$
y=\log \left(x^{2}+\sqrt{x^{4}+1}\right)
$$

12. *. Differentiate $\sqrt{-\log (\cos x)}$.
13. *. Calculate and simplify the derivative of $\log \left(x+\sqrt{x^{2}+4}\right)$.
14. *. Evaluate the derivative of $g(x)=\log \left(e^{x^{2}}+\sqrt{1+x^{4}}\right)$.
15. *. Evaluate the derivative of the following function at $x=1: g(x)=$ $\log \left(\frac{2 x-1}{2 x+1}\right)$.
16. Evaluate the derivative of the function $f(x)=\log \left(\sqrt{\frac{\left(x^{2}+5\right)^{3}}{x^{4}+10}}\right)$.
17. Evaluate $f^{\prime}(2)$ if $f(x)=\log (g(x h(x))), h(2)=2, h^{\prime}(2)=3, g(4)=3, g^{\prime}(4)=$ 5.
18. *. Differentiate the function

$$
g(x)=\pi^{x}+x^{\pi}
$$

19. Differentiate $f(x)=x^{x}$.
20. *. Find $f^{\prime}(x)$ if $f(x)=x^{x}+\log _{10} x$.
21. Differentiate $f(x)=\sqrt[4]{\frac{\left(x^{4}+12\right)\left(x^{4}-x^{2}+2\right)}{x^{3}}}$.
22. Differentiate $f(x)=(x+1)\left(x^{2}+1\right)^{2}\left(x^{3}+1\right)^{3}\left(x^{4}+1\right)^{4}\left(x^{5}+1\right)^{5}$.
23. Differentiate $f(x)=\left(\frac{5 x^{2}+10 x+15}{3 x^{4}+4 x^{3}+5}\right)\left(\frac{1}{10(x+1)}\right)$.
24. *. Let $f(x)=(\cos x)^{\sin x}$, with domain $0<x<\frac{\pi}{2}$. Find $f^{\prime}(x)$.
25. *. Find the derivative of $(\tan (x))^{x}$, when $x$ is in the interval $(0, \pi / 2)$.
26. *. Find $f^{\prime}(x)$ if $f(x)=\left(x^{2}+1\right)^{\left(x^{2}+1\right)}$
27. *. Differentiate $f(x)=\left(x^{2}+1\right)^{\sin (x)}$.
28. *. Let $f(x)=x^{\cos ^{3}(x)}$, with domain $(0, \infty)$. Find $f^{\prime}(x)$.
29. *. Differentiate $f(x)=(3+\sin (x))^{x^{2}-3}$.

## Exercises - Stage 3

30. Let $f(x)$ and $g(x)$ be differentiable functions, with $f(x)>0$. Evaluate $\frac{\mathrm{d}}{\mathrm{d} x}\left\{[f(x)]^{g(x)}\right\}$.
31. Let $f(x)$ be a function whose range includes only positive numbers. Show that the curves $y=f(x)$ and $y=\log (f(x))$ have horizontal tangent lines at the same values of $x$.

### 2.11ム Implicit Differentiation

### 2.11.1 Implicit Differentiation

Implicit differentiation is a simple trick that is used to compute derivatives of functions either

- when you don't know an explicit formula for the function, but you know an equation that the function obeys or
- even when you have an explicit, but complicated, formula for the function, and the function obeys a simple equation.

The trick is just to differentiate both sides of the equation and then solve for the derivative we are seeking. In fact we have already done this, without using the name "implicit differentiation", when we found the derivative of $\log x$ in the previous section. There we knew that the function $f(x)=\log x$ satisfied the equation $e^{f(x)}=x$ for all $x$.

That is, the functions $e^{f(x)}$ and $x$ are in fact the same function and so have the same derivative. So we had

$$
\frac{\mathrm{d}}{\mathrm{~d} x} e^{f(x)}=\frac{\mathrm{d}}{\mathrm{~d} x} x=1
$$

We then used the chain rule to get $\frac{\mathrm{d}}{\mathrm{d} x} e^{f(x)}=e^{f(x)} f^{\prime}(x)$, which told us that $f^{\prime}(x)$ obeys the equation

$$
\begin{aligned}
e^{f(x)} f^{\prime}(x) & =1 \\
f^{\prime}(x) & =e^{-f(x)}=e^{-\log x}=\frac{1}{x}
\end{aligned}
$$

The typical way to get used to implicit differentiation is to play with problems involving tangent lines to curves. So here are a few examples finding the equations of tangent lines to curves. Recall, from Theorem 2.3.4, that, in general, the tangent line to the curve $y=f(x)$ at $\left(x_{0}, y_{0}\right)$ is $y=f\left(x_{0}\right)+f^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)=y_{0}+f^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)$.

Example 2.11.1 Finding a tangent line using implicit differentiation.
Find the equation of the tangent line to $y=y^{3}+x y+x^{3}$ at $x=1$.
This is a very standard sounding example, but made a little complicated by the fact that the curve is given by a cubic equation - which means we cannot solve directly for $y$ in terms of $x$ or vice versa. So we really do need implicit differentiation.

- First notice that when $x=1$ the equation, $y=y^{3}+x y+x^{3}$, of the curve simplifies to $y=y^{3}+y+1$ or $y^{3}=-1$, which we can solve ${ }^{a}: y=-1$. So we know that the curve passes through $(1,-1)$ when $x=1$.
- Now, to find the slope of the tangent line at $(1,-1)$, pretend that our curve is $y=f(x)$ so that $f(x)$ obeys

$$
f(x)=f(x)^{3}+x f(x)+x^{3}
$$

for all $x$. Differentiating both sides gives

$$
f^{\prime}(x)=3 f(x)^{2} f^{\prime}(x)+f(x)+x f^{\prime}(x)+3 x^{2}
$$

- At this point we could isolate for $f^{\prime}(x)$ and write it in terms of $f(x)$ and $x$, but since we only want answers when $x=1$, let us substitute in $x=1$ and $f(1)=-1$ (since the curve passes through $(1,-1)$ ) and clean things up before doing anything else.
- Subbing in $x=1, f(1)=-1$ gives

$$
f^{\prime}(1)=3 f^{\prime}(1)-1+f^{\prime}(1)+3 \quad \text { and so } f^{\prime}(1)=-\frac{2}{3}
$$

- The equation of the tangent line is

$$
y=y_{0}+f^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)=-1-\frac{2}{3}(x-1)=-\frac{2}{3} x-\frac{1}{3}
$$

We can further clean up the equation of the line to write it as $2 x+3 y=-1$.
$a$ This type of luck rarely happens in the "real world". But it happens remarkably frequently in textbooks, problem sets and tests.

In the previous example we replace $y$ by $f(x)$ in the middle of the computation. We don't actually have to do this. When we are writing out our solution we can remember that $y$ is a function of $x$. So we can start with

$$
y=y^{3}+x y+x^{3}
$$

and differentiate remembering that $y \equiv y(x)$

$$
y^{\prime}=3 y^{2} y^{\prime}+x y^{\prime}+y+3 x^{2}
$$

And now substitute $x=1, y=-1$ to get

$$
\begin{array}{ll}
y^{\prime}(1)=3 \cdot y^{\prime}(1)+y^{\prime}(1)-1+3 & \text { and so } \\
y^{\prime}(1)=-\frac{2}{3} &
\end{array}
$$

The next one is at the same time a bit easier (because it is a quadratic) and a bit harder (because we are asked for the tangent at a general point on the curve, not a specific one).

## Example 2.11.2 Another tangent line through implicit differentiation.

Let $\left(x_{0}, y_{0}\right)$ be a point on the ellipse $3 x^{2}+5 y^{2}=7$. Find the equation for the tangent lines when $x=1$ and $y$ is positive. Then find an equation for the tangent line to the ellipse at a general point $\left(x_{0}, y_{0}\right)$.
Since we are not given an specific point $x_{0}$ we are going to have to be careful with the second half of this question.

- When $x=1$ the equation simplifies to

$$
\begin{aligned}
3+5 y^{2} & =7 \\
5 y^{2} & =4 \\
y & = \pm \frac{2}{\sqrt{5}} .
\end{aligned}
$$

We are only interested in positive $y$, so our point on the curve is $(1,2 / \sqrt{5})$.

- Now we use implicit differentiation to find $\frac{\mathrm{d} y}{\mathrm{~d} x}$ at this point. First we pretend that we have solved the curve explicitly, for some interval of $x$ 's, as $y=f(x)$. The
equation becomes

$$
\begin{array}{rlr}
3 x^{2}+5 f(x)^{2} & =7 & \text { now differentiate } \\
6 x+10 f(x) f^{\prime}(x) & =0 & \\
f^{\prime}(x) & =-\frac{3 x}{5 f(x)} &
\end{array}
$$

- When $x=1, y=2 / \sqrt{5}$ this becomes

$$
f^{\prime}(1)=-\frac{3}{5 \cdot 2 / \sqrt{5}}=-\frac{3}{2 \sqrt{5}}
$$

So the tangent line passes through $(1,2 / \sqrt{5})$ and has slope $-\frac{3}{2 \sqrt{5}}$. Hence the tangent line has equation

$$
\begin{array}{rlr}
y & =y_{0}+f^{\prime}\left(x_{0}\right)\left(x-x_{0}\right) & \\
& =\frac{2}{\sqrt{5}}-\frac{3}{2 \sqrt{5}}(x-1) & \\
& =\frac{7-3 x}{2 \sqrt{5}} &
\end{array}
$$

Now we should go back and do the same but for a general point on the curve $\left(x_{0}, y_{0}\right)$ :

- A good first step here is to sketch the curve. Since this is an ellipse, it is pretty straight-forward.


- Notice that there are two points on the ellipse - the extreme right and left points $\left(x_{0}, y_{0}\right)= \pm\left(\sqrt{\frac{7}{3}}, 0\right)$ - at which the tangent line is vertical. In those two cases, the tangent line is just $x=x_{0}$.
- Since this is a quadratic for $y$, we could solve it explicitly to get

$$
y= \pm \sqrt{\frac{7-3 x^{2}}{5}}
$$

and choose the positive or negative branch as appropriate. Then we could differentiate to find the slope and put things together to get the tangent line.
But even in this relatively easy case, it is computationally cleaner, and hence less vulnerable to mechanical errors, to use implicit differentiation. So that's what we'll do.

- Now we could again "pretend" that we have solved the equation for the ellipse for $y=f(x)$ near $\left(x_{0}, y_{0}\right)$, but let's not do that. Instead (as we did just before this example) just remember that when we differentiate $y$ is really a function of $x$. So starting from

$$
\begin{array}{rlr}
3 x^{2}+5 y^{2} & =7 & \text { differentiating gives } \\
6 x+5 \cdot 2 y \cdot y^{\prime} & =0 &
\end{array}
$$

We can then solve this for $y^{\prime}$ :

$$
y^{\prime}=-\frac{3 x}{5 y}
$$

where $y^{\prime}$ and $y$ are both functions of $x$.

- Hence at the point $\left(x_{0}, y_{0}\right)$ we have

$$
\left.y^{\prime}\right|_{\left(x_{0}, y_{0}\right)}=-\frac{3 x_{0}}{5 y_{0}}
$$

This is the slope of the tangent line at $\left(x_{0}, y_{0}\right)$ and so its equation is

$$
\begin{aligned}
y & =y_{0}+y^{\prime} \cdot\left(x-x_{0}\right) \\
& =y_{0}-\frac{3 x_{0}}{5 y_{0}}\left(x-x_{0}\right)
\end{aligned}
$$

We can simplify this by multiplying through by $5 y_{0}$ to get

$$
5 y_{0} y=5 y_{0}^{2}-3 x_{0} x+3 x_{0}^{2}
$$

We can clean this up more by moving all the terms that contain $x$ or $y$ to the left-hand side and everything else to the right:

$$
3 x_{0} x+5 y_{0} y=3 x_{0}^{2}+5 y_{0}^{2}
$$

But there is one more thing we can do, our original equation is $3 x^{2}+5 y^{2}=7$ for all points on the curve, so we know that $3 x_{0}^{2}+5 y_{0}^{2}=7$. This cleans up the right-hand side.

$$
3 x_{0} x+5 y_{0} y=7
$$

- In deriving this formula for the tangent line at $\left(x_{0}, y_{0}\right)$ we have assumed that $y_{0} \neq 0$. But in fact the final answer happens to also work when $y_{0}=0$ (which means $x_{0}= \pm \sqrt{\frac{7}{3}}$, so that the tangent line is $x=x_{0}$.
We can also check that our answer for general $\left(x_{0}, y_{0}\right)$ reduces to our answer for $x_{0}=1$.
- When $x_{0}=1$ we worked out that $y_{0}=2 / \sqrt{5}$.
- Plugging this into our answer above gives

$$
\begin{aligned}
3 x_{0} x+5 y_{0} y & =7 & \text { sub in }\left(x_{0}, y_{0}\right)=(1,2 / \sqrt{5}): \\
3 x+5 \frac{2}{\sqrt{5}} y & =7 & \text { clean up a little } \\
3 x+2 \sqrt{5} y & =7 &
\end{aligned}
$$

as required.
Example 2.11.2

Example 2.11.3 A more involved example.
At which points does the curve $x^{2}-x y+y^{2}=3$ cross the $x$-axis? Are the tangent lines to the curve at those points parallel?
This is a 2 part question - first the $x$-intercepts and then we need to examine tangent lines.

- Finding where the curve crosses the $x$-axis is straight forward. It does so when $y=0$. This means $x$ satisfies

$$
x^{2}-x \cdot 0+0^{2}=3 \quad \text { so } x= \pm \sqrt{3}
$$

So the curve crosses the $x$-axis at two points $( \pm \sqrt{3}, 0)$.

- Now we need to find the tangent lines at those points. But we don't actually need the lines, just their slopes. Again we can pretend that near one of those points the curve is $y=f(x)$. Applying $\frac{\mathrm{d}}{\mathrm{d} x}$ to both sides of $x^{2}-x f(x)+f(x)^{2}=3$ gives

$$
2 x-f(x)-x f^{\prime}(x)+2 f(x) f^{\prime}(x)=0
$$

etc etc.

- But let us stop "pretending". Just make sure we remember that $y$ is a function of $x$ when we differentiate:

$$
\begin{aligned}
x^{2}-x y+y^{2} & =3 \quad \text { start with the curve, and differentiate } \\
2 x-x y^{\prime}-y+2 y y^{\prime} & =0
\end{aligned}
$$

Now substitute in the first point, $x=+\sqrt{3}, y=0$ :

$$
\begin{aligned}
2 \sqrt{3}-\sqrt{3} y^{\prime}+0 & =0 \\
y^{\prime} & =2
\end{aligned}
$$

And now do the second point $x=-\sqrt{3}, y=0$ :

$$
\begin{aligned}
-2 \sqrt{3}+\sqrt{3} y^{\prime}+0 & =0 \\
y^{\prime} & =2
\end{aligned}
$$

Thus the slope is the same at $x=\sqrt{3}$ and $x=-\sqrt{3}$ and the tangent lines are parallel.


Okay - let's get away from curves and do something a little different.
Example 2.11.4 A rough game of baseball.
You are standing at the origin. At time zero a pitcher throws a ball at your head ${ }^{a}$.


Figure 2.11.5
The position of the (centre of the) ball at time $t$ is $x(t)=d-v t$, where $d$ is the distance from your head to the pitcher's mound and $v$ is the ball's velocity. Your eye sees the ball filling ${ }^{b}$ an angle $2 \theta(t)$ with

$$
\sin (\theta(t))=\frac{r}{d-v t}
$$

where $r$ is the radius of the baseball. The question is "How fast is $\theta$ growing at time $t$ ?" That is, what is $\frac{\mathrm{d} \theta}{\mathrm{d} t}$ ?

- We don't know (yet) how to solve this equation to find $\theta(t)$ explicitly. So we use implicit differentiation.
- To do so we apply $\frac{\mathrm{d}}{\mathrm{d} t}$ to both sides of our equation. This gives

$$
\cos (\theta(t)) \cdot \theta^{\prime}(t)=\frac{r v}{(d-v t)^{2}}
$$

- Then we solve for $\theta^{\prime}(t)$ :

$$
\theta^{\prime}(t)=\frac{r v}{(d-v t)^{2} \cos (\theta(t))}
$$

- As is often the case, when using implicit differentiation, this answer is not very satisfying because it contains $\theta(t)$, for which we still do not have an explicit formula. However in this case we can get an explicit formula for $\cos (\theta(t))$, without having an explicit formula for $\theta(t)$, just by looking at the right-angled triangle in Figure 2.11.5, above.
- The hypotenuse of that triangle has length $d-v t$. By Pythagoras, the length of the side of the triangle adjacent of the angle $\theta(t)$ is $\sqrt{(d-v t)^{2}-r^{2}}$. So

$$
\cos (\theta(t))=\frac{\sqrt{(d-v t)^{2}-r^{2}}}{d-v t}
$$

and

$$
\theta^{\prime}(t)=\frac{r v}{(d-v t) \sqrt{(d-v t)^{2}-r^{2}}}
$$

$a$ It seems that it is not a friendly game today.
$b$ This is the "visual angle" or "angular size".

Okay - just one more tangent-to-the-curve example and then we'll go on to something different.

Example 2.11.6 The astroid (no that is not a typo).
Let $\left(x_{0}, y_{0}\right)$ be a point on the astroid ${ }^{a b}$

$$
x^{\frac{2}{3}}+y^{\frac{2}{3}}=1 .
$$

Find an equation for the tangent line to the astroid at $\left(x_{0}, y_{0}\right)$.

- As was the case in examples above we can rewrite the equation of the astroid near $\left(x_{0}, y_{0}\right)$ in the form $y=f(x)$, with an explicit $f(x)$, by solving the equation $x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$. But again, it is computationally cleaner, and hence less vulnerable to mechanical errors, to use implicit differentiation. So that's what we'll do.
- First up, since $\left(x_{0}, y_{0}\right)$ lies on the curve, it satisfies

$$
x_{0}^{\frac{2}{3}}+y_{0}^{\frac{2}{3}}=1
$$

- Now, no pretending that $y=f(x)$, this time - just make sure we remember when we differentiate that $y$ changes with $x$.

$$
x^{\frac{2}{3}}+y^{\frac{2}{3}}=1
$$

Start with the curve, and then differentiate

$$
\frac{2}{3} x^{-\frac{1}{3}}+\frac{2}{3} y^{-\frac{1}{3}} y^{\prime}=0
$$

- Note the derivative of $x^{\frac{2}{3}}$, namely $\frac{2}{3} x^{-\frac{1}{3}}$, and the derivative of $y^{\frac{2}{3}}$, namely $\frac{2}{3} y^{-\frac{1}{3}} y^{\prime}$, are defined only when $x \neq 0$ and $y \neq 0$. We are interested in the case that $x=x_{0}$ and $y=y_{0}$. So we better assume that $x_{0} \neq 0$ and $y_{0} \neq 0$. Probably something weird happens when $x_{0}=0$ or $y_{0}=0$. We'll come back to this shortly.
- To continue on, we set $x=x_{0}, y=y_{0}$ in the equation above, and then solve for $y^{\prime}$ :

$$
\frac{2}{3} x_{0}^{-\frac{1}{3}}+\frac{2}{3} y_{0}^{-\frac{1}{3}} y^{\prime}(x)=0 \Longrightarrow y^{\prime}\left(x_{0}\right)=-\left(\frac{y_{0}}{x_{0}}\right)^{\frac{1}{3}}
$$

This is the slope of the tangent line and its equation is

$$
y=y_{0}+f^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)=y_{0}-\left(\frac{y_{0}}{x_{0}}\right)^{\frac{1}{3}}\left(x-x_{0}\right)
$$

Now let's think a little bit about what the tangent line slope of $-\sqrt[3]{\frac{y_{0}}{x_{0}}}$ tells us about the astroid.

- First, as a preliminary observation, note that since $x_{0}^{\frac{2}{3}} \geq 0$ and $y_{0}^{\frac{2}{3}} \geq 0$ the equation $x_{0}^{\frac{2}{3}}+y_{0}^{\frac{2}{3}}=1$ of the astroid forces $0 \leq x_{0}^{\frac{2}{3}}, y_{0}^{\frac{2}{3}} \leq 1$ and hence $-1 \leq$ $x_{0}, y_{0} \leq 1$.
- For all $x_{0}, y_{0}>0$ the slope $-\sqrt[3]{\frac{y_{0}}{x_{0}}}<0$. So at all points on the astroid that are in the first quadrant, the tangent line has negative slope, i.e. is "leaning backwards".
- As $x_{0}$ tends to zero, $y_{0}$ tends to $\pm 1$ and the tangent line slope tends to infinity. So at points on the astroid near $(0, \pm 1)$, the tangent line is almost vertical.
- As $y_{0}$ tends to zero, $x_{0}$ tends to $\pm 1$ and the tangent line slope tends to zero. So at points on the astroid near $( \pm 1,0)$, the tangent line is almost horizontal.

Here is a figure illustrating all this.



Sure enough, as we speculated earlier, something weird does happen to the astroid when $x_{0}$ or $y_{0}$ is zero. The astroid is pointy, and does not have a tangent there.
$a \quad$ Here is where is the astroid comes from. Imagine two circles, one of radius $1 / 4$ and one of radius 1. Paint a red dot on the smaller circle. Then imagine the smaller circle rolling around the inside of the larger circle. The curve traced by the red dot is our astroid. Google "astroid" (be careful about the spelling) to find animations showing this.
$b$ The astroid was first discussed by Johann Bernoulli in 1691-92. It also appears in the work of Leibniz.

Example 2.11.6

### 2.11.2 Exercises

## Exercises - Stage 1

1. If we implicitly differentiate $x^{2}+y^{2}=1$, we get the equation $2 x+2 y y^{\prime}=0$. In the step where we differentiate $y^{2}$ to obtain $2 y y^{\prime}$, which rule(s) below are we using? (a) power rule, (b) chain rule, (c) quotient rule, (d) derivatives of exponential functions
2. Using the picture below, estimate $\frac{\mathrm{d} y}{\mathrm{~d} x}$ at the three points where the curve crosses the $y$-axis.


Remark: for this curve, one value of $x$ may correspond to multiple values of $y$. So, we cannot express this curve as $y=f(x)$ for any function $x$. This is one typical situation where we might use implicit differentiation.
3. Consider the unit circle, formed by all points $(x, y)$ that satisfy $x^{2}+y^{2}=1$.

a Is there a function $f(x)$ so that $y=f(x)$ completely describes the unit circle? That is, so that the points $(x, y)$ that make the equation $y=f(x)$ true are exactly the same points that make the equation $x^{2}+y^{2}=1$ true?
b Is there a function $f^{\prime}(x)$ so that $y=f^{\prime}(x)$ completely describes the slope of the unit circle? That is, so that for every point $(x, y)$ on the unit circle, the slope of the tangent line to the circle at that point is given by $f^{\prime}(x)$ ?
c Use implicit differentiation to find an expression for $\frac{\mathrm{d} y}{\mathrm{~d} x}$. Simplify until the expression is a function in terms of $x$ only (not $y$ ), or explain why this is impossible.

## Exercises - Stage 2

4. *. Find $\frac{\mathrm{d} y}{\mathrm{~d} x}$ if $x y+e^{x}+e^{y}=1$.
5. *. If $e^{y}=x y^{2}+x$, compute $\frac{\mathrm{d} y}{\mathrm{~d} x}$.
6. *. If $x^{2} \tan (\pi y / 4)+2 x \log (y)=16$, then find $y^{\prime}$ at the points where $y=1$.
7. *. If $x^{3}+y^{4}=\cos \left(x^{2}+y\right)$ compute $\frac{\mathrm{d} y}{\mathrm{~d} x}$.
8. *. If $x^{2} e^{y}+4 x \cos (y)=5$, then find $y^{\prime}$ at the points where $y=0$.
9. *. If $x^{2}+y^{2}=\sin (x+y)$ compute $\frac{\mathrm{d} y}{\mathrm{~d} x}$.
10. *. If $x^{2} \cos (y)+2 x e^{y}=8$, then find $y^{\prime}$ at the points where $y=0$.
11. At what points on the ellipse $x^{2}+3 y^{2}=1$ is the tangent line parallel to the line $y=x ?$
12. *. For the curve defined by the equation $\sqrt{x y}=x^{2} y-2$, find the slope of the tangent line at the point $(1,4)$.
13. *. If $x^{2} y^{2}+x \sin (y)=4$, find $\frac{\mathrm{d} y}{\mathrm{~d} x}$.

## Exercises - Stage 3

14. *. If $x^{2}+(y+1) e^{y}=5$, then find $y^{\prime}$ at the points where $y=0$.
15. For what values of $x$ do the circle $x^{2}+y^{2}=1$ and the ellipse $x^{2}+3 y^{2}=1$ have parallel tangent lines?

### 2.12」 Inverse Trigonometric Functions

One very useful application of implicit differentiation is to find the derivatives of inverse functions. We have already used this approach to find the derivative of the inverse of the exponential function - the logarithm.

We are now going to consider the problem of finding the derivatives of the inverses of trigonometric functions. Now is a very good time to go back and reread Section 0.6 on inverse functions - especially Definition 0.6.4. Most importantly, given a function $f(x)$, its inverse function $f^{-1}(x)$ only exists, with domain $D$, when $f(x)$ passes the "horizontal line test", which says that for each $Y$ in $D$ the horizontal line $y=Y$ intersects the graph $y=f(x)$ exactly once. (That is, $f(x)$ is a one-to-one function.)

Let us start by playing with the sine function and determine how to restrict the domain of $\sin x$ so that its inverse function exists.

Example 2.12.1 The inverse of $\sin x$.
Let $y=f(x)=\sin (x)$. We would like to find the inverse function which takes $y$ and returns to us a unique $x$-value so that $\sin (x)=y$.


- For each real number $Y$, the number of $x$-values that obey $\sin (x)=Y$, is exactly the number of times the horizontal straight line $y=Y$ intersects the graph of $\sin (x)$.
- When $-1 \leq Y \leq 1$, the horizontal line intersects the graph infinitely many times. This is illustrated in the figure above by the line $y=0.3$.
- On the other hand, when $Y<-1$ or $Y>1$, the line $y=Y$ never intersects the graph of $\sin (x)$. This is illustrated in the figure above by the line $y=-1.2$.

This is exactly the horizontal line test and it shows that the sine function is not one-to-one.
Now consider the function

$$
y=\sin (x) \quad \text { with domain }-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}
$$

This function has the same formula but the domain has been restricted so that, as we'll now show, the horizontal line test is satisfied.


As we saw above when $|Y|>1$ no $x$ obeys $\sin (x)=Y$ and, for each $-1 \leq Y \leq 1$, the line $y=Y$ (illustrated in the figure above with $y=0.3$ ) crosses the curve $y=\sin (x)$
infinitely many times, so that there are infinitely many $x$ 's that obey $f(x)=\sin x=Y$. However exactly one of those crossings (the dot in the figure) has $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. That is, for each $-1 \leq Y \leq 1$, there is exactly one $x$, call it $X$, that obeys both

$$
\sin X=Y
$$

and

$$
-\frac{\pi}{2} \leq X \leq \frac{\pi}{2}
$$

That unique value, $X$, is typically denoted $\arcsin (Y)$. That is

$$
\sin (\arcsin (Y))=Y \quad \text { and } \quad-\frac{\pi}{2} \leq \arcsin (Y) \leq \frac{\pi}{2}
$$

Renaming $Y \rightarrow x$, the inverse function $\arcsin (x)$ is defined for all $-1 \leq x \leq 1$ and is determined by the equation

## Equation 2.12.2

$$
\sin (\arcsin (x))=x \quad \text { and } \quad-\frac{\pi}{2} \leq \arcsin (x) \leq \frac{\pi}{2}
$$

Note that many texts will use $\sin ^{-1}(x)$ to denote arcsine, however we will use $\arcsin (x)$ since we feel that it is clearer ${ }^{a}$; the reader should recognise both.
$a \quad$ The main reason being that people frequently confuse $\sin ^{-1}(x)$ with $(\sin (x))^{-1}=\frac{1}{\sin x}$. We feel that prepending the prefix "arc" less likely to lead to such confusion. The notations asin $(x)$ and $\operatorname{Arcsin}(x)$ are also used.

Example 2.12.2

Example 2.12.3 More on the inverse of $\sin x$.
Since

$$
\sin \frac{\pi}{2}=1 \quad \sin \frac{\pi}{6}=\frac{1}{2}
$$

and $-\frac{\pi}{2} \leq \frac{\pi}{6}, \frac{\pi}{2} \leq \frac{\pi}{2}$, we have

$$
\arcsin 1=\frac{\pi}{2} \quad \arcsin \frac{1}{2}=\frac{\pi}{6}
$$

Even though

$$
\sin (2 \pi)=0
$$

it is not true that $\arcsin 0=2 \pi$, and it is not true that $\arcsin (\sin (2 \pi))=2 \pi$, because $2 \pi$ is not between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. More generally

$$
\arcsin (\sin (x))=\text { the unique angle } \theta \text { between }-\frac{\pi}{2} \text { and } \frac{\pi}{2} \text { obeying } \sin \theta=\sin x
$$

$$
=x \quad \text { if and only if }-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}
$$

So, for example, $\arcsin \left(\sin \left(\frac{11 \pi}{16}\right)\right)$ cannot be $\frac{11 \pi}{16}$ because $\frac{11 \pi}{16}$ is bigger than $\frac{\pi}{2}$. So how do we find the correct answer? Start by sketching the graph of $\sin (x)$.


It looks like the graph of $\sin x$ is symmetric about $x=\frac{\pi}{2}$. The mathematical way to say that "the graph of $\sin x$ is symmetric about $x=\frac{\pi}{2}$ " is " $\sin \left(\frac{\pi}{2}-\theta\right)=\sin \left(\frac{\pi}{2}+\theta\right)$ " for all $\theta$. That is indeed true ${ }^{a}$.
Now $\frac{11 \pi}{16}=\frac{\pi}{2}+\frac{3 \pi}{16}$ so

$$
\sin \left(\frac{11 \pi}{16}\right)=\sin \left(\frac{\pi}{2}+\frac{3 \pi}{16}\right)=\sin \left(\frac{\pi}{2}-\frac{3 \pi}{16}\right)=\sin \left(\frac{5 \pi}{16}\right)
$$

and, since $\frac{5 \pi}{16}$ is indeed between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$,

$$
\arcsin \left(\sin \left(\frac{11 \pi}{16}\right)\right)=\frac{5 \pi}{16} \quad\left(\text { and not } \frac{11 \pi}{16}\right)
$$

$a$ Indeed both are equal to $\cos \theta$. You can see this by playing with the trig identities in Appendix A.8.


### 2.12.1 Derivatives of Inverse Trig Functions

Now that we have explored the arcsine function we are ready to find its derivative. Lets call

$$
\arcsin (x)=\theta(x)
$$

so that the derivative we are seeking is $\frac{\mathrm{d} \theta}{\mathrm{d} x}$. The above equation is (after taking sine of both sides) equivalent to

$$
\sin (\theta)=x
$$

Now differentiate this using implicit differentiation (we just have to remember that $\theta$ varies with $x$ and use the chain rule carefully):

$$
\begin{aligned}
\cos (\theta) \cdot \frac{\mathrm{d} \theta}{\mathrm{~d} x} & =1 \\
\frac{\mathrm{~d} \theta}{\mathrm{~d} x} & =\frac{1}{\cos (\theta)} \\
\frac{\mathrm{d}}{\mathrm{~d} x} \arcsin x & =\frac{1}{\cos (\arcsin x)}
\end{aligned}
$$

$$
\text { substitute } \theta=\arcsin x
$$

This doesn't look too bad, but it's not really very satisfying because the right hand side is expressed in terms of $\arcsin (x)$ and we do not have an explicit formula for $\arcsin (x)$.

However even without an explicit formula for $\arcsin (x)$, it is a simple matter to get an explicit formula for $\cos (\arcsin (x))$, which is all we need. Just draw a right-angled triangle with one angle being $\arcsin (x)$. This is done in the figure below ${ }^{1}$.


Since $\sin (\theta)=x$ (see 2.12.2), we have made the side opposite the angle $\theta$ of length $x$ and the hypotenuse of length 1 . Then, by Pythagoras, the side adjacent to $\theta$ has length $\sqrt{1-x^{2}}$ and so

$$
\cos (\arcsin (x))=\cos (\theta)=\sqrt{1-x^{2}}
$$

which in turn gives us the answer we need:

$$
\frac{\mathrm{d}}{\mathrm{~d} x} \arcsin (x)=\frac{1}{\sqrt{1-x^{2}}}
$$

The definitions for arccos, arctan and arccot are developed in the same way. Here are the graphs that are used.


1 The figure is drawn for the case that $0 \leq \arcsin (x) \leq \frac{\pi}{2}$. Virtually the same argument works for the case $-\frac{\pi}{2} \leq \arcsin (x) \leq 0$



The definitions for the remaining two inverse trigonometric functions may also be developed in the same way ${ }^{23}$. But it's a little easier to use

$$
\csc x=\frac{1}{\sin x} \quad \sec x=\frac{1}{\cos x}
$$

## Definition 2.12.4

$\arcsin x$ is defined for $|x| \leq 1$. It is the unique number obeying

$$
\sin (\arcsin (x))=x \quad \text { and } \quad-\frac{\pi}{2} \leq \arcsin (x) \leq \frac{\pi}{2}
$$

$\arccos x$ is defined for $|x| \leq 1$. It is the unique number obeying

$$
\cos (\arccos (x))=x \quad \text { and } \quad 0 \leq \arccos (x) \leq \pi
$$

2 In fact, there are two different widely used definitions of arcsec $x$. Under our definition, below, $\theta=\operatorname{arcsec} x$ takes values in $0 \leq \theta \leq \pi$. Some people, perfectly legitimately, define $\theta=\operatorname{arcsec} x$ to take values in the union of $0 \leq \theta<\frac{\pi}{2}$ and $\pi \leq \theta<\frac{3 \pi}{2}$. Our definition is sometimes called the "trigonometry friendly" definition. The definition itself has the advantage of simplicity. The other definition is sometimes called the "calculus friendly" definition. It eliminates some absolute values and hence simplifies some computations. Similarly there are two different widely used definitions of $\operatorname{arccsc} x$.
3 One could also define $\operatorname{arccot}(x)=\arctan (1 / x)$ with $\operatorname{arccot}(0)=\frac{\pi}{2}$. We have chosen not to do so, because the definition we have chosen is both continuous and standard.
$\arctan x$ is defined for all $x \in \mathbb{R}$. It is the unique number obeying

$$
\tan (\arctan (x))=x \quad \text { and } \quad-\frac{\pi}{2}<\arctan (x)<\frac{\pi}{2}
$$

$\operatorname{arccsc} x=\arcsin \frac{1}{x}$ is defined for $|x| \geq 1$. It is the unique number obeying

$$
\csc (\operatorname{arccsc}(x))=x \quad \text { and } \quad-\frac{\pi}{2} \leq \operatorname{arccsc}(x) \leq \frac{\pi}{2}
$$

Because $\csc (0)$ is undefined, $\operatorname{arccsc}(x)$ never takes the value 0.
$\operatorname{arcsec} x=\arccos \frac{1}{x}$ is defined for $|x| \geq 1$. It is the unique number obeying

$$
\sec (\operatorname{arcsec}(x))=x \quad \text { and } \quad 0 \leq \operatorname{arcsec}(x) \leq \pi
$$

Because $\sec (\pi / 2)$ is undefined, $\operatorname{arcsec}(x)$ never takes the value $\pi / 2$.
$\operatorname{arccot} x$ is defined for all $x \in \mathbb{R}$. It is the unique number obeying

$$
\cot (\operatorname{arccot}(x))=x \quad \text { and } \quad 0<\operatorname{arccot}(x)<\pi
$$

Example 2.12.5 The derivative of $\arccos x$.
To find the derivative of arccos we can follow the same steps:

- Write $\arccos (x)=\theta(x)$ so that $\cos \theta=x$ and the desired derivative is $\frac{\mathrm{d} \theta}{\mathrm{d} x}$.
- Differentiate implicitly, remembering that $\theta$ is a function of $x$ :

$$
\begin{aligned}
-\sin \theta \frac{\mathrm{d} \theta}{\mathrm{~d} x} & =1 \\
\frac{\mathrm{~d} \theta}{\mathrm{~d} x} & =-\frac{1}{\sin \theta} \\
\frac{\mathrm{~d}}{\mathrm{~d} x} \arccos x & =-\frac{1}{\sin (\arccos x)} .
\end{aligned}
$$

- To simplify this expression, again draw the relevant triangle

from which we see

$$
\sin (\arccos x)=\sin \theta=\sqrt{1-x^{2}}
$$

- Thus

$$
\frac{\mathrm{d}}{\mathrm{~d} x} \arccos x=-\frac{1}{\sqrt{1-x^{2}}} .
$$

Example 2.12.6 The derivative of $\arctan x$.
Very similar steps give the derivative of $\arctan x$ :

- Start with $\theta=\arctan x$, so $\tan \theta=x$.
- Differentiate implicitly:

$$
\begin{aligned}
\sec ^{2} \theta \frac{\mathrm{~d} \theta}{\mathrm{~d} x} & =1 \\
\frac{\mathrm{~d} \theta}{\mathrm{~d} x} & =\frac{1}{\sec ^{2} \theta}=\cos ^{2} \theta \\
\frac{\mathrm{~d}}{\mathrm{~d} x} \arctan x & =\cos ^{2}(\arctan x)
\end{aligned}
$$

- To simplify this expression, we draw the relevant triangle

from which we see

$$
\cos ^{2}(\arctan x)=\cos ^{2} \theta=\frac{1}{1+x^{2}}
$$

- Thus

$$
\frac{\mathrm{d}}{\mathrm{~d} x} \arctan x=\frac{1}{1+x^{2}}
$$

An almost identical computation gives the derivative of arccot $x$ :

- Start with $\theta=\operatorname{arccot} x$, so $\cot \theta=x$.
- Differentiate implicitly:

$$
-\csc ^{2} \theta \frac{\mathrm{~d} \theta}{\mathrm{~d} x}=1
$$

$$
\frac{\mathrm{d}}{\mathrm{~d} x} \operatorname{arccot} x=\frac{\mathrm{d} \theta}{\mathrm{~d} x}=-\frac{1}{\csc ^{2} \theta}=-\sin ^{2} \theta=-\frac{1}{1+x^{2}}
$$

from the triangle


Example 2.12.6

Example 2.12.7 The derivative of $\operatorname{arccsc} x$.
To find the derivative of arccsc we can use its definition and the chain rule.

$$
\begin{array}{rlrl}
\theta & =\operatorname{arccsc} x & \text { take cosecant of both sides } \\
\csc \theta & =x & \text { but } \csc \theta=\frac{1}{\sin \theta}, \text { so flip both sides } \\
\sin \theta & =\frac{1}{x} & & \text { now take arcsine of both sides } \\
\theta & =\arcsin \left(\frac{1}{x}\right) &
\end{array}
$$

Now just differentiate, carefully using the chain rule :

$$
\begin{aligned}
\frac{\mathrm{d} \theta}{\mathrm{~d} x} & =\frac{\mathrm{d}}{\mathrm{~d} x} \arcsin \left(\frac{1}{x}\right) \\
& =\frac{1}{\sqrt{1-x^{-2}}} \cdot \frac{-1}{x^{2}}
\end{aligned}
$$

To simplify further we will factor $x^{-2}$ out of the square root. We need to be a little careful doing that. Take another look at examples 1.5.6 and 1.5.7 and the discussion between them before proceeding.

$$
\begin{aligned}
& =\frac{1}{\sqrt{x^{-2}\left(x^{2}-1\right)}} \cdot \frac{-1}{x^{2}} \\
& =\frac{1}{\left|x^{-1}\right| \cdot \sqrt{x^{2}-1}} \cdot \frac{-1}{x^{2}} \quad \text { note that } x^{2} \cdot\left|x^{-1}\right|=|x| . \\
& =-\frac{1}{|x| \sqrt{x^{2}-1}}
\end{aligned}
$$

In the same way we can find the derivative of the remaining inverse trig function. We
just use its definition, a derivative we already know and the chain rule.

$$
\frac{\mathrm{d}}{\mathrm{~d} x} \operatorname{arcsec}(x)=\frac{\mathrm{d}}{\mathrm{~d} x} \arccos \left(\frac{1}{x}\right)=-\frac{1}{\sqrt{1-\frac{1}{x^{2}}}} \cdot\left(-\frac{1}{x^{2}}\right)=\frac{1}{|x| \sqrt{x^{2}-1}}
$$

By way of summary, we have

## Theorem 2.12.8

The derivatives of the inverse trigonometric functions are

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x} \arcsin (x) & =\frac{1}{\sqrt{1-x^{2}}} & \frac{\mathrm{~d}}{\mathrm{~d} x} \operatorname{arccsc}(x) & =-\frac{1}{|x| \sqrt{x^{2}-1}} \\
\frac{\mathrm{~d}}{\mathrm{~d} x} \arccos (x) & =-\frac{1}{\sqrt{1-x^{2}}} & \frac{\mathrm{~d}}{\mathrm{~d} x} \operatorname{arcsec}(x) & =\frac{1}{|x| \sqrt{x^{2}-1}} \\
\frac{\mathrm{~d}}{\mathrm{~d} x} \arctan (x) & =\frac{1}{1+x^{2}} & \frac{\mathrm{~d}}{\mathrm{~d} x} \operatorname{arccot}(x) & =-\frac{1}{1+x^{2}}
\end{aligned}
$$

### 2.12.2 $\leadsto$ Exercises

## Exercises - Stage 1

1. Give the domains of each of the following functions.
(a) $f(x)=\arcsin (\cos x)$
(b) $g(x)=\operatorname{arccsc}(\cos x)$
(c) $h(x)=\sin (\arccos x)$
2. A particle starts moving at time $t=10$, and it bobs up and down, so that its height at time $t \geq 10$ is given by $\cos t$. True or false: the particle has height 1 at time $t=\arccos (1)$.
3. The curve $y=f(x)$ is shown below, for some function $f$. Restrict $f$ to the largest possible interval containing 0 over which it is one-to-one, and sketch the curve $y=f^{-1}(x)$.

4. Let $a$ be some constant. Where does the curve $y=a x+\cos x$ have a horizontal tangent line?
5. Define a function $f(x)=\arcsin x+\operatorname{arccsc} x$. What is the domain of $f(x)$ ? Where is $f(x)$ differentiable?

## Exercises - Stage 2

6. Differentiate $f(x)=\arcsin \left(\frac{x}{3}\right)$. What is the domain of $f(x)$ ?
7. Differentiate $f(t)=\frac{\arccos t}{t^{2}-1}$. What is the domain of $f(t)$ ?
8. Differentiate $f(x)=\operatorname{arcsec}\left(-x^{2}-2\right)$. What is the domain of $f(x)$ ?
9. Differentiate $f(x)=\frac{1}{a} \arctan \left(\frac{x}{a}\right)$, where $a$ is a nonzero constant. What is the domain of $f(x)$ ?
10. Differentiate $f(x)=x \arcsin x+\sqrt{1-x^{2}}$. What is the domain of $f(x)$ ?
11. For which values of $x$ is the tangent line to $y=\arctan \left(x^{2}\right)$ horizontal?
12. Evaluate $\frac{\mathrm{d}}{\mathrm{d} x}\{\arcsin x+\arccos x\}$.
13. *. Find the derivative of $y=\arcsin \left(\frac{1}{x}\right)$.
14. *. Find the derivative of $y=\arctan \left(\frac{1}{x}\right)$.
15. *. Calculate and simplify the derivative of $\left(1+x^{2}\right) \arctan x$.
16. Show that $\frac{\mathrm{d}}{\mathrm{d} x}\{\sin (\arctan (x))\}=\left(x^{2}+1\right)^{-3 / 2}$.
17. Show that $\frac{\mathrm{d}}{\mathrm{d} x}\{\cot (\arcsin (x))\}=\frac{-1}{x^{2} \sqrt{1-x^{2}}}$.
18. *. Determine all points on the curve $y=\arcsin x$ where the tangent line is parallel to the line $y=2 x+9$.
19. For which values of $x$ does the function $f(x)=\arctan (\csc x)$ have a horizontal tangent line?

## Exercises - Stage 3

20. *. Let $f(x)=x+\cos x$, and let $g(y)=f^{-1}(y)$ be the inverse function. Determine $g^{\prime}(y)$.
21. *. $f(x)=2 x-\sin (x)$ is one-to-one. Find $\left(f^{-1}\right)^{\prime}(\pi-1)$.
22. *. $f(x)=e^{x}+x$ is one-to-one. Find $\left(f^{-1}\right)^{\prime}(e+1)$.
23. Differentiate $f(x)=[\sin x+2]^{\operatorname{arcsec} x}$. What is the domain of this function?
24. Suppose you can't remember whether the derivative of arcsine is $\frac{1}{\sqrt{1-x^{2}}}$ or $\frac{1}{\sqrt{x^{2}-1}}$. Describe how the domain of arcsine suggests that one of these is wrong.
25. Evaluate $\lim _{x \rightarrow 1}\left((x-1)^{-1}\left(\arctan x-\frac{\pi}{4}\right)\right)$.
26. Suppose $f(2 x+1)=\frac{5 x-9}{3 x+7}$. Evaluate $f^{-1}(7)$.
27. Suppose $f^{-1}(4 x-1)=\frac{2 x+3}{x+1}$. Evaluate $f(0)$.
28. Suppose a curve is defined implicitly by

$$
\arcsin (x+2 y)=x^{2}+y^{2}
$$

Solve for $y^{\prime}$ in terms of $x$ and $y$.

### 2.13^ The Mean Value Theorem

Consider the following situation. Two towns are separated by a 120 km long stretch of road. The police in town $A$ observe a car leaving at 1 pm . Their colleagues in town $B$ see the car arriving at 2 pm . After a quick phone call between the two police stations, the driver is issued a fine for going $120 \mathrm{~km} / \mathrm{h}$ at some time between 1 pm and 2 pm . It is intuitively obvious ${ }^{1}$ that, because his average velocity was $120 \mathrm{~km} / \mathrm{h}$, the driver must have been going at least $120 \mathrm{~km} / \mathrm{h}$ at some point. From a knowledge of the average velocity of the car, we are able to deduce something about an instantaneous velocity ${ }^{2}$

Let us turn this around a little bit. Consider the premise of a 90 s action film ${ }^{3}$ a bus must travel at a velocity of no less than $80 \mathrm{~km} / \mathrm{h}$. Being a bus, it is unable to go faster than, say, $120 \mathrm{~km} / \mathrm{h}$. The film runs for about 2 hours, and let's assume that there is about thirty minutes of non-action - so the bus' velocity is constrained between 80 and $120 \mathrm{~km} / \mathrm{h}$ for a total of 1.5 hours.

It is again obvious that the bus must have travelled between $80 \times 1.5=120$ and $120 \times 1.5=180 \mathrm{~km}$ during the film. This time, from a knowledge of the instantaneous rate of change of position - the derivative - throughout a 90 minute time interval, we are able to say something about the net change of position during the 90 minutes.

In both of these scenarios we are making use of a piece of mathematics called the Mean Value Theorem. It says that, under appropriate hypotheses, the average rate of change $\frac{f(b)-f(a)}{b-a}$ of a function over an interval is achieved exactly by the instantaneous rate of change $f^{\prime}(c)$ of the function at some ${ }^{4}$ (unknown) point $a \leq c \leq b$. We shall get to a precise statement in Theorem 2.13.5. We start working up to it by first considering the special case in which $f(a)=f(b)$.

1 Unfortunately there are many obvious things that are decidedly false - for example "There are more rational numbers than integers." or "Viking helmets had horns on them".
2 Recall that speed and velocity are not the same. Velocity specifies the direction of motion as well as the rate of change. Objects moving along a straight line have velocities that are positive or negative numbers indicating which direction the object is moving along the line. Speed, on the other hand, is the distance travelled per unit time and is always a non-negative number - it is the absolute value of velocity.
3 The sequel won a Raspberry award for "Worst remake or sequel".
4 There must be at least one such point - there could be more than one - but there cannot be zero.

### 2.13.1 $\quad$ Rolle's Theorem

## Theorem 2.13.1 Rolle's theorem.

Let $a$ and $b$ be real numbers with $a<b$. And let $f$ be a function so that

- $f(x)$ is continuous on the closed interval $a \leq x \leq b$,
- $f(x)$ is differentiable on the open interval $a<x<b$, and
- $f(a)=f(b)$
then there is a $c$ strictly between $a$ and $b$, i.e. obeying $a<c<b$, such that

$$
f^{\prime}(c)=0 .
$$

Again, like the two scenarios above, this theorem says something intuitively obvious. Consider - if you throw a ball straight up into the air and then catch it, at some time in between the throw and the catch it must be stationary. Translating this into mathematical statements, let $s(t)$ be the height of the ball above the ground in metres, and let $t$ be time from the moment the ball is thrown in seconds. Then we have

$$
\begin{array}{ll}
s(0)=1 & \text { we release the ball at about hip-height } \\
s(4)=1 & \text { we catch the ball } 4 s \text { later at hip-height }
\end{array}
$$

Then we know there is some time in between - say at $t=c$ - when the ball is stationary (in this case when the ball is at the top of its trajectory). I.e.

$$
v(c)=s^{\prime}(c)=0
$$

Rolle's theorem guarantees that for any differentiable function that starts and ends at the same value, there will always be at least one point between the start and finish where the derivative is zero.


There can, of course, also be multiple points at which the derivative is zero - but there must always be at least one. Notice, however, the theorem ${ }^{5}$
does not tell us the value of $c$, just that such a $c$ must exist.

5 Notice this is very similar to the intermediate value theorem (see Theorem 1.6.12)

Example 2.13.2 A simple application of Rolle's theorem.
We can use Rolle's theorem to show that the function

$$
f(x)=\sin (x)-\cos (x)
$$

has a point $c$ between 0 and $\frac{3 \pi}{2}$ so that $f^{\prime}(c)=0$.
To apply Rolle's theorem we first have to show the function satisfies the conditions of the theorem on the interval $\left[0, \frac{3 \pi}{2}\right]$.

- Since $f$ is the sum of sine and cosine it is continuous on the interval and also differentiable on the interval.
- Further, since

$$
\begin{aligned}
f(0) & =\sin 0-\cos 0=0-1=-1 \\
f\left(\frac{3 \pi}{2}\right) & =\sin \frac{3 \pi}{2}-\cos \frac{3 \pi}{2}=-1-0=-1
\end{aligned}
$$

we can now apply Rolle's theorem.

- Rolle's theorem implies that there must be a point $c \in(0,3 \pi / 2)$ so that $f^{\prime}(c)=0$.

While Rolle's theorem doesn't tell us the value of $c$, this example is sufficiently simple that we can find it directly.

$$
\begin{aligned}
f^{\prime}(x) & =\cos x+\sin x \\
f^{\prime}(c) & =\cos c+\sin c=0 \\
\sin c & =-\cos c \\
\tan c & =-1
\end{aligned} \quad \text { rearrange }
$$

Hence $c=\frac{3 \pi}{4}$. We have sketched the function and the relevant points below.


A more substantial application of Rolle's theorem (in conjunction with the intermediate value theorem - Theorem 1.6.12) is to show that a function does not have
multiple zeros in an interval:
Example 2.13.3 Showing an equation has exactly 1 solution.
Show that the equation $2 x-1=\sin (x)$ has exactly 1 solution.

- Start with a rough sketch of each side of the equation


This seems like it should be true.

- Notice that the problem we are trying to solve is equivalent to showing that the function

$$
f(x)=2 x-1-\sin (x)
$$

has only a single zero.

- Since $f(x)$ is the sum of a polynomial and a sine function, it is continuous and differentiable everywhere. Thus we can apply both the IVT and Rolle's theorem.
- Notice that $f(0)=-1$ and $f(2)=4-1-\sin (2)=3-\sin (2) \geq 2$, since $-1 \leq \sin (2) \leq 1$. Thus by the IVT we know there is at least one number $c$ between 0 and 2 so that $f(c)=0$.
- But our job is only half done - this shows that there is at least one zero, but it does not tell us there is no more than one. We have more work to do, and Rolle's theorem is the tool we need.
- Consider what would happen if $f(x)$ is zero in 2 places - that is, there are numbers $a, b$ so that $f(a)=f(b)=0$.
- Since $f(x)$ is differentiable everywhere and $f(a)=f(b)=0$, we can apply Rolle's theorem.
- Hence we know there is a point $c$ between $a$ and $b$ so that $f^{\prime}(c)=0$.
- But let us examine $f^{\prime}(x)$ :

$$
f^{\prime}(x)=2-\cos x
$$

Since $-1 \leq \cos x \leq 1$, we must have that $f^{\prime}(x) \geq 1$.

- But this contradicts Rolle's theorem which tells us there must be a point at which the derivative is zero.

Thus the function cannot be zero at two different places - otherwise we'd have a contradiction.

We can actually nail down the value of $c$ using the bisection approach we used in example 1.6.15. If we do this carefully we find that $c \approx 0.887862 \ldots$

Example 2.13.3

### 2.13.2 $\leadsto$ Back to the MVT

Rolle's theorem can be generalised in a straight-forward way; given a differentiable function $f(x)$ we can still say something about $\frac{\mathrm{d} f}{\mathrm{~d} x}$, even if $f(a) \neq f(b)$. Consider the following sketch:


Figure 2.13.4

All we have done is tilt the picture so that $f(a)<f(b)$. Now we can no longer guarantee that there will be a point on the graph where the tangent line is horizontal, but there will be a point where the tangent line is parallel to the secant joining ( $a, f(a)$ ) to $(b, f(b))$.

To state this in terms of our first scenario back at the beginning of this section, suppose that you are driving along the $x$-axis. At time $t=a$ you are at $x=f(a)$ and at time $t=b$ you are at $x=f(b)$. For simplicity, let's suppose that $b>a$ and $f(b) \geq f(a)$, just like in the above sketch. Then during the time interval in question you travelled a net distance of $f(b)-f(a)$. It took you $b-a$ units of time to travel that distance, so your average velocity was $\frac{f(b)-f(a)}{b-a}$. You may very well have been going faster than $\frac{f(b)-f(a)}{b-a}$ part of the time and slower than $\frac{f(b)-f(a)}{b-a}$ part of the time. But it is reasonable to guess that at some time between $t=a$ and $t=b$ your instantaneous velocity was exactly $\frac{f(b)-f(a)}{b-a}$. The mean value theorem says that, under reasonable assumptions about $f$, this is indeed the case.

Theorem 2.13.5 The mean value theorem.
Let $a$ and $b$ be real numbers with $a<b$. And let $f(x)$ be a function so that

- $f(x)$ is continuous on the closed interval $a \leq x \leq b$, and
- $f(x)$ is differentiable on the open interval $a<x<b$
then there is a $c \in(a, b)$, such that

$$
f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}
$$

which we can also express as

$$
f(b)=f(a)+f^{\prime}(c)(b-a)
$$

Let us start to explore the mean value theorem - which is very frequently known as the MVT. A simple example to start:

## Example 2.13.6 Apply MVT to a polynomial.

Consider the polynomial $f(x)=3 x^{2}-4 x+2$ on $[-1,1]$.

- Since $f$ is a polynomial it is continuous on the interval and also differentiable on the interval. Hence we can apply the MVT.
- The MVT tells us that there is a point $c \in(-1,1)$ so that

$$
f^{\prime}(c)=\frac{f(1)-f(-1)}{1-(-1)}=\frac{1-9}{2}=-4
$$

This example is sufficiently simple that we can find the point $c$ and the corresponding tangent line:

- The derivative is

$$
f^{\prime}(x)=6 x-4
$$

- So we need to solve $f^{\prime}(c)=-4$ :

$$
6 c-4=-4
$$

which tells us that $c=0$.

- The tangent line has slope -4 and passes through $(0, f(0))=(0,2)$, and so is given by

$$
y=-4 x+2
$$

- The secant line joining $(-1, f(-1))=(-1,9)$ to $(1, f(1))=(1,1)$ is just

$$
y=5-4 x
$$

- Here is a sketch of the curve and the two lines:


Example 2.13.6

Example 2.13.7 MVT, speed and distance.
We can return to our initial car-motivated examples. Say you are driving along a straight road in a car that can go at most $80 \mathrm{~km} / \mathrm{h}$. How far can you go in 2 hours? the answer is easy, but we can also solve this using MVT.

- Let $s(t)$ be the position of the car in $k m$ at time $t$ measured in hours.
- Then $s(0)=0$ and $s(2)=q$, where $q$ is the quantity that we need to bound.
- We are told that $\left|s^{\prime}(t)\right| \leq 80$, or equivalently

$$
-80 \leq s^{\prime}(t) \leq 80
$$

- By the MVT there is some $c$ between 0 and 2 so that

$$
s^{\prime}(c)=\frac{q-0}{2}=\frac{q}{2}
$$

- Now since $-80 \leq s^{\prime}(c) \leq 80$ we must have $-80 \leq q / 2 \leq 80$ and hence $-160 \leq$ $q=s(2) \leq 160$.

More generally if we have some information about the derivative, then we can use the MVT to leverage this information to tell us something about the function.

Example 2.13.8 Using MVT to bound a function.
Let $f(x)$ be a differentiable function so that

$$
f(1)=10 \quad \text { and } \quad-1 \leq f^{\prime}(x) \leq 2 \text { everywhere }
$$

Obtain upper and lower bounds on $f(5)$.
Okay - what do we do?

- Since $f(x)$ is differentiable we can use the MVT.
- Say $f(5)=q$, then the MVT tells us that there is some $c$ between 1 and 5 such that

$$
f^{\prime}(c)=\frac{q-10}{5-1}=\frac{q-10}{4}
$$

- But we know that $-1 \leq f^{\prime}(c) \leq 2$, so

$$
\begin{aligned}
&-1 \leq f^{\prime}(c) \leq 2 \\
&-1 \leq \frac{q-10}{4} \leq 2 \\
&-4 \leq q-10 \leq 8 \\
& 6 \leq q \leq 18
\end{aligned}
$$

- Thus we must have $6 \leq f(5) \leq 18$.


### 2.13.3 $\leftrightarrow$ (Optional) - Why is the MVT True

We won't give a real proof for this theorem, but we'll look at a picture which shows why it is true. Here is the picture. It contains a sketch of the graph of $f(x)$, with $x$ running from $a$ to $b$, as well as a line segment which is the secant of the graph from the point $(a, f(a))$ to the point $(b, f(b))$. The slope of the secant is exactly $\frac{f(b)-f(a)}{b-a}$.


Remember that we are looking for a point, $(c, f(c))$, on the graph of $f(x)$ with the property that $f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}$, i.e. with the property that the slope of the tangent line at $(c, f(c))$ is the same as the slope of the secant. So imagine that you start moving the secant upward, carefully keeping the moved line segment parallel to the secant. So the slope of the moved line segment is always exactly $\frac{f(b)-f(a)}{b-a}$. When we first start moving the line segment it is not tangent to the curve - it crosses the curve. This is illustrated in the figure by the second line segment from the bottom. If we move the line segment too far it does not touch the curve at all. This is illustrated in the figure by the top segment. But if we stop moving the line segment just before it stops intersecting the curve at all, we get exactly the tangent line to the curve at the point on the curve that is farthest from the secant. This tangent line has exactly the desired slope. This is illustrated in the figure by the third line segment from the bottom.

### 2.13.4 Be Careful with Hypotheses

The mean value theorem has hypotheses - $f(x)$ has to be continuous for $a \leq x \leq b$ and has to be differentiable for $a<x<b$. If either hypothesis is violated, the conclusion of the mean value theorem can fail. That is, the curve $y=f(x)$ need not have a tangent line at some $x=c$ between $a$ and $b$ whose slope, $f^{\prime}(c)$, is the same as the slope, $\frac{f(b)-f(a)}{b-a}$, of the secant joining the points $(a, f(a))$ and $(b, f(b))$ on the curve. If $f^{\prime}(x)$ fails to exist for even a single value of $x$ between $a$ and $b$, all bets are off. The following two examples illustrate this.

Example 2.13.9 MVT doesn't work here.
For the first "bad" example, $a=0, b=2$ and

$$
f(x)=\left\{\begin{array}{ll}
0 & \text { if } x \leq 1 \\
1 & \text { if } x>1
\end{array}\right\}(b, f(b))
$$

For this example, $f^{\prime}(x)=0$ at every $x$ where it is defined. That is, at every $x \neq 1$. But the slope of the secant joining $(a, f(a))=(0,0)$ and $(b, f(b))=(2,1)$ is $\frac{1}{2}$.

Example 2.13.9

Example 2.13.10 MVT doesn't work here either.
For the second "bad" example, $a=-1, b=1$ and $f(x)=|x|$. For this function
$f^{\prime}(x)= \begin{cases}-1 & \text { if } x<0 \\ \text { undefined } & \text { if } x=0 \\ 1 & \text { if } x>0\end{cases}$


For this example, $f^{\prime}(x)= \pm 1$ at every $x$ where it is defined. That is, at every $x \neq 0$.
$\uparrow$ But the slope of the secant joining $(a, f(a))=(-1,1)$ and $(b, f(b))=(1,1)$ is 0 .

Example 2.13.11 MVT does work on this one.
Here is one "good" example, where the hypotheses of the mean value theorem are satisfied. Let $f(x)=x^{2}$. Then $f^{\prime}(x)=2 x$. For any $a<b$,

$$
\frac{f(b)-f(a)}{b-a}=\frac{b^{2}-a^{2}}{b-a}=b+a
$$

So $f^{\prime}(c)=2 c$ is exactly $\frac{f(b)-f(a)}{b-a}$ when $c=\frac{a+b}{2}$, which, in this example, happens to be exactly half way between $x=a$ and $x=b$.


Recall from Section 2.3 that if $f^{\prime}(c)>0$, then $f(x)$ is increasing at $x=c$. A simple consequence of the mean value theorem is that if you know the sign of $f^{\prime}(c)$ for all $c$ 's between $a$ and $b$, with $b>a$, then $f(b)-f(a)=f^{\prime}(c)(b-a)$ must have the same sign.

## Corollary 2.13.12 Consequences of the mean value theorem.

Let $A$ and $B$ be real numbers with $A<B$. Let function $f(x)$ be defined and continuous on the closed interval $A \leq x \leq B$ and be differentiable on the open interval $A<x<B$.
a If $f^{\prime}(c)=0$ for all $A<c<B$, then $f(b)=f(a)$ for all $A \leq a<b \leq B$.

- That is, $f(x)$ is constant on $A \leq x \leq B$.
b If $f^{\prime}(c) \geq 0$ for all $A<c<B$, then $f(b) \geq f(a)$ for all $A \leq a \leq b \leq B$.
- That is, $f(x)$ is increasing on $A \leq x \leq B$.
c If $f^{\prime}(c) \leq 0$ for all $A<c<B$, then $f(b) \leq f(a)$ for all $A \leq a \leq b \leq B$.
- That is, $f(x)$ is decreasing on $A \leq x \leq B$.

It is not hard to see why the above is true:

- Say $f^{\prime}(x)=0$ at every point in the interval $[A, B]$. Now pick any $a, b \in[A, B]$ with $a<b$. Then the MVT tells us that there is $c \in(a, b)$ so that

$$
f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}
$$

If $f(b) \neq f(a)$ then we must have that $f^{\prime}(c) \neq 0$ - contradicting what we are told about $f^{\prime}(x)$. Thus we must have that $f(b)=f(a)$.

- Similarly, say $f^{\prime}(x) \geq 0$ at every point in the interval $[A, B]$. Now pick any $a, b \in[A, B]$ with $a<b$. Then the MVT tells us that there is $c \in(a, b)$ so that

$$
f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}
$$

Since $b>a$, the denominator is positive. Now if $f(b)<f(a)$ the numerator would be negative, making the right-hand side negative, and contradicting what we are told about $f^{\prime}(x)$. Hence we must have $f(b) \geq f(a)$.

A nice corollary of the above corollary is the following:

If $f^{\prime}(x)=g^{\prime}(x)$ for all $x$ in the open interval $(a, b)$, then $f-g$ is a constant on $(a, b)$. That is $f(x)=g(x)+c$, where $c$ is some constant.

We can prove this by setting $h(x)=f(x)-g(x)$. Then $h^{\prime}(x)=0$ and so the previous corollary tells us that $h(x)$ is constant.

Example 2.13.14 Summing arcsin and arccos.
Using this corollary we can prove results like the following:

$$
\arcsin x+\arccos x=\frac{\pi}{2} \quad \text { for all }-1<x<1
$$

How does this work? Let $f(x)=\arcsin x+\arccos x$. Then

$$
f^{\prime}(x)=\frac{1}{\sqrt{1-x^{2}}}+\frac{-1}{\sqrt{1-x^{2}}}=0
$$

Thus $f$ must be a constant. To find out which constant, we can just check its value at a convenient point, like $x=0$.

$$
\arcsin (0)+\arccos (0)=\pi / 2+0=\pi / 2
$$

Since the function is constant, this must be the value.

### 2.13.5 $\leftrightarrows$ Exercises

## Exercises - Stage 1

1. Suppose a particular caribou has a top speed of 70 kph , and in one year it migrates 5000 km . What do you know about the amount of time the caribou spent travelling during its migration?
2. Suppose a migrating sandhill crane flew 240 kilometres in one day. What does the mean value theorem tell you about its speed during that day?
3. Below is the graph of $y=f(x)$, where $x$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Mark on the graph the approximate location of a value $c$ guaranteed by the mean value theorem.

4. Give a function $f(x)$ with the properties that:

- $f(x)$ is differentiable on the open interval $0<x<10$
- $f(0)=0, f(10)=10$
but for all $c \in(0,10), f^{\prime}(c)=0$.

5. For each of the parts below, sketch a function $f(x)$ (different in each part) that is continuous and differentiable over all real numbers, with $f(1)=$ $f(2)=0$, and with the listed property, or explain why no such function exists.
a $f^{\prime}(c)=0$ for no point $c \in(1,2)$
b $f^{\prime}(c)=0$ for exactly one point $c \in(1,2)$
c $f^{\prime}(c)=0$ for exactly five points $c \in(1,2)$
d $f^{\prime}(c)=0$ for infinitely many points $c \in(1,2)$
6. Suppose you want to show that a point exists where the function $f(x)=\sqrt{|x|}$ has a tangent line with slope $\frac{1}{13}$. Find the mistake(s) in the following work, and give a correct proof.

The function $f(x)$ is continuous and differentiable over all real numbers, so the mean value theorem applies. $f(-4)=2$ and $f(9)=3$, so by the mean value theorem, there exists some $c \in(-4,9)$ such that $f^{\prime}(x)=\frac{3-2}{9-(-4)}=\frac{1}{13}$.

## Exercises - Stage 2

7. *. Let $f(x)=x^{2}-2 \pi x+\cos (x)-1$. Show that there exists a real number $c$ such that $f^{\prime}(c)=0$.
8. *. Let $f(x)=e^{x}+(1-e) x^{2}-1$. Show that there exists a real number $c$ such that $f^{\prime}(c)=0$.
9. *. Let $f(x)=\sqrt{3+\sin (x)}+(x-\pi)^{2}$. Show that there exists a real number $c$ such that $f^{\prime}(c)=0$.
10. *. Let $f(x)=x \cos (x)-x \sin (x)$. Show that there exists a real number $c$ such that $f^{\prime}(c)=0$.
11. How many roots does the function $f(x)=3 x-\sin x$ have?
12. How many roots does the function $f(x)=\frac{(4 x+1)^{4}}{16}+x$ have?
13. How many roots does the function $f(x)=x^{3}+\sin \left(x^{5}\right)$ have?
14. How many positive-valued solutions does the equation $e^{x}=4 \cos (2 x)$ have?
15. *. Let $f(x)=3 x^{5}-10 x^{3}+15 x+a$, where $a$ is some constant.
a Prove that, regardless of the value $a, f^{\prime}(x)>0$ for all $x$ in $(-1,1)$.
b Prove that, regardless of the value $a, f(x)=3 x^{5}-10 x^{3}+15 x+a$ has at most one root in $[-1,1]$.
16. *. Find the point promised by the Mean Value Theorem for the function $e^{x}$ on the interval $[0, T]$.
17. Use Corollary 2.13.12 and Theorem 2.12 .8 to show that $\operatorname{arcsec} x=C-\operatorname{arccsc} x$ for some constant $C$; then find $C$.

## Exercises - Stage 3

18. *. Suppose $f(0)=0$ and $f^{\prime}(x)=\frac{1}{1+e^{-f(x)}}$. Prove that $f(100)<100$. Remark: an equation relating a function to its own derivative is called a differential equation. We'll see some very basic differential equations in Section 3.3.
19. Let $f(x)=2 x+\sin x$. What is the largest interval containing $x=0$ over which $f(x)$ is one-to-one? What are the domain and range of $f^{-1}(x)$ ?
20. Let $f(x)=\frac{x}{2}+\sin x$. What is the largest interval containing $x=0$ over which $f(x)$ is one-to-one? What are the domain and range of $f^{-1}(x)$, if we restrict $f$ to this interval?
21. Suppose $f(x)$ and $g(x)$ are functions that are continuous over the interval $[a, b]$ and differentiable over the interval $(a, b)$. Suppose further that $f(a)<g(a)$ and $g(b)<f(b)$. Show that there exists some $c \in[a, b]$ with $f^{\prime}(c)>g^{\prime}(c)$.
22. Suppose $f(x)$ is a function that is differentiable over all real numbers, and $f^{\prime}(x)$ has precisely two roots. What is the maximum number of distinct roots that $f(x)$ may have?
23. How many roots does $f(x)=\sin x+x^{2}+5 x+1$ have?

### 2.14^ Higher Order Derivatives

### 2.14.1 Higher Order Derivatives

The operation of differentiation takes as input one function, $f(x)$, and produces as output another function, $f^{\prime}(x)$. Now $f^{\prime}(x)$ is once again a function. So we can differentiate it again, assuming that it is differentiable, to create a third function, called the second derivative of $f$. And we can differentiate the second derivative again to create a fourth function, called the third derivative of $f$. And so on.

## Definition 2.14.1

- $f^{\prime \prime}(x)$ and $f^{(2)}(x)$ and $\frac{\mathrm{d}^{2} f}{\mathrm{~d} x^{2}}(x)$ all mean $\frac{\mathrm{d}}{\mathrm{d} x}\left(\frac{\mathrm{~d}}{\mathrm{~d} x} f(x)\right)$
- $f^{\prime \prime \prime}(x)$ and $f^{(3)}(x)$ and $\frac{\mathrm{d}^{3} f}{\mathrm{~d} x^{3}}(x)$ all mean $\frac{\mathrm{d}}{\mathrm{d} x}\left(\frac{\mathrm{~d}}{\mathrm{~d} x}\left(\frac{\mathrm{~d}}{\mathrm{~d} x} f(x)\right)\right)$
- $f^{(4)}(x)$ and $\frac{\mathrm{d}^{4} f}{\mathrm{~d} x^{4}}(x)$ both mean $\frac{\mathrm{d}}{\mathrm{d} x}\left(\frac{\mathrm{~d}}{\mathrm{~d} x}\left(\frac{\mathrm{~d}}{\mathrm{~d} x}\left(\frac{\mathrm{~d}}{\mathrm{~d} x} f(x)\right)\right)\right)$
- and so on.

Here is a simple example. Then we'll think a little about the significance of second order derivatives. Then we'll do a more a computationally complex example.

Example 2.14.2 Derivatives of $x^{n}$.
Let $n$ be a natural number and let $f(x)=x^{n}$. Then

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x} x^{n} & =n x^{n-1} \\
\frac{\mathrm{~d}^{2}}{\mathrm{~d} x^{2}} x^{n} & =\frac{\mathrm{d}}{\mathrm{~d} x}\left(n x^{n-1}\right)=n(n-1) x^{n-2}
\end{aligned}
$$

$$
\frac{\mathrm{d}^{3}}{\mathrm{~d} x^{3}} x^{n}=\frac{\mathrm{d}}{\mathrm{~d} x}\left(n(n-1) x^{n-2}\right)=n(n-1)(n-2) x^{n-3}
$$

Each time we differentiate, we bring down the exponent, which is exactly one smaller than the previous exponent brought down, and we reduce the exponent by one. By the time we have differentiated $n-1$ times, the exponent has decreased to $n-(n-1)=1$ and we have brought down the factors $n(n-1)(n-2) \cdots 2$. So

$$
\frac{\mathrm{d}^{n-1}}{\mathrm{~d} x^{n-1}} x^{n}=n(n-1)(n-2) \cdots 2 x
$$

and

$$
\frac{\mathrm{d}^{n}}{\mathrm{~d} x^{n}} x^{n}=n(n-1)(n-2) \cdots 1
$$

The product of the first $n$ natural numbers, $1 \cdot 2 \cdot 3 \cdots \cdot n$, is called " $n$ factorial" and is denoted $n$ !. So we can also write

$$
\frac{\mathrm{d}^{n}}{\mathrm{~d} x^{n}} x^{n}=n!
$$

If $m>n$, then

$$
\frac{\mathrm{d}^{m}}{\mathrm{~d} x^{m}} x^{n}=0
$$

Example 2.14.2

Example 2.14.3 Position, velocity and acceleration.
Recall that the derivative $v^{\prime}(a)$ is the (instantaneous) rate of change of the function $v(t)$ at $t=a$. Suppose that you are walking on the $x$-axis and that $x(t)$ is your $x$-coordinate at time $t$. Also suppose, for simplicity, that you are moving from left to right. Then $v(t)=x^{\prime}(t)$ is your velocity at time $t$ and $v^{\prime}(a)=x^{\prime \prime}(a)$ is the rate at which your velocity is changing at time $t=a$. It is called your acceleration. In particular, if $x^{\prime \prime}(a)>0$, then your velocity is increasing, i.e. you are speeding up, at time $a$. If $x^{\prime \prime}(a)<0$, then your velocity is decreasing, i.e. you are slowing down, at time $a$. That's one interpretation of the second derivative.

Example 2.14.4 2.11.1 continued.
Find $y^{\prime \prime}$ if $y=y^{3}+x y+x^{3}$.
Solution This problem concerns some function $y(x)$ that is not given to us explicitly. All that we are told is that $y(x)$ satisfies

$$
\begin{equation*}
y(x)=y(x)^{3}+x y(x)+x^{3} \tag{E1}
\end{equation*}
$$

for all $x$. We are asked to find $y^{\prime \prime}(x)$. We cannot solve this equation to get an explicit formula for $y(x)$. So we use implicit differentiation, as we did in Example 2.11.1. That is, we apply $\frac{\mathrm{d}}{\mathrm{d} x}$ to both sides of (E1). This gives

$$
\begin{equation*}
y^{\prime}(x)=3 y(x)^{2} y^{\prime}(x)+y(x)+x y^{\prime}(x)+3 x^{2} \tag{E2}
\end{equation*}
$$

which we can solve for $y^{\prime}(x)$, by moving all $y^{\prime}(x)^{\prime}$ 's to the left hand side, giving

$$
\left[1-x-3 y(x)^{2}\right] y^{\prime}(x)=y(x)+3 x^{2}
$$

and then dividing across.

$$
\begin{equation*}
y^{\prime}(x)=\frac{y(x)+3 x^{2}}{1-x-3 y(x)^{2}} \tag{E3}
\end{equation*}
$$

To get $y^{\prime \prime}(x)$, we have two options.
Method 1. Apply $\frac{\mathrm{d}}{\mathrm{d} x}$ to both sides of (E2). This gives

$$
y^{\prime \prime}(x)=3 y(x)^{2} y^{\prime \prime}(x)+6 y(x) y^{\prime}(x)^{2}+2 y^{\prime}(x)+x y^{\prime \prime}(x)+6 x
$$

We can now solve for $y^{\prime \prime}(x)$, giving

$$
\begin{equation*}
y^{\prime \prime}(x)=\frac{6 x+2 y^{\prime}(x)+6 y(x) y^{\prime}(x)^{2}}{1-x-3 y(x)^{2}} \tag{E4}
\end{equation*}
$$

Then we can substitute in (E3), giving

$$
\begin{aligned}
& y^{\prime \prime}(x)=2 \frac{3 x+\frac{y(x)+3 x^{2}}{1-x-3 y(x)^{2}}+3 y(x)\left(\frac{y(x)+3 x^{2}}{1-x-3 y(x)^{2}}\right)^{2}}{1-x-3 y(x)^{2}} \\
& =2 \frac{3 x\left[1-x-3 y(x)^{2}\right]^{2}+\left[y(x)+3 x^{2}\right]\left[1-x-3 y(x)^{2}\right]+3 y(x)\left[y(x)+3 x^{2}\right]^{2}}{\left[1-x-3 y(x)^{2}\right]^{3}}
\end{aligned}
$$

Method 2. Alternatively, we can also differentiate (E3).

$$
\begin{aligned}
& y^{\prime \prime}(x)=\frac{\left[y^{\prime}(x)+6 x\right]\left[1-x-3 y(x)^{2}\right]-\left[y(x)+3 x^{2}\right]\left[-1-6 y(x) y^{\prime}(x)\right]}{\left[1-x-3 y(x)^{2}\right]^{2}} \\
& =\frac{\left[\frac{y(x)+3 x^{2}}{1-x-3 y(x)^{2}}+6 x\right]\left[1-x-3 y(x)^{2}\right]-\left[y(x)+3 x^{2}\right]\left[-1-6 y(x) \frac{y(x)+3 x^{2}}{1-x-3 y(x)^{2}}\right]}{\left[1-x-3 y(x)^{2}\right]^{2}} \\
& =\frac{2\left[y(x)+3 x^{2}\right]\left[1-x-3 y(x)^{2}\right]+6 x\left[1-x-3 y(x)^{2}\right]^{2}+6 y(x)\left[y(x)+3 x^{2}\right]^{2}}{\left[1-x-3 y(x)^{2}\right]^{3}}
\end{aligned}
$$

Remark 1. We have now computed $y^{\prime \prime}(x)$ - sort of. The answer is in terms of $y(x)$, which we don't know. Since we cannot get an explicit formula for $y(x)$, there's not a great deal that we can do, in general.
Remark 2. Even though we cannot solve $y=y^{3}+x y+x^{3}$ explicitly for $y(x)$, for general $x$, it is sometimes possible to solve equations like this for some special values
of $x$. In fact, we saw in Example 2.11 .1 that when $x=1$, the given equation reduces to $y(1)=y(1)^{3}+1 \cdot y(1)+1^{3}$, or $y(1)^{3}=-1$, which we can solve to get $y(1)=-1$. Substituting into (E2), as we did in Example 2.11.1 gives

$$
y^{\prime}(1)=\frac{-1+3}{1-1-3(-1)^{2}}=-\frac{2}{3}
$$

and substituting into (E4) gives

$$
y^{\prime \prime}(1)=\frac{6+2\left(-\frac{2}{3}\right)+6(-1)\left(-\frac{2}{3}\right)^{2}}{1-1-3(-1)^{2}}=\frac{6-\frac{4}{3}-\frac{8}{3}}{-3}=-\frac{2}{3}
$$

(It's a fluke that, in this example, $y^{\prime}(1)$ and $y^{\prime \prime}(1)$ happen to be equal.) So we now know that, even though we can't solve $y=y^{3}+x y+x^{3}$ explicitly for $y(x)$, the graph of the solution passes through $(1,-1)$ and has slope $-\frac{2}{3}$ (i.e. is sloping downwards by between $30^{\circ}$ and $45^{\circ}$ ) there and, furthermore, the slope of the graph decreases as $x$ increases through $x=1$.


Here is a sketch of the part of the graph very near $(1,-1)$. The tangent line to the graph at $(1,-1)$ is also shown. Note that the tangent line is sloping down to the right, as we expect, and that the graph lies below the tangent line near $(1,-1)$. That's because the slope $f^{\prime}(x)$ is decreasing (becoming more negative) as $x$ passes through 1.

Example 2.14.4

## Warning 2.14.5

Many people will suppress the $(x)$ in $y(x)$ when doing computations like those in Example 2.14.4. This gives shorter, easier to read formulae, like $y^{\prime}=\frac{y+3 x^{2}}{1-x-3 y^{2}}$. If you do this, you must never forget that $y$ is a function of $x$ and is not a constant. If you do forget, you'll make the very serious error of saying that $\frac{\mathrm{d} y}{\mathrm{~d} x}=0$, which is false.

### 2.14.2 $\leadsto$ Exercises

## Exercises - Stage 1

1. What is the 180th derivative of the function $f(x)=e^{x}$ ?
2. Suppose $f(x)$ is a differentiable function, with $f^{\prime}(x)>0$ and $f^{\prime \prime}(x)>0$ for every $x \in(a, b)$. Which of the following must be true?
i $f(x)$ is positive over $(a, b)$
ii $f(x)$ is increasing over $(a, b)$
iii $f(x)$ is increasing at a constant rate over $(a, b)$
iv $f(x)$ is increasing faster and faster over $(a, b)$
v $f^{\prime \prime \prime \prime}(x)>0$ for some $x \in(a, b)$
3. Let $f(x)=a x^{15}$ for some constant $a$. Which value of $a$ results in $f^{(15)}(x)=3$ ?
4. Find the mistake(s) in the following work, and provide a corrected answer.

Suppose $-14 x^{2}+2 x y+y^{2}=1$. We find $\frac{\mathrm{d}^{2} y}{\mathrm{~d}^{2}}$ at the point $(1,3)$.
Differentiating implicitly:

$$
-28 x+2 y+2 x y^{\prime}+2 y y^{\prime}=0
$$

Plugging in $x=1, y=3$ :

$$
\begin{aligned}
-28+6+2 y^{\prime}+6 y^{\prime} & =0 \\
y^{\prime} & =\frac{11}{4}
\end{aligned}
$$

Differentiating:

$$
y^{\prime \prime}=0
$$

## Exercises - Stage 2

5. Let $f(x)=(\log x-1) x$. Evaluate $f^{\prime \prime}(x)$.
6. Evaluate $\frac{\mathrm{d}^{2}}{\mathrm{~d} x^{2}}\{\arctan x\}$.
7. The unit circle consists of all point $x^{2}+y^{2}=1$. Give an expression for $\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}$ in
terms of $y$.
8. Suppose the position of a particle at time $t$ is given by $s(t)=\frac{e^{t}}{t^{2}+1}$. Find the acceleration of the particle at time $t=1$.
9. Evaluate $\frac{\mathrm{d}^{3}}{\mathrm{~d} x^{3}}\left\{\log \left(5 x^{2}-12\right)\right\}$.
10. The height of a particle at time $t$ seconds is given by $h(t)=-\cos t$. Is the particle speeding up or slowing down at $t=1$ ?
11. The height of a particle at time $t$ seconds is given by $h(t)=t^{3}-t^{2}-5 t+10$. Is the particle's motion getting faster or slower at $t=1$ ?
12. Suppose a curve is defined implicitly by

$$
x^{2}+x+y=\sin (x y)
$$

What is $\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}$ at the point $(0,0)$ ?
13. Which statements below are true, and which false?
a $\frac{\mathrm{d}^{4}}{\mathrm{~d} x^{4}} \sin x=\sin x$
b $\frac{\mathrm{d}^{4}}{\mathrm{~d} x^{4}} \cos x=\cos x$
c $\frac{\mathrm{d}^{4}}{\mathrm{~d} x^{4}} \tan x=\tan x$

## Exercises - Stage 3

14. A function $f(x)$ satisfies $f^{\prime}(x)<0$ and $f^{\prime \prime}(x)>0$ over $(a, b)$. Which of the following curves below might represent $y=f(x)$ ?

15. Let $f(x)=2^{x}$. What is $f^{(n)}(x)$, if $n$ is a whole number?
16. Let $f(x)=a x^{3}+b x^{2}+c x+d$, where $a, b, c$, and $d$ are nonzero constants. What is the smallest integer $n$ so that $\frac{\mathrm{d}^{n} f}{\mathrm{~d} x^{n}}=0$ for all $x$ ?
17. *.

$$
f(x)=e^{x+x^{2}} \quad h(x)=1+x+\frac{3}{2} x^{2}
$$

a Find the first and second derivatives of both functions
b Evaluate both functions and their first and second derivatives at 0 .
c Show that for all $x>0, f(x)>h(x)$.
Remark: for some applications, we only need to know that a function is "big enough." Since $f(x)$ is a difficult function to evaluate, it may be useful to know that it is bigger than $h(x)$ when $x$ is positive.
18. *. The equation $x^{3} y+y^{3}=10 x$ defines $y$ implicitly as a function of $x$ near the point $(1,2)$.
a Compute $y^{\prime}$ at this point.
b It can be shown that $y^{\prime \prime}$ is negative when $x=1$. Use this fact and your answer to 2.14.2.18.a to make a sketch showing the relationship of the curve to its tangent line at $(1,2)$.
19. Let $g(x)=f(x) e^{x}$. In Question 2.7.3.12, Section 2.7, we learned that $g^{\prime}(x)=$ $\left[f(x)+f^{\prime}(x)\right] e^{x}$.
a What is $g^{\prime \prime}(x)$ ?
b What is $g^{\prime \prime \prime}(x)$ ?
c Based on your answers above, guess a formula for $g^{(4)}(x)$. Check it by differentiating.
20. Suppose $f(x)$ is a function whose first $n$ derivatives exist over all real numbers, and $f^{(n)}(x)$ has precisely $m$ roots. What is the maximum number of roots that $f(x)$ may have?
21. How many roots does the function $f(x)=(x+1) \log (x+1)+\sin x-x^{2}$ have?
22. *. Let $f(x)=x|x|$.
a Show that $f(x)$ is differentiable at $x=0$, and find $f^{\prime}(0)$.
b Find the second derivative of $f(x)$. Explicitly state, with justification, the point(s) at which $f^{\prime \prime}(x)$ does not exist, if any.

### 2.15^ (Optional) - Is $\lim _{x \rightarrow c} f^{\prime}(x)$ Equal to $f^{\prime}(c)$ ?

Consider the function

$$
f(x)= \begin{cases}\frac{\sin x^{2}}{x} & \text { if } x \neq 0 \\ 0 & \text { if } x=0\end{cases}
$$

For any $x \neq 0$ we can easily use our differentiation rules to find

$$
f^{\prime}(x)=\frac{2 x^{2} \cos x^{2}-\sin x^{2}}{x^{2}}
$$

But for $x=0$ none of our usual differentation rules apply. So how do we find $f^{\prime}(0)$ ? One obviously legitimate strategy is to directly apply the Definition 2.2 .1 of the derivative. As an alternative, we will now consider the question "Can one find $f^{\prime}(0)$ by taking the limit of $f^{\prime}(x)$ as $x$ tends to zero?". There is bad news and there is good news.

- The bad news is that, even for functions $f(x)$ that are differentiable for all $x, f^{\prime}(x)$ need not be continuous. That is, it is not always true that $\lim _{x \rightarrow 0} f^{\prime}(x)=f^{\prime}(0)$. We will see a function for which $\lim _{x \rightarrow 0} f^{\prime}(x) \neq f^{\prime}(0)$ in Example 2.15.1, below.
- The good news is that Theorem 2.15.2, below provides conditions which are sufficient to guarantee that $f(x)$ is differentiable at $x=0$ and that $\lim _{x \rightarrow 0} f^{\prime}(x)=$ $f^{\prime}(0)$.

Example 2.15.1
Consider the function

$$
f(x)= \begin{cases}x^{2} \sin \frac{1}{x} & \text { if } x \neq 0 \\ 0 & \text { if } x=0\end{cases}
$$

For $x \neq 0$ we have, by the product and chain rules,

$$
\begin{aligned}
f^{\prime}(x) & =2 x \sin \frac{1}{x}+x^{2}\left(\cos \frac{1}{x}\right)\left(-\frac{1}{x^{2}}\right) \\
& =2 x \sin \frac{1}{x}-\cos \frac{1}{x}
\end{aligned}
$$

As $\left|\sin \frac{1}{x}\right| \leq 1$, we have

$$
\lim _{x \rightarrow 0} 2 x \sin \frac{1}{x}=0
$$

On the other hand, as $x$ tends to zero, $\frac{1}{x}$ goes to $\pm \infty$. So

$$
\lim _{x \rightarrow 0} \cos \frac{1}{x}=D N E \Longrightarrow \lim _{x \rightarrow 0} f^{\prime}(x)=D N E
$$

We will now see that, despite this, $f^{\prime}(0)$ is perfectly well defined. By definition

$$
\begin{aligned}
f^{\prime}(0) & =\lim _{h \rightarrow 0} \frac{f(h)-f(0)}{h} \\
& =\lim _{h \rightarrow 0} \frac{h^{2} \sin \frac{1}{h}-0}{h} \\
& =\lim _{h \rightarrow 0} h \sin \frac{1}{h} \\
& =0 \quad \text { since }\left|\sin \frac{1}{h}\right| \leq 1
\end{aligned}
$$

So $f^{\prime}(0)$ exists, but is not equal to $\lim _{x \rightarrow 0} f^{\prime}(x)$, which does not exist.

Now for the good news.

## Theorem 2.15.2

Let $a<c<b$. Assume that

- the function $f(x)$ is continous on the interval $a<x<b$ and
- is differentiable at every $x$ in the intervals $a<x<c$ and $c<x<b$ and
- the limit $\lim _{x \rightarrow c} f^{\prime}(x)$ exists.

Then $f$ is differentiable at $x=c$ and

$$
f^{\prime}(c)=\lim _{x \rightarrow c} f^{\prime}(x)
$$

Proof. By hypothesis, there is a number $L$ such that

$$
\lim _{x \rightarrow c} f^{\prime}(x)=L
$$

By definition

$$
f^{\prime}(c)=\lim _{h \rightarrow 0} \frac{f(c+h)-f(c)}{h}
$$

By the Mean Value Theorem (Theorem 2.13.5) there is, for each $h$, an (unknown) number $x_{h}$ between $c$ and $c+h$ such that $f^{\prime}\left(x_{h}\right)=\frac{f(c+h)-f(c)}{h}$. So

$$
f^{\prime}(c)=\lim _{h \rightarrow 0} f^{\prime}\left(x_{h}\right)
$$

As $h$ tends to zero, $c+h$ tends to $c$, and so $x_{h}$ is forced to tend to $c$, and $f^{\prime}\left(x_{h}\right)$ is forced to tend to $L$ so that

$$
f^{\prime}(c)=\lim _{h \rightarrow 0} f^{\prime}\left(x_{h}\right)=L
$$

In the next example we evaluate $f^{\prime}(0)$ by applying Theorem 2.15.2.

## Example 2.15.3

Let

$$
f(x)= \begin{cases}\frac{\sin x^{2}}{x} & \text { if } x \neq 0 \\ 0 & \text { if } x=0\end{cases}
$$

We have already observed above that, for $x \neq 0$,

$$
f^{\prime}(x)=\frac{2 x^{2} \cos x^{2}-\sin x^{2}}{x^{2}}=2 \cos x^{2}-\frac{\sin x^{2}}{x^{2}}
$$

We use Theorem 2.15.2 with $c=0$ to show that $f(x)$ is differentiable at $x=0$ and to evaluate $f^{\prime}(0)$. That theorem has two hypotheses that we have not yet verified, namely the continuity of $f(x)$ at $x=0$, and the existence of the limit $\lim _{x \rightarrow 0} f^{\prime}(x)$. We verify them now.

- We already know, by Lemma 2.8.1, that $\lim _{h \rightarrow 0} \frac{\sin h}{h}=1$. So

$$
\lim _{x \rightarrow 0} \frac{\sin x^{2}}{x^{2}}=\lim _{h \rightarrow 0^{+}} \frac{\sin h}{h} \quad \text { with } h=x^{2}
$$

$$
=1
$$

and

$$
\lim _{x \rightarrow 0} f(x)=\lim _{x \rightarrow 0} \frac{\sin x^{2}}{x}=\lim _{x \rightarrow 0} x \frac{\sin x^{2}}{x^{2}}=\lim _{x \rightarrow 0} x \lim _{x \rightarrow 0} \frac{\sin x^{2}}{x^{2}}=0 \times 1=0
$$

and $f(x)$ is continuous at $x=0$.

- The limit of the derivative is

$$
\lim _{x \rightarrow 0} f^{\prime}(x)=\lim _{x \rightarrow 0}\left[2 \cos x^{2}-\frac{\sin x^{2}}{x^{2}}\right]=2 \times 1-1=1
$$

So, by Theorem 2.15.2, $f(x)$ is differentiable at $x=0$ and $f^{\prime}(0)=1$.

In Section 2.2 we defined the derivative at $x=a, f^{\prime}(a)$, of an abstract function $f(x)$, to be its instantaneous rate of change at $x=a$ :

$$
f^{\prime}(a)=\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a}
$$

This abstract definition, and the whole theory that we have developed to deal with it, turns out be extremely useful simply because "instantaneous rate of change" appears in a huge number of settings. Here are a few examples.

- If you are moving along a line and $x(t)$ is your position on the line at time $t$, then your rate of change of position, $x^{\prime}(t)$, is your velocity. If, instead, $v(t)$ is your velocity at time $t$, then your rate of change of velocity, $v^{\prime}(t)$, is your acceleration. We shall explore this further in Section 3.1.
- If $P(t)$ is the size of some population (say the number of humans on the earth) at time $t$, then $P^{\prime}(t)$ is the rate at which the size of that population is changing. It is called the net birth rate. We shall explore it further in Section 3.3.3.
- Radiocarbon dating, a procedure used to determine the age of, for example, archaeological materials, is based on an understanding of the rate at which an unstable isotope of carbon decays. We shall look at this procedure in Section 3.3.1
- A capacitor is an electrical component that is used to repeatedly store and release electrical charge (say electrons) in an electronic circuit. If $Q(t)$ is the charge on a capacitor at time $t$, then $Q^{\prime}(t)$ is the instantaneous rate at which charge is flowing into the capacitor. That's called the current. The standard unit of charge is the coulomb. One coulomb is the magnitude of the charge of approximately $6.241 \times 10^{18}$ electrons. The standard unit for current is the amp. One amp represents one coulomb per second.

Applications of Derivatives

## 3.1 • Velocity and Acceleration

### 3.1.1 $\leadsto$ Velocity and Acceleration

If you are moving along the $x$-axis and your position at time $t$ is $x(t)$, then your velocity at time $t$ is $v(t)=x^{\prime}(t)$ and your acceleration at time $t$ is $a(t)=v^{\prime}(t)=x^{\prime \prime}(t)$.

## Example 3.1.1 Velocity as derivative of position.

Suppose that you are moving along the $x$-axis and that at time $t$ your position is given by

$$
x(t)=t^{3}-3 t+2
$$

We're going to try and get a good picture of what your motion is like. We can learn quite a bit just by looking at the sign of the velocity $v(t)=x^{\prime}(t)$ at each time $t$.

- If $x^{\prime}(t)>0$, then at that instant $x$ is increasing, i.e. you are moving to the right.
- If $x^{\prime}(t)=0$, then at that instant you are not moving at all.
- If $x^{\prime}(t)<0$, then at that instant $x$ is decreasing, i.e. you are moving to the left.

From the given formula for $x(t)$ it is straight forward to work out the velocity

$$
v(t)=x^{\prime}(t)=3 t^{2}-3=3\left(t^{2}-1\right)=3(t+1)(t-1)
$$

This is zero only when $t=-1$ and when $t=+1$; at no other value ${ }^{a}$ of $t$ can this polynomial be equal zero. Consequently in any time interval that does not include either $t=-1$ or $t=+1, v(t)$ takes only a single sign ${ }^{b}$. So

- For all $t<-1$, both $(t+1)$ and $(t-1)$ are negative (sub in, for example, $t=-10$ ) so the product $v(t)=x^{\prime}(t)=3(t+1)(t-1)>0$.
- For all $-1<t<1$, the factor $(t+1)>0$ and the factor $(t-1)<0$ (sub in, for example, $t=0$ ) so the product $v(t)=x^{\prime}(t)=3(t+1)(t-1)<0$.
- For all $t>1$, both $(t+1)$ and $(t-1)$ are positive (sub in, for example, $t=+10$ ) so the product $v(t)=x^{\prime}(t)=3(t+1)(t-1)>0$.

The figure below gives a summary of the sign information we have about $t-1, t+1$ and $x^{\prime}(t)$.


It is now easy to put together a mental image of your trajectory.

- For $t$ large and negative (i.e. far in the past), $x(t)$ is large and negative and $v(t)$ is large and positive. For example ${ }^{c}$, when $t=-10^{6}, x(t) \approx t^{3}=-10^{18}$ and $v(t) \approx 3 t^{2}=3 \cdot 10^{12}$. So you are moving quickly to the right.
- For $t<-1, v(t)=x^{\prime}(t)>0$ so that $x(t)$ is increasing and you are moving to the right.
- At $t=-1, v(-1)=0$ and you have come to a halt at position $x(-1)=(-1)^{3}-$ $3(-1)+2=4$.
- For $-1<t<1, v(t)=x^{\prime}(t)<0$ so that $x(t)$ is decreasing and you are moving to the left.
- At $t=+1, v(1)=0$ and you have again come to a halt, but now at position $x(1)=1^{3}-3+2=0$.
- For $t>1, v(t)=x^{\prime}(t)>0$ so that $x(t)$ is increasing and you are again moving to the right.
- For $t$ large and positive (i.e. in the far future), $x(t)$ is large and positive and $v(t)$ is large and positive. For example ${ }^{d}$, when $t=10^{6}, x(t) \approx t^{3}=10^{18}$ and $v(t) \approx 3 t^{2}=3 \cdot 10^{12}$. So you are moving quickly to the right.

Here is a sketch of the graphs of $x(t)$ and $v(t)$. The heavy lines in the graphs indicate when you are moving to the right - that is where $v(t)=x^{\prime}(t)$ is positive.



And here is a schematic picture of the whole trajectory.

$a$ This is because the equation $a b=0$ is only satisfied for real numbers $a$ and $b$ when either $a=0$ or $b=0$ or both $a=b=0$. Hence if a polynomial is the product of two (or more) factors, then it is only zero when at least one of those factors is zero. There are more complicated mathematical environments in which you have what are called "zero divisors" but they are beyond the scope of this course.
$b \quad$ This is because if $v\left(t_{a}\right)<0$ and $v\left(t_{b}\right)>0$ then, by the intermediate value theorem, the continuous function $v(t)=x^{\prime}(t)$ must take the value 0 for some $t$ between $t_{a}$ and $t_{b}$.
$c \quad$ Notice here we are using the fact that when $t$ is very large $t^{3}$ is much bigger than $t^{2}$ and $t^{1}$. So we can approximate the value of the polynomial $x(t)$ by the largest term - in this case $t^{3}$. We can do similarly with $v(t)$ - the largest term is $3 t^{2}$.
$d$ We are making a similar rough approximation here.
Example 3.1.1

## Example 3.1.2 Position and velocity from acceleration.

In this example we are going to figure out how far a body falling from rest will fall in a given time period.

- We should start by defining some variables and their units. Denote
- time in seconds by $t$,
- mass in kilograms by $m$,
- distance fallen (in metres) at time $t$ by $s(t)$, velocity (in $\mathrm{m} / \mathrm{sec}$ ) by $v(t)=s^{\prime}(t)$ and acceleration (in $\mathrm{m} / \mathrm{sec}^{2}$ ) by $a(t)=v^{\prime}(t)=s^{\prime \prime}(t)$.

It makes sense to choose a coordinate system so that the body starts to fall at $t=0$.

- We will use Newton's second law of motion
the force applied to the body at time $t=m \cdot a(t)$.
together with the assumption that the only force acting on the body is gravity (in particular, no air resistance). Note that near the surface of the Earth,
the force due to gravity acting on a body of mass $m=m \cdot g$.
The constant $g$, called the acceleration of gravity ${ }^{a}$, is about $9.8 \mathrm{~m} / \mathrm{sec}^{2}$.
- Since the body is falling from rest, we know that its initial velocity is zero. That is

$$
v(0)=0 .
$$

Newton's second law then implies that

$$
\begin{array}{rlr}
m \cdot a(t) & =\text { force due to gravity } & \\
m \cdot v^{\prime}(t) & =m \cdot g \quad \text { cancel the } m \\
v^{\prime}(t) & =g &
\end{array}
$$

- In order to find the velocity, we need to find a function of $t$ whose derivative is constant. We are simply going to guess such a function and then we will verify that our guess has all of the desired properties. It's easy to guess a function whose derivative is the constant $g$. Certainly $g t$ has the correct derivative. So does

$$
v(t)=g t+c
$$

for any constant $c$. One can then verify ${ }^{b}$ that $v^{\prime}(t)=g$. Using the fact that $v(0)=0$ we must then have $c=0$ and so

$$
v(t)=g t .
$$

- Since velocity is the derivative of position, we know that

$$
s^{\prime}(t)=v(t)=g \cdot t
$$

To find $s(t)$ we are again going to guess and check. It's not hard to see that we can use

$$
s(t)=\frac{g}{2} t^{2}+c
$$

where again $c$ is some constant. Again we can verify that this works simply by differentiating ${ }^{c}$. Since we have defined $s(t)$ to be the distance fallen, it follows that $s(0)=0$ which in turn tells us that $c=0$. Hence

$$
\begin{array}{rlr}
s(t) & =\frac{g}{2} t^{2} & \text { but } g=9.8, \text { so } \\
& =4.9 t^{2}, &
\end{array}
$$

which is exactly the $s(t)$ used way back in Section 1.2.
$a \quad$ It is also called the standard acceleration due to gravity or standard gravity. For those of you who prefer imperial units (or US customary units), it is about $32 \mathrm{ft} / \mathrm{sec}^{2}, 77165$ cubits $/$ minute ${ }^{2}$, or 631353 furlongs/hour ${ }^{2}$.
$b$ While it is clear that this satisfies the equation we want, it is less clear that it is the only function that works. To see this, assume that there are two functions $f(t)$ and $h(t)$ which both satisfy $v^{\prime}(t)=g$. Then $f^{\prime}(t)=h^{\prime}(t)=g$ and so $f^{\prime}(t)-h^{\prime}(t)=0$. Equivalently $\frac{\mathrm{d}}{\mathrm{d} t}(f(t)-h(t))=0$. The only function whose derivative is zero everywhere is the constant function (see Section 2.13 and Theorem 2.13.12). Thus $f(t)-h(t)=$ constant. So all the functions that satisfy $v^{\prime}(t)=g$ must be of the form $g t+$ constant.
$c$ To show that any solution of $s^{\prime}(t)=g v$ must be of this form we can use the same reasoning we used to get $v(t)=g t+$ constant.

Let's now do a similar but more complicated example.
Example 3.1.3 Stopping distance of a braking car.
A car's brakes can decelerate the car at $64000 \mathrm{~km} / \mathrm{hr}^{2}$. How fast can the car be driven if it must be able to stop within a distance of 50 m ?
Solution Before getting started, notice that there is a small "trick" in this problem several quantities are stated but their units are different. The acceleration is stated in kilometres per hour ${ }^{2}$, but the distance is stated in metres. Whenever we come across a "real world" problem ${ }^{a}$ we should be careful of the units used.

- We should first define some variables and their units. Denote
- time (in hours) by $t$,
- the position of the car (in kilometres) at time $t$ by $x(t)$, and
- the velocity (in kilometres per hour) by is $v(t)$.

We can also choose a coordinate system such that $x(0)=0$ and the car starts braking at time $t=0$.

- Now let us rewrite the information in the problem in terms of these variables.
- We are told that, at maximum braking, the acceleration $v^{\prime}(t)=x^{\prime \prime}(t)$ of the car is -64000 .
- We need to determine the maximum initial velocity $v(0)$ so that the stopping distance is at most $50 \mathrm{~m}=0.05 \mathrm{~km}$ (being careful with our units). Let us call the stopping distance $x_{\text {stop }}$ which is really $x\left(t_{\text {stop }}\right)$ where $t_{\text {stop }}$ is the stopping time.
- In order to determine $x_{\text {stop }}$ we first need to determine $t_{\text {stop }}$, which we will do by assuming maximum braking from a, yet to be determined, initial velocity of $v(0)=q \mathrm{~m} / \mathrm{sec}$.
- Assuming that the car undergoes a constant acceleration at this maximum braking power, we have

$$
v^{\prime}(t)=-64000
$$

This equation is very similar to the ones we had to solve in Example 3.1.2 just above.
As we did there ${ }^{b}$, we are going to just guess $v(t)$. First, we just guess one function whose derivative is -64000 , namely $-64000 t$. Next we observe that, since the derivative of a constant is zero, any function of the form

$$
v(t)=-64000 t+c
$$

with constant $c$, has the correct derivative. Finally, the requirement that the initial velocity $v(0)=q^{\prime \prime}$ forces $c=q$, so

$$
v(t)=q-64000 t
$$

- From this we can easily determine the stopping time $t_{\text {stop }}$, when the initial velocity is $q$, since this is just when $v(t)=0$ :

$$
\begin{aligned}
0=v\left(t_{\text {stop }}\right) & =q-64000 \cdot t_{\text {stop }} \quad \text { and so } \\
t_{\text {stop }} & =\frac{q}{64000}
\end{aligned}
$$

- Armed with the stopping time, how do we get at the stopping distance? We need to find the formula satisfied by $x(t)$. Again (as per Example 3.1.2) we make use of the fact that

$$
x^{\prime}(t)=v(t)=q-64000 t
$$

So we need to guess a function $x(t)$ so that $x^{\prime}(t)=q-64000 t$. It is not hard to see that

$$
x(t)=q t-32000 t^{2}+\text { constant }
$$

works. Since we know that $x(0)=0$, this constant is just zero and

$$
x(t)=q t-32000 t^{2}
$$

- We are now ready to compute the stopping distance (in terms of the, still yet to be determined, initial velocity $q$ ):

$$
\begin{aligned}
x_{\text {stop }} & =x\left(t_{\text {stop }}\right)=q t_{\text {stop }}-32000 t_{\text {stop }}^{2} \\
& =\frac{q^{2}}{64000}-\frac{32000 q^{2}}{64000^{2}} \\
& =\frac{q^{2}}{64000}\left(1-\frac{1}{2}\right) \\
& =\frac{q^{2}}{2 \times 64000}
\end{aligned}
$$

Notice that the stopping distance is a quadratic function of the initial velocity if you go twice as fast, you need four times the distance to stop.

- But we are told that the stopping distance must be less than $50 \mathrm{~m}=0.05 \mathrm{~km}$. This means that

$$
\begin{aligned}
x_{\text {stop }}=\frac{q^{2}}{2 \times 64000} & \leq \frac{5}{100} \\
q^{2} & \leq \frac{2 \times 64000 \times 5}{100}=\frac{64000 \times 10}{100}=6400
\end{aligned}
$$

Thus we must have $q \leq 80$. Hence the initial velocity can be no greater than $80 \mathrm{~km} / \mathrm{h}$.
$a \quad$ Well - "realer world" would perhaps be a betterer term.
$b$ Now is a good time to go back and have a read of that example.
Example 3.1.3

### 3.1.2 Exercises

## Exercises - Stage 1

1. Suppose you throw a ball straight up in the air, and its height from $t=0$ to $t=4$ is given by $h(t)=-4.9 t^{2}+19.6 t$. True or false: at time $t=2$, the acceleration of the ball is 0 .
2. Suppose an object is moving with a constant acceleration. It takes ten seconds to accelerate from $1 \frac{\mathrm{~m}}{\mathrm{~s}}$ to $2 \frac{\mathrm{~m}}{\mathrm{~s}}$. How long does it take to accelerate from $2 \frac{\mathrm{~m}}{\mathrm{~s}}$ to $3 \frac{\mathrm{~m}}{\mathrm{~s}}$ ? How long does it take to accelerate from $3 \frac{\mathrm{~m}}{\mathrm{~s}}$ to $13 \frac{\mathrm{~m}}{\mathrm{~s}}$ ?
3. Let $s(t)$ be the position of a particle at time $t$. True or false: if $s^{\prime \prime}(a)>0$ for some $a$, then the particle's speed is increasing when $t=a$.
4. Let $s(t)$ be the position of a particle at time $t$. True or false: if $s^{\prime}(a)>0$ and $s^{\prime \prime}(a)>0$ for some $a$, then the particle's speed is increasing when $t=a$.

Exercises - Stage 2 For this section, we will ask you a number of questions that have to do with objects falling on Earth. Unless otherwise stated, you should assume that an object falling through the air has an acceleration due to gravity of 9.8 meters per second per second.
5. A flower pot rolls out of a window 10 m above the ground. How fast is it falling just as it smacks into the ground?
6. You want to know how deep a well is, so you drop a stone down and count the seconds until you hear it hit bottom.
a If the stone took $x$ seconds to hit bottom, how deep is the well?
b Suppose you think you dropped the stone down the well, but actually you tossed it down, so instead of an initial velocity of 0 metres per second, you accidentally imparted an initial speed of 1 metres per second. What is the actual depth of the well, if the stone fell for $x$ seconds?
7. You toss a key to your friend, standing two metres away. The keys initially move towards your friend at 2 metres per second, but slow at a rate of 0.25 metres per second per second. How much time does your friend have to react to catch the keys? That is-how long are the keys flying before they reach your friend?
8. A car is driving at 100 kph , and it brakes with a deceleration of $50000 \frac{\mathrm{~km}}{\mathrm{hr}^{2}}$. How long does the car take to come to a complete stop?
9. You are driving at 120 kph , and need to stop in 100 metres. How much deceleration do your brakes need to provide? You may assume the brakes cause a constant deceleration.
10. You are driving at 100 kph , and apply the brakes steadily, so that your car decelerates at a constant rate and comes to a stop in exactly 7 seconds. What was your speed one second before you stopped?
11. About 8.5 minutes after liftoff, the US space shuttle has reached orbital velocity, 17500 miles per hour. Assuming its acceleration was constant, how far did it travel in those 8.5 minutes?
Source: https://www.nasa.gov/mission_pages/shuttle/shuttlemissions/ sts121/launch/qa-leinbach.html
12. A pitching machine has a dial to adjust the speed of the pitch. You rotate it so that it pitches the ball straight up in the air. How fast should the ball exit the machine, in order to stay in the air exactly 10 seconds?
You may assume that the ball exits from ground level, and is acted on only by gravity, which causes a constant deceleration of 9.8 metres per second.
13. A peregrine falcon can dive at a speed of 325 kph . If you were to drop a stone, how high up would you have to be so that the stone reached the same speed in its fall?
14. You shoot a cannon ball into the air with initial velocity $v_{0}$, and then gravity brings it back down (neglecting all other forces). If the cannon ball made it to a height of 100 m , what was $v_{0}$ ?
15. Suppose you are driving at 120 kph , and you start to brake at a deceleration of 50000 kph per hour. For three seconds you steadily increase your deceleration to 60000 kph per hour. (That is, for three seconds, the rate of change of your deceleration is constant.) How fast are you driving at the end of those three seconds?

## Exercises - Stage 3

16. You jump up from the side of a trampoline with an initial upward velocity of 1 metre per second. While you are in the air, your deceleration is a constant 9.8 metres per second per second due to gravity. Once you hit the trampoline, as you fall your speed decreases by 4.9 metres per second
per second. How many seconds pass between the peak of your jump and the lowest part of your fall on the trampoline?

17. Suppose an object is moving so that its velocity doubles every second. Give an expression for the acceleration of the object.

## 3.2 $\triangle$ Related Rates

### 3.2.1 ~ Related Rates

Consider the following problem
A spherical balloon is being inflated at a rate of $13 \mathrm{~cm}^{3} / \mathrm{sec}$. How fast is the radius changing when the balloon has radius 15 cm ?
There are several pieces of information in the statement:

- The balloon is spherical
- The volume is changing at a rate of $13 \mathrm{~cm}^{3} / \mathrm{sec}$ - so we need variables for volume (in $\mathrm{cm}^{3}$ ) and time (in sec). Good choices are $V$ and $t$.
- We are asked for the rate at which the radius is changing - so we need a variable for radius and units. A good choice is $r$, measured in cm - since volume is measured in $\mathrm{cm}^{3}$.
Since the balloon is a sphere we know ${ }^{1}$ that

$$
V=\frac{4}{3} \pi r^{3}
$$

1 If you don't know the formula for the volume of a sphere, now is a good time to revise by looking at Appendix A. 11 .

Since both the volume and radius are changing with time, both $V$ and $r$ are implicitly functions of time; we could really write

$$
V(t)=\frac{4}{3} \pi r(t)^{3} .
$$

We are told the rate at which the volume is changing and we need to find the rate at which the radius is changing. That is, from a knowledge of $\frac{d V}{d t}$, find the related rate ${ }^{2}$ $\frac{\mathrm{d} r}{\mathrm{~d} t}$.

In this case, we can just differentiate our equation by $t$ to get

$$
\frac{\mathrm{d} V}{\mathrm{~d} t}=4 \pi r^{2} \frac{\mathrm{~d} r}{\mathrm{~d} t}
$$

This can then be rearranged to give

$$
\frac{\mathrm{d} r}{\mathrm{~d} t}=\frac{1}{4 \pi r^{2}} \frac{\mathrm{~d} V}{\mathrm{~d} t}
$$

Now we were told that $\frac{\mathrm{d} V}{\mathrm{~d} t}=13$, so

$$
\frac{\mathrm{d} r}{\mathrm{~d} t}=\frac{13}{4 \pi r^{2}}
$$

We were also told that the radius is 15 cm , so at that moment in time

$$
\frac{\mathrm{d} r}{\mathrm{~d} t}=\frac{13}{\pi 4 \times 15^{2}}
$$

This is a very typical example of a related rate problem. This section is really just a collection of problems, but all will follow a similar pattern.

- The statement of the problem will tell you quantities that must be related (above it was volume, radius and, implicitly, time).
- Typically a little geometry (or some physics or...) will allow you to relate these quantities (above it was the formula that links the volume of a sphere to its radius).
- Implicit differentiation will then allow you to link the rate of change of one quantity to another.

Another balloon example

## Example 3.2.1 A rising balloon.

Consider a helium balloon rising vertically from a fixed point 200 m away from you. You are trying to work out how fast it is rising. Now - computing the velocity directly is difficult, but you can measure angles. You observe that when it is at an angle of $\pi / 4$

2 Related rate problems are problems in which you are given the rate of change of one quantity and are to determine the rate of change of another, related, quantity.
its angle is changing by 0.05 radians per second.

- Start by drawing a picture with the relevant variables

- So denote the angle to be $\theta$ (in radians), the height of the balloon (in m) by $h$ and time (in seconds) by $t$. Then trigonometry tells us

$$
h=200 \cdot \tan \theta
$$

- Differentiating allows us to relate the rates of change

$$
\frac{\mathrm{d} h}{\mathrm{~d} t}=200 \sec ^{2} \theta \cdot \frac{\mathrm{~d} \theta}{\mathrm{~d} t}
$$

- We are told that when $\theta=\pi / 4$ we observe $\frac{\mathrm{d} \theta}{\mathrm{d} t}=0.05$, so

$$
\begin{aligned}
\frac{\mathrm{d} h}{\mathrm{~d} t} & =200 \cdot \sec ^{2}(\pi / 4) \cdot 0.05 \\
& =200 \cdot 0.05 \cdot(\sqrt{2})^{2} \\
& =200 \cdot \frac{5}{100} \cdot 2 \\
& =20 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

- So the balloon is rising at a rate of $20 \mathrm{~m} / \mathrm{s}$.

The following problem is perhaps the classic related rate problem.

## Example 3.2.2 A sliding ladder

A 5 m ladder is leaning against a wall. The floor is quite slippery and the base of the ladder slides out from the wall at a rate of $1 \mathrm{~m} / \mathrm{s}$. How fast is the top of the ladder sliding down the wall when the base of the ladder is 3 m from the wall?

- A good first step is to draw a picture stating all relevant quantities. This will also help us define variables and units.

- So now define $x(t)$ to be the distance between the bottom of the ladder and the wall, at time $t$, and let $y(t)$ be the distance between the top of the ladder and the ground at time $t$. Measure time in seconds, but both distances in meters.
- We can relate the quantities using Pythagoras:

$$
x^{2}+y^{2}=5^{2}
$$

- Differentiating with respect to time then gives

$$
2 x \frac{\mathrm{~d} x}{\mathrm{~d} t}+2 y \frac{\mathrm{~d} y}{\mathrm{~d} t}=0
$$

- We know that $\frac{\mathrm{d} x}{\mathrm{~d} t}=1$ and $x=3$, so

$$
6 \cdot 1+2 y \frac{\mathrm{~d} y}{\mathrm{~d} t}=0
$$

but we need to determine $y$ before we can go further. Thankfully we know that $x^{2}+y^{2}=25$ and $x=3$, so $y^{2}=25-9=16$ and ${ }^{a}$ so $y=4$.

- So finally putting everything together

$$
\begin{aligned}
6 \cdot 1+8 \frac{\mathrm{~d} y}{\mathrm{~d} t} & =0 \\
\frac{\mathrm{~d} y}{\mathrm{~d} t} & =-\frac{3}{4} \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Thus the top of the ladder is sliding towards the floor at a rate of $\frac{3}{4} \mathrm{~m} / \mathrm{s}$.
$a \quad$ Since the ladder isn't buried in the ground, we can discard the solution $y=-4$.


The next example is complicated by the rates of change being stated not just as "the rate of change per unit time" but instead being stated as "the percentage rate of change per unit time". If a quantity $f$ is changing with rate $\frac{\mathrm{d} f}{\mathrm{~d} t}$, then we can say that
$f$ is changing at a rate of $100 \cdot \frac{\frac{\mathrm{~d} f}{\mathrm{~d} t}}{f}$ percent.
Thus if, at time $t, f$ has rate of change $r \%$, then

$$
100 \frac{f^{\prime}(t)}{f(t)}=r \Longrightarrow f^{\prime}(t)=\frac{r}{100} f(t)
$$

so that if $h$ is a very small time increment

$$
\frac{f(t+h)-f(t)}{h} \approx \frac{r}{100} f(t) \Longrightarrow f(t+h) \approx f(t)+\frac{r h}{100} f(t)
$$

That is, over a very small time interval $h, f$ increases by the fraction $\frac{r h}{100}$ of its value at time $t$.

So armed with this, let's look at the problem.
Example 3.2.3 Percentage rate of change of $R=P Q$.
The quantities $P, Q$ and $R$ are functions of time and are related by the equation $R=P Q$. Assume that $P$ is increasing instantaneously at the rate of $8 \%$ per year (meaning that $100 \frac{P^{\prime}}{P}=8$ ) and that $Q$ is decreasing instantaneously at the rate of $2 \%$ per year (meaning that $100 \frac{Q^{\prime}}{Q}=-2$ ). Determine the percentage rate of change for $R$. Solution This one is a little different - we are given the variables and the formula, so no picture drawing or defining required. Though we do need to define a time variable - let $t$ denote time in years.

- Since $R(t)=P(t) \cdot Q(t)$ we can differentiate with respect to $t$ to get

$$
\frac{\mathrm{d} R}{\mathrm{~d} t}=P Q^{\prime}+Q P^{\prime}
$$

- But we need the percentage change in $R$, namely

$$
100 \frac{R^{\prime}}{R}=100 \frac{P Q^{\prime}+Q P^{\prime}}{R}
$$

but $R=P Q$, so rewrite it as

$$
\begin{aligned}
& =100 \frac{P Q^{\prime}+Q P^{\prime}}{P Q} \\
& =100 \frac{P Q^{\prime}}{P Q}+100 \frac{Q P^{\prime}}{P Q} \\
& =100 \frac{Q^{\prime}}{Q}+100 \frac{P^{\prime}}{P}
\end{aligned}
$$

so we have stated the instantaneous percentage rate of change in $R$ as the sum of the percentage rate of change in $P$ and $Q$.

- We know the percentage rate of change of $P$ and $Q$, so

$$
100 \frac{R^{\prime}}{R}=-2+8=6
$$

That is, the instantaneous percentage rate of change of $R$ is $6 \%$ per year.
Example 3.2.3
Yet another falling object example.
Example 3.2.4 The shadow of a falling ball.
A ball is dropped from a height of 49 m above level ground. The height of the ball at time $t$ is $h(t)=49-4.9 t^{2} \mathrm{~m}$. A light, which is also 49 m above the ground, is 10 m to the left of the ball's original position. As the ball descends, the shadow of the ball caused by the light moves across the ground. How fast is the shadow moving one second after the ball is dropped?
Solution There is quite a bit going on in this example, so read carefully.

- First a diagram; the one below is perhaps a bit over the top.

- Let's call $s(t)$ the distance from the shadow to the point on the ground directly underneath the ball.
- By similar triangles we see that

$$
\frac{4.9 t^{2}}{10}=\frac{49-4.9 t^{2}}{s(t)}
$$

We can then solve for $s(t)$ by just multiplying both sides by $\frac{10}{4.9 t^{2}} s(t)$. This gives

$$
s(t)=10 \frac{49-4.9 t^{2}}{4.9 t^{2}}=\frac{100}{t^{2}}-10
$$

- Differentiating with respect to $t$ will then give us the rates,

$$
s^{\prime}(t)=-2 \frac{100}{t^{3}}
$$

- So, at $t=1, s^{\prime}(1)=-200 \mathrm{~m} / \mathrm{sec}$. That is, the shadow is moving to the left at $200 \mathrm{~m} / \mathrm{sec}$.

Example 3.2.4
A more nautical example.

Example 3.2.5 The distance between moving boats.
Two boats spot each other in the ocean at midday - Boat $A$ is 15 km west of Boat $B$. Boat $A$ is travelling east at $3 \mathrm{~km} / \mathrm{h}$ and boat $B$ is travelling north at $4 \mathrm{~km} / \mathrm{h}$. At 3 pm how fast is the distance between the boats changing.

- First we draw a picture.

- Let $x(t)$ be the distance at time $t$, in km , from boat $A$ to the original position of boat $B$ (i.e. to the position of boat $B$ at noon). And let $y(t)$ be the distance at time $t$, in km, of boat $B$ from its original position. And let $z(t)$ be the distance between the two boats at time $t$.
- Additionally we are told that $x^{\prime}=-3$ and $y^{\prime}=4$ - notice that $x^{\prime}<0$ since that distance is getting smaller with time, while $y^{\prime}>0$ since that distance is increasing with time.
- Further at $3 p m$ boat $A$ has travelled 9 km towards the original position of boat $B$, so $x=15-9=6$, while boat $B$ has travelled 12 km away from its original position, so $y=12$.
- The distances $x, y$ and $z$ form a right-angled triangle, and Pythagoras tells us that

$$
z^{2}=x^{2}+y^{2} .
$$

At 3 pm we know $x=6, y=12$ so

$$
\begin{aligned}
z^{2} & =36+144=180 \\
z & =\sqrt{180}=6 \sqrt{5}
\end{aligned}
$$

- Differentiating then gives

$$
2 z \frac{\mathrm{~d} z}{\mathrm{~d} t}=2 x \frac{\mathrm{~d} x}{\mathrm{~d} t}+2 y \frac{\mathrm{~d} y}{\mathrm{~d} t}
$$

$$
\begin{align*}
& =12 \cdot(-3)+24 \cdot(4)  \tag{4}\\
& =60 .
\end{align*}
$$

Dividing through by $2 z=12 \sqrt{5}$ then gives

$$
\frac{\mathrm{d} z}{\mathrm{~d} t}=\frac{60}{12 \sqrt{5}}=\frac{5}{\sqrt{5}}=\sqrt{5}
$$

So the distance between the boats is increasing at $\sqrt{5} \mathrm{~km} / \mathrm{h}$.
Example 3.2.5
One last one before we move on to another topic.
Example 3.2.6 Fuel level in a cylindrical tank.
Consider a cylindrical fuel tank of radius $r$ and length $L$ (in some appropriate units) that is lying on its side. Suppose that fuel is being pumped into the tank at a rate $q$. At what rate is the fuel level rising?


Solution If the tank were vertical everything would be much easier. Unfortunately the tank is on its side, so we are going to have to work a bit harder to establish the relation between the depth and volume. Also notice that we have not been supplied with units for this problem - so we do not need to state the units of our variables.

- Again - draw a picture. Here is an end view of the tank; the shaded part of the circle is filled with fuel.

- Let us denote by $V(t)$ the volume of fuel in the tank at time $t$ and by $h(t)$ the
fuel level at time $t$.
- We have been told that $V^{\prime}(t)=q$ and have been asked to determine $h^{\prime}(t)$. While it is possible to do so by finding a formula relating $V(t)$ and $h(t)$, it turns out to be quite a bit easier to first find a formula relating $V$ and the angle $\theta$ shown in the end view. We can then translate this back into a formula in terms of $h$ using the relation

$$
h(t)=r-r \cos \theta(t) .
$$

Once we know $\theta^{\prime}(t)$, we can easily obtain $h^{\prime}(t)$ by differentiating the above equation.

- The computation that follows below gets a little involved in places, so we will drop the " $(t)$ " on the variables $V, h$ and $\theta$. The reader must never forget that these three quantities are really functions of time, while $r$ and $L$ are constants that do not depend on time.
- The volume of fuel is $L$ times the cross-sectional area filled by the fuel. That is,

$$
V=L \times \text { Area of }
$$



While we do not have a canned formula for the area of a chord of a circle like this, it is easy to express the area of the chord in terms of two areas that we can compute.

$$
\begin{aligned}
V & =L \times \text { Area of } \\
& =L \times[\text { Area of }
\end{aligned}
$$



- The piece of pie

is the fraction $\frac{2 \theta}{2 \pi}$ of the full circle, so its area is $\frac{2 \theta}{2 \pi} \pi r^{2}=\theta r^{2}$.
- The triangle

has height $r \cos \theta$ and base $2 r \sin \theta$ and hence has area $\frac{1}{2}(r \cos \theta)(2 r \sin \theta)=$ $r^{2} \sin \theta \cos \theta=\frac{r^{2}}{2} \sin (2 \theta)$, where we have used a double-angle formula (see Appendix A.14).
- Subbing these two areas into the above expression for $V$ gives

$$
V=L \times\left[\theta r^{2}-\frac{r^{2}}{2} \sin 2 \theta\right]=\frac{L r^{2}}{2}[2 \theta-\sin 2 \theta]
$$

## Oof!

- Now we can differentiate to find the rate of change. Recalling that $V=V(t)$ and $\theta=\theta(t)$, while $r$ and $L$ are constants,

$$
\begin{aligned}
V^{\prime} & =\frac{L r^{2}}{2}\left[2 \theta^{\prime}-2 \cos 2 \theta \cdot \theta^{\prime}\right] \\
& =L r^{2} \cdot \theta^{\prime} \cdot[1-\cos 2 \theta]
\end{aligned}
$$

Solving this for $\theta^{\prime}$ and using $V^{\prime}=q$ gives

$$
\theta^{\prime}=\frac{q}{L r^{2}(1-\cos 2 \theta)}
$$

This is the rate at which $\theta$ is changing, but we need the rate at which $h$ is changing. We get this from

$$
\begin{aligned}
h & =r-r \cos \theta & \text { differentiating this gives } \\
h^{\prime} & =r \sin \theta \cdot \theta^{\prime} &
\end{aligned}
$$

Substituting our expression for $\theta^{\prime}$ into the expression for $h^{\prime}$ gives

$$
h^{\prime}=r \sin \theta \cdot \frac{q}{L r^{2}(1-\cos 2 \theta)}
$$

- We can clean this up a bit more - recall more double-angle formulas ${ }^{a}$

$$
\begin{array}{rlr}
h^{\prime} & =r \sin \theta \cdot \frac{q}{L r^{2}(1-\cos 2 \theta)} & \text { substitute } \cos 2 \theta=1-2 \sin ^{2} \theta \\
& =r \sin \theta \cdot \frac{q}{L r^{2} \cdot 2 \sin ^{2} \theta} & \text { now cancel } r \text { 's and a } \sin \theta \\
& =\frac{q}{2 L r \sin \theta} &
\end{array}
$$

- But we can clean this up even more - instead of writing this rate in terms of $\theta$ it is more natural to write it in terms of $h$ (since the initial problem is stated in terms of $h$ ). From the triangle

and Pythagoras we have

$$
\sin \theta=\frac{\sqrt{r^{2}-(r-h)^{2}}}{r}=\frac{\sqrt{2 r h-h^{2}}}{r}
$$

and hence

$$
h^{\prime}=\frac{q}{2 L \sqrt{2 r h-h^{2}}} .
$$

- As a check, notice that $h^{\prime}$ becomes undefined when $h<0$ and also when $h>2 r$, because then the argument of the square root in the denominator is negative. Both make sense - the fuel level in the tank must obey $0 \leq h \leq 2 r$.

[^7]Example 3.2.6

### 3.2.2 $\leadsto$ Exercises

## Exercises - Stage 1

1. Suppose the quantities $P$ and $Q$ are related by the formula $P=Q^{3}$. $P$ and $Q$ are changing with respect to time, $t$. Given this information, which of the following are problems you could solve?
i Given $\frac{\mathrm{d} P}{\mathrm{~d} t}(0)$, find $\frac{\mathrm{d} Q}{\mathrm{~d} t}(0)$. (Remember: the notation $\frac{\mathrm{d} P}{\mathrm{~d} t}(0)$ means the derivative of $P$ with respect to $t$ at the time $t=0$.)
ii Given $\frac{\mathrm{d} P}{\mathrm{~d} t}(0)$ and the value of $Q$ when $t=0$, find $\frac{\mathrm{d} Q}{\mathrm{~d} t}(0)$.
iii Given $\frac{\mathrm{d} Q}{\mathrm{~d} t}(0)$, find $\frac{\mathrm{d} P}{\mathrm{~d} t}(0)$.
iv Given $\frac{\mathrm{d} Q}{\mathrm{~d} t}(0)$ and the value of $P$ when $t=0$, find $\frac{\mathrm{d} P}{\mathrm{~d} t}(0)$.

Exercises - Stage 2 For problems 3.2.2.2 through 3.2.2.4, the relationship between several variables is explicitly given. Use this information to relate their rates of change.For Questions 3.2.2.5 through 3.2.2.9, look for a way to use the Pythagorean Theorem.For Questions 3.2.2.10 through 3.2.2.14, look for tricks from trigonometry.For Questions 3.2.2.15 through 3.2.2.20, you'll need to know formulas for volume or area.
2. *. A point is moving on the unit circle $\left\{(x, y): x^{2}+y^{2}=1\right\}$ in the $x y-$ plane. At $(2 / \sqrt{5}, 1 / \sqrt{5})$, its $y$-coordinate is increasing at rate 3 . What is the rate of change of its $x$-coordinate?
3. *. The quantities $P, Q$ and $R$ are functions of time and are related by the equation $R=P Q$. Assume that $P$ is increasing instantaneously at the rate of $8 \%$ per year and that $Q$ is decreasing instantaneously at the rate of $2 \%$ per year. That is, $\frac{P^{\prime}}{P}=0.08$ and $\frac{Q^{\prime}}{Q}=-0.02$. Determine the percentage rate of change for $R$.
4. *. Three quantities, $F, P$ and $Q$ all depend upon time $t$ and are related by the equation

$$
F=\frac{P}{Q}
$$

a Assume that at a particular moment in time $P=25$ and $P$ is increasing at the instantaneous rate of 5 units $/ \mathrm{min}$. At the same moment, $Q=5$ and $Q$ is increasing at the instantaneous rate of 1 unit $/ \mathrm{min}$. What is the instantaneous rate of change in $F$ at this moment?
b Assume that at another moment in time $P$ is increasing at the instantaneous rate of $10 \%$ and $Q$ is decreasing at the instantaneous rate $5 \%$. What can you conclude about the rate of change of $F$ at this moment?
5. *. Two particles move in the Cartesian plane. Particle A travels on the $x$-axis starting at $(10,0)$ and moving towards the origin with a speed of 2 units per second. Particle B travels on the $y$-axis starting at $(0,12)$ and moving towards the origin with a speed of 3 units per second. What is the rate of change of the distance between the two particles when particle A reaches the point $(4,0)$ ?
6. *. Two particles $A$ and $B$ are placed on the Cartesian plane at $(0,0)$ and $(3,0)$ respectively. At time 0 , both start to move in the $+y$ direction. Particle $A$ moves at 3 units per second, while $B$ moves at 2 units per second. How fast is the distance between the particles changing when particle $A$ is at a distance of 5 units from $B$.
7. *. Ship A is 400 miles directly south of Hawaii and is sailing south at 20 miles/hour. Ship B is 300 miles directly east of Hawaii and is sailing west at 15 miles/hour. At what rate is the distance between the ships changing?
8. *. Two tall sticks are vertically planted into the ground, separated by a distance of 30 cm . We simultaneously put two snails at the base of each stick. The two snails then begin to climb their respective sticks. The first snail is moving with a speed of 25 cm per minute, while the second snail is moving with a speed of 15 cm per minute. What is the rate of change of the distance between the two snails when the first snail reaches 100 cm above the ground?
9. *. A 20 m long extension ladder leaning against a wall starts collapsing in on itself at a rate of $2 \mathrm{~m} / \mathrm{s}$, while the foot of the ladder remains a constant 5 m from the wall. How fast is the ladder moving down the wall after 3.5 seconds?
10. A watering trough has a cross section shaped like an isosceles trapezoid. The trough is 2 metres long, 50 cm high, 1 metre wide at the top, and 60 cm wide at the bottom.


A pig is drinking water from the trough at a rate of 3 litres per minute. When the height of the water is 25 cm , how fast is the height decreasing?
11. A tank is 5 metres long, and has a trapezoidal cross section with the dimensions shown below.


A hose is filling the tank up at a rate of one litre per second. How fast is the height of the water increasing when the water is 10 centimetres deep?
12. A rocket is blasting off, 2 kilometres away from you. You and the rocket start at the same height. The height of the rocket in kilometres, $t$ hours after liftoff, is given by

$$
h(t)=61750 t^{2}
$$

How fast (in radians per second) is your line of sight rotating to keep looking at the rocket, one minute after liftoff?
13. *. A high speed train is traveling at $2 \mathrm{~km} / \mathrm{min}$ along a straight track. The train is moving away from a movie camera which is located 0.5 km from the track.
a How fast is the distance between the train and the camera increasing when they are 1.3 km apart?
b Assuming that the camera is always pointed at the train, how fast (in radians per min) is the camera rotating when the train and the camera are 1.3 km apart?
14. A clock has a minute hand that is 10 cm long, and an hour hand that is 5 cm long. Let $D$ be the distance between the tips of the two hands. How fast is $D$ decreasing at 4:00?

15. *. Find the rate of change of the area of the annulus $\left\{(x, y): r^{2} \leq x^{2}+y^{2} \leq\right.$ $\left.R^{2}\right\}$. (i.e. the points inside the circle of radius $R$ but outside the circle of radius $r$ ) if $R=3 \mathrm{~cm}, r=1 \mathrm{~cm}, \frac{\mathrm{~d} R}{\mathrm{~d} t}=2 \frac{\mathrm{~cm}}{\mathrm{~s}}$, and $\frac{\mathrm{d} r}{\mathrm{~d} t}=7 \frac{\mathrm{~cm}}{\mathrm{~s}}$.

16. Two spheres are centred at the same point. The radius $R$ of the bigger sphere at time $t$ is given by $R(t)=10+2 t$, while the radius $r$ of the smaller sphere is given by $r(t)=6 t, t \geq 0$. How fast is the volume between the spheres (inside the big sphere and outside the small sphere) changing when the bigger sphere has a radius twice as large as the smaller?
17. You attach two sticks together at their ends, and stick the other ends in the mud. One stick is 150 cm long, and the other is 200 cm .


The structure starts out being 1.4 metres high at its peak, but the sticks slide, and the height decreases at a constant rate of three centimetres per minute. How quickly is the area of the triangle (formed by the two sticks and the level ground) changing when the height of the structure is 120 cm ?
18. The circular lid of a salt shaker has radius 8 . There is a cut-out to allow the salt to pour out of the lid, and a door that rotates around to cover the cut-out. The door is a quarter-circle of radius 7 cm . The cut-out has the shape of a quarter-annulus with outer radius 6 cm and inner radius 1 cm . If the uncovered area of the cut-out is $A \mathrm{~cm}^{2}$, then the salt flows out at $\frac{1}{5} A \mathrm{~cm}^{3}$ per second.


Recall: an annulus is the set of points inside one circle and outside another, like a flat doughnut (see Question 3.2.2.15).

annulus

quarter annulus

While pouring out salt, you spin the door around the lid at a constant rate of $\frac{\pi}{6}$ radians per second, covering more and more of the cut-out. When exactly half of the cut-out is covered, how fast is the flow of salt changing?
19. A cylindrical sewer pipe with radius 1 metre has a vertical rectangular door that slides in front of it to block the flow of water, as shown below. If the uncovered area of the pipe is $A \mathrm{~m}^{2}$, then the flow of water through the pipe is $\frac{1}{5} A$ cubic metres per second.
The door slides over the pipe, moving vertically at a rate of 1 centimetre per second. How fast is the flow of water changing when the door covers the top 25 centimetres of the pipe?

20. A martini glass is shaped like a cone, with top diameter 10 cm and side length 10 cm .


When the liquid in the glass is 7 cm high, it is evaporating at a rate of 5 mL per minute. How fast is the height of the liquid decreasing?

## Exercises - Stage 3

21. A floating buoy is anchored to the bottom of a river. As the river flows, the buoy is pulled in the direction of flow until its 2-metre rope is taut. A sensor at the anchor reads the angle $\theta$ between the rope and the riverbed, as shown in the diagram below. This data is used to measure the depth $D$ of water in the river, which depends on time.

a If $\theta=\frac{\pi}{4}$ and $\frac{\mathrm{d} \theta}{\mathrm{d} t}=0.25 \frac{\mathrm{rad}}{\mathrm{hr}}$, how fast is the depth $D$ of the water changing?
b A measurement shows $\frac{\mathrm{d} \theta}{\mathrm{d} t}=0$, but $\frac{\mathrm{d} D}{\mathrm{~d} t} \neq 0$. Under what circumstances does this occur?
c A measurement shows $\frac{\mathrm{d} \theta}{\mathrm{d} t}>0$, but $\frac{\mathrm{d} D}{\mathrm{~d} t}<0$. Under what circumstances does this occur?
22. A point is moving in the $x y$-plane along the quadrilateral shown below.

a When the point is at $(0,-2)$, it is moving to the right. An observer stationed at the origin must turn at a rate of one radian per second to keep looking directly at the point. How fast is the point moving?
b When the point is at $(0,2)$, its $x$-coordinate is increasing at a rate of one unit per second. How fast it its $y$-coordinate changing? How fast is the point moving?
23. You have a cylindrical water bottle 20 cm high, filled with water. Its cross section is a circle of radius 5 . You slowly smoosh the sides, so the cross section becomes an ellipse with major axis (widest part) $2 a$ and minor axis (skinniest
part) $2 b$.


After $t$ seconds of smooshing the bottle, $a=5+t \mathrm{~cm}$. The perimeter of the cross section is unchanged as the bottle deforms. The perimeter of an ellipse is actually quite difficult to calculate, but we will use an approximation derived by Ramanujan and assume that the perimeter $p$ of our ellipse is

$$
p \approx \pi[3(a+b)-\sqrt{(a+3 b)(3 a+b)}] .
$$

The area of an ellipse is $\pi a b$.
a Give an equation that relates $a$ and $b$ (and no other variables).
b Give an expression for the volume of the bottle as it is being smooshed, in terms of $a$ and $b$ (and no other variables).
c Suppose the bottle was full when its cross section was a circle. How fast is the water spilling out when $a$ is twice as big as $b$ ?
24. The quantities $A, B, C$, and $D$ all depend on time, and are related by the formula

$$
A B=\log \left(C^{2}+D^{2}+1\right)
$$

At time $t=10$, the following values are known:

- $A=0$
- $\frac{\mathrm{d} A}{\mathrm{~d} t}=2$ units per second

What is $B$ when $t=10$ ?

### 3.3 Exponential Growth and Decay - a First Look at Differential Equations

A differential equation is an equation for an unknown function that involves the derivative of the unknown function. For example, Newton's law of cooling says:

The rate of change of temperature of an object is proportional to the difference in temperature between the object and its surroundings.

We can write this more mathematically using a differential equation - an equation for the unknown function $T(t)$ that also involves its derivative $\frac{\mathrm{d} T}{\mathrm{~d} t}(t)$. If we denote by $T(t)$ the temperature of the object at time $t$ and by $A$ the temperature of its surroundings, Newton's law of cooling says that there is some constant of proportionality, $K$, such that

$$
\frac{\mathrm{d} T}{\mathrm{~d} t}(t)=K[T(t)-A]
$$

Differential equations play a central role in modelling a huge number of different phenomena, including the motion of particles, electromagnetic radiation, financial options, ecosystem populations and nerve action potentials. Most universities offer half a dozen different undergraduate courses on various aspects of differential equations. We are barely going to scratch the surface of the subject. At this point we are going to restrict ourselves to a few very simple differential equations for which we can just guess the solution. In particular, we shall learn how to solve systems obeying Newton's law of cooling in Section 3.3.2, below. But first, here is another slightly simpler example.

### 3.3.1 Carbon Dating

Scientists can determine the age of objects containing organic material by a method called carbon dating or radiocarbon dating ${ }^{1}$. Cosmic rays hitting the atmosphere convert nitrogen into a radioactive isotope of carbon, ${ }^{14} C$, with a half-life of about 5730 years ${ }^{2}$. Vegetation absorbs carbon dioxide from the atmosphere through photosynthesis and animals acquire ${ }^{14} C$ by eating plants. When a plant or animal dies, it stops replacing its carbon and the amount of ${ }^{14} \mathrm{C}$ begins to decrease through radioactive decay. More precisely, let $Q(t)$ denote the amount of ${ }^{14} C$ in the plant or animal $t$ years after it dies. The number of radioactive decays per unit time, at time $t$, is proportional to the amount of ${ }^{14} C$ present at time $t$, which is $Q(t)$. Thus

[^8]
## Equation 3.3.1 Radioactive decay.

$$
\frac{\mathrm{d} Q}{\mathrm{~d} t}(t)=-k Q(t)
$$

Here $k$ is a constant of proportionality that is determined by the half-life. We shall explain what half-life is and also determine the value of $k$ in Example 3.3.3, below. Before we do so, let's think about the sign in equation 3.3.1.

- Recall that $Q(t)$ denotes a quantity, namely the amount of ${ }^{14} C$ present at time $t$. There cannot be a negative amount of ${ }^{14} C$, nor can this quantity be zero (otherwise we wouldn't use carbon dating, so we must have $Q(t)>0$.
- As the time $t$ increases, $Q(t)$ decreases, because ${ }^{14} C$ is being continuously converted into ${ }^{14} N$ by radioactive decay ${ }^{3}$. Thus $\frac{\mathrm{d} Q}{\mathrm{~d} t}(t)<0$.
- The signs $Q(t)>0$ and $\frac{\mathrm{d} Q}{\mathrm{~d} t}(t)<0$ are consistent with equation 3.3.1 provided the constant of proportionality $k>0$.
- In equation 3.3.1, we chose to call the constant of proportionality " $-k$ ". We did so in order to make $k>0$. We could just as well have chosen to call the constant of proportionality " $K$ ". That is, we could have replaced equation 3.3.1 by $\frac{\mathrm{d} Q}{\mathrm{~d} t}(t)=K Q(t)$. The constant of proportionality $K$ would have to be negative, (and $K$ and $k$ would be related by $K=-k$ ).

Now, let's guess some solutions to equation 3.3.1. We wish to guess a function $Q(t)$ whose derivative is just a constant times itself. Here is a short table of derivatives. It is certainly not complete, but it contains the most important derivatives that we know.

| $F(t)$ | 1 | $t^{a}$ | $\sin t$ | $\cos t$ | $\tan t$ | $e^{t}$ | $\log t$ | $\arcsin t$ | $\arctan t$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\frac{\mathrm{~d}}{\mathrm{~d} t} F(t)$ | 0 | $a t^{a-1}$ | $\cos t$ | $-\sin t$ | $\sec ^{2} t$ | $e^{t}$ | $\frac{1}{t}$ | $\frac{1}{\sqrt{1-t^{2}}}$ | $\frac{1}{1+t^{2}}$ |

There is exactly one function in this table whose derivative is just a (nonzero) constant times itself. Namely, the derivative of $e^{t}$ is exactly $e^{t}=1 \times e^{t}$. This is almost, but not quite what we want. We want the derivative of $Q(t)$ to be the constant $-k$ (rather than the constant 1) times $Q(t)$. We want the derivative to "pull a constant" out of our guess. That is exactly what happens when we differentiate $e^{a t}$, where $a$ is a constant. Differentiating gives


3 The precise transition is ${ }^{14} C \rightarrow{ }^{14} N+e^{-}+\bar{\nu}_{e}$ where $e^{-}$is an electron and $\bar{\nu}_{e}$ is an electron neutrino.
i.e. "pulls the constant $a$ out of $e^{a t "}$.

We have succeeded in guessing a single function, namely $e^{-k t}$, that obeys equation 3.3.1. Can we guess any other solutions? Yes. If $C$ is any constant, $C e^{-k t}$ also obeys equation 3.3.1:

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\left(C e^{-k t}\right)=C \frac{\mathrm{~d}}{\mathrm{~d} t} e^{-k t}=C e^{-k t}(-k)=-k\left(C e^{-k t}\right)
$$

You can try guessing some more solutions, but you won't find any, because with a little trickery we can prove that a function $Q(t)$ obeys equation 3.3.1 if and only if $Q(t)$ is of the form $C e^{-k t}$, where $C$ is some constant.

The trick ${ }^{4}$ is to imagine that $Q(t)$ is any (at this stage, unknown) solution to equation 3.3.1 and to compare $Q(t)$ and our known solution $e^{-k t}$ by studying the ratio $Q(t) / e^{-k t}$. We will show that $Q(t)$ obeys equation 3.3 .1 if and only if the ratio $Q(t) / e^{-k t}$ is a constant, i.e. if and only if the derivative of the ratio is zero. By the product rule

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\left[Q(t) / e^{-k t}\right]=\frac{\mathrm{d}}{\mathrm{~d} t}\left[e^{k t} Q(t)\right]=k e^{k t} Q(t)+e^{k t} Q^{\prime}(t)
$$

Since $e^{k t}$ is never 0 , the right hand side is zero if and only if $k Q(t)+Q^{\prime}(t)=0$; that is $Q^{\prime}(t)=-k Q(t)$. Thus

$$
\frac{\mathrm{d}}{\mathrm{~d} t} Q(t)=-k Q(t) \Longleftrightarrow \frac{\mathrm{d}}{\mathrm{~d} t}\left[Q(t) / e^{-k t}\right]=0
$$

as required.
We have succeed in finding all functions that obey 3.3.1. That is we have found the general solution to 3.3.1. This is worth stating as a theorem.

## Theorem 3.3.2

A differentiable function $Q(t)$ obeys the differential equation

$$
\frac{\mathrm{d} Q}{\mathrm{~d} t}(t)=-k Q(t)
$$

if and only if there is a constant $C$ such that

$$
Q(t)=C e^{-k t}
$$

Before we start to apply the above theorem, we take this opportunity to remind the reader that in this text we will use $\log x$ with no base to indicate the natural logarithm. That is

$$
\log x=\log _{e} x=\ln x
$$

Both of the notations $\log (x)$ and $\ln (x)$ are used widely and the reader should be comfortable with both.

4 Notice that is very similar to what we needed in Example 3.1.2, except that here the constant is multiplicative rather than additive. That is const $\times f(t)$ rather than const $+f(t)$.

Example 3.3.3 Carbon dating and half-life.
In this example, we determine the value of the constant of proportionality $k$ in equation 3.3.1 that corresponds to the half-life of ${ }^{14} C$, which is 5730 years.

- Imagine that some plant or animal contains a quantity $Q_{0}$ of ${ }^{14} C$ at its time of death. Let's choose the zero point of time $t=0$ to be the instant that the plant or animal died.
- Denote by $Q(t)$ the amount of ${ }^{14} C$ in the plant or animal $t$ years after it died. Then $Q(t)$ must obey both equation 3.3.1 and $Q(0)=Q_{0}$.
- Since $Q(t)$ must obey equation 3.3.1, Theorem 3.3.2 tells us that there must be a constant $C$ such that $Q(t)=C e^{-k t}$. To also have $Q_{0}=Q(0)=C e^{-k \times 0}$, the constant $C$ must be $Q_{0}$. That is, $Q(t)=Q_{0} e^{-k t}$ for all $t \geq 0$.
- By definition, the half-life of ${ }^{14} C$ is the length of time that it takes for half of the ${ }^{14} C$ to decay. That is, the half-life $t_{1 / 2}$ is determined by

$$
\begin{aligned}
Q\left(t_{1 / 2}\right)=\frac{1}{2} Q(0) & =\frac{1}{2} Q_{0} & \text { but we know } Q(t)=Q_{0} e^{-k t} \\
Q_{0} e^{-k t_{1 / 2}} & =\frac{1}{2} Q_{0} & \text { now cancel } Q_{0} \\
e^{-k t_{1 / 2}} & =\frac{1}{2} &
\end{aligned}
$$

Taking the logarithm of both sides gives

$$
\begin{aligned}
-k t_{1 / 2} & =\log \frac{1}{2}=-\log 2 \quad \text { and so } \\
k & =\frac{\log 2}{t_{1 / 2}}
\end{aligned}
$$

We are told that, for ${ }^{14} C$, the half-life $t_{1 / 2}=5730$, so

$$
k=\frac{\log 2}{5730}=0.000121 \quad \text { to } 6 \text { digits }
$$

From the work in the above example we have accumulated enough new facts to make a corollary to Theorem 3.3.2.

## Corollary 3.3.4

The function $Q(t)$ satisfies the equation

$$
\frac{\mathrm{d} Q}{\mathrm{~d} t}=-k Q(t)
$$

if and only if

$$
Q(t)=Q(0) \cdot e^{-k t}
$$

The half-life is defined to be the time $t_{1 / 2}$ which obeys

$$
Q\left(t_{1 / 2}\right)=\frac{1}{2} \cdot Q(0)
$$

The half-life is related to the constant $k$ by

$$
t_{1 / 2}=\frac{\log 2}{k}
$$

Now here is a typical problem that is solved using Corollary 3.3.4.
Example 3.3.5 Determining the age of an artefact.
A particular piece of parchment contains about $64 \%$ as much ${ }^{14} C$ as plants do today. Estimate the age of the parchment.
Solution Let $Q(t)$ denote the amount of ${ }^{14} C$ in the parchment $t$ years after it was first created.
By equation 3.3.1 and Example 3.3.3,

$$
\frac{\mathrm{d} Q}{\mathrm{~d} t}=-k Q(t) \quad \text { with } k=\frac{\log 2}{5730}=0.000121
$$

By Corollary 3.3.4

$$
Q(t)=Q(0) \cdot e^{-k t}
$$

The time at which $Q(t)$ reaches $0.64 Q(0)$ is determined by

$$
\begin{array}{rlrl}
Q(t) & =0.64 Q(0) & \text { but } Q(t)=Q(0) e^{-k t} \\
Q(0) e^{-k t} & =0.64 Q(0) & & \text { cancel } Q(0) \\
e^{-k t} & =0.64 & & \text { take logarithms } \\
-k t & =\log 0.64 & & \\
t & =\frac{\log 0.64}{-k}=\frac{\log 0.64}{-0.000121}=3700 & \text { to 2significant digits. }
\end{array}
$$

That is, the parchment ${ }^{a}$ is about 37 centuries old.
$\uparrow a \quad$ The British Museum has an Egyptian mathematical text from the seventeenth century B.C.

We have stated that the half-life of ${ }^{14} C$ is 5730 years. How can this be determined? We can explain this using the following example.

Example 3.3.6 Computing a half-life.
A scientist in a B-grade science fiction film is studying a sample of the rare and fictitious element, implausium ${ }^{a}$. With great effort he has produced a sample of pure implausium. The next day - 17 hours later - he comes back to his lab and discovers that his sample is now only $37 \%$ pure. What is the half-life of the element?
Solution We can again set up our problem using Corollary 3.3.4. Let $Q(t)$ denote the quantity of implausium at time $t$, measured in hours. Then we know

$$
Q(t)=Q(0) \cdot e^{-k t}
$$

We also know that

$$
Q(17)=0.37 Q(0)
$$

That enables us to determine $k$ via

$$
\begin{array}{rlrl}
Q(17)=0.37 Q(0) & =Q(0) e^{-17 k} \quad \quad \text { divide both sides by } Q(0) \\
0.37 & =e^{-17 k} &
\end{array}
$$

and so

$$
k=-\frac{\log 0.37}{17}=0.05849
$$

We can then convert this to the half life using Corollary 3.3.4:

$$
t_{1 / 2}=\frac{\log 2}{k} \approx 11.85 \text { hours }
$$

While this example is entirely fictitious, one really can use this approach to measure the half-life of materials.


### 3.3.2 Newton's Law of Cooling

Recall Newton's law of cooling from the start of this section:
The rate of change of temperature of an object is proportional to the difference in temperature between the object and its surroundings.

The temperature of the surroundings is sometimes called the ambient temperature. We then translated this statement into the following differential equation

Equation 3.3.7 Newton's law of cooling.

$$
\frac{\mathrm{d} T}{\mathrm{~d} t}(t)=K[T(t)-A]
$$

where $T(t)$ is the temperature of the object at time $t, A$ is the temperature of its surroundings, and $K$ is a constant of proportionality. This mathematical model of temperature change works well when studying a small object in a large, fixed temperature, environment. For example, a hot cup of coffee in a large room ${ }^{5}$.

Before we worry about solving this equation, let's think a little about the sign of the constant of proportionality. At any time $t$, there are three possibilities.

- If $T(t)>A$, that is, if the body is warmer than its surroundings, we would expect heat to flow from the body into its surroundings and so we would expect the body to cool off so that $\frac{\mathrm{d} T}{\mathrm{~d} t}(t)<0$. For this expectation to be consistent with equation 3.3.7, we need $K<0$.
- If $T(t)<A$, that is the body is cooler than its surroundings, we would expect heat to flow from the surroundings into the body and so we would expect the body to warm up so that $\frac{\mathrm{d} T}{\mathrm{~d} t}(t)>0$. For this expectation to be consistent with equation 3.3.7, we again need $K<0$.
- Finally if $T(t)=A$, that is the body and its environment have the same temperature, we would not expect any heat to flow between the two and so we would expect that $\frac{\mathrm{d} T}{\mathrm{~d} t}(t)=0$. This does not impose any condition on $K$.

In conclusion, we would expect $K<0$. Of course, we could have chosen to call the constant of proportionality $-k$, rather than $K$. Then the differential equation would be $\frac{\mathrm{d} T}{\mathrm{~d} t}=-k(T-A)$ and we would expect $k>0$.

Now to find the general solution to equation 3.3.7. Since this equation is so similar in form to equation 3.3.1, we might expect a similar solution. Start by trying $T(t)=C e^{K t}$ and let's see what goes wrong. Substitute it into the equation:

$$
\begin{aligned}
\frac{\mathrm{d} T}{\mathrm{~d} t} & =K(T(t)-A) \\
K C e^{K t} & =K C e^{K T}-K A
\end{aligned}
$$

$$
? 0=-K A ? \quad \text { the constant } A \text { causes problems! }
$$

5 It does not work so well when the object is of a similar size to its surroundings since the temperature of the surroundings will rise as the object cools. It also fails when there are phase transitions involved - for example, an ice-cube melting in a warm room does not obey Newton's law of cooling.

Let's try something a little different - recall that the derivative of a constant is zero. So we can add or subtract a constant from $T(t)$ without changing its derivative. Set $Q(t)=T(t)+B$, then

$$
\begin{array}{rlr}
\frac{\mathrm{d} Q}{\mathrm{~d} t}(t) & =\frac{\mathrm{d} T}{\mathrm{~d} t}(t) \quad \quad \text { by Newton's law of cooling } \\
& =K(T(t)-A)=K(Q(t)-B-A) \quad
\end{array}
$$

So if we choose $B=-A$ then we will have

$$
\frac{\mathrm{d} Q}{\mathrm{~d} t}(t)=K Q(t)
$$

which is exactly the same form as equation 3.3.1, but with $K=-k$. So by Theorem 3.3.2

$$
Q(t)=Q(0) e^{K t}
$$

We can translate back to $T(t)$, since $Q(t)=T(t)-A$ and $Q(0)=T(0)-A$. This gives us the solution.

## Corollary 3.3.8

A differentiable function $T(t)$ obeys the differential equation

$$
\frac{\mathrm{d} T}{\mathrm{~d} t}(t)=K[T(t)-A]
$$

if and only if

$$
T(t)=[T(0)-A] e^{K t}+A
$$

Just before we put this into action, we remind the reader that $\log x=\log _{e} x=\ln x$.
Example 3.3.9 Warming iced tea.
The temperature of a glass of iced tea is initially $5^{\circ}$. After 5 minutes, the tea has heated to $10^{\circ}$ in a room where the air temperature is $30^{\circ}$.
a Determine the temperature as a function of time.
b What is the temperature after 10 minutes?
c Determine when the tea will reach a temperature of $20^{\circ}$.
Solution Part (a)

- Denote by $T(t)$ the temperature of the tea $t$ minutes after it was removed from the fridge, and let $A=30$ be the ambient temperature.
- By Newton's law of cooling,

$$
\frac{\mathrm{d} T}{\mathrm{~d} t}=K(T-A)=K(T-30)
$$

for some, as yet unknown, constant of proportionality $K$.

- By Corollary 3.3.8,

$$
T(t)=[T(0)-30] e^{K t}+30=30-25 e^{K t}
$$

since the initial temperature $T(0)=5$.

- This solution is not complete because it still contains an unknown constant, namely $K$. We have not yet used the given data that $T(5)=10$. We can use it to determine $K$. At $t=5$,

$$
\begin{aligned}
T(5) & =30-25 e^{5 K}=10 & & \text { rearrange } \\
e^{5 K} & =\frac{20}{25} & & \\
5 K & =\log \frac{20}{25} & & \text { and so } \\
K & =\frac{1}{5} \log \frac{4}{5}=-0.044629 & & \text { to } 6 \text { digits }
\end{aligned}
$$

## Part (b)

- To find the temperature at 10 minutes we can just use the solution we have determined above.

$$
\begin{aligned}
T(10) & =30-25 e^{10 K} \\
& =30-25 e^{10 \times \frac{1}{5} \log \frac{4}{5}} \\
& =30-25 e^{2 \log \frac{4}{5}}=30-25 e^{\log \frac{16}{25}} \\
& =30-16=14^{\circ}
\end{aligned}
$$

Part (c)

- We can find when the temperature is $20^{\circ}$ by solving $T(t)=20$ :

$$
\begin{array}{rlr}
20 & =30-25 e^{K t} & \\
e^{K t} & =\frac{10}{25}=\frac{2}{5} & \text { rearrange } \\
K t & =\log \frac{2}{5} & \\
t & =\frac{\log \frac{2}{5}}{K} & \\
& =20.5 \text { minutes } & \text { to } 1 \text { decimal place }
\end{array}
$$

A slightly more gruesome example.
Example 3.3.10 Determining a time from temperatures.
A dead body is discovered at $3: 45 \mathrm{pm}$ in a room where the temperature is $20^{\circ} \mathrm{C}$. At that time the temperature of the body is $27^{\circ} \mathrm{C}$. Two hours later, at $5: 45 \mathrm{pm}$, the temperature of the body is $25.3^{\circ} \mathrm{C}$. What was the time of death? Note that the normal (adult human) body temperature is $37^{\circ}$.
Solution We will assume ${ }^{a}$ that the body's temperature obeys Newton's law of cooling.

- Denote by $T(t)$ the temperature of the body at time $t$, with $t=0$ corresponding to $3: 45 \mathrm{pm}$. We wish to find the time of death - call it $t_{d}$.
- There is a lot of data in the statement of the problem; we are told that
- the ambient temperature: $A=20$
- the temperature of the body when discovered: $T(0)=27$
- the temperature of the body 2 hours later: $T(2)=25.3$
- assuming the person was a healthy adult right up until he died, the temperature at the time of death: $T\left(t_{d}\right)=37$.
- Since we assume the temperature of the body obeys Newton's law of cooling, we use Corollary 3.3.8 to find,

$$
T(t)=[T(0)-A] e^{K t}+A=20+7 e^{K t}
$$

Two unknowns remain, $K$ and $t_{d}$.

- We can find the constant $K$ by using $T(2)=25.3$ :

$$
\begin{array}{rlrl}
25.3=T(2) & =20+7 e^{2 K} & \text { rearrange } \\
7 e^{2 K} & =5.3 & & \text { rearrange a bit more } \\
2 K & =\log \left(\frac{5.3}{7}\right) & & \\
K & =\frac{1}{2} \log \left(\frac{5.3}{7}\right)=-0.139 & & \text { to } 3 \text { decimal places }
\end{array}
$$

- Since we know ${ }^{b}$ that $t_{d}$ is determined by $T\left(t_{d}\right)=37$, we have

$$
\begin{array}{rlrl}
37=T\left(t_{d}\right) & =20+7 e^{-0.139 t_{d}} & & \text { rearrange } \\
e^{-0.139 t_{d}} & =\frac{17}{7} & & \\
-0.139 t_{d} & =\log \left(\frac{17}{7}\right) & \\
t_{d} & =-\frac{1}{0.139} \log \left(\frac{17}{7}\right) & & \\
& =-6.38 & & \text { to } 2 \text { decimal places }
\end{array}
$$

Now 6.38 hours is 6 hours and $0.38 \times 60=23$ minutes. So the time of death was 6 hours and 23 minutes before $3: 45 \mathrm{pm}$, which is $9: 22 \mathrm{am}$.
$a \quad$ We don't know any other method!
$b$ Actually, we are assuming again.

A slightly tricky example - we need to determine the ambient temperature from three measurements at different times.

Example 3.3.11 Finding the temperature outside.
A glass of room-temperature water is carried out onto a balcony from an apartment where the temperature is $22^{\circ} \mathrm{C}$. After one minute the water has temperature $26^{\circ} \mathrm{C}$ and after two minutes it has temperature $28^{\circ} \mathrm{C}$. What is the outdoor temperature?
Solution We will assume that the temperature of the thermometer obeys Newton's law of cooling.

- Let $A$ be the outdoor temperature and $T(t)$ be the temperature of the water $t$ minutes after it is taken outside.
- By Newton's law of cooling,

$$
T(t)=A+(T(0)-A) e^{K t}
$$

by Corollary 3.3.8. Notice there are 3 unknowns here $-A, T(0)$ and $K-$ so we need three pieces of information to find them all.

- We are told $T(0)=22$, so

$$
T(t)=A+(22-A) e^{K t}
$$

- We are also told $T(1)=26$, which gives

$$
\begin{array}{rlr}
26 & =A+(22-A) e^{K} & \text { rearrange things } \\
e^{K} & =\frac{26-A}{22-A} &
\end{array}
$$

- Finally, $T(2)=28$, so

$$
\begin{array}{rlr}
28 & =A+(22-A) e^{2 K} & \text { rearrange } \\
e^{2 K} & =\frac{28-A}{22-A} & \text { but } e^{K}=\frac{26-A}{22-A}, \text { so } \\
\left(\frac{26-A}{22-A}\right)^{2} & =\frac{28-A}{22-A} & \text { multiply through by }(22-A)^{2} \\
(26-A)^{2} & =(28-A)(22-A) &
\end{array}
$$

We can expand out both sides and collect up terms to get

$$
\begin{aligned}
\underbrace{26^{2}}_{=676}-52 A+A^{2} & =\underbrace{28 \times 22}_{=616}-50 A+A^{2} \\
60 & =2 A \\
30 & =A
\end{aligned}
$$

So the temperature outside is $30^{\circ}$.

### 3.3.3 $\leadsto$ Population Growth

Suppose that we wish to predict the size $P(t)$ of a population as a function of the time $t$. In the most naive model of population growth, each couple produces $\beta$ offspring (for some constant $\beta$ ) and then dies. Thus over the course of one generation $\beta \frac{P(t)}{2}$ children are produced and $P(t)$ parents die so that the size of the population grows from $P(t)$ to

$$
P\left(t+t_{g}\right)=\underbrace{P(t)+\beta \frac{P(t)}{2}}_{\text {parents }+ \text { offspring }}-\underbrace{P(t)}_{\text {parents die }}=\frac{\beta}{2} P(t)
$$

where $t_{g}$ denotes the lifespan of one generation. The rate of change of the size of the population per unit time is

$$
\frac{P\left(t+t_{g}\right)-P(t)}{t_{g}}=\frac{1}{t_{g}}\left[\frac{\beta}{2} P(t)-P(t)\right]=b P(t)
$$

where $b=\frac{\beta-2}{2 t_{g}}$ is the net birthrate per member of the population per unit time. If we approximate

$$
\frac{P\left(t+t_{g}\right)-P(t)}{t_{g}} \approx \frac{\mathrm{~d} P}{\mathrm{~d} t}(t)
$$

we get the differential equation

Equation 3.3.12 Simple population model.

$$
\frac{\mathrm{d} P}{\mathrm{~d} t}=b P(t)
$$

By Corollary 3.3.4, with $-k$ replaced by $b$,

$$
P(t)=P(0) \cdot e^{b t}
$$

This is called the Malthusian ${ }^{6}$ growth model. It is, of course, very simplistic. One of its main characteristics is that, since $P(t+T)=P(0) \cdot e^{b(t+T)}=P(t) \cdot e^{b T}$, every time you add $T$ to the time, the population size is multiplied by $e^{b T}$. In particular, the population size doubles every $\frac{\log 2}{b}$ units of time. The Malthusian growth model can be a reasonably good model only when the population size is very small compared to its environment ${ }^{7}$. A more sophisticated model of population growth, that takes into account the "carrying capacity of the environment" is considered in the optional subsection below.

6 This is named after Rev. Thomas Robert Malthus. He described this model in a 1798 paper called "An essay on the principle of population".
7 That is, the population has plenty of food and space to grow.

Example 3.3.13 A simple prediction of future population.
In 1927 the population of the world was about 2 billion. In 1974 it was about 4 billion. Estimate when it reached 6 billion. What will the population of the world be in 2100 , assuming the Malthusian growth model?
Solution We follow our usual pattern for dealing with such problems.

- Let $P(t)$ be the world's population $t$ years after 1927. Note that 1974 corresponds to $t=1974-1927=47$.
- We are assuming that $P(t)$ obeys equation 3.3.12. So, by Corollary 3.3.4 with $-k$ replaced by $b$,

$$
P(t)=P(0) \cdot e^{b t}
$$

Notice that there are 2 unknowns here - $b$ and $P(0)$ - so we need two pieces of information to find them.

- We are told $P(0)=2$, so

$$
P(t)=2 \cdot e^{b t}
$$

- We are also told $P(47)=4$, which gives

$$
\begin{aligned}
4 & =2 \cdot e^{47 b} & \text { clean up } \\
e^{47 b} & =2 & \text { take the log and clean up } \\
b & =\frac{\log 2}{47}=0.0147 & \text { to } 3 \text { significant digits }
\end{aligned}
$$

- We now know $P(t)$ completely, so we can easily determine the predicted population ${ }^{a}$ in 2100, i.e. at $t=2100-1927=173$.

$$
P(173)=2 e^{173 b}=2 e^{173 \times 0.0147}=25.4 \text { billion }
$$

- Finally, our crude model predicts that the population is 6 billion at the time $t$ that obeys

$$
\begin{aligned}
P(t) & =2 e^{b t}=6 & \text { clean up } \\
e^{b t} & =3 & \text { take the } \log \text { and clean up } \\
t & =\frac{\log 3}{b}=47 \frac{\log 3}{\log 2}=74.5 &
\end{aligned}
$$

which corresponds ${ }^{b}$ to the middle of 2001.
$a$ The 2015 Revision of World Population, a publication of the United Nations, predicts that the world's population in 2100 will be about 11 billion. They are predicting a reduction in the world population growth rate due to lower fertility rates, which the Malthusian growth model does not take into account.
$\uparrow \quad b \quad$ The world population really reached 6 billion in about 1999.

Applications of derivatives

### 3.3.3.1M (Optional) - Logistic Population Growth

Logistic growth adds one more wrinkle to the simple population model. It assumes that the population only has access to limited resources. As the size of the population grows the amount of food available to each member decreases. This in turn causes the net birth rate $b$ to decrease. In the logistic growth model $b=b_{0}\left(1-\frac{P}{K}\right)$, where $K$ is called the carrying capacity of the environment, so that

$$
P^{\prime}(t)=b_{0}\left(1-\frac{P(t)}{K}\right) P(t)
$$

We can learn quite a bit about the behaviour of solutions to differential equations like this, without ever finding formulae for the solutions, just by watching the sign of $P^{\prime}(t)$. For concreteness, we'll look at solutions of the differential equation

$$
\frac{\mathrm{d} P}{\mathrm{~d} t}(t)=(6000-3 P(t)) P(t)
$$

We'll sketch the graphs of four functions $P(t)$ that obey this equation.

- For the first function, $P(0)=0$.
- For the second function, $P(0)=1000$.
- For the third function, $P(0)=2000$.
- For the fourth function, $P(0)=3000$.

The sketches will be based on the observation that $(6000-3 P) P=3(2000-P) P$

- is zero for $P=0,2000$,
- is strictly positive for $0<P<2000$ and
- is strictly negative for $P>2000$.

Consequently

$$
\frac{\mathrm{d} P}{\mathrm{~d} t}(t) \begin{cases}=0 & \text { if } P(t)=0 \\ >0 & \text { if } 0<P(t)<2000 \\ =0 & \text { if } P(t)=2000 \\ <0 & \text { if } P(t)>2000\end{cases}
$$

Thus if $P(t)$ is some function that obeys $\frac{\mathrm{d} P}{\mathrm{~d} t}(t)=(6000-3 P(t)) P(t)$, then as the graph of $P(t)$ passes through $(t, P(t))$
the graph has $\begin{cases}\text { slope zero, } & \text { i.e. is horizontal, if } P(t)=0 \\ \text { positive slope, } & \text { i.e. is increasing, if } 0<P(t)<2000 \\ \text { slope zero, } & \text { i.e. is horizontal, if } P(t)=2000 \\ \text { negative slope, } & \text { i.e. is decreasing, if } P(t)>2000\end{cases}$
as illustrated in the figure


As a result,

- if $P(0)=0$, the graph starts out horizontally. In other words, as $t$ starts to increase, $P(t)$ remains at zero, so the slope of the graph remains at zero. The population size remains zero for all time. As a check, observe that the function $P(t)=0$ obeys $\frac{\mathrm{d} P}{\mathrm{~d} t}(t)=(6000-3 P(t)) P(t)$ for all $t$.
- Similarly, if $P(0)=2000$, the graph again starts out horizontally. So $P(t)$ remains at 2000 and the slope remains at zero. The population size remains 2000 for all time. Again, the function $P(t)=2000$ obeys $\frac{\mathrm{d} P}{\mathrm{~d} t}(t)=(6000-3 P(t)) P(t)$ for all $t$.
- If $P(0)=1000$, the graph starts out with positive slope. So $P(t)$ increases with $t$. As $P(t)$ increases towards 2000, the slope $(6000-3 P(t)) P(t)$, while remaining positive, gets closer and closer to zero. As the graph approaches height 2000, it becomes more and more horizontal. The graph cannot actually cross from below 2000 to above 2000, because to do so, it would have to have strictly positive slope for some value of $P$ above 2000, which is not allowed.
- If $P(0)=3000$, the graph starts out with negative slope. So $P(t)$ decreases with $t$. As $P(t)$ decreases towards 2000, the slope $(6000-3 P(t)) P(t)$, while remaining negative, gets closer and closer to zero. As the graph approaches height 2000, it becomes more and more horizontal. The graph cannot actually cross from above 2000 to below 2000, because to do so, it would have to have negative slope for some value of $P$ below 2000. which is not allowed.

These curves are sketched in the figure below. We conclude that for any initial population size $P(0)$, except $P(0)=0$, the population size approaches 2000 as $t \rightarrow \infty$.


Applications of derivatives

### 3.3.4 $円$ Exercises

## Exercises for § 3.3.1

## Exercises - Stage 1

1. Which of the following is a differential equation for an unknown function $y$ of $x$ ?
(a) $y=\frac{\mathrm{d} y}{\mathrm{~d} x}$
(b) $\frac{\mathrm{d} y}{\mathrm{~d} x}=3[y-5]$
(c) $y=3\left[y-\frac{\mathrm{d} x}{\mathrm{~d} x}\right]$
(d) $e^{x}=e^{y}+1$
(e) $y=10 e^{x}$
2. Which of the following functions $Q(t)$ satisfy the differential equation $Q(t)=5 \frac{\mathrm{~d} Q}{\mathrm{~d} t} ?$
(a) $Q(t)=0$
(b) $Q(t)=5 e^{t}$
(c) $Q(t)=e^{5 t}$
(d) $Q(t)=e^{t / 5}$
(e) $Q(t)=e^{t / 5}+1$
3. Suppose a sample starts out with $C$ grams of a radioactive isotope, and the amount of the radioactive isotope left in the sample at time $t$ is given by

$$
Q(t)=C e^{-k t}
$$

for some positive constant $k$. When will $Q(t)=0$ ?

## Exercises - Stage 2

4. *. Consider a function of the form $f(x)=A e^{k x}$ where $A$ and $k$ are constants. If $f(0)=5$ and $f(7)=\pi$, find the constants $A$ and $k$.
5. $\quad$. Find the function $y(t)$ if $\frac{\mathrm{d} y}{\mathrm{~d} t}+3 y=0, y(1)=2$.
6. A sample of bone belongs to an animal that died 10,000 years ago. If the bone contained $5 \mu \mathrm{~g}$ of Carbon-14 when the animal died, how much Carbon-14 do you expect it to have now?
7. A sample containing one gram of Radium- 226 was stored in a lab 100 years ago; now the sample only contains 0.9576 grams of Radium- 226 . What is the half-life of Radium-226?
8. *. The mass of a sample of Polonium-210, initially 6 grams, decreases at a rate proportional to the mass. After one year, 1 gram remains. What is the half-life (the time it takes for the sample to decay to half its original mass)?
9. Radium- 221 has a half-life of 30 seconds. How long does it take for only $0.01 \%$ of an original sample to be left?

## Exercises - Stage 3

10. Polonium- 210 has a half life of 138 days. What percentage of a sample of Polonium-210 decays in a day?
11. A sample of ore is found to contain $7.2 \pm 0.3 \mu \mathrm{~g}$ of Uranium-232, the half-life of which is between 68.8 and 70 years. How much Uranium- 232 will remain undecayed in the sample in 10 years?

## Exercises for § 3.3.2

## Exercises - Stage 1

1. Which of the following functions $T(t)$ satisfy the differential equation $\frac{\mathrm{d} T}{\mathrm{~d} t}=$ $5[T-20]$ ?
(a) $T(t)=20$
(b) $T(t)=20 e^{5 t}-20$
(c) $T(t)=e^{5 t}+20$
(d) $T(t)=20 e^{5 t}+20$
2. At time $t=0$, an object is placed in a room, of temperature $A$. After $t$ seconds, Newton's Law of Cooling gives the temperature of the object is as

$$
T(t)=35 e^{K t}-10
$$

What is the temperature of the room? Is the room warmer or colder than the object?
3. A warm object is placed in a cold room. The temperature of the object, over time, approaches the temperature of the room it is in. The temperature of the object at time $t$ is given by

$$
T(t)=[T(0)-A] e^{K t}+A
$$

Can $K$ be a positive number? Can $K$ be a negative number? Can $K$ be zero?
4. Suppose an object obeys Newton's Law of Cooling, and its temperature is given by

$$
T(t)=[T(0)-A] e^{k t}+A
$$

for some constant $k$. At what time is $T(t)=A$ ?

## Exercises - Stage 2

5. A piece of copper at room temperature $\left(25^{\circ}\right)$ is placed in a boiling pot of water. After 10 seconds, it has heated to $90^{\circ}$. When will it be $99.9^{\circ}$ ?
6. Today is a chilly day. We heated up a stone to $500^{\circ} \mathrm{C}$ in a bonfire, then took it out and left it outside, where the temperature is $0^{\circ} \mathrm{C}$. After 10 minutes outside of the bonfire, the stone had cooled to a still-untouchable $100^{\circ} \mathrm{C}$. Now the stone is at a cozy $50^{\circ} \mathrm{C}$. How long ago was the stone taken out of the fire?

## Exercises - Stage 3

7. *. Isaac Newton drinks his coffee with cream. To be exact, 9 parts coffee to 1 part cream. His landlady pours him a cup of coffee at $95^{\circ} \mathrm{C}$ into which Newton stirs cream taken from the icebox at $5^{\circ} \mathrm{C}$. When he drinks the mixture ten minutes later, he notes that it has cooled to $54^{\circ} \mathrm{C}$. Newton wonders if his coffee would be hotter (and by how much) if he waited until just before drinking it to add the cream. Analyze this question, assuming that:
i The temperature of the dining room is constant at $22^{\circ} \mathrm{C}$.
ii When a volume $V_{1}$ of liquid at temperature $T_{1}$ is mixed with a volume $V_{2}$ at temperature $T_{2}$, the temperature of the mixture is $\frac{V_{1} T_{1}+V_{2} T_{2}}{V_{1}+V_{2}}$.
iii Newton's Law of Cooling: The temperature of an object cools at a rate proportional to the difference in temperature between the object and its surroundings.
iv The constant of proportionality is the same for the cup of coffee with cream as for the cup of pure coffee.
8. *. The temperature of a glass of iced tea is initially $5^{\circ}$. After 5 minutes, the tea has heated to $10^{\circ}$ in a room where the air temperature is $30^{\circ}$.
a Use Newton's law of cooling to obtain a differential equation for the temperature $T(t)$ at time $t$.
b Determine when the tea will reach a temperature of $20^{\circ}$.
9. Suppose an object is changing temperature according to Newton's Law of Cooling, and its temperature at time $t$ is given by

$$
T(t)=0.8^{k t}+15
$$

Is $k$ positive or negative?

## Exercises for § 3.3.3

## Exercises - Stage 1

1. Let a population at time $t$ be given by the Malthusian model,

$$
P(t)=P(0) e^{b t} \text { for some positive constant } b
$$

Evaluate $\lim _{t \rightarrow \infty} P(t)$. Does this model make sense for large values of $t$ ?

## Exercises - Stage 2

2. In the 1950s, pure-bred wood bison were thought to be extinct. However, a small population was found in Canada. For decades, a captive breeding program has been working to increase their numbers, and from time to time wood bison are released to the wild. Suppose in 2015, a released herd numbered 121 animals, and a year later, there were $136{ }^{a}$. If the wood bison adhere to the Malthusian model (a big assumption!), and if there are no more releases of captive animals, how many animals will the herd have in 2020?

$a$ These numbers are loosely based on animals actually released near Shageluk, Alaska in 2015. Watch the first batch being released here.
3. A founding colony of 1,000 bacteria is placed in a petri dish of yummy bacteria food. After an hour, the population has doubled. Assuming the Malthusian model, how long will it take for the colony to triple its original population?
4. A single pair of rats comes to an island after a shipwreck. They multiply according to the Malthusian model. In 1928, there were 1,000 rats on the island, and the next year there were 1500 . When was the shipwreck?
5. A farmer wants to farm cochineals, which are insects used to make red dye. The farmer raises a small number of cochineals as a test. In three months, a test population of cochineals will increase from 200 individuals to 1000, given ample space and food.
The farmer's plan is to start with an initial population of $P(0)$ cochineals, and after a year have $1000000+P(0)$ cochineals, so that one million can be harvested, and $P(0)$ saved to start breeding again. What initial population $P(0)$ does the Malthusian model suggest?

## Exercises - Stage 3

6. Let $f(t)=100 e^{k t}$, for some constant $k$.
a If $f(t)$ is the amount of a decaying radioactive isotope in a sample at time $t$, what is the amount of the isotope in the sample when $t=0$ ? What is the sign of $k$ ?
b If $f(t)$ is the number of individuals in a population that is growing according to the Malthusian model, how many individuals are there when $t=0$ ? What is the sign of $k$ ?
c If $f(t)$ is the temperature of an object at time $t$, given by Newton's Law of Cooling, what is the ambient temperature surrounding the object? What is the sign of $k$ ?

## $\leadsto$ Further problems for § 3.3

1. *. Find $f(2)$ if $f^{\prime}(x)=\pi f(x)$ for all $x$, and $f(0)=2$.
2. Which functions $T(t)$ satisfy the differential equation $\frac{\mathrm{d} T}{\mathrm{~d} t}=7 T+9$ ?
3. *. It takes 8 days for $20 \%$ of a particular radioactive material to decay. How long does it take for 100 grams of the material to decay to 40 grams?
4. A glass of boiling water is left in a room. After 15 minutes, it has cooled to $85^{\circ}$ C , and after 30 minutes it is $73^{\circ} \mathrm{C}$. What temperature is the room?
5. *. A 25 -year-old graduate of UBC is given $\$ 50,000$ which is invested at $5 \%$ per year compounded continuously. The graduate also intends to deposit money continuously at the rate of $\$ 2000$ per year. Assuming that the interest rate remains $5 \%$, the amount $A(t)$ of money at time $t$ satisfies the equation

$$
\frac{\mathrm{d} A}{\mathrm{~d} t}=0.05 A+2000
$$

a Solve this equation and determine the amount of money in the account when the graduate is 65 .
b At age 65, the graduate will withdraw money continuously at the rate of $W$ dollars per year. If the money must last until the person is 85 , what is the largest possible value of $W$ ?
6. *. An investor puts $\$ 120,000$ which into a bank account which pays $6 \%$ annual interest, compounded continuously. She plans to withdraw money continuously from the account at the rate of $\$ 9000$ per year. If $A(t)$ is the amount of money at
time $t$, then

$$
\frac{\mathrm{d} A}{\mathrm{~d} t}=0.06 A-9000
$$

a Solve this equation for $A(t)$.
b When will the money run out?
7. *. A particular bacterial culture grows at a rate proportional to the number of bacteria present. If the size of the culture triples every nine hours, how long does it take the culture to double?
8. *. An object falls under gravity near the surface of the earth and its motion is impeded by air resistance proportional to its speed. Its velocity $v$ satisfies the differential equation

$$
\frac{d v}{d t}=-g-k v
$$

where $g$ and $k$ are positive constants.
a Find the velocity of the object as a function of time $t$, given that it was $v_{0}$ at $t=0$.
b Find $\lim _{t \rightarrow \infty} v(t)$.

## 3.4^ Approximating Functions Near a Specified Point Taylor Polynomials

Suppose that you are interested in the values of some function $f(x)$ for $x$ near some fixed point $a$. When the function is a polynomial or a rational function we can use some arithmetic (and maybe some hard work) to write down the answer. For example:

$$
\begin{aligned}
f(x) & =\frac{x^{2}-3}{x^{2}-2 x+4} \\
f(1 / 5) & =\frac{\frac{1}{25}-3}{\frac{1}{25}-\frac{2}{5}+4}=\frac{\frac{1-75}{25}}{\frac{1-10+100}{25}} \\
& =\frac{-74}{91}
\end{aligned}
$$

Tedious, but we can do it. On the other hand if you are asked to compute $\sin (1 / 10)$ then what can we do? We know that a calculator can work it out

$$
\sin (1 / 10)=0.09983341 \ldots
$$

but how does the calculator do this? How did people compute this before calculators ${ }^{1}$ ? A hint comes from the following sketch of $\sin (x)$ for $x$ around 0 .

1 Originally the word "calculator" referred not to the software or electronic (or even mechanical) device we think of today, but rather to a person who performed calculations.


The above figure shows that the curves $y=x$ and $y=\sin x$ are almost the same when $x$ is close to 0 . Hence if we want the value of $\sin (1 / 10)$ we could just use this approximation $y=x$ to get

$$
\sin (1 / 10) \approx 1 / 10
$$

Of course, in this case we simply observed that one function was a good approximation of the other. We need to know how to find such approximations more systematically.

More precisely, say we are given a function $f(x)$ that we wish to approximate close to some point $x=a$, and we need to find another function $F(x)$ that

- is simple and easy to compute ${ }^{2}$
- is a good approximation to $f(x)$ for $x$ values close to $a$.

Further, we would like to understand how good our approximation actually is. Namely we need to be able to estimate the error $|f(x)-F(x)|$.

There are many different ways to approximate a function and we will discuss one family of approximations: Taylor polynomials. This is an infinite family of ever improving approximations, and our starting point is the very simplest.

### 3.4.1 Zeroth Approximation - the Constant Approximation

The simplest functions are those that are constants. And our zeroth ${ }^{3}$ approximation will be by a constant function. That is, the approximating function will have the form $F(x)=A$, for some constant $A$. Notice that this function is a polynomial of degree zero.

To ensure that $F(x)$ is a good approximation for $x$ close to $a$, we choose $A$ so that $f(x)$ and $F(x)$ take exactly the same value when $x=a$.

$$
F(x)=A \quad \text { so } \quad F(a)=A=f(a) \Longrightarrow A=f(a)
$$

Our first, and crudest, approximation rule is

2 It is no good approximating a function with something that is even more difficult to work with.
3 It barely counts as an approximation at all, but it will help build intuition. Because of this, and the fact that a constant is a polynomial of degree 0 , we'll start counting our approximations from zero rather than 1.

## Equation 3.4.1 Constant approximation.

$$
f(x) \approx f(a)
$$

An important point to note is that we need to know $f(a)$ - if we cannot compute that easily then we are not going to be able to proceed. We will often have to choose $a$ (the point around which we are approximating $f(x)$ ) with some care to ensure that we can compute $f(a)$.

Here is a figure showing the graphs of a typical $f(x)$ and approximating function $F(x)$.


At $x=a, f(x)$ and $F(x)$ take the same value. For $x$ very near $a$, the values of $f(x)$ and $F(x)$ remain close together. But the quality of the approximation deteriorates fairly quickly as $x$ moves away from $a$. Clearly we could do better with a straight line that follows the slope of the curve. That is our next approximation.

But before then, an example:

Example 3.4.2 A (weak) approximation of $e^{0.1}$.
Use the constant approximation to estimate $e^{0.1}$.
Solution First set $f(x)=e^{x}$.

- Now we first need to pick a point $x=a$ to approximate the function. This point needs to be close to 0.1 and we need to be able to evaluate $f(a)$ easily. The obvious choice is $a=0$.
- Then our constant approximation is just

$$
\begin{aligned}
F(x) & =f(0)=e^{0}=1 \\
F(0.1) & =1
\end{aligned}
$$

Note that $e^{0.1}=1.105170918 \ldots$..., so even this approximation isn't too bad..

### 3.4.2 $\leadsto$ First Approximation - the Linear Approximation

Our first ${ }^{4}$ approximation improves on our zeroth approximation by allowing the approximating function to be a linear function of $x$ rather than just a constant function. That is, we allow $F(x)$ to be of the form $A+B x$, for some constants $A$ and $B$.

To ensure that $F(x)$ is a good approximation for $x$ close to $a$, we still require that $f(x)$ and $F(x)$ have the same value at $x=a$ (that was our zeroth approximation). Our additional requirement is that their tangent lines at $x=a$ have the same slope - that the derivatives of $f(x)$ and $F(x)$ are the same at $x=a$. Hence

$$
\begin{aligned}
& F(x)=A+B x \quad \Longrightarrow \quad F(a)=A+B a=f(a) \\
& F^{\prime}(x)=B \quad \Longrightarrow \quad F^{\prime}(a)=\quad B=f^{\prime}(a)
\end{aligned}
$$

So we must have $B=f^{\prime}(a)$. Substituting this into $A+B a=f(a)$ we get $A=$ $f(a)-a f^{\prime}(a)$. So we can write

$$
\begin{aligned}
F(x) & =A+B x=\overbrace{f(a)-a f^{\prime}(a)}^{A}+f^{\prime}(a) \cdot x \\
& =f(a)+f^{\prime}(a) \cdot(x-a)
\end{aligned}
$$

We write it in this form because we can now clearly see that our first approximation is just an extension of our zeroth approximation. This first approximation is also often called the linear approximation of $f(x)$ about $x=a$.

## Equation 3.4.3 Linear approximation.

$$
f(x) \approx f(a)+f^{\prime}(a)(x-a)
$$

We should again stress that in order to form this approximation we need to know $f(a)$ and $f^{\prime}(a)$ - if we cannot compute them easily then we are not going to be able to proceed.

Recall, from Theorem 2.3.4, that $y=f(a)+f^{\prime}(a)(x-a)$ is exactly the equation of the tangent line to the curve $y=f(x)$ at $a$. Here is a figure showing the graphs of a typical $f(x)$ and the approximating function $F(x)$.

4 Recall that we started counting from zero.


Observe that the graph of $f(a)+f^{\prime}(a)(x-a)$ remains close to the graph of $f(x)$ for a much larger range of $x$ than did the graph of our constant approximation, $f(a)$. One can also see that we can improve this approximation if we can use a function that curves down rather than being perfectly straight. That is our next approximation.

But before then, back to our example:

Example 3.4.4 A better approximation of $e^{0.1}$.
Use the linear approximation to estimate $e^{0.1}$.
Solution First set $f(x)=e^{x}$ and $a=0$ as before.

- To form the linear approximation we need $f(a)$ and $f^{\prime}(a)$ :

$$
\begin{array}{rlrl}
f(x) & =e^{x} & f(0) & =1 \\
f^{\prime}(x) & =e^{x} & f^{\prime}(0) & =1
\end{array}
$$

- Then our linear approximation is

$$
\begin{aligned}
F(x) & =f(0)+x f^{\prime}(0)=1+x \\
F(0.1) & =1.1
\end{aligned}
$$

Recall that $e^{0.1}=1.105170918 \ldots$, so the linear approximation is almost correct to 3 digits.

It is worth doing another simple example here.

Example 3.4.5 A linear approximation of $\sqrt{4.1}$.
Use a linear approximation to estimate $\sqrt{4.1}$.
Solution First set $f(x)=\sqrt{x}$. Hence $f^{\prime}(x)=\frac{1}{2 \sqrt{x}}$. Then we are trying to approximate $f(4.1)$. Now we need to choose a sensible $a$ value.

- We need to choose $a$ so that $f(a)$ and $f^{\prime}(a)$ are easy to compute.
- We could try $a=4.1$ - but then we need to compute $f(4.1)$ and $f^{\prime}(4.1)$ -
which is our original problem and more!
- We could try $a=0$ - then $f(0)=0$ and $f^{\prime}(0)=D N E$.
- Setting $a=1$ gives us $f(1)=1$ and $f^{\prime}(1)=\frac{1}{2}$. This would work, but we can get a better approximation by choosing $a$ is closer to 4.1.
- Indeed we can set $a$ to be the square of any rational number and we'll get a result that is easy to compute.
- Setting $a=4$ gives $f(4)=2$ and $f^{\prime}(4)=\frac{1}{4}$. This seems good enough.
- Substitute this into equation 3.4.3 to get

$$
\begin{aligned}
f(4.1) & \approx f(4)+f^{\prime}(4) \cdot(4.1-4) \\
& =2+\frac{0.1}{4}=2+0.025=2.025
\end{aligned}
$$

Notice that the true value is $\sqrt{4.1}=2.024845673 \ldots$...

### 3.4.3 Second Approximation - the Quadratic Approximation

We next develop a still better approximation by now allowing the approximating function be to a quadratic function of $x$. That is, we allow $F(x)$ to be of the form $A+B x+C x^{2}$, for some constants $A, B$ and $C$. To ensure that $F(x)$ is a good approximation for $x$ close to $a$, we choose $A, B$ and $C$ so that

- $f(a)=F(a)$ (just as in our zeroth approximation),
- $f^{\prime}(a)=F^{\prime}(a)$ (just as in our first approximation), and
- $f^{\prime \prime}(a)=F^{\prime \prime}(a)$ - this is a new condition.

These conditions give us the following equations

$$
\begin{array}{rlrlrl}
F(x) & =A+B x+C x^{2} & \Longrightarrow & & F(a)=A+B a+C a^{2}=f(a) \\
F^{\prime}(x) & =B+2 C x & \Longrightarrow & & F^{\prime}(a)= & B+2 C a=f^{\prime}(a) \\
F^{\prime \prime}(x) & =2 C & \Longrightarrow & F^{\prime \prime}(a)= & 2 C=f^{\prime \prime}(a)
\end{array}
$$

Solve these for $C$ first, then $B$ and finally $A$.

$$
\begin{aligned}
& C=\frac{1}{2} f^{\prime \prime}(a) \\
& B=f^{\prime}(a)-2 C a=f^{\prime}(a)-a f^{\prime \prime}(a) \\
& A=f(a)-B a-C a^{2}=f(a)-a\left[f^{\prime}(a)-a f^{\prime \prime}(a)\right]-\frac{1}{2} f^{\prime \prime}(a) a^{2}
\end{aligned}
$$

Then put things back together to build up $F(x)$ :

$$
\left.\left.\left.\begin{array}{rlrl}
F(x)=f(a)-f^{\prime}(a) a & +\frac{1}{2} f^{\prime \prime}(a) a^{2} & & \text { (this line is } A) \\
& +f^{\prime}(a) x & -f^{\prime \prime}(a) a x &
\end{array}\right) \text { (this line is } B x\right) ~ 子 \begin{array}{rlr}
2 & & \text { (this line is } \left.C x^{2}\right) \\
& +\frac{1}{2} f^{\prime \prime}(a) x^{2} &
\end{array}\right)
$$

Oof! We again write it in this form because we can now clearly see that our second approximation is just an extension of our first approximation.

Our second approximation is called the quadratic approximation:

## Equation 3.4.6 Quadratic approximation.

$$
f(x) \approx f(a)+f^{\prime}(a)(x-a)+\frac{1}{2} f^{\prime \prime}(a)(x-a)^{2}
$$

Here is a figure showing the graphs of a typical $f(x)$ and approximating function $F(x)$.


This new approximation looks better than both the first and second.
Now there is actually an easier way to derive this approximation, which we show you now. Let us rewrite ${ }^{5}$
$F(x)$ so that it is easy to evaluate it and its derivatives at $x=a$ :

$$
F(x)=\alpha+\beta \cdot(x-a)+\gamma \cdot(x-a)^{2}
$$

Then

$$
\begin{array}{rlrl}
F(x) & =\alpha+\beta \cdot(x-a)+\gamma \cdot(x-a)^{2} & F(a) & =\alpha=f(a) \\
F^{\prime}(x) & =\beta+2 \gamma \cdot(x-a) & F^{\prime}(a) & =\beta=f^{\prime}(a)
\end{array}
$$

5 Any polynomial of degree two can be written in this form. For example, when $a=1,3+2 x+x^{2}=$ $6+4(x-1)+(x-1)^{2}$.

$$
F^{\prime \prime}(x)=2 \gamma
$$

$$
F^{\prime \prime}(a)=2 \gamma=f^{\prime \prime}(a)
$$

And from these we can clearly read off the values of $\alpha, \beta$ and $\gamma$ and so recover our function $F(x)$. Additionally if we write things this way, then it is quite clear how to extend this to a cubic approximation and a quartic approximation and so on.

Return to our example:
Example 3.4.7 An even better approximation of $e^{0.1}$.
Use the quadratic approximation to estimate $e^{0.1}$.
Solution Set $f(x)=e^{x}$ and $a=0$ as before.

- To form the quadratic approximation we need $f(a), f^{\prime}(a)$ and $f^{\prime \prime}(a)$ :

$$
\begin{array}{rlrl}
f(x) & =e^{x} & f(0) & =1 \\
f^{\prime}(x) & =e^{x} & f^{\prime}(0) & =1 \\
f^{\prime \prime}(x) & =e^{x} & f^{\prime \prime}(0) & =1
\end{array}
$$

- Then our quadratic approximation is

$$
\begin{aligned}
F(x) & =f(0)+x f^{\prime}(0)+\frac{1}{2} x^{2} f^{\prime \prime}(0)=1+x+\frac{x^{2}}{2} \\
F(0.1) & =1.105
\end{aligned}
$$

Recall that $e^{0.1}=1.105170918 \ldots$, so the quadratic approximation is quite accurate with very little effort.

Before we go on, let us first introduce (or revise) some notation that will make our discussion easier.

### 3.4.4 Whirlwind Tour of Summation Notation

In the remainder of this section we will frequently need to write sums involving a large number of terms. Writing out the summands explicitly can become quite impractical - for example, say we need the sum of the first 11 squares:

$$
1+2^{2}+3^{2}+4^{2}+5^{2}+6^{2}+7^{2}+8^{2}+9^{2}+10^{2}+11^{2}
$$

This becomes tedious. Where the pattern is clear, we will often skip the middle few terms and instead write

$$
1+2^{2}+\cdots+11^{2}
$$

A far more precise way to write this is using $\Sigma$ (capital-sigma) notation. For example, we can write the above sum as

$$
\sum_{k=1}^{11} k^{2}
$$

This is read as
The sum from $k$ equals 1 to 11 of $k^{2}$.
More generally

## Definition 3.4.8

Let $m \leq n$ be integers and let $f(x)$ be a function defined on the integers. Then we write

$$
\sum_{k=m}^{n} f(k)
$$

to mean the sum of $f(k)$ for $k$ from $m$ to $n$ :

$$
f(m)+f(m+1)+f(m+2)+\cdots+f(n-1)+f(n) .
$$

Similarly we write

$$
\sum_{i=m}^{n} a_{i}
$$

to mean

$$
a_{m}+a_{m+1}+a_{m+2}+\cdots+a_{n-1}+a_{n}
$$

for some set of coefficients $\left\{a_{m}, \ldots, a_{n}\right\}$.
Consider the example

$$
\sum_{k=3}^{7} \frac{1}{k^{2}}=\frac{1}{3^{2}}+\frac{1}{4^{2}}+\frac{1}{5^{2}}+\frac{1}{6^{2}}+\frac{1}{7^{2}}
$$

It is important to note that the right hand side of this expression evaluates to a number ${ }^{6}$; it does not contain " $k$ ". The summation index $k$ is just a "dummy" variable and it does not have to be called $k$. For example

$$
\sum_{k=3}^{7} \frac{1}{k^{2}}=\sum_{i=3}^{7} \frac{1}{i^{2}}=\sum_{j=3}^{7} \frac{1}{j^{2}}=\sum_{\ell=3}^{7} \frac{1}{\ell^{2}}
$$

Also the summation index has no meaning outside the sum. For example

$$
k \sum_{k=3}^{7} \frac{1}{k^{2}}
$$

has no mathematical meaning; It is gibberish ${ }^{7}$.
6 Some careful addition shows it is $\frac{46181}{176400}$.
7 Or possibly gobbledygook. For a discussion of statements without meaning and why one should avoid them we recommend the book "Bendable learnings: the wisdom of modern management" by Don Watson.

### 3.4.5 Still Better Approximations - Taylor Polynomials

We can use the same strategy to generate still better approximations by polynomials ${ }^{8}$ of any degree we like. As was the case with the approximations above, we determine the coefficients of the polynomial by requiring, that at the point $x=a$, the approximation and its first $n$ derivatives agree with those of the original function.

Rather than simply moving to a cubic polynomial, let us try to write things in a more general way. We will consider approximating the function $f(x)$ using a polynomial, $T_{n}(x)$, of degree $n$ - where $n$ is a non-negative integer. As we discussed above, the algebra is easier if we write

$$
\begin{aligned}
T_{n}(x) & =c_{0}+c_{1}(x-a)+c_{2}(x-a)^{2}+\cdots+c_{n}(x-a)^{n} \\
& =\sum_{k=0}^{n} c_{k}(x-a)^{k}
\end{aligned}
$$

## using $\Sigma$ notation

The above form ${ }^{9}{ }^{10}$ makes it very easy to evaluate this polynomial and its derivatives at $x=a$. Before we proceed, we remind the reader of some notation (see Notation 2.2.8):

- Let $f(x)$ be a function and $k$ be a positive integer. We can denote its $k^{\text {th }}$ derivative with respect to $x$ by

$$
\frac{\mathrm{d}^{k} f}{\mathrm{~d} x^{k}} \quad\left(\frac{\mathrm{~d}}{\mathrm{~d} x}\right)^{k} f(x) \quad f^{(k)}(x)
$$

Additionally we will need

## Definition 3.4.9 Factorial.

Let $n$ be a positive integer ${ }^{a}$, then $n$-factorial, denoted $n$ !, is the product

$$
n!=n \times(n-1) \times \cdots \times 3 \times 2 \times 1
$$

Further, we use the convention that

$$
0!=1
$$

The first few factorials are

$$
1!=1 \quad 2!=2 \quad 3!=6
$$

8 Polynomials are generally a good choice for an approximating function since they are so easy to work with. Depending on the situation other families of functions may be more appropriate. For example if you are approximating a periodic function, then sums of sines and cosines might be a better choice; this leads to Fourier series.
9 Any polynomial in $x$ of degree $n$ can also be expressed as a polynomial in $(x-a)$ of the same degree $n$ and vice versa. So $T_{n}(x)$ really still is a polynomial of degree $n$.
10 Furthermore when $x$ is close to $a,(x-a)^{k}$ decreases very quickly as $k$ increases, which often makes the "high $k$ " terms in $T_{n}(x)$ very small. This can be a considerable advantage when building up approximations by adding more and more terms. If we were to rewrite $T_{n}(x)$ in the form $\sum_{k=0}^{n} b_{k} x^{k}$ the "high $k$ " terms would typically not be very small when $x$ is close to $a$.

$$
4!=24 \quad 5!=120 \quad 6!=720
$$

$a \quad$ It is actually possible to define the factorial of positive real numbers and even negative numbers but it requires more advanced calculus and is outside the scope of this course. The interested reader should look up the Gamma function.

Now consider $T_{n}(x)$ and its derivatives:

$$
\begin{array}{rlrl}
T_{n}(x) & =c_{0}+c_{1}(x-a)+c_{2}(x-a)^{2}+c_{3}(x-a)^{3} & +\cdots+c_{n}(x-a)^{n} \\
T_{n}^{\prime}(x) & =c_{1} & +2 c_{2}(x-a)+3 c_{3}(x-a)^{2}+\cdots+n c_{n}(x-a)^{n-1} \\
T_{n}^{\prime \prime}(x) & = & 2 c_{2}+6 c_{3}(x-a)+\cdots+n(n-1) c_{n}(x-a)^{n-2} \\
T_{n}^{\prime \prime \prime}(x) & = & 6 c_{3} & +\cdots+n(n-1)(n-2) c_{n}(x-a)^{n-3} \\
\vdots & & n!\cdot c_{n}
\end{array}
$$

Now notice that when we substitute $x=a$ into the above expressions only the constant terms survive and we get

$$
\begin{aligned}
T_{n}(a) & =c_{0} \\
T_{n}^{\prime}(a) & =c_{1} \\
T_{n}^{\prime \prime}(a) & =2 \cdot c_{2} \\
T_{n}^{\prime \prime \prime}(a) & =6 \cdot c_{3} \\
\vdots & \\
T_{n}^{(n)}(a) & =n!\cdot c_{n}
\end{aligned}
$$

So now if we want to set the coefficients of $T_{n}(x)$ so that it agrees with $f(x)$ at $x=a$ then we need

$$
T_{n}(a)=c_{0}=f(a) \quad c_{0}=f(a)=\frac{1}{0!} f(a)
$$

We also want the first $n$ derivatives of $T_{n}(x)$ to agree with the derivatives of $f(x)$ at $x=a$, so

$$
\begin{array}{ll}
T_{n}^{\prime}(a)=c_{1}=f^{\prime}(a) & c_{1}=f^{\prime}(a)=\frac{1}{1!} f^{\prime}(a) \\
T_{n}^{\prime \prime}(a)=2 \cdot c_{2}=f^{\prime \prime}(a) & c_{2}=\frac{1}{2} f^{\prime \prime}(a)=\frac{1}{2!} f^{\prime \prime}(a) \\
T_{n}^{\prime \prime \prime}(a)=6 \cdot c_{3}=f^{\prime \prime \prime}(a) & c_{3}=\frac{1}{6} f^{\prime \prime \prime}(a)=\frac{1}{3!} f^{\prime \prime \prime}(a)
\end{array}
$$

More generally, making the $k^{\text {th }}$ derivatives agree at $x=a$ requires :

$$
T_{n}^{(k)}(a)=k!\cdot c_{k}=f^{(k)}(a) \quad c_{k}=\frac{1}{k!} f^{(k)}(a)
$$

And finally the $n^{\text {th }}$ derivative:

$$
T_{n}^{(n)}(a)=n!\cdot c_{n}=f^{(n)}(a) \quad c_{n}=\frac{1}{n!} f^{(n)}(a)
$$

Putting this all together we have

## Equation 3.4.10 Taylor polynomial.

$$
\begin{aligned}
f(x) \approx T_{n}(x)= & f(a)+f^{\prime}(a)(x-a)+\frac{1}{2} f^{\prime \prime}(a) \cdot(x-a)^{2}+\cdots \\
& \quad+\frac{1}{n!} f^{(n)}(a) \cdot(x-a)^{n} \\
= & \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a) \cdot(x-a)^{k}
\end{aligned}
$$

Let us formalise this definition.

## Definition 3.4.11 Taylor polynomial.

Let $a$ be a constant and let $n$ be a non-negative integer. The $n^{\text {th }}$ degree Taylor polynomial for $f(x)$ about $x=a$ is

$$
T_{n}(x)=\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a) \cdot(x-a)^{k} .
$$

The special case $a=0$ is called a Maclaurin ${ }^{a}$ polynomial.
$a \quad$ The polynomials are named after Brook Taylor who devised a general method for constructing them in 1715. Slightly later, Colin Maclaurin made extensive use of the special case $a=0$ (with attribution of the general case to Taylor) and it is now named after him. The special case of $a=0$ was worked on previously by James Gregory and Isaac Newton, and some specific cases were known to the 14th century Indian mathematician Madhava of Sangamagrama.

Before we proceed with some examples, a couple of remarks are in order.

- While we can compute a Taylor polynomial about any $a$-value (providing the derivatives exist), in order to be a useful approximation, we must be able to compute $f(a), f^{\prime}(a), \cdots, f^{(n)}(a)$ easily. This means we must choose the point $a$ with care. Indeed for many functions the choice $a=0$ is very natural - hence the prominence of Maclaurin polynomials.
- If we have computed the approximation $T_{n}(x)$, then we can readily extend this to the next Taylor polynomial $T_{n+1}(x)$ since

$$
T_{n+1}(x)=T_{n}(x)+\frac{1}{(n+1)!} f^{(n+1)}(a) \cdot(x-a)^{n+1}
$$

This is very useful if we discover that $T_{n}(x)$ is an insufficient approximation, because then we can produce $T_{n+1}(x)$ without having to start again from scratch.

### 3.4.6 Some Examples

Let us return to our running example of $e^{x}$ :
Example 3.4.12 Taylor approximations of $e^{x}$.
The constant, linear and quadratic approximations we used above were the first few Maclaurin polynomial approximations of $e^{x}$. That is

$$
T_{0}(x)=1 \quad T_{1}(x)=1+x \quad T_{2}(x)=1+x+\frac{x^{2}}{2}
$$

Since $\frac{\mathrm{d}}{\mathrm{d} x} e^{x}=e^{x}$, the Maclaurin polynomials are very easy to compute. Indeed this invariance under differentiation means that

$$
\begin{array}{ll}
f^{(n)}(x)=e^{x} & n=0,1,2, \ldots \\
f^{(n)}(0)=1 &
\end{array}
$$

Substituting this into equation 3.4.10 we get

$$
T_{n}(x)=\sum_{k=0}^{n} \frac{1}{k!} x^{k}
$$

Thus we can write down the seventh Maclaurin polynomial very easily:

$$
T_{7}(x)=1+x+\frac{x^{2}}{2}+\frac{x^{3}}{6}+\frac{x^{4}}{24}+\frac{x^{5}}{120}+\frac{x^{6}}{720}+\frac{x^{7}}{5040}
$$

The following figure contains sketches of the graphs of $e^{x}$ and its Taylor polynomials $T_{n}(x)$ for $n=0,1,2,3,4$.


Also notice that if we use $T_{7}(1)$ to approximate the value of $e^{1}$ we obtain:

$$
\begin{aligned}
e^{1} \approx T_{7}(1) & =1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\frac{1}{720}+\frac{1}{5040} \\
& =\frac{685}{252}=2.718253968 \ldots
\end{aligned}
$$

The true value of $e$ is $2.718281828 \ldots$, so the approximation has an error of about $3 \times 10^{-5}$.
Under the assumption that the accuracy of the approximation improves with $n$ (an assumption we examine in Subsection 3.4.9 below) we can see that the approximation of $e$ above can be improved by adding more and more terms. Indeed this is how the expression for $e$ in equation 2.7.4 in Section 2.7 comes about.

Example 3.4.12
Now that we have examined Maclaurin polynomials for $e^{x}$ we should take a look at $\log x$. Notice that we cannot compute a Maclaurin polynomial for $\log x$ since it is not defined at $x=0$.

Example 3.4.13 Taylor approximation of $\log x$.
Compute the $5^{\text {th }}$ Taylor polynomial for $\log x$ about $x=1$.
Solution We have been told $a=1$ and fifth degree, so we should start by writing down the function and its first five derivatives:

$$
\begin{array}{rlrl}
f(x) & =\log x & f(1) & =\log 1=0 \\
f^{\prime}(x) & =\frac{1}{x} & f^{\prime}(1) & =1 \\
f^{\prime \prime}(x) & =\frac{-1}{x^{2}} & f^{\prime \prime}(1) & =-1 \\
f^{\prime \prime \prime}(x) & =\frac{2}{x^{3}} & f^{\prime \prime \prime}(1) & =2 \\
f^{(4)}(x) & =\frac{-6}{x^{4}} & f^{(4)}(1) & =-6 \\
f^{(5)}(x) & =\frac{24}{x^{5}} & f^{(5)}(1)=24
\end{array}
$$

Substituting this into equation 3.4.10 gives

$$
\begin{aligned}
T_{5}(x)= & 0+1 \cdot(x-1)+\frac{1}{2} \cdot(-1) \cdot(x-1)^{2}+\frac{1}{6} \cdot 2 \cdot(x-1)^{3} \\
& +\frac{1}{24} \cdot(-6) \cdot(x-1)^{4}+\frac{1}{120} \cdot 24 \cdot(x-1)^{5} \\
= & (x-1)-\frac{1}{2}(x-1)^{2}+\frac{1}{3}(x-1)^{3}-\frac{1}{4}(x-1)^{4}+\frac{1}{5}(x-1)^{5}
\end{aligned}
$$

Again, it is not too hard to generalise the above work to find the Taylor polynomial of degree $n$ : With a little work one can show that

$$
T_{n}(x)=\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}(x-1)^{k}
$$

For cosine:
Example 3.4.14 Maclaurin polynomial for $\cos x$.
Find the 4th degree Maclaurin polynomial for $\cos x$.
Solution We have $a=0$ and we need to find the first 4 derivatives of $\cos x$.

$$
\begin{array}{rlrl}
f(x) & =\cos x & f(0) & =1 \\
f^{\prime}(x) & =-\sin x & f^{\prime}(0) & =0 \\
f^{\prime \prime}(x) & =-\cos x & f^{\prime \prime}(0) & =-1 \\
f^{\prime \prime \prime}(x) & =\sin x & f^{\prime \prime \prime}(0) & =0 \\
f^{(4)}(x) & =\cos x & f^{(4)}(0) & =1
\end{array}
$$

Substituting this into equation 3.4.10 gives

$$
\begin{aligned}
T_{4}(x) & =1+1 \cdot(0) \cdot x+\frac{1}{2} \cdot(-1) \cdot x^{2}+\frac{1}{6} \cdot 0 \cdot x^{3}+\frac{1}{24} \cdot(1) \cdot x^{4} \\
& =1-\frac{x^{2}}{2}+\frac{x^{4}}{24}
\end{aligned}
$$

Notice that since the $4^{\text {th }}$ derivative of $\cos x$ is $\cos x$ again, we also have that the fifth derivative is the same as the first derivative, and the sixth derivative is the same as the second derivative and so on. Hence the next four derivatives are

$$
\begin{array}{ll}
f^{(4)}(x)=\cos x & f^{(4)}(0)=1 \\
f^{(5)}(x)=-\sin x & f^{(5)}(0)=0 \\
f^{(6)}(x)=-\cos x & f^{(6)}(0)=-1 \\
f^{(7)}(x)=\sin x & f^{(7)}(0)=0 \\
f^{(8)}(x)=\cos x & f^{(8)}(0)=1
\end{array}
$$

Using this we can find the $8^{\text {th }}$ degree Maclaurin polynomial:

$$
T_{8}(x)=1-\frac{x^{2}}{2}+\frac{x^{4}}{24}-\frac{x^{6}}{6!}+\frac{x^{8}}{8!}
$$

Continuing this process gives us the $2 n^{\text {th }}$ Maclaurin polynomial

$$
T_{2 n}(x)=\sum_{k=0}^{n} \frac{(-1)^{k}}{(2 k)!} \cdot x^{2 k}
$$

## Warning 3.4.15

The above formula only works when x is measured in radians, because all of our derivative formulae for trig functions were developed under the assumption that angles are measured in radians.

Below we plot $\cos x$ against its first few Maclaurin polynomial approximations:

$\cos x \approx 1$

$\cos x \approx 1-\frac{1}{2!} x^{2}$

$\cos x \approx 1-\frac{1}{2!} x^{2}+\frac{1}{4!} x^{4}$

$\cos x \approx 1-\frac{1}{2!} x^{2}+\frac{1}{4!} x^{4}-\frac{1}{6!} x^{6}$

The above work is quite easily recycled to get the Maclaurin polynomial for sine:
Example 3.4.16 Maclaurin polynomial for $\sin x$.
Find the 5 th degree Maclaurin polynomial for $\sin x$.
Solution We could simply work as before and compute the first five derivatives of $\sin x$. But set $g(x)=\sin x$ and notice that $g(x)=-f^{\prime}(x)$, where $f(x)=\cos x$. Then we have

$$
\begin{aligned}
g(0) & =-f^{\prime}(0)=0 \\
g^{\prime}(0) & =-f^{\prime \prime}(0)=1 \\
g^{\prime \prime}(0) & =-f^{\prime \prime \prime}(0)=0 \\
g^{\prime \prime \prime}(0) & =-f^{(4)}(0)=-1 \\
g^{(4)}(0) & =-f^{(5)}(0)=0 \\
g^{(5)}(0) & =-f^{(6)}(0)=1
\end{aligned}
$$

Hence the required Maclaurin polynomial is

$$
T_{5}(x)=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}
$$

Just as we extended to the $2 n^{\text {th }}$ Maclaurin polynomial for cosine, we can also extend our work to compute the $(2 n+1)^{\text {th }}$ Maclaurin polynomial for sine:

$$
T_{2 n+1}(x)=\sum_{k=0}^{n} \frac{(-1)^{k}}{(2 k+1)!} \cdot x^{2 k+1}
$$

## Warning 3.4.17

The above formula only works when x is measured in radians, because all of our derivative formulae for trig functions were developed under the assumption that angles are measured in radians.

Below we plot $\sin x$ against its first few Maclaurin polynomial approximations.


To get an idea of how good these Taylor polynomials are at approximating sin and $\cos$, let's concentrate on $\sin x$ and consider $x$ 's whose magnitude $|x| \leq 1$. There are tricks that you can employ ${ }^{11}$ to evaluate sine and cosine at values of $x$ outside this

11 If you are writing software to evaluate $\sin x$, you can always use the trig identity $\sin (x)=\sin (x-$ $2 n \pi)$, to easily restrict to $|x| \leq \pi$. You can then use the trig identity $\sin (x)=-\sin (x \pm \pi)$ to reduce to $|x| \leq \frac{\pi}{2}$. Finally you can use the trig identity $\left.\sin (x)=\mp \cos \left(\frac{\pi}{2} \pm x\right)\right)$ to reduce to $|x| \leq \frac{\pi}{4}<1$.
range.
If $|x| \leq 1$ radians ${ }^{12}$, then the magnitudes of the successive terms in the Taylor polynomials for $\sin x$ are bounded by

$$
\begin{array}{rlrl}
|x| & \leq 1 & \frac{1}{3!}|x|^{3} \leq \frac{1}{6} & \frac{1}{5!}|x|^{5} \leq \frac{1}{120} \approx 0.0083 \\
\frac{1}{7!}|x|^{7} \leq \frac{1}{7!} \approx 0.0002 & \frac{1}{9!}|x|^{9} \leq \frac{1}{9!} \approx 0.000003 & \frac{1}{11!}|x|^{11} \leq \frac{1}{11!} \approx 0.000000025
\end{array}
$$

From these inequalities, and the graphs on the previous pages, it certainly looks like, for $x$ not too large, even relatively low degree Taylor polynomials give very good approximations. In Section 3.4.9 we'll see how to get rigorous error bounds on our Taylor polynomial approximations.

### 3.4.7 Estimating Change and $\Delta x, \Delta y$ Notation

Suppose that we have two variables $x$ and $y$ that are related by $y=f(x)$, for some function $f$. One of the most important applications of calculus is to help us understand what happens to $y$ when we make a small change in $x$.

## Definition 3.4.18

Let $x, y$ be variables related by a function $f$. That is $y=f(x)$. Then we denote a small change in the variable $x$ by $\Delta x$ (read as "delta $x$ "). The corresponding small change in the variable $y$ is denoted $\Delta y$ (read as "delta $y$ ").

$$
\Delta y=f(x+\Delta x)-f(x)
$$

In many situations we do not need to compute $\Delta y$ exactly and are instead happy with an approximation. Consider the following example.

Example 3.4.19 Estimate the increase in cost for a given change in production.
Let $x$ be the number of cars manufactured per week in some factory and let $y$ the cost of manufacturing those $x$ cars. Given that the factory currently produces $a$ cars per week, we would like to estimate the increase in cost if we make a small change in the number of cars produced.
Solution We are told that $a$ is the number of cars currently produced per week; the cost of production is then $f(a)$.

- Say the number of cars produced is changed from $a$ to $a+\Delta x$ (where $\Delta x$ is some small number.
- As $x$ undergoes this change, the costs change from $y=f(a)$ to $f(a+\Delta x)$. Hence

$$
\Delta y=f(a+\Delta x)-f(a)
$$

12 Recall that the derivative formulae that we used to derive the Taylor polynomials are valid only when $x$ is in radians. The restriction $-1 \leq x \leq 1$ radians translates to angles bounded by $\frac{180}{\pi} \approx 57^{\circ}$.

- We can estimate this change using a linear approximation. Substituting $x=$ $a+\Delta x$ into the equation 3.4.3 yields the approximation

$$
f(a+\Delta x) \approx f(a)+f^{\prime}(a)(a+\Delta x-a)
$$

and consequently the approximation

$$
\Delta y=f(a+\Delta x)-f(a) \approx f(a)+f^{\prime}(a) \Delta x-f(a)
$$

simplifies to the following neat estimate of $\Delta y$ :

Equation 3.4.20 Linear approximation of $\Delta y$.

$$
\Delta y \approx f^{\prime}(a) \Delta x
$$

- In the automobile manufacturing example, when the production level is a cars per week, increasing the production level by $\Delta x$ will cost approximately $f^{\prime}(a) \Delta x$. The additional cost per additional car, $f^{\prime}(a)$, is called the "marginal cost" of a car.
- If we instead use the quadratic approximation (given by equation 3.4.6) then we estimate

$$
f(a+\Delta x) \approx f(a)+f^{\prime}(a) \Delta x+\frac{1}{2} f^{\prime \prime}(a) \Delta x^{2}
$$

and so

$$
\Delta y=f(a+\Delta x)-f(a) \approx f(a)+f^{\prime}(a) \Delta x+\frac{1}{2} f^{\prime \prime}(a) \Delta x^{2}-f(a)
$$

which simplifies to

Equation 3.4.21 Quadratic approximation of $\Delta y$.

$$
\Delta y \approx f^{\prime}(a) \Delta x+\frac{1}{2} f^{\prime \prime}(a) \Delta x^{2}
$$

### 3.4.8 $\leadsto$ Further Examples

In this subsection we give further examples of computation and use of Taylor approximations.

## Example 3.4.22 Estimating tan $46^{\circ}$.

Estimate $\tan 46^{\circ}$, using the constant-, linear- and quadratic-approximations (equations 3.4.1, 3.4.3 and 3.4.6).
Solution Note that we need to be careful to translate angles measured in degrees to radians.

- Set $f(x)=\tan x, x=46 \frac{\pi}{180}$ radians and $a=45 \frac{\pi}{180}=\frac{\pi}{4}$ radians. This is a good choice for $a$ because
- $a=45^{\circ}$ is close to $x=46^{\circ}$. As noted above, it is generally the case that the closer $x$ is to $a$, the better various approximations will be.
- We know the values of all trig functions at $45^{\circ}$.
- Now we need to compute $f$ and its first two derivatives at $x=a$. It is a good time to recall the special $1: 1: \sqrt{2}$ triangle


So

$$
\begin{array}{rlrl}
f(x) & =\tan x & f(\pi / 4) & =1 \\
f^{\prime}(x) & =\sec ^{2} x=\frac{1}{\cos ^{2} x} & f^{\prime}(\pi / 4) & =\frac{1}{1 / \sqrt{2}^{2}}=2 \\
f^{\prime \prime}(x) & =\frac{2 \sin x}{\cos ^{3} x} & f^{\prime \prime}(\pi / 4) & =\frac{2 / \sqrt{2}}{1 / \sqrt{2}^{3}}=4
\end{array}
$$

- As $x-a=46 \frac{\pi}{180}-45 \frac{\pi}{180}=\frac{\pi}{180}$ radians, the three approximations are

$$
\begin{aligned}
f(x) & \approx f(a) \\
& =1 \\
f(x) & \approx f(a)+f^{\prime}(a)(x-a) \quad=1+2 \frac{\pi}{180} \\
& =1.034907 \\
f(x) & \approx f(a)+f^{\prime}(a)(x-a)+\frac{1}{2} f^{\prime \prime}(a)(x-a)^{2}
\end{aligned}
$$

$$
=1.035516
$$

For comparison purposes, $\tan 46^{\circ}$ really is 1.035530 to 6 decimal places.

## Warning 3.4.23

All of our derivative formulae for trig functions were developed under the assumption that angles are measured in radians. Those derivatives appeared in the approximation formulae that we used in Example 3.4.22, so we were obliged to express $x-a$ in radians.

Example 3.4.24 Error inferring a height from an angle.
Suppose that you are ten meters from a vertical pole. You were contracted to measure the height of the pole. You can't take it down or climb it. So you measure the angle subtended by the top of the pole. You measure $\theta=30^{\circ}$, which gives

$$
h=10 \tan 30^{\circ}=\frac{10}{\sqrt{3}} \approx 5.77 \mathrm{~m}
$$

This is just standard trigonometry - if we know the angle exactly then we know the height exactly.
However, in the "real world" angles are hard to measure with such precision. If the contract requires you the measurement of the pole to be accurate within 10 cm , how accurate does your measurement of the angle $\theta$ need to be?
Solution For simplicity ${ }^{a}$, we are going to assume that the pole is perfectly straight and perfectly vertical and that your distance from the pole was exactly 10 m .

- Write $\theta=\theta_{0}+\Delta \theta$ where $\theta$ is the exact angle, $\theta_{0}$ is the measured angle and $\Delta \theta$ is the error.
- Similarly write $h=h_{0}+\Delta h$, where $h$ is the exact height and $h_{0}=\frac{10}{\sqrt{3}}$ is the computed height. Their difference, $\Delta h$, is the error.
- Then

$$
\begin{array}{rlr}
h_{0} & =10 \tan \theta_{0} & h_{0}+\Delta h=10 \tan \left(\theta_{0}+\Delta \theta\right) \\
\Delta h & =10 \tan \left(\theta_{0}+\Delta \theta\right)-10 \tan \theta_{0} &
\end{array}
$$

We could attempt to solve this equation for $\Delta \theta$ in terms of $\Delta h$ - but it is far simpler to approximate $\Delta h$ using the linear approximation in equation 3.4.20.

- To use equation 3.4.20, replace $y$ with $h, x$ with $\theta$ and $a$ with $\theta_{0}$. Our function $f(\theta)=10 \tan \theta$ and $\theta_{0}=30^{\circ}=\pi / 6$ radians. Then

$$
\Delta y \approx f^{\prime}(a) \Delta x \quad \text { becomes } \quad \Delta h \approx f^{\prime}\left(\theta_{0}\right) \Delta \theta
$$

Since $f(\theta)=10 \tan \theta, f^{\prime}(\theta)=10 \sec ^{2} \theta$ and

$$
f^{\prime}\left(\theta_{0}\right)=10 \sec ^{2}(\pi / 6)=10 \cdot\left(\frac{2}{\sqrt{3}}\right)^{2}=\frac{40}{3}
$$

- Putting things together gives

$$
\Delta h \approx f^{\prime}\left(\theta_{0}\right) \Delta \theta \quad \text { becomes } \quad \Delta h \approx \frac{40}{3} \Delta \theta
$$

We can then solve this equation for $\Delta \theta$ in terms of $\Delta h$ :

$$
\Delta \theta \approx \frac{3}{40} \Delta h
$$

- We are told that we must have $|\Delta h|<0.1$, so we must have

$$
|\Delta \theta| \leq \frac{3}{400}
$$

This is measured in radians, so converting back to degrees

$$
\frac{3}{400} \cdot \frac{180}{\pi}=0.43^{\circ}
$$

$a \quad$ Mathematicians love assumptions that let us tame the real world.

## Definition 3.4.25

Suppose that you measure, approximately, some quantity. Suppose that the exact value of that quantity is $Q_{0}$ and that your measurement yielded $Q_{0}+\Delta Q$. Then $|\Delta Q|$ is called the absolute error of the measurement and $100 \frac{|\Delta Q|}{Q_{0}}$ is called the percentage error of the measurement. As an example, if the exact value is 4 and the measured value is 5 , then the absolute error is $|5-4|=1$ and the percentage error is $100 \frac{|5-4|}{4}=25$. That is, the error, 1 , was $25 \%$ of the exact value, 4 .

Example 3.4.26 Error inferring the area and volume from the radius.
Suppose that the radius of a sphere has been measured with a percentage error of at most $\varepsilon \%$. Find the corresponding approximate percentage errors in the surface area and volume of the sphere.
Solution We need to be careful in this problem to convert between absolute and percentage errors correctly.

- Suppose that the exact radius is $r_{0}$ and that the measured radius is $r_{0}+\Delta r$.
- Then the absolute error in the measurement is $|\Delta r|$ and, by definition, the percentage error is $100 \frac{|\Delta r|}{r_{0}}$. We are told that $100 \frac{|\Delta r|}{r_{0}} \leq \varepsilon$.
- The surface area ${ }^{a}$ of a sphere of radius $r$ is $A(r)=4 \pi r^{2}$. The error in the surface area computed with the measured radius is

$$
\begin{aligned}
\Delta A & =A\left(r_{0}+\Delta r\right)-A\left(r_{0}\right) \approx A^{\prime}\left(r_{0}\right) \Delta r \\
& =8 \pi r_{0} \Delta r
\end{aligned}
$$

where we have made use of the linear approximation, equation 3.4.20.

- The corresponding percentage error is then

$$
100 \frac{|\Delta A|}{A\left(r_{0}\right)} \approx 100 \frac{\left|A^{\prime}\left(r_{0}\right) \Delta r\right|}{A\left(r_{0}\right)}=100 \frac{8 \pi r_{0}|\Delta r|}{4 \pi r_{0}^{2}}=2 \times 100 \frac{|\Delta r|}{r_{0}} \leq 2 \varepsilon
$$

- The volume of a sphere ${ }^{b}$ of radius $r$ is $V(r)=\frac{4}{3} \pi r^{3}$. The error in the volume computed with the measured radius is

$$
\begin{aligned}
\Delta V & =V\left(r_{0}+\Delta r\right)-V\left(r_{0}\right) \approx V^{\prime}\left(r_{0}\right) \Delta r \\
& =4 \pi r_{0}^{2} \Delta r
\end{aligned}
$$

where we have again made use of the linear approximation, equation 3.4.20.

- The corresponding percentage error is

$$
100 \frac{|\Delta V|}{V\left(r_{0}\right)} \approx 100 \frac{\left|V^{\prime}\left(r_{0}\right) \Delta r\right|}{V\left(r_{0}\right)}=100 \frac{4 \pi r_{0}^{2}|\Delta r|}{4 \pi r_{0}^{3} / 3}=3 \times 100 \frac{|\Delta r|}{r_{0}} \leq 3 \varepsilon
$$

We have just computed an approximation to $\Delta V$. This problem is actually sufficiently simple that we can compute $\Delta V$ exactly:

$$
\Delta V=V\left(r_{0}+\Delta r\right)-V\left(r_{0}\right)=\frac{4}{3} \pi\left(r_{0}+\Delta r\right)^{3}-\frac{4}{3} \pi r_{0}^{3}
$$

- Applying $(a+b)^{3}=a^{3}+3 a^{2} b+3 a b^{2}+b^{3}$ with $a=r_{0}$ and $b=\Delta r$, gives

$$
\begin{aligned}
V\left(r_{0}+\Delta r\right)-V\left(r_{0}\right) & =\frac{4}{3} \pi\left[r_{0}^{3}+3 r_{0}^{2} \Delta r+3 r_{0}(\Delta r)^{2}+(\Delta r)^{3}\right]-\frac{4}{3} \pi r_{0}^{3} \\
& =\frac{4}{3} \pi\left[3 r_{0}^{2} \Delta r+3 r_{0}(\Delta r)^{2}+(\Delta r)^{3}\right]
\end{aligned}
$$

- Thus the difference between the exact error and the linear approximation to the error is obtained by retaining only the last two terms in the square brackets. This has magnitude

$$
\frac{4}{3} \pi\left|3 r_{0}(\Delta r)^{2}+(\Delta r)^{3}\right|=\frac{4}{3} \pi\left|3 r_{0}+\Delta r\right|(\Delta r)^{2}
$$

or in percentage terms

$$
\begin{aligned}
100 \cdot \frac{1}{\frac{4}{3} \pi r_{0}^{3}} \cdot \frac{4}{3} \pi\left|3 r_{0}(\Delta r)^{2}+(\Delta r)^{3}\right| & =100\left|3 \frac{\Delta r^{2}}{r_{0}^{2}}+\frac{\Delta r^{3}}{r_{0}^{3}}\right| \\
& =\left(100 \frac{3 \Delta r}{r_{0}}\right) \cdot\left(\frac{\Delta r}{r_{0}}\right)\left|1+\frac{\Delta r}{3 r_{0}}\right| \\
& \leq 3 \varepsilon\left(\frac{\varepsilon}{100}\right) \cdot\left(1+\frac{\varepsilon}{300}\right)
\end{aligned}
$$

Since $\varepsilon$ is small, we can assume that $1+\frac{\varepsilon}{300} \approx 1$. Hence the difference between the exact error and the linear approximation of the error is roughly a factor of $\frac{\varepsilon}{100}$ smaller than the linear approximation $3 \varepsilon$.

- As an aside, notice that if we argue that $\Delta r$ is very small and so we can ignore terms involving $(\Delta r)^{2}$ and $(\Delta r)^{3}$ as being really really small, then we obtain

$$
\begin{aligned}
V\left(r_{0}+\Delta r\right)-V\left(r_{0}\right) & =\frac{4}{3} \pi[3 r_{0}^{2} \Delta r \underbrace{+3 r_{0}(\Delta r)^{2}+(\Delta r)^{3}}_{\text {really really small }}] \\
& \approx \frac{4}{3} \pi \cdot 3 r_{0}^{2} \Delta r=4 \pi r_{0}^{2} \Delta r
\end{aligned}
$$

which is precisely the result of our linear approximation above.
$a \quad$ We do not expect you to remember the surface areas of solids for this course.
$b$ We do expect you to remember the formula for the volume of a sphere.

Example 3.4.27 Percentage error inferring a height.
To compute the height $h$ of a lamp post, the length $s$ of the shadow of a two meter pole is measured. The pole is 6 m from the lamp post. If the length of the shadow was measured to be 4 m , with an error of at most one cm , find the height of the lamp post and estimate the percentage error in the height.
Solution We should first draw a picture ${ }^{a}$


- By similar triangles we see that

$$
\frac{2}{s}=\frac{h}{6+s}
$$

from which we can isolate $h$ as a function of $s$ :

$$
h=\frac{2(6+s)}{s}=\frac{12}{s}+2
$$

- The length of the shadow was measured to be $s_{0}=4 \mathrm{~m}$. The corresponding height of the lamp post is

$$
h_{0}=\frac{12}{4}+2=5 m
$$

- If the error in the measurement of the length of the shadow was $\Delta s$, then the exact shadow length was $s=s_{0}+\Delta s$ and the exact lamp post height is $h=f\left(s_{0}+\Delta s\right)$, where $f(s)=\frac{12}{s}+2$. The error in the computed lamp post height is

$$
\Delta h=h-h_{0}=f\left(s_{0}+\Delta s\right)-f\left(s_{0}\right)
$$

- We can then make a linear approximation of this error using equation 3.4.20:

$$
\Delta h \approx f^{\prime}\left(s_{0}\right) \Delta s=-\frac{12}{s_{0}^{2}} \Delta s=-\frac{12}{4^{2}} \Delta s
$$

- We are told that $|\Delta s| \leq \frac{1}{100} \mathrm{~m}$. Consequently, approximately,

$$
|\Delta h| \leq \frac{12}{4^{2}} \frac{1}{100}=\frac{3}{400}
$$

The percentage error is then approximately

$$
100 \frac{|\Delta h|}{h_{0}} \leq 100 \frac{3}{400 \times 5}=0.15 \%
$$

### 3.4.9 $\sim$ The Error in the Taylor Polynomial Approximations

Any time you make an approximation, it is desirable to have some idea of the size of the error you introduced. That is, we would like to know the difference $R(x)$ between the original function $f(x)$ and our approximation $F(x)$ :

$$
R(x)=f(x)-F(x)
$$

Of course if we know $R(x)$ exactly, then we could recover $f(x)=F(x)+R(x)$ - so this is an unrealistic hope. In practice we would simply like to bound $R(x)$ :

$$
|R(x)|=|f(x)-F(x)| \leq M
$$

where (hopefully) $M$ is some small number. It is worth stressing that we do not need the tightest possible value of $M$, we just need a relatively easily computed $M$ that isn't too far off the true value of $|f(x)-F(x)|$.

We will now develop a formula for the error introduced by the constant approximation, equation 3.4.1 (developed back in Section 3.4.1)

$$
f(x) \approx f(a)=T_{0}(x) \quad 0^{\text {th }} \text { Taylor polynomial }
$$

The resulting formula can be used to get an upper bound on the size of the error $|R(x)|$.
The main ingredient we will need is the Mean-Value Theorem (Theorem 2.13.5) so we suggest you quickly revise it. Consider the following obvious statement:

$$
\begin{array}{rlr}
f(x) & =f(x) & \text { now some sneaky manipulations } \\
& =f(a)+(f(x)-f(a)) & \\
& =\underbrace{f(a)}_{=T_{0}(x)}+(f(x)-f(a)) \cdot \underbrace{\frac{x-a}{x-a}}_{=1} \\
& =T_{0}(x)+\underbrace{\frac{f(x)-f(a)}{x-a}}_{\text {looks familiar }} \cdot(x-a)
\end{array}
$$

Indeed, this equation is important in the discussion that follows, so we'll highlight it

Equation 3.4.28 We will need it again soon.

$$
f(x)=T_{0}(x)+\left[\frac{f(x)-f(a)}{x-a}\right](x-a)
$$

The coefficient $\frac{f(x)-f(a)}{x-a}$ of $(x-a)$ is the average slope of $f(t)$ as $t$ moves from $t=a$ to $t=x$. We can picture this as the slope of the secant joining the points $(a, f(a))$ and $(x, f(x))$ in the sketch below.


As $t$ moves from $a$ to $x$, the instantaneous slope $f^{\prime}(t)$ keeps changing. Sometimes $f^{\prime}(t)$ might be larger than the average slope $\frac{f(x)-f(a)}{x-a}$, and sometimes $f^{\prime}(t)$ might be smaller than the average slope $\frac{f(x)-f(a)}{x-a}$. However, by the Mean-Value Theorem (Theorem 2.13.5), there must be some number $c$, strictly between $a$ and $x$, for which $f^{\prime}(c)=\frac{f(x)-f(a)}{x-a}$ exactly.

Substituting this into formula 3.4.28 gives

Equation 3.4.29 Towards the error.

$$
f(x)=T_{0}(x)+f^{\prime}(c)(x-a) \quad \text { for some } c \text { strictly between } a \text { and } x
$$

Notice that this expression as it stands is not quite what we want. Let us massage this around a little more into a more useful form

## Equation 3.4.30 The error in constant approximation.

$$
f(x)-T_{0}(x)=f^{\prime}(c) \cdot(x-a) \quad \text { for some } c \text { strictly between } a \text { and } x
$$

Notice that the MVT doesn't tell us the value of $c$, however we do know that it lies strictly between $x$ and $a$. So if we can get a good bound on $f^{\prime}(c)$ on this interval then we can get a good bound on the error.

Example 3.4.31 Error in the approximation in 3.4.2.
Let us return to Example 3.4.2, and we'll try to bound the error in our approximation of $e^{0.1}$.

- Recall that $f(x)=e^{x}, a=0$ and $T_{0}(x)=e^{0}=1$.
- Then by equation 3.4.30

$$
e^{0.1}-T_{0}(0.1)=f^{\prime}(c) \cdot(0.1-0) \quad \text { with } 0<c<0.1
$$

- Now $f^{\prime}(c)=e^{c}$, so we need to bound $e^{c}$ on $(0,0.1)$. Since $e^{c}$ is an increasing function, we know that

$$
e^{0}<f^{\prime}(c)<e^{0.1} \quad \text { when } 0<c<0.1
$$

So one is tempted to write that

$$
\begin{aligned}
\left|e^{0.1}-T_{0}(0.1)\right| & =|R(x)|=\left|f^{\prime}(c)\right| \cdot(0.1-0) \\
& <e^{0.1} \cdot 0.1
\end{aligned}
$$

And while this is true, it is rather circular. We have just bounded the error in our approximation of $e^{0.1}$ by $\frac{1}{10} e^{0.1}$ - if we actually knew $e^{0.1}$ then we wouldn't need to estimate it!

- While we don't know $e^{0.1}$ exactly, we do know ${ }^{a}$ that $1=e^{0}<e^{0.1}<e^{1}<3$. This gives us

$$
|R(0.1)|<3 \times 0.1=0.3
$$

That is - the error in our approximation of $e^{0.1}$ is no greater than 0.3 . Recall that we don't need the error exactly, we just need a good idea of how large it actually is.

- In fact the real error here is

$$
\left|e^{0.1}-T_{0}(0.1)\right|=\left|e^{0.1}-1\right|=0.1051709 \ldots
$$

so we have over-estimated the error by a factor of 3 .
But we can actually go a little further here - we can bound the error above and below. If we do not take absolute values, then since

$$
e^{0.1}-T_{0}(0.1)=f^{\prime}(c) \cdot 0.1 \quad \text { and } 1<f^{\prime}(c)<3
$$

we can write

$$
1 \times 0.1 \leq\left(e^{0.1}-T_{0}(0.1)\right) \leq 3 \times 0.1
$$

SO

$$
\begin{aligned}
T_{0}(0.1)+0.1 & \leq e^{0.1} \leq T_{0}(0.1)+0.3 \\
1.1 & \leq e^{0.1} \leq 1.3
\end{aligned}
$$

So while the upper bound is weak, the lower bound is quite tight.
$a$ Oops! Do we really know that $e<3$ ? We haven't proved it. We will do so soon.


There are formulae similar to equation 3.4.29, that can be used to bound the error in our other approximations; all are based on generalisations of the MVT. The next one - for linear approximations - is

$$
f(x)=\underbrace{f(a)+f^{\prime}(a)(x-a)}_{=T_{1}(x)}+\frac{1}{2} f^{\prime \prime}(c)(x-a)^{2} \quad \text { for some } c \text { strictly between } a \text { and } x
$$

which we can rewrite in terms of $T_{1}(x)$ :

Equation 3.4.32 The error in linear approximation.

$$
f(x)-T_{1}(x)=\frac{1}{2} f^{\prime \prime}(c)(x-a)^{2} \quad \text { for some } c \text { strictly between } a \text { and } x
$$

It implies that the error that we make when we approximate $f(x)$ by $T_{1}(x)=$ $f(a)+f^{\prime}(a)(x-a)$ is exactly $\frac{1}{2} f^{\prime \prime}(c)(x-a)^{2}$ for some $c$ strictly between $a$ and $x$.

More generally

$$
f(x)=\underbrace{f(a)+f^{\prime}(a) \cdot(x-a)+\cdots+\frac{1}{n!} f^{(n)}(a) \cdot(x-a)^{n}}_{=T_{n}(x)}+\frac{1}{(n+1)!} f^{(n+1)}(c) \cdot(x-a)^{n+1}
$$

for some $c$ strictly between $a$ and $x$. Again, rewriting this in terms of $T_{n}(x)$ gives

## Equation 3.4.33

$$
f(x)-T_{n}(x)=\frac{1}{(n+1)!} f^{(n+1)}(c) \cdot(x-a)^{n+1} \quad \text { for some } c \text { strictly between } a \text { and } x
$$

That is, the error introduced when $f(x)$ is approximated by its Taylor polynomial of degree $n$, is precisely the last term of the Taylor polynomial of degree $n+1$, but with the derivative evaluated at some point between $a$ and $x$, rather than exactly at $a$. These error formulae are proven in the optional Section 3.4.10 later in this chapter.

Example 3.4.34 Approximate $\sin 46^{\circ}$ and estimate the error.
Approximate $\sin 46^{\circ}$ using Taylor polynomials about $a=45^{\circ}$, and estimate the resulting error.

## Solution

- Start by defining $f(x)=\sin x$ and

$$
\begin{array}{rlr}
a & =45^{\circ}=45 \frac{\pi}{180} \text { radians } \quad x=46^{\circ}=46 \frac{\pi}{180} \text { radians } \\
x-a & =\frac{\pi}{180} \text { radians }
\end{array}
$$

- The first few derivatives of $f$ at $a$ are

$$
\begin{array}{rlrl}
f(x) & =\sin x & f(a)=\frac{1}{\sqrt{2}} \\
f^{\prime}(x) & =\cos x & \\
f^{\prime}(a) & =\frac{1}{\sqrt{2}} & \\
f^{\prime \prime}(x) & =-\sin x & \\
f^{\prime \prime}(a) & =-\frac{1}{\sqrt{2}} & f^{(3)}(a)=-\frac{1}{\sqrt{2}}
\end{array}
$$

- The constant, linear and quadratic Taylor approximations for $\sin (x)$ about $\frac{\pi}{4}$ are

$$
\begin{array}{ll}
T_{0}(x)=f(a) & =\frac{1}{\sqrt{2}} \\
T_{1}(x)=T_{0}(x)+f^{\prime}(a) \cdot(x-a) & =\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right) \\
T_{2}(x)=T_{1}(x)+\frac{1}{2} f^{\prime \prime}(a) \cdot(x-a)^{2} & =\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right)-\frac{1}{2 \sqrt{2}}\left(x-\frac{\pi}{4}\right)^{2}
\end{array}
$$

- So the approximations for $\sin 46^{\circ}$ are

$$
\begin{aligned}
\sin 46^{\circ} & \approx T_{0}\left(\frac{46 \pi}{180}\right)=\frac{1}{\sqrt{2}} \\
& =0.70710678 \\
\sin 46^{\circ} & \approx T_{1}\left(\frac{46 \pi}{180}\right)=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\left(\frac{\pi}{180}\right) \\
& =0.71944812 \\
\sin 46^{\circ} & \approx T_{2}\left(\frac{46 \pi}{180}\right)=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\left(\frac{\pi}{180}\right)-\frac{1}{2 \sqrt{2}}\left(\frac{\pi}{180}\right)^{2} \\
& =0.71934042
\end{aligned}
$$

- The errors in those approximations are (respectively)

$$
\begin{aligned}
& \text { error in } 0.70710678=f^{\prime}(c)(x-a)=\cos c \cdot\left(\frac{\pi}{180}\right) \\
& \text { error in } 0.71944812=\frac{1}{2} f^{\prime \prime}(c)(x-a)^{2}=-\frac{1}{2} \cdot \sin c \cdot\left(\frac{\pi}{180}\right)^{2} \\
& \text { error in } 0.71923272=\frac{1}{3!} f^{(3)}(c)(x-a)^{3}=-\frac{1}{3!} \cdot \cos c \cdot\left(\frac{\pi}{180}\right)^{3}
\end{aligned}
$$

In each of these three cases $c$ must lie somewhere between $45^{\circ}$ and $46^{\circ}$.

- Rather than carefully estimating $\sin c$ and $\cos c$ for $c$ in that range, we make use of a simpler (but much easier bound). No matter what $c$ is, we know that $|\sin c| \leq 1$ and $|\cos c| \leq 1$. Hence

$$
\begin{aligned}
\mid \text { error in } 0.70710678 \mid & \leq\left(\frac{\pi}{180}\right) \quad<0.018 \\
\mid \text { error in } 0.71944812 \mid & \leq \frac{1}{2}\left(\frac{\pi}{180}\right)^{2}<0.00015 \\
\mid \text { error in } 0.71934042 \mid & \leq \frac{1}{3!}\left(\frac{\pi}{180}\right)^{3}<0.0000009
\end{aligned}
$$

Example 3.4.34

Example 3.4.35 Showing $e<3$.
In Example 3.4.31 above we used the fact that $e<3$ without actually proving it. Let's do so now.

- Consider the linear approximation of $e^{x}$ about $a=0$.

$$
T_{1}(x)=f(0)+f^{\prime}(0) \cdot x=1+x
$$

So at $x=1$ we have

$$
e \approx T_{1}(1)=2
$$

- The error in this approximation is

$$
e^{x}-T_{1}(x)=\frac{1}{2} f^{\prime \prime}(c) \cdot x^{2}=\frac{e^{c}}{2} \cdot x^{2}
$$

So at $x=1$ we have

$$
e-T_{1}(1)=\frac{e^{c}}{2}
$$

where $0<c<1$.

- Now since $e^{x}$ is an increasing ${ }^{a}$ function, it follows that $e^{c}<e$. Hence

$$
e-T_{1}(1)=\frac{e^{c}}{2}<\frac{e}{2}
$$

Moving the $\frac{e}{2}$ to the left hand side and the $T_{1}(1)$ to the right hand side gives

$$
\frac{e}{2} \leq T_{1}(1)=2
$$

So $e<4$.

- This isn't as tight as we would like - so now do the same with the quadratic approximation with $a=0$ :

$$
e^{x} \approx T_{2}(x)=1+x+\frac{x^{2}}{2}
$$

So when $x=1$ we have

$$
e \approx T_{2}(1)=1+1+\frac{1}{2}=\frac{5}{2}
$$

- The error in this approximation is

$$
e^{x}-T_{2}(x)=\frac{1}{3!} f^{\prime \prime \prime}(c) \cdot x^{3}=\frac{e^{c}}{6} \cdot x^{3}
$$

So at $x=1$ we have

$$
e-T_{2}(1)=\frac{e^{c}}{6}
$$

where $0<c<1$.

- Again since $e^{x}$ is an increasing function we have $e^{c}<e$. Hence

$$
e-T_{2}(1)=\frac{e^{c}}{6}<\frac{e}{6}
$$

That is

$$
\frac{5 e}{6}<T_{2}(1)=\frac{5}{2}
$$

So $e<3$ as required.
$a \quad$ Since the derivative of $e^{x}$ is $e^{x}$ which is positive everywhere, the function is increasing everywhere.
$\begin{array}{lc} \\ & \text { Example 3.4.35 }\end{array}$

Example 3.4.36 More on $e^{x}$.
We wrote down the general $n^{\text {th }}$ degree Maclaurin polynomial approximation of $e^{x}$ in Example 3.4.12 above.

- Recall that

$$
T_{n}(x)=\sum_{k=0}^{n} \frac{1}{k!} x^{k}
$$

- The error in this approximation is (by equation 3.4.33)

$$
e^{x}-T_{n}(x)=\frac{1}{(n+1)!} e^{c}
$$

where $c$ is some number between 0 and $x$.

- So setting $x=1$ in this gives

$$
e-T_{n}(1)=\frac{1}{(n+1)!} e^{c}
$$

where $0<c<1$.

- Since $e^{x}$ is an increasing function we know that $1=e^{0}<e^{c}<e^{1}<3$, so the above expression becomes

$$
\frac{1}{(n+1)!} \leq e-T_{n}(1)=\frac{1}{(n+1)!} e^{c} \leq \frac{3}{(n+1)!}
$$

- So when $n=9$ we have

$$
\frac{1}{10!} \leq e-\left(1+1+\frac{1}{2}+\cdots+\frac{1}{9!}\right) \leq \frac{3}{10!}
$$

- Now $1 / 10!<3 / 10!<10^{-6}$, so the approximation of $e$ by

$$
e \approx 1+1+\frac{1}{2}+\cdots+\frac{1}{9!}=\frac{98641}{36288}=2.718281 \ldots
$$

is correct to 6 decimal places.

- More generally we know that using $T_{n}(1)$ to approximate $e$ will have an error of at most $\frac{3}{(n+1)!}$ - so it converges very quickly.

Example 3.4.37 Example 3.4.24 Revisited.
Recall ${ }^{a}$ that in Example 3.4.24 (measuring the height of the pole), we used the linear approximation

$$
f\left(\theta_{0}+\Delta \theta\right) \approx f\left(\theta_{0}\right)+f^{\prime}\left(\theta_{0}\right) \Delta \theta
$$

with $f(\theta)=10 \tan \theta$ and $\theta_{0}=30 \frac{\pi}{180}$ to get

$$
\Delta h=f\left(\theta_{0}+\Delta \theta\right)-f\left(\theta_{0}\right) \approx f^{\prime}\left(\theta_{0}\right) \Delta \theta \quad \text { which implies that } \quad \Delta \theta \approx \frac{\Delta h}{f^{\prime}\left(\theta_{0}\right)}
$$

- While this procedure is fairly reliable, it did involve an approximation. So that you could not $100 \%$ guarantee to your client's lawyer that an accuracy of 10 cm was achieved.
- On the other hand, if we use the exact formula 3.4.29, with the replacements $x \rightarrow \theta_{0}+\Delta \theta$ and $a \rightarrow \theta_{0}$

$$
f\left(\theta_{0}+\Delta \theta\right)=f\left(\theta_{0}\right)+f^{\prime}(c) \Delta \theta \quad \text { for some } c \text { between } \theta_{0} \text { and } \theta_{0}+\Delta \theta
$$

in place of the approximate formula 3.4.3, this legality is taken care of:

$$
\Delta h=f\left(\theta_{0}+\Delta \theta\right)-f\left(\theta_{0}\right)=f^{\prime}(c) \Delta \theta \quad \text { for some } c \text { between } \theta_{0} \text { and } \theta_{0}+\Delta \theta
$$

We can clean this up a little more since in our example $f^{\prime}(\theta)=10 \sec ^{2} \theta$. Thus for some $c$ between $\theta_{0}$ and $\theta_{0}+\Delta \theta$ :

$$
|\Delta h|=10 \sec ^{2}(c)|\Delta \theta|
$$

- Of course we do not know exactly what $c$ is. But suppose that we know that the angle was somewhere between $25^{\circ}$ and $35^{\circ}$. In other words suppose that, even though we don't know precisely what our measurement error was, it was certainly no more than $5^{\circ}$.
- Now on the range $25^{\circ}<c<35^{\circ}, \sec (c)$ is an increasing and positive function. Hence on this range

$$
1.217 \cdots=\sec ^{2} 25^{\circ} \leq \sec ^{2} c \leq \sec ^{2} 35^{\circ}=1.490 \ldots<1.491
$$

So

$$
12.17 \cdot|\Delta \theta| \leq|\Delta h|=10 \sec ^{2}(c) \cdot|\Delta \theta| \leq 14.91 \cdot|\Delta \theta|
$$

- Since we require $|\Delta h|<0.1$, we need $14.91|\Delta \theta|<0.1$, that is

$$
|\Delta \theta|<\frac{0.1}{14.91}=0.0067 \ldots
$$

So we must measure angles with an accuracy of no less than 0.0067 radians which is

$$
\frac{180}{\pi} \cdot 0.0067=0.38^{\circ}
$$

Hence a measurement error of $0.38^{\circ}$ or less is acceptable.
$a \quad$ Now is a good time to go back and re-read it.
$\uparrow \quad$ Example 3.4.37

### 3.4.10 (Optional) - Derivation of the Error Formulae

In this section we will derive the formula for the error that we gave in equation 3.4.33 - namely

$$
R_{n}(x)=f(x)-T_{n}(x)=\frac{1}{(n+1)!} f^{(n+1)}(c) \cdot(x-a)^{n+1}
$$

for some $c$ strictly between $a$ and $x$, and where $T_{n}(x)$ is the $n^{\text {th }}$ degree Taylor polynomial approximation of $f(x)$ about $x=a$ :

$$
T_{n}(x)=\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a) .
$$

Recall that we have already proved a special case of this formula for the constant approximation using the Mean-Value Theorem (Theorem 2.13.5). To prove the general case we need the following generalisation ${ }^{13}$ of that theorem:

13 It is not a terribly creative name for the generalisation, but it is an accurate one.

## Theorem 3.4.38 Generalised Mean-Value Theorem.

Let the functions $F(x)$ and $G(x)$ both be defined and continuous on $a \leq x \leq b$ and both be differentiable on $a<x<b$. Furthermore, suppose that $G^{\prime}(x) \neq 0$ for all $a<x<b$. Then, there is a number $c$ obeying $a<c<b$ such that

$$
\frac{F(b)-F(a)}{G(b)-G(a)}=\frac{F^{\prime}(c)}{G^{\prime}(c)}
$$

Notice that setting $G(x)=x$ recovers the original Mean-Value Theorem. It turns out that this theorem is not too difficult to prove from the MVT using some sneaky algebraic manipulations:

## Proof.

- First we construct a new function $h(x)$ as a linear combination of $F(x)$ and $G(x)$ so that $h(a)=h(b)=0$. Some experimentation yields

$$
h(x)=[F(b)-F(a)] \cdot[G(x)-G(a)]-[G(b)-G(a)] \cdot[F(x)-F(a)]
$$

- Since $h(a)=h(b)=0$, the Mean-Value theorem (actually Rolle's theorem) tells us that there is a number $c$ obeying $a<c<b$ such that $h^{\prime}(c)=0$ :

$$
\begin{aligned}
h^{\prime}(x) & =[F(b)-F(a)] \cdot G^{\prime}(x)-[G(b)-G(a)] \cdot F^{\prime}(x) \\
0 & =[F(b)-F(a)] \cdot G^{\prime}(c)-[G(b)-G(a)] \cdot F^{\prime}(c)
\end{aligned}
$$

Now move the $G^{\prime}(c)$ terms to one side and the $F^{\prime}(c)$ terms to the other:

$$
[F(b)-F(a)] \cdot G^{\prime}(c)=[G(b)-G(a)] \cdot F^{\prime}(c) .
$$

- Since we have $G^{\prime}(x) \neq 0$, we know that $G^{\prime}(c) \neq 0$. Further the Mean-Value theorem ensures ${ }^{a}$ that $G(a) \neq G(b)$. Hence we can move terms about to get

$$
\begin{aligned}
{[F(b)-F(a)] } & =[G(b)-G(a)] \cdot \frac{F^{\prime}(c)}{G^{\prime}(c)} \\
\frac{F(b)-F(a)}{G(b)-G(a)} & =\frac{F^{\prime}(c)}{G^{\prime}(c)}
\end{aligned}
$$

as required.

$a \quad$ Otherwise if $G(a)=G(b)$ the MVT tells us that there is some point $c$ between $a$ and $b$ so that $G^{\prime}(c)=0$.

Armed with the above theorem we can now move on to the proof of the Taylor remainder formula.

Proof of equation 3.4.33. We begin by proving the remainder formula for $n=1$. That is

$$
f(x)-T_{1}(x)=\frac{1}{2} f^{\prime \prime}(c) \cdot(x-a)^{2}
$$

- Start by setting

$$
F(x)=f(x)-T_{1}(x) \quad G(x)=(x-a)^{2}
$$

Notice that, since $T_{1}(a)=f(a)$ and $T_{1}^{\prime}(x)=f^{\prime}(a)$,

$$
\begin{array}{rlrl}
F(a) & =0 & G(a) & =0 \\
F^{\prime}(x) & =f^{\prime}(x)-f^{\prime}(a) & G^{\prime}(x) & =2(x-a)
\end{array}
$$

- Now apply the generalised MVT with $b=x$ : there exists a point $q$ between $a$ and $x$ such that

$$
\begin{aligned}
\frac{F(x)-F(a)}{G(x)-G(a)} & =\frac{F^{\prime}(q)}{G^{\prime}(q)} \\
\frac{F(x)-0}{G(x)-0} & =\frac{f^{\prime}(q)-f^{\prime}(a)}{2(q-a)} \\
2 \cdot \frac{F(x)}{G(x)} & =\frac{f^{\prime}(q)-f^{\prime}(a)}{q-a}
\end{aligned}
$$

- Consider the right-hand side of the above equation and set $g(x)=f^{\prime}(x)$. Then we have the term $\frac{g(q)-g(a)}{q-a}$ - this is exactly the form needed to apply the MVT. So now apply the standard MVT to the right-hand side of the above equation - there is some $c$ between $q$ and $a$ so that

$$
\frac{f^{\prime}(q)-f^{\prime}(a)}{q-a}=\frac{g(q)-g(a)}{q-a}=g^{\prime}(c)=f^{\prime \prime}(c)
$$

Notice that here we have assumed that $f^{\prime \prime}(x)$ exists.

- Putting this together we have that

$$
\begin{aligned}
2 \cdot \frac{F(x)}{G(x)} & =\frac{f^{\prime}(q)-f^{\prime}(a)}{q-a}=f^{\prime \prime}(c) \\
2 \frac{f(x)-T_{1}(x)}{(x-a)^{2}} & =f^{\prime \prime}(c) \\
f(x)-T_{1}(x) & =\frac{1}{2!} f^{\prime \prime}(c) \cdot(x-a)^{2}
\end{aligned}
$$

as required.

Oof! We have now proved the cases $n=1$ (and we did $n=0$ earlier).
To proceed - assume we have proved our result for $n=1,2, \cdots, k$. We realise that we haven't done this yet, but bear with us. Using that assumption we will prove the result is true for $n=k+1$. Once we have done that, then

- we have proved the result is true for $n=1$, and
- we have shown if the result is true for $n=k$ then it is true for $n=k+1$

Hence it must be true for all $n \geq 1$. This style of proof is called mathematical induction. You can think of the process as something like climbing a ladder:

- prove that you can get onto the ladder (the result is true for $n=1$ ), and
- if I can stand on the current rung, then I can step up to the next rung (if the result is true for $n=k$ then it is also true for $n=k+1$ )

Hence I can climb as high as like.

- Let $k>0$ and assume we have proved

$$
f(x)-T_{k}(x)=\frac{1}{(k+1)!} f^{(k+1)}(c) \cdot(x-a)^{k+1}
$$

for some $c$ between $a$ and $x$.

- Now set

$$
F(x)=f(x)-T_{k+1}(x) \quad G(x)=(x-a)^{k+1}
$$

and notice that, since $T_{k+1}(a)=f(a)$,

$$
F(a)=f(a)-T_{k+1}(a)=0 \quad G(a)=0 \quad G^{\prime}(x)=(k+1)(x-a)^{k}
$$

and apply the generalised MVT with $b=x$ : hence there exists a $q$ between $a$ and $x$ so that

$$
\begin{array}{rlr}
\frac{F(x)-F(a)}{G(x)-G(a)} & =\frac{F^{\prime}(q)}{G^{\prime}(q)} & \text { which becomes } \\
\frac{F(x)}{(x-a)^{k+1}} & =\frac{F^{\prime}(q)}{(k+1)(q-a)^{k}} & \text { rearrange } \\
F(x) & =\frac{(x-a)^{k+1}}{(k+1)(q-a)^{k}} \cdot F^{\prime}(q) &
\end{array}
$$

- We now examine $F^{\prime}(q)$. First carefully differentiate $F(x)$ :

$$
F^{\prime}(x)=\frac{\mathrm{d}}{\mathrm{~d} x}\left[f(x)-\left(f(a)+f^{\prime}(a)(x-a)+\frac{1}{2} f^{\prime \prime}(a)(x-a)^{2}+\cdots\right.\right.
$$

$$
\begin{aligned}
& \left.\left.+\frac{1}{k!} f^{(k)}(x-a)^{k}\right)\right] \\
=f^{\prime}(x)-\left(f^{\prime}(a)+\frac{2}{2} f^{\prime \prime}(a)(x-a)\right. & +\frac{3}{3!} f^{\prime \prime \prime}(a)(x-a)^{2}+\cdots \\
& \left.+\frac{k}{k!} f^{(k)}(a)(x-a)^{k-1}\right) \\
=f^{\prime}(x)-\left(f^{\prime}(a)+f^{\prime \prime}(a)(x-a)+\right. & \frac{1}{2} f^{\prime \prime \prime}(a)(x-a)^{2}+\cdots \\
& \left.+\frac{1}{(k-1)!} f^{(k)}(a)(x-a)^{k-1}\right)
\end{aligned}
$$

Now notice that if we set $f^{\prime}(x)=g(x)$ then this becomes

$$
\begin{aligned}
F^{\prime}(x)=g(x)-\left(g(a)+g^{\prime}(a)(x-a)+\right. & \frac{1}{2} g^{\prime \prime}(a)(x-a)^{2}+\cdots \\
& \left.+\frac{1}{(k-1)!} g^{(k-1)}(a)(x-a)^{k-1}\right)
\end{aligned}
$$

So $F^{\prime}(x)$ is then exactly the remainder formula but for a degree $k-1$ approximation to the function $g(x)=f^{\prime}(x)$.

- Hence the function $F^{\prime}(q)$ is the remainder when we approximate $f^{\prime}(q)$ with a degree $k-1$ Taylor polynomial. The remainder formula, equation 3.4.33, then tells us that there is a number $c$ between $a$ and $q$ so that

$$
\begin{aligned}
& F^{\prime}(q)= g(q)-\left(g(a)+g^{\prime}(a)(q-a)+\frac{1}{2} g^{\prime \prime}(a)(q-a)^{2}+\cdots\right. \\
&\left.+\frac{1}{(k-1)!} g^{(k-1)}(a)(q-a)^{k-1}\right) \\
&=\frac{1}{k!} g^{(k)}(c)(q-a)^{k}=\frac{1}{k!} f^{(k+1)}(c)(q-a)^{k}
\end{aligned}
$$

Notice that here we have assumed that $f^{(k+1)}(x)$ exists.

- Now substitute this back into our equation above

$$
\begin{aligned}
F(x) & =\frac{(x-a)^{k+1}}{(k+1)(q-a)^{k}} \cdot F^{\prime}(q) \\
& =\frac{(x-a)^{k+1}}{(k+1)(q-a)^{k}} \cdot \frac{1}{k!} f^{(k+1)}(c)(q-a)^{k} \\
& =\frac{1}{(k+1) k!} \cdot f^{(k+1)}(c) \cdot \frac{(x-a)^{k+1}(q-a)^{k}}{(q-a)^{k}} \\
& =\frac{1}{(k+1)!} \cdot f^{(k+1)}(c) \cdot(x-a)^{k+1}
\end{aligned}
$$

as required.

So we now know that

- if, for some $k$, the remainder formula (with $n=k$ ) is true for all $k$ times differentiable functions,
- then the remainder formula is true (with $n=k+1$ ) for all $k+1$ times differentiable functions.

Repeatedly applying this for $k=1,2,3,4, \cdots$ (and recalling that we have shown the remainder formula is true when $n=0,1$ ) gives equation 3.4.33 for all $n=$ $0,1,2, \cdots$.

### 3.4.11 $円$ Exercises

Exercises for § 3.4.1

## Exercises - Stage 1

1. The graph below shows three curves. The black curve is $y=f(x)$, the red curve is $y=g(x)=1+2 \sin (1+x)$, and the blue curve is $y=h(x)=0.7$. If you want to estimate $f(0)$, what might cause you to use $g(0)$ ? What might cause you to use $h(0)$ ?


Exercises - Stage 2 In this and following sections, we will ask you to approximate the value of several constants, such as $\log (0.93)$. A valid question to consider is why we would ask for approximations of these constants that take lots of time, and are less accurate than what you get from a calculator.

One answer to this question is historical: people were approximating logarithms before they had calculators, and these are some of the ways they did that. Pretend you're on a desert island without any of your usual devices and that you want to make a number of quick and dirty approximate evaluations.

Another reason to make these approximations is technical: how does the calculator get such a good approximation of $\log (0.93)$ ? The techniques you will learn later on in this chapter give very accurate formulas for approximating functions like $\log x$ and $\sin x$, which are sometimes used in calculators.

A third reason to make simple approximations of expressions that a calculator could evaluate is to provide a reality check. If you have a ballpark guess for your answer, and your calculator gives you something wildly different, you know to double-check that you typed everything in correctly.For now, questions like Question 3.4.11.2 through Question 3.4.11.4 are simply for you to practice the fundamental ideas we're learning.
2. Use a constant approximation to estimate the value of $\log (x)$ when $x=0.93$. Sketch the curve $y=f(x)$ and your constant approximation. (Remember that in CLP-1 we use $\log x$ to mean the natural logarithm of $x$, $\log _{e} x$.)
3. Use a constant approximation to estimate $\arcsin (0.1)$.
4. Use a constant approximation to estimate $\sqrt{3} \tan (1)$.

## Exercises - Stage 3

5. Use a constant approximation to estimate the value of $10.1^{3}$. Your estimation should be something you can calculate in your head.

## $\leadsto$ Exercises for § 3.4.2

## Exercises - Stage 1

1. Suppose $f(x)$ is a function, and we calculated its linear approximation near $x=5$ to be $f(x) \approx 3 x-9$.
a What is $f(5)$ ?
b What is $f^{\prime}(5)$ ?
c What is $f(0)$ ?
2. The curve $y=f(x)$ is shown below. Sketch the linear approximation of $f(x)$ about $x=2$.

3. What is the linear approximation of the function $f(x)=2 x+5$ about $x=a$ ?

## Exercises - Stage 2

4. Use a linear approximation to estimate $\log (x)$ when $x=0.93$. Sketch the curve $y=f(x)$ and your linear approximation.
(Remember that in CLP-1 we use $\log x$ to mean the natural logarithm of $x, \log _{e} x$.)
5. Use a linear approximation to estimate $\sqrt{5}$.
6. Use a linear approximation to estimate $\sqrt[5]{30}$

## Exercises - Stage 3

7. Use a linear approximation to estimate $10.1^{3}$, then compare your estimation with the actual value.
8. Imagine $f(x)$ is some function, and you want to estimate $f(b)$. To do this, you choose a value $a$ and take an approximation (linear or constant) of $f(x)$ about $a$. Give an example of a function $f(x)$, and values $a$ and $b$, where the constant approximation gives a more accurate estimation of $f(b)$ than the linear approximation.
9. The function

$$
L(x)=\frac{1}{4} x+\frac{4 \pi-\sqrt{27}}{12}
$$

is the linear approximation of $f(x)=\arctan x$ about what point $x=a$ ?

## H Exercises for § 3.4.3

## Exercises - Stage 1

1. The quadratic approximation of a function $f(x)$ about $x=3$ is

$$
f(x) \approx-x^{2}+6 x
$$

What are the values of $f(3), f^{\prime}(3), f^{\prime \prime}(3)$, and $f^{\prime \prime \prime}(3)$ ?
2. Give a quadratic approximation of $f(x)=2 x+5$ about $x=a$.

## Exercises - Stage 2

3. Use a quadratic approximation to estimate $\log (0.93)$.
(Remember that in CLP-1 we use $\log x$ to mean the natural logarithm of $x, \log _{e} x$.)
4. Use a quadratic approximation to estimate $\cos \left(\frac{1}{15}\right)$.
5. Calculate the quadratic approximation of $f(x)=e^{2 x}$ about $x=0$.
6. Use a quadratic approximation to estimate $5^{\frac{4}{3}}$.
7. Evaluate the expressions below.
a $\sum_{n=5}^{30} 1$
b $\sum_{n=1}^{3}\left[2(n+3)-n^{2}\right]$
c $\sum_{n=1}^{10}\left[\frac{1}{n}-\frac{1}{n+1}\right]$
$\mathrm{d} \sum_{n=1}^{4} \frac{5 \cdot 2^{n}}{4^{n+1}}$
8. Write the following in sigma notation:
a $1+2+3+4+5$
b $2+4+6+8$
c $3+5+7+9+11$
d $9+16+25+36+49$
e $9+4+16+5+25+6+36+7+49+8$
f $8+15+24+35+48$
g $3-6+9-12+15-18$

## Exercises - Stage 3

9. Use a quadratic approximation of $f(x)=2 \arcsin x$ about $x=0$ to approximate $f(1)$. What number are you approximating?
10. Use a quadratic approximation of $e^{x}$ to estimate $e$ as a decimal.
11. Group the expressions below into collections of equivalent expressions.
a $\sum_{n=1}^{10} 2 n$
b $\sum_{n=1}^{10} 2^{n}$
c $\sum_{n=1}^{10} n^{2}$
d $2 \sum_{n=1}^{10} n$
e $2 \sum_{n=2}^{11}(n-1)$
$\mathrm{f} \sum_{n=5}^{14}(n-4)^{2}$
g $\frac{1}{4} \sum_{n=1}^{10}\left(\frac{4^{n+1}}{2^{n}}\right)$

## M Exercises for § 3.4.4

## Exercises - Stage 1

1. The 3rd degree Taylor polynomial for a function $f(x)$ about $x=1$ is

$$
T_{3}(x)=x^{3}-5 x^{2}+9 x
$$

What is $f^{\prime \prime}(1) ?$
2. The $n$th degree Taylor polynomial for $f(x)$ about $x=5$ is

$$
T_{n}(x)=\sum_{k=0}^{n} \frac{2 k+1}{3 k-9}(x-5)^{k}
$$

What is $f^{(10)}(5) ?$

## Exercises - Stage 3

3. The 4th-degree Maclaurin polynomial for $f(x)$ is

$$
T_{4}(x)=x^{4}-x^{3}+x^{2}-x+1
$$

What is the third-degree Maclaurin polynomial for $f(x)$ ?
4. The 4 th degree Taylor polynomial for $f(x)$ about $x=1$ is

$$
T_{4}(x)=x^{4}+x^{3}-9
$$

What is the third degree Taylor polynomial for $f(x)$ about $x=1$ ?
5. For any even number $n$, suppose the $n$th degree Taylor polynomial for $f(x)$ about $x=5$ is

$$
\sum_{k=0}^{n / 2} \frac{2 k+1}{3 k-9}(x-5)^{2 k}
$$

What is $f^{(10)}(5) ?$
6. The third-degree Taylor polynomial for $f(x)=x^{3}\left[2 \log x-\frac{11}{3}\right]$ about $x=a$ is

$$
T_{3}(x)=-\frac{2}{3} \sqrt{e^{3}}+3 e x-6 \sqrt{e} x^{2}+x^{3}
$$

What is $a$ ?

## Exercises for § 3.4.5

## Exercises - Stage 1

1. Give the 16 th degree Maclaurin polynomial for $f(x)=\sin x+\cos x$.
2. Give the 100 th degree Taylor polynomial for $s(t)=4.9 t^{2}-t+10$ about $t=5$.
3. Write the $n$ th-degree Taylor polynomial for $f(x)=2^{x}$ about $x=1$ in sigma notation.
4. Find the 6 th degree Taylor polynomial of $f(x)=x^{2} \log x+2 x^{2}+5$ about $x=1$, remembering that $\log x$ is the natural logarithm of $x, \log _{e} x$.
5. Give the $n$th degree Maclaurin polynomial for $\frac{1}{1-x}$ in sigma notation.

## Exercises - Stage 3

6. Calculate the 3rd-degree Taylor Polynomial for $f(x)=x^{x}$ about $x=1$.
7. Use a 5 th-degree Maclaurin polynomial for $6 \arctan x$ to approximate $\pi$.
8. Write the 100th-degree Taylor polynomial for $f(x)=x(\log x-1)$ about $x=1$ in sigma notation.
9. Write the $(2 n)$ th-degree Taylor polynomial for $f(x)=\sin x$ about $x=\frac{\pi}{4}$ in sigma notation.
10. Estimate the sum below

$$
1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{157!}
$$

by interpreting it as a Maclaurin polynomial.
11. Estimate the sum below

$$
\sum_{k=0}^{100} \frac{(-1)^{k}}{2 k!}\left(\frac{5 \pi}{4}\right)^{2 k}
$$

by interpreting it as a Maclaurin polynomial.

## Exercises for § 3.4.6

## Exercises - Stage 1

1. In the picture below, label the following:

$$
f(x) \quad f(x+\Delta x) \quad \Delta x \quad \Delta y
$$


2. At this point in the book, every homework problem takes you about 5 minutes. Use the terms you learned in this section to answer the question: if you spend 15 minutes more, how many more homework problems will you finish?

## Exercises - Stage 2

3. Let $f(x)=\arctan x$.
a Use a linear approximation to estimate $f(5.1)-f(5)$.
b Use a quadratic approximation to estimate $f(5.1)-f(5)$.
4. When diving off a cliff from $x$ metres above the water, your speed as you hit the water is given by

$$
s(x)=\sqrt{19.6 x} \frac{\mathrm{~m}}{\mathrm{sec}}
$$

Your last dive was from a height of 4 metres.
a Use a linear approximation of $\Delta y$ to estimate how much faster you will be falling when you hit the water if you jump from a height of 5 metres.
b A diver makes three jumps: the first is from $x$ metres, the second from $x+\Delta x$ metres, and the third from $x+2 \Delta x$ metres, for some fixed positive
values of $x$ and $\Delta x$. Which is bigger: the increase in terminal speed from the first to the second jump, or the increase in terminal speed from the second to the third jump?

## $\Perp$ Exercises for § 3.4.7

## Exercises - Stage 1

1. Let $f(x)=7 x^{2}-3 x+4$. Suppose we measure $x$ to be $x_{0}=2$ but that the real value of $x$ is $x_{0}+\Delta x$. Suppose further that the error in our measurement is $\Delta x=1$. Let $\Delta y$ be the change in $f(x)$ corresponding to a change of $\Delta x$ in $x_{0}$. That is, $\Delta y=f\left(x_{0}+\Delta x\right)-f\left(x_{0}\right)$. True or false: $\Delta y=f^{\prime}(2)(1)=25$
2. Suppose the exact amount you are supposed to tip is $\$ 5.83$, but you approximate and tip $\$ 6$. What is the absolute error in your tip? What is the percent error in your tip?
3. Suppose $f(x)=3 x^{2}-5$. If you measure $x$ to be 10 , but its actual value is 11 , estimate the resulting error in $f(x)$ using the linear approximation, and then the quadratic approximation.

## Exercises - Stage 2

4. A circular pen is being built on a farm. The pen must contain $A_{0}$ square metres, with an error of no more than $2 \%$. Estimate the largest percentage error allowable on the radius.
5. A circle with radius 3 has a sector cut out of it. It's a smallish sector, no more than a quarter of the circle. You want to find out the area of the sector.

a Suppose the angle of the sector is $\theta$. What is the area of the sector?
b Unfortunately, you don't have a protractor, only a ruler. So, you measure the chord made by the sector (marked $d$ in the diagram above). What is $\theta$ in terms of $d$ ?
c Suppose you measured $d=0.7$, but actually $d=0.68$. Estimate the absolute error in your calculation of the area removed.
6. A conical tank, standing on its pointy end, has height 2 metres and radius 0.5 metres. Estimate change in volume of the water in the tank associated to a change in the height of the water from 50 cm to 45 cm .


## Exercises - Stage 3

7. A sample begins with precisely $1 \mu \mathrm{~g}$ of a radioactive isotope, and after 3 years is measured to have $0.9 \mu \mathrm{~g}$ remaining. If this measurement is correct to within $0.05 \mu \mathrm{~g}$, estimate the corresponding accuracy of the half-life calculated using it.

## Exercises for § 3.4.8

## Exercises - Stage 1

1. Suppose $f(x)$ is a function that we approximated by $F(x)$. Further, suppose $f(10)=-3$, while our approximation was $F(10)=5$. Let $R(x)=f(x)-F(x)$.
a True or false: $|R(10)| \leq 7$
b True or false: $|R(10)| \leq 8$
c True or false: $|R(10)| \leq 9$
d True or false: $|R(10)| \leq 100$
2. Let $f(x)=e^{x}$, and let $T_{3}(x)$ be the third-degree Maclaurin polynomial for $f(x)$,

$$
T_{3}(x)=1+x+\frac{1}{2} x^{2}+\frac{1}{3!} x^{3}
$$

Use Equation 3.4.33 to give a reasonable bound on the error $\left|f(2)-T_{3}(2)\right|$. Then, find the error $\left|f(2)-T_{3}(2)\right|$ using a calculator.
3. Let $f(x)=5 x^{3}-24 x^{2}+e x-\pi^{4}$, and let $T_{5}(x)$ be the fifth-degree Taylor polynomial for $f(x)$ about $x=1$. Give the best bound you can on the error $|f(37)-T(37)|$.
4. You and your friend both want to approximate $\sin (33)$. Your friend uses the first-degree Maclaurin polynomial for $f(x)=\sin x$, while you use the zerothdegree (constant) Maclaurin polynomial for $f(x)=\sin x$. Who has a better approximation, you or your friend?

## Exercises - Stage 2

5. Suppose a function $f(x)$ has sixth derivative

$$
f^{(6)}(x)=\frac{6!(2 x-5)}{x+3}
$$

Let $T_{5}(x)$ be the 5th-degree Taylor polynomial for $f(x)$ about $x=11$.
Give a bound for the error $\left|f(11.5)-T_{5}(11.5)\right|$.
6. Let $f(x)=\tan x$, and let $T_{2}(x)$ be the second-degree Taylor polynomial for $f(x)$ about $x=0$. Give a reasonable bound on the error $|f(0.1)-T(0.1)|$ using Equation 3.4.33.
7. Let $f(x)=\log (1-x)$, and let $T_{5}(x)$ be the fifth-degree Maclaurin polynomial for $f(x)$. Use Equation 3.4.33 to give a bound on the error $\left|f\left(-\frac{1}{4}\right)-T_{5}\left(-\frac{1}{4}\right)\right|$.
(Remember $\log x=\log _{e} x$, the natural logarithm of $x$.)
8. Let $f(x)=\sqrt[5]{x}$, and let $T_{3}(x)$ be the third-degree Taylor polynomial for $f(x)$ about $x=32$. Give a bound on the error $\left|f(30)-T_{3}(30)\right|$.
9. Let

$$
f(x)=\sin \left(\frac{1}{x}\right)
$$

and let $T_{1}(x)$ be the first-degree Taylor polynomial for $f(x)$ about $x=\frac{1}{\pi}$. Give a bound on the error $\left|f(0.01)-T_{1}(0.01)\right|$, using Equation 3.4.33. You may leave your answer in terms of $\pi$.
Then, give a reasonable bound on the error $\left|f(0.01)-T_{1}(0.01)\right|$.
10. Let $f(x)=\arcsin x$, and let $T_{2}(x)$ be the second-degree Maclaurin polynomial for $f(x)$. Give a reasonable bound on the error $\left|f\left(\frac{1}{2}\right)-T_{2}\left(\frac{1}{2}\right)\right|$ using Equation 3.4.33. What is the exact value of the error $\left|f\left(\frac{1}{2}\right)-T_{2}\left(\frac{1}{2}\right)\right|$ ?

## Exercises - Stage 3

11. Let $f(x)=\log (x)$, and let $T_{n}(x)$ be the $n$ th-degree Taylor polynomial for $f(x)$ about $x=1$. You use $T_{n}(1.1)$ to estimate $\log (1.1)$. If your estimation needs to
have an error of no more than $10^{-4}$, what is an acceptable value of $n$ to use?
12. Give an estimation of $\sqrt[7]{2200}$ using a Taylor polynomial. Your estimation should have an error of less than 0.001 .
13. Use Equation 3.4.33 to show that

$$
\frac{4241}{5040} \leq \sin (1) \leq \frac{4243}{5040}
$$

14. In this question, we use the remainder of a Maclaurin polynomial to approximate $e$.
a Write out the 4th degree Maclaurin polynomial $T_{4}(x)$ of the function $e^{x}$.
b Compute $T_{4}(1)$.
c Use your answer from 3.4.11.14.b to conclude $\frac{326}{120}<e<\frac{325}{119}$.

## Further problems for § 3.4

## Exercises - Stage 1

1. *. Consider a function $f(x)$ whose third-degree Maclaurin polynomial is $4+$ $3 x^{2}+\frac{1}{2} x^{3}$. What is $f^{\prime}(0)$ ? What is $f^{\prime \prime}(0)$ ?
2. *. Consider a function $h(x)$ whose third-degree Maclaurin polynomial is $1+$ $4 x-\frac{1}{3} x^{2}+\frac{2}{3} x^{3}$. What is $h^{(3)}(0) ?$
3. *. The third-degree Taylor polynomial of $h(x)$ about $x=2$ is $3+\frac{1}{2}(x-2)+$ $2(x-2)^{3}$.
What is $h^{\prime}(2)$ ? What is $h^{\prime \prime}(2)$ ?

## Exercises - Stage 2

4. *. The function $f(x)$ has the property that $f(3)=2, f^{\prime}(3)=4$ and $f^{\prime \prime}(3)=-10$.
a Use the linear approximation to $f(x)$ centred at $x=3$ to approximate $f(2.98)$.
b Use the quadratic approximation to $f(x)$ centred at $x=3$ to approximate $f(2.98)$.
5. *. Use the tangent line to the graph of $y=x^{1 / 3}$ at $x=8$ to find an approximate value for $10^{1 / 3}$. Is the approximation too large or too small?
6. *. Estimate $\sqrt{2}$ using a linear approximation.
7. *. Estimate $\sqrt[3]{26}$ using a linear approximation.
8. *. Estimate $(10.1)^{5}$ using a linear approximation.
9. *. Estimate $\sin \left(\frac{101 \pi}{100}\right)$ using a linear approximation. (Leave your answer in terms of $\pi$.)
10. *. Use a linear approximation to estimate $\arctan (1.1)$, using $\arctan 1=\frac{\pi}{4}$.
11. *. Use a linear approximation to estimate $(2.001)^{3}$. Write your answer in the form $n / 1000$ where $n$ is an integer.
12. *. Using a suitable linear approximation, estimate $(8.06)^{2 / 3}$. Give your answer as a fraction in which both the numerator and denominator are integers.
13. *. Find the third-order Taylor polynomial for $f(x)=(1-3 x)^{-1 / 3}$ around $x=0$.
14. *. Consider a function $f(x)$ which has $f^{(3)}(x)=\frac{x}{22-x^{2}}$. Show that when we approximate $f(2)$ using its second degree Taylor polynomial at $a=1$, the absolute value of the error is less than $\frac{1}{50}=0.02$.
15. *. Consider a function $f(x)$ which has $f^{(4)}(x)=\frac{\cos \left(x^{2}\right)}{3-x}$. Show that when we approximate $f(0.5)$ using its third-degree Maclaurin polynomial, the absolute value of the error is less than $\frac{1}{500}=0.002$.
16. *. Consider a function $f(x)$ which has $f^{(3)}(x)=\frac{e^{-x}}{8+x^{2}}$. Show that when we approximate $f(1)$ using its second degree Maclaurin polynomial, the absolute value of the error is less than $1 / 40$.
17. *.
a By using an appropriate linear approximation for $f(x)=x^{1 / 3}$, estimate $5^{2 / 3}$.
b Improve your answer in 3.4.11.17. a by making a quadratic approximation.
c Obtain an error estimate for your answer in 3.4.11.17.a (not just by comparing with your calculator's answer for $5^{2 / 3}$ ).

## Exercises - Stage 3

18. The 4th degree Maclaurin polynomial for $f(x)$ is

$$
T_{4}(x)=5 x^{2}-9
$$

What is the third degree Maclaurin polynomial for $f(x)$ ?
19. *. The equation $y^{4}+x y=x^{2}-1$ defines $y$ implicitly as a function of $x$ near the point $x=2, y=1$.
a Use the tangent line approximation at the given point to estimate the value of $y$ when $x=2.1$.
b Use the quadratic approximation at the given point to estimate the value of $y$ when $x=2.1$.
c Make a sketch showing how the curve relates to the tangent line at the given point.
20. *. The equation $x^{4}+y+x y^{4}=1$ defines $y$ implicitly as a function of $x$ near the point $x=-1, y=1$.
a Use the tangent line approximation at the given point to estimate the value of $y$ when $x=-0.9$.
b Use the quadratic approximation at the given point to get another estimate of $y$ when $x=-0.9$.
c Make a sketch showing how the curve relates to the tangent line at the given point.
21. *. Given that $\log 10 \approx 2.30259$, estimate $\log 10.3$ using a suitable tangent line approximation. Give an upper and lower bound for the error in your approximation by using a suitable error estimate.
22. *. Consider $f(x)=e^{e^{x}}$.
a Give the linear approximation for $f$ near $x=0$ (call this $L(x)$ ).
b Give the quadratic approximation for $f$ near $x=0$ (call this $Q(x)$ ).
c Prove that $L(x)<Q(x)<f(x)$ for all $x>0$.
d Find an interval of length at most 0.01 that is guaranteed to contain the number $e^{0.1}$.

### 3.54 Optimisation

One important application of differential calculus is to find the maximum (or minimum) value of a function. This often finds real world applications in problems such as the following.

Example 3.5.1 Enclosing a paddock.
A farmer has 400 m of fencing materials. What is the largest rectangular paddock that can be enclosed?
Solution We will describe a general approach to these sorts of problems in Sections 3.5.2 and 3.5.3 below, but here we can take a stab at starting the problem.

- Begin by defining variables and their units (more generally we might draw a picture too); let the dimensions of the paddock be $x$ by $y$ metres.
- The area enclosed is then $A m^{2}$ where

$$
A=x \cdot y
$$

At this stage we cannot apply the calculus we have developed since the area is a function of two variables and we only know how to work with functions of a single variable. We need to eliminate one variable.

- We know that the perimeter of the rectangle (and hence the dimensions $x$ and $y$ ) are constrained by the amount of fencing materials the farmer has to hand:

$$
2 x+2 y \leq 400
$$

and so we have

$$
y \leq 200-x
$$

Clearly the area of the paddock is maximised when we use all the fencing possible, so

$$
y=200-x
$$

- Now substitute this back into our expression for the area

$$
A=x \cdot(200-x)
$$

Since the area cannot be negative (and our lengths $x, y$ cannot be negative either), we must also have

$$
0 \leq x \leq 200
$$

- Thus the question of the largest paddock enclosed becomes the problem of finding the maximum value of

$$
A=x \cdot(200-x) \quad \text { subject to the constraint } 0 \leq x \leq 200
$$

The above example is sufficiently simple that we can likely determine the answer by several different methods. In general, we will need more systematic methods for solving problems of the form

Find the maximum value of $y=f(x)$ subject to $a \leq x \leq b$
To do this we need to examine what a function looks like near its maximum and minimum values.

### 3.5.1 Local and Global Maxima and Minima

We start by asking:
Suppose that the maximum (or minimum) value of $f(x)$ is $f(c)$ then what does that tell us about $c$ ?

Notice that we have not yet made the ideas of maximum and minimum very precise. For the moment think of maximum as "the biggest value" and minimum as "the smallest value".

## Warning 3.5.2

It is important to distinguish between "the smallest value" and "the smallest magnitude". For example, because

$$
-5<-1
$$

the number -5 is smaller than -1 . But the magnitude of -1 , which is $|-1|=1$, is smaller than the magnitude of -5 , which is $|-5|=5$. Thus the smallest number in the set $\{-1,-5\}$ is -5 , while the number in the set $\{-1,-5\}$ that has the smallest magnitude is -1 .

Now back to thinking about what happens around a maximum. Suppose that the maximum value of $f(x)$ is $f(c)$, then for all "nearby" points, the function should be smaller.


Consider the derivative of $f^{\prime}(c)$ :

$$
f^{\prime}(c)=\lim _{h \rightarrow 0} \frac{f(c+h)-f(c)}{h}
$$

Split the above limit into the left and right limits:

- Consider points to the right of $x=c$, For all $h>0$,

$$
\begin{aligned}
f(c+h) & \leq f(c) & \text { which implies that } \\
f(c+h)-f(c) & \leq 0 & \text { which also implies } \\
\frac{f(c+h)-f(c)}{h} & \leq 0 & \text { since } \frac{\text { negative }}{\text { positive }}=\text { negative. }
\end{aligned}
$$

But now if we squeeze $h \rightarrow 0$ we get

$$
\lim _{h \rightarrow 0^{+}} \frac{f(c+h)-f(c)}{h} \leq 0
$$

(provided the limit exists).

- Consider points to the left of $x=c$. For all $h<0$,

$$
\begin{array}{rlr}
f(c+h) & \leq f(c) & \text { which implies that } \\
f(c+h)-f(c) & \leq 0 & \text { which also implies } \\
\frac{f(c+h)-f(c)}{h} & \geq 0 & \text { since } \frac{\text { negative }}{\text { negative }}=\text { positive. }
\end{array}
$$

But now if we squeeze $h \rightarrow 0$ we get

$$
\lim _{h \rightarrow 0^{-}} \frac{f(c+h)-f(c)}{h} \geq 0
$$

(provided the limit exists).

- So if the derivative $f^{\prime}(c)$ exists, then the above right- and left-hand limits must agree, which forces $f^{\prime}(c)=0$.

Thus we can conclude that
If the maximum value of $f(x)$ is $f(c)$ and $f^{\prime}(c)$ exists, then $f^{\prime}(c)=0$.
Using similar reasoning one can also see that
If the minimum value of $f(x)$ is $f(c)$ and $f^{\prime}(c)$ exists, then $f^{\prime}(c)=0$.
Notice two things about the above reasoning:

- Firstly, in order for the argument to work we only need that $f(x)<f(c)$ for $x$ close to $c$ - it does not matter what happens for $x$ values far from $c$.
- Secondly, in the above argument we needed to consider $f(x)$ for $x$ both to the left of and to the right of $c$. If the function $f(x)$ is defined on a closed interval $[a, b]$, then the above argument only applies when $a<c<b$ - not when $c$ is either of the endpoints $a$ and $b$.
Consider the function below


This function has only 1 maximum value (the middle green point in the graph) and 1 minimum value (the rightmost blue point), however it has 4 points at which the derivative is zero. In the small intervals around those points where the derivative is zero, we can see that function is locally a maximum or minimum, even if it is not the global maximum or minimum. We clearly need to be more careful distinguishing between these cases.

## Definition 3.5.3

Let $I$ be an interval, like $(a, b)$ or $[a, b]$ for example, and let the function $f(x)$ be defined for all $x \in I$. Now let $c \in I$. Then

- we say that $f(x)$ has a global (or absolute) minimum on the interval $I$ at the point $x=c$ if $f(x) \geq f(c)$ for all $x \in I$.
- Similarly, we say that $f(x)$ has a global (or absolute) maximum on $I$ at $x=c$ if $f(x) \leq f(c)$ for all $x \in I$.
- We say that $f(x)$ has a local ${ }^{a}$ minimum on $I$ at $x=c$ if $f(x) \geq f(c)$ for all $x \in I$ that are near $c$. Precisely, if there is a $\delta>0$ such that $f(x) \geq f(c)$ for all $x \in I$ that are within a distance $\delta$ of $c$.
- Similarly, we say that $f(x)$ has a local maximum on $I$ at $x=c$ if $f(x) \leq f(c)$ for all $x \in I$ that are near $c$. Precisely, if there is a $\delta>0$ such that $f(x) \leq f(c)$ for all $x \in I$ that are within a distance $\delta$ of $c$.

The global maxima and minima of a function are called the global extrema of the function, while the local maxima and minima are called the local extrema.
$a$ Beware that, while many textbooks use these definitions of local minimum and maximum, some textbooks exclude the endpoints $a, b$ of the interval $[a, b]$ from their definitions. Our definitions allow the endpoints $a$ and $b$ to be local minima and maxima. Note that, under our definitions, every global minimum (maximum) is also a local minimum (maximum).

Consider again the function we showed in the figure above


It has 3 local maxima and 3 local minima on the interval $[a, b]$. The global maximum occurs at the middle green point (which is also a local maximum), and the global minimum occurs at the rightmost blue point (which is also a local minimum).

Using the above definition we can summarise what we have learned above as the following theorem ${ }^{1}$ :

## Theorem 3.5.4

Let the function $f(x)$ be defined on the interval $I$ and let $a, b, c$ be points in $I$ with $a<c<b$. If $f(x)$ has a local maximum or local minimum at $x=c$ and if $f^{\prime}(c)$ exists, then $f^{\prime}(c)=0$.

- It is often (but not always) the case that, when $f(x)$ has a local maximum at $x=c$, the function $f(x)$ increases strictly as $x$ approaches $c$ from the left and decreases strictly as $x$ leaves $c$ to the right. That is, $f^{\prime}(x)>0$ for $x$ just to the left of $c$ and $f^{\prime}(x)<0$ for $x$ just to the right of $c$. Then, it is often the case, because $f^{\prime}(x)$ is decreasing as $x$ increases through $c$, that $f^{\prime \prime}(c)<0$.
- Conversely, if $f^{\prime}(c)=0$ and $f^{\prime \prime}(c)<0$, then, just to the right of $c, f^{\prime}(x)$ must be negative, so that $f(x)$ is decreasing, and just to the left of $c, f^{\prime}(x)$ must be positive, so that $f(x)$ is increasing. So $f(x)$ has a local maximum at $c$.
- Similarly, it is often the case that, when $f(x)$ has a local minimum at $x=c$, $f^{\prime}(x)<0$ for $x$ just to the left of $c$ and $f^{\prime}(x)>0$ for $x$ just to the right of $c$ and $f^{\prime \prime}(x)>0$.

1 This is one of several important mathematical contributions made by Pierre de Fermat, a French government lawyer and amateur mathematician, who lived in the first half of the seventeenth century.

- Conversely, if $f^{\prime}(c)=0$ and $f^{\prime \prime}(c)>0$, then, just to the right of $c, f^{\prime}(x)$ must be positive, so that $f(x)$ is increasing, and, just to the left of $c, f^{\prime}(x)$ must be negative, so that $f(x)$ is decreasing. So $f(x)$ has a local minimum at $c$.


## Theorem 3.5.5

Let $f(x)$ be defined on the interval $I$ and let $a, b, c \in I$ with $a<c<b$. If $f^{\prime}(c)=0$ and $f^{\prime \prime}(c)<0$, then $f(x)$ has a local maximum at $c$. If $f^{\prime}(c)=0$ and $f^{\prime \prime}(c)>0$, then $f(x)$ has a local minimum at $c$. Note the strict inequalities.

Theorem 3.5.4 says that, when $f(x)$ has a local maximum or minimum at $x=c$, there are three possibilities.

- The derivative $f^{\prime}(c)=0$. This case is illustrated in the following figure.



Observe that, in this example, $f^{\prime}(x)$ changes continuously from negative to positive at the local minimum, taking the value zero at the local minimum (the red dot).

- The derivative $f^{\prime}(c)$ does not exist. This case is illustrated in the following figure.



Observe that, in this example, $f^{\prime}(x)$ changes discontinuously from negative to positive at the local minimum $(x=0)$ and $f^{\prime}(0)$ does not exist.

- The point $c$ is an endpoint of the interval $I$. This case is also illustrated in the above figure. The endpoints $a$ and $b$ are both local maxima. But $f^{\prime}(a)$ and $f^{\prime}(b)$ are not zero.

This theorem demonstrates that the points at which the derivative is zero or does not exist are very important. It simplifies the discussion that follows if we give these points names.

## Definition 3.5.6

Let $f(x)$ be a function that is defined on the interval $a<x<b$ and let $a<c<b$. Then

- if $f^{\prime}(c)$ exists and is zero we call $x=c$ a critical point of the function, and
- if $f^{\prime}(c)$ does not exist then we call $x=c$ a singular point ${ }^{a}$ of the function.

$a \quad$ For $c$ to be a local maximum or minimum of $f$, the function $f$ must obviously be defined at $c$. So here we are considering only points $c$ in the domain of $f$. We will later, in §3.6.2, extend the definition of singular points of $f$ to points that are not in the domain of $f$.


## Warning 3.5.7

Note that some people (and texts) will combine both of these cases and call $x=c$ a critical point when either the derivative is zero or does not exist. The reader should be aware of the lack of convention on this point ${ }^{a}$ and should be careful to understand whether the more inclusive definition of critical point is being used, or if the text is using the more precise definition that distinguishes critical and singular points.
$a \quad$ No pun intended.

We'll now look at a few simple examples involving local maxima and minima, critical points and singular points. Then we will move on to global maxima and minima.

Example 3.5.8 Local max and min of $x^{3}-6 x$.
In this example, we'll look for local maxima and minima of the function $f(x)=x^{3}-6 x$ on the interval $-2 \leq x \leq 3$.

- First compute the derivative

$$
f^{\prime}(x)=3 x^{2}-6
$$

Since this is a polynomial it is defined everywhere on the domain and so there will not be any singular points. So we now look for critical points.

- To do so we look for zeroes of the derivative

$$
f^{\prime}(x)=3 x^{2}-6=3\left(x^{2}-2\right)=3(x-\sqrt{2})(x+\sqrt{2}) .
$$

This derivative takes the value 0 at two different values of $x$. Namely $x=c_{-}=$ $-\sqrt{2}$ and $x=c_{+}=\sqrt{2}$. Here is a sketch of the graph of $f(x)$.


From the figure we see that

- $f(x)$ has a local minimum at the endpoint $x=-2$ (i.e. we have $f(x) \geq f(-2)$ whenever $x \geq-2$ is close to -2 ) and
- $f(x)$ has a local minimum at $x=c_{+}$(i.e. we have $f(x) \geq f\left(c_{+}\right)$whenever $x$ is close to $c_{+}$) and
- $f(x)$ has a local maximum at $x=c_{-}$(i.e. we have $f(x) \leq f\left(c_{-}\right)$whenever $x$ is close to $c_{-}$) and
- $f(x)$ has a local maximum at the endpoint $x=3$ (i.e. we have $f(x) \leq f(3)$ whenever $x \leq 3$ is close to 3 ) and
- the global minimum of $f(x)$, for $x$ in the interval $-2 \leq x \leq 3$, is at $x=c_{+}$ (i.e. we have $f(x) \geq f\left(c_{+}\right)$whenever $-2 \leq x \leq 3$ ) and
- the global maximum of $f(x)$, for $x$ in the interval $-2 \leq x \leq 3$, is at $x=3$ (i.e. we have $f(x) \leq f(3)$ whenever $-2 \leq x \leq 3$ ).
- Note that we have carefully constructed this example to illustrate that the global maximum (or minimum) of a function on an interval may or may not also be a critical point of the function.

Example 3.5.8

Example 3.5.9 Local max and min of $x^{3}$.
In this example, we'll look for local maxima and minima of the function $f(x)=x^{3}$ on the interval $-1<x<1$.

- First compute the derivative:

$$
f^{\prime}(x)=3 x^{2} .
$$

Again, this is a polynomial and so defined on all of the domain. The function will not have singular points, but may have critical points.

- The derivative is zero only when $x=0$, so $x=c=0$ is the only critical point of the function.
- The graph of $f(x)$ is sketched below. From that sketch we see that $f(x)$ has neither a local maximum nor a local minimum at $x=c$ despite the fact that $f^{\prime}(c)=0$ - we have $f(x)<f(c)=0$ for all $x<c=0$ and $f(x)>f(c)=0$ for all $x>c=0$.

- Note that this example has been constructed to illustrate that a critical point (or singular point) of a function need not be a local maximum or minimum for the function.
- Reread Theorem 3.5.4. It says ${ }^{a}$ "Let $\cdots$. If $f(x)$ has a local maximum/minimum at $x=c$ and if $f^{\prime}(c)$ exists, then $f^{\prime}(c)=0$ ". It does not say that "if $f^{\prime}(c)=0$ then $f$ has a local maximum/minimum at $x=c^{\prime \prime}$.
$a$ A very common error of logic that people make is "Affirming the consequent". When the statement "if P then Q " is true, observing Q does not imply P. ("Affirming the consequent" eliminates "not" from the previous sentence.) For example, "If he is Shakespeare then he is dead." and "That man is dead." does not imply "He must be Shakespeare.". Or you may have also seen someone use this reasoning: "If a person is a genius before their time then they are misunderstood." "I am misunderstood." "So I must be a genius before my time.".

Example 3.5.9

Example 3.5.10 Local max and min of $|x|$ and $x^{2 / 3}$.
In this example, we'll look for local maxima and minima of the function

$$
f(x)=|x|= \begin{cases}x & \text { if } x \geq 0 \\ -x & \text { if } x<0\end{cases}
$$

on the interval $-1<x<1$ and we'll also look for local maxima and minima of the function

$$
g(x)=x^{2 / 3}
$$

on the interval $-1<x<1$.

- Again, start by computing the derivatives (reread Example 2.2.10):

$$
\begin{aligned}
& f^{\prime}(x)= \begin{cases}1 & \text { if } x>0 \\
\text { undefined } & \text { if } x=0 \\
-1 & \text { if } x<0\end{cases} \\
& g^{\prime}(x)= \begin{cases}\frac{2}{3} x^{-1 / 3} & \text { if } x \neq 0 \\
\text { undefined } & \text { if } x=0\end{cases}
\end{aligned}
$$

- These derivatives never take the value 0 , so the functions $f(x)$ and $g(x)$ do not have any critical points. However both derivatives do not exist at the point $x=0$, so that point is a singular point for both $f(x)$ and $g(x)$.
- Here is a sketch of the graph of $f(x)$

and a sketch of the graph of $g(x)$.


From the figures we see that both $f(x)$ and $g(x)$ have a local (and in fact global) minimum at $x=0$ despite the fact that $x=0$ is not a critical point.

- Reread Theorem 3.5.4 yet again. It says "Let $\cdots$. If $f(x)$ has a local maximum or local minimum at $x=c$ and if $f$ is differentiable at $x=c$, then $f^{\prime}(c)=0$ ". It says nothing about what happens at points where the derivative does not exist. Indeed that is why we have to consider both critical points and singular points when we look for maxima and minima.

Example 3.5.10

### 3.5.2 $\leadsto$ Finding Global Maxima and Minima

We now have a technique for finding local maxima and minima - just look at endpoints of the interval of interest and for values of $x$ for which either $f^{\prime}(x)=0$ or $f^{\prime}(x)$ does not exist. What about finding global maxima and minima? We'll start by stating explicitly that, under appropriate hypotheses, global maxima and minima are guaranteed to exist.

## Theorem 3.5.11

Let the function $f(x)$ be defined and continuous on the closed, finite interval ${ }^{a}$ $-\infty<a \leq x \leq b<\infty$. Then $f(x)$ attains a maximum and a minimum at least once. That is, there exist numbers $a \leq x_{m}, x_{M} \leq b$ such that

$$
f\left(x_{m}\right) \leq f(x) \leq f\left(x_{M}\right) \quad \text { for all } a \leq x \leq b
$$

$a$ The hypotheses that $f(x)$ be continuous and that the interval be finite and closed are all essential. We suggest that you find three functions $f_{1}(x), f_{2}(x)$ and $f_{3}(x)$ with $f_{1}$ defined but not continuous on $0 \leq x \leq 1, f_{2}$ defined and continuous on $-\infty<x<\infty$, and $f_{3}$ defined and continuous on $0<x<1$, and with none of $f_{1}, f_{2}$ and $f_{3}$ attaining either a global maximum or a global minimum.

So let's again consider the question
Suppose that the maximum (or minimum) value of $f(x)$, for $a \leq x \leq b$, is $f(c)$. What does that tell us about $c$ ?

If $c$ obeys $a<c<b$ (note the strict inequalities), then $f$ has a local maximum (or minimum) at $x=c$ and Theorem 3.5.4 tells us that either $f^{\prime}(c)=0$ or $f^{\prime}(c)$ does not exist. The only other place that a maximum or minimum can occur are at the ends of the interval. We can summarise this as:

## Theorem 3.5.12

If $f(x)$ has a global maximum or global minimum, for $a \leq x \leq b$, at $x=c$ then there are 3 possibilities. Either

- $f^{\prime}(c)=0$, or
- $f^{\prime}(c)$ does not exist, or
- $c=a$ or $c=b$.

That is, a global maximum or minimum must occur either at a critical point, a singular point or at the endpoints of the interval.

This theorem provides the basis for a method to find the maximum and minimum values of $f(x)$ for $a \leq x \leq b$ :

## Corollary 3.5.13

Let $f(x)$ be a function on the interval $a \leq x \leq b$. Then to find the global maximum and minimum of the function:

- Make a list of all values of $c$, with $a \leq c \leq b$, for which
- $f^{\prime}(c)=0$, or
- $f^{\prime}(c)$ does not exist, or
- $c=a$ or $c=b$.

That is - compute the function at all the critical points, singular points, and endpoints.

- Evaluate $f(c)$ for each $c$ in that list. The largest (or smallest) of those values is the largest (or smallest) value of $f(x)$ for $a \leq x \leq b$.

Let's now demonstrate how to use this strategy. The function in this first example is not too simple - but it is a good example of a function that contains both a singular point and a critical point.

Example 3.5.14 Find max and min of $2 x^{5 / 3}+3 x^{2 / 3}$.
Find the largest and smallest values of the function $f(x)=2 x^{5 / 3}+3 x^{2 / 3}$ for $-1 \leq x \leq 1$.
Solution We will apply the method in Corollary 3.5.13. It is perhaps easiest to find the values at the endpoints of the intervals and then move on to the values at any critical or singular points.

- Before we get into things, notice that we can rewrite the function by factoring it:

$$
f(x)=2 x^{5 / 3}+3 x^{2 / 3}=x^{2 / 3} \cdot(2 x+3)
$$

- Let's compute the function at the endpoints of the interval:

$$
\begin{aligned}
f(1) & =2+3=5 \\
f(-1) & =2 \cdot(-1)^{5 / 3}+3 \cdot(-1)^{2 / 3}=-2+3=1
\end{aligned}
$$

- To compute the function at the critical and singular points we first need to find the derivative:

$$
\begin{aligned}
f^{\prime}(x) & =2 \cdot \frac{5}{3} x^{2 / 3}+3 \cdot \frac{2}{3} x^{-1 / 3} \\
& =\frac{10}{3} x^{2 / 3}+2 x^{-1 / 3} \\
& =\frac{10 x+6}{3 x^{1 / 3}}
\end{aligned}
$$

- Notice that the numerator and denominator are defined for all $x$. The only place the derivative is undefined is when the denominator is zero. Hence the only singular point is at $x=0$. The corresponding function value is

$$
f(0)=0
$$

- To find the critical points we need to solve $f^{\prime}(x)=0$ :

$$
0=\frac{10 x+6}{3 x^{1 / 3}}
$$

Hence we must have $10 x=-6$ or $x=-3 / 5$. The corresponding function value is

$$
\begin{array}{rlr}
f(x) & =x^{2 / 3} \cdot(2 x+3) \quad \text { recall this from above, then } \\
f(-3 / 5) & =(-3 / 5)^{2 / 3} \cdot\left(2 \cdot \frac{-3}{5}+3\right) & \\
& =\left(\frac{9}{25}\right)^{1 / 3} \cdot \frac{-6+15}{5} & \\
& =\left(\frac{9}{25}\right)^{1 / 3} \cdot \frac{9}{5} \approx 1.28 &
\end{array}
$$

Note that if we do not want to approximate the root (if, for example, we do not have a calculator handy), then we can also write

$$
\begin{aligned}
f(-3 / 5) & =\left(\frac{9}{25}\right)^{1 / 3} \cdot \frac{9}{5} \\
& =\left(\frac{9}{25}\right)^{1 / 3} \cdot \frac{9}{25} \cdot 5 \\
& =5 \cdot\left(\frac{9}{25}\right)^{4 / 3}
\end{aligned}
$$

Since $0<9 / 25<1$, we know that $0<\left(\frac{9}{25}\right)^{4 / 3}<1$, and hence

$$
0<f(-3 / 5)=5 \cdot\left(\frac{9}{25}\right)^{4 / 3}<5
$$

- We summarise our work in this table

| $c$ | $-\frac{3}{5}$ | 0 | -1 | 1 |
| :---: | :---: | :---: | :---: | :---: |
| type | critical point | singular point | endpoint | endpoint |
| $f(c)$ | $\frac{9}{5} \sqrt[3]{\frac{9}{25}} \approx 1.28$ | 0 | 1 | 5 |

- The largest value of $f$ in the table is 5 and the smallest value of $f$ in the table is 0.
- Thus on the interval $-1 \leq x \leq 1$ the global maximum of $f$ is 5 , and is taken at $x=1$, while the global minimum value of $f(x)$ is 0 , and is taken at $x=0$.
- For completeness we also sketch the graph of this function on the same interval.


Later (in Section 3.6) we will see how to construct such a sketch without using a calculator or computer.

### 3.5.3 $\quad$ Max/Min Examples

As noted at the beginning of this section, the problem of finding maxima and minima is a very important application of differential calculus in the real world. We now turn to a number of examples of this process. But to guide the reader we will describe a general procedure to follow for these problems.

1 Read - read the problem carefully. Work out what information is given in the statement of the problem and what we are being asked to compute.

2 Diagram - draw a diagram. This will typically help you to identify what you know about the problem and what quantities you need to work out.

3 Variables - assign variables to the quantities in the problem along with their units. It is typically a good idea to make sensible choices of variable names: $A$ for area, $h$ for height, $t$ for time etc.

4 Relations - find relations between the variables. By now you should know the quantity we are interested in (the one we want to maximise or minimise) and we need to establish a relation between it and the other variables.

5 Reduce - the relation down to a function of one variable. In order to apply the calculus we know, we must have a function of a single variable. To do this we need to use all the information we have to eliminate variables. We should also work out the domain of the resulting function.

6 Maximise or minimise - we can now apply the methods of Corollary 3.5.13 to find the maximum or minimum of the quantity we need (as the problem dictates).

7 Be careful - make sure your answer makes sense. Make sure quantities are physical. For example, lengths and areas cannot be negative.

8 Answer the question - be sure your answer really answers the question asked in the problem.

Let us start with a relatively simple problem:
Example 3.5.15 Constructing a container of maximal volume.
A closed rectangular container with a square base is to be made from two different materials. The material for the base costs $\$ 5$ per square meter, while the material for the other five sides costs $\$ 1$ per square meter. Find the dimensions of the container which has the largest possible volume if the total cost of materials is $\$ 72$.
Solution We can follow the steps we outlined above to find the solution.

- We need to determine the area of the two types of materials used and the corresponding total cost.
- Draw a picture of the box.


The more useful picture is the unfolded box on the right.

- In the picture we have already introduced two variables. The square base has side-length $b$ metres and it has height $h$ metres. Let the area of the base be $A_{b}$ and the area of the other fives sides be $A_{s}$ (both in $m^{2}$ ), and the total cost be $C$ (in dollars). Finally let the volume enclosed be $V \mathrm{~m}^{3}$.
- Some simple geometry tells us that

$$
\begin{aligned}
A_{b} & =b^{2} \\
A_{s} & =4 b h+b^{2} \\
V & =b^{2} h \\
C & =5 \cdot A_{b}+1 \cdot A_{s}=5 b^{2}+4 b h+b^{2}=6 b^{2}+4 b h
\end{aligned}
$$

- To eliminate one of the variables we use the fact that the total cost is $\$ 72$.

$$
\begin{array}{rlrl}
C & =6 b^{2}+4 b h=72 & \text { rearrange } \\
4 b h & =72-6 b^{2} & & \text { isolate } h \\
h & =\frac{72-6 b^{2}}{4 b}=\frac{3}{2} \cdot \frac{12-b^{2}}{b} &
\end{array}
$$

Substituting this into the volume gives

$$
V=b^{2} h=\frac{3 b}{2}\left(12-b^{2}\right)=18 b-\frac{3}{2} b^{3}
$$

Now note that since $b$ is a length it cannot be negative, so $b \geq 0$. Further since volume cannot be negative, we must also have

$$
12-b^{2} \geq 0
$$

and so $b \leq \sqrt{12}$.

- Now we can apply Corollary 3.5.13 on the above expression for the volume with $0 \leq b \leq \sqrt{12}$. The endpoints give:

$$
\begin{aligned}
V(0) & =0 \\
V(\sqrt{12}) & =0
\end{aligned}
$$

The derivative is

$$
V^{\prime}(b)=18-\frac{9 b^{2}}{2}
$$

Since this is a polynomial there are no singular points. However we can solve $V^{\prime}(b)=0$ to find critical points:

$$
\begin{aligned}
18-\frac{9 b^{2}}{2} & =0 \quad \text { divide by } 9 \text { and multiply by } 2 \\
4-b^{2} & =0
\end{aligned}
$$

Hence $b= \pm 2$. Thus the only critical point in the domain is $b=2$. The corresponding volume is

$$
\begin{aligned}
V(2) & =18 \times 2-\frac{3}{2} \times 2^{3} \\
& =36-12=24
\end{aligned}
$$

So by Corollary 3.5.13, the maximum volume is when 24 when $b=2$ and

$$
h=\frac{3}{2} \cdot \frac{12-b^{2}}{b}=\frac{3}{2} \frac{12-4}{2}=6 .
$$

- All our quantities make sense; lengths, areas and volumes are all non-negative.
- Checking the question again, we see that we are asked for the dimensions of the container (rather than its volume) so we can answer with

The container with dimensions $2 \times 2 \times 6 m$ will be the largest possible.
Example 3.5.15

Example 3.5.16 Constructing another box.
A rectangular sheet of cardboard is 6 inches by 9 inches. Four identical squares are cut from the corners of the cardboard, as shown in the figure below, and the remaining piece is folded into an open rectangular box. What should the size of the cut out squares be in order to maximize the volume of the box?
Solution This one is quite similar to the previous one, so we perhaps don't need to go into so much detail.

- After reading carefully we produce the following picture:

- Let the height of the box be $x$ inches, and the base be $\ell \times w$ inches. The volume of the box is then $V$ cubic inches.
- Some simple geometry tells us that $\ell=9-2 x, w=6-2 x$ and so

$$
\begin{aligned}
V & =x(9-2 x)(6-2 x) \text { cubic inches } \\
& =54 x-30 x^{2}+4 x^{3} .
\end{aligned}
$$

Notice that since all lengths must be non-negative, we must have

$$
x, \ell, w \geq 0
$$

and so $0 \leq x \leq 3$ (if $x>3$ then $w<0$ ).

- We can now apply Corollary 3.5.13. First the endpoints of the interval give

$$
V(0)=0 \quad V(3)=0
$$

The derivative is

$$
\begin{aligned}
V^{\prime}(x) & =54-60 x+12 x^{2} \\
& =6\left(9-10 x+2 x^{2}\right)
\end{aligned}
$$

Since this is a polynomial there are no singular points. To find critical points we solve $V^{\prime}(x)=0$ to get

$$
\begin{aligned}
x_{ \pm} & =\frac{10 \pm \sqrt{100-4 \times 2 \times 9}}{4} \\
& =\frac{10 \pm \sqrt{28}}{4}=\frac{10 \pm 2 \sqrt{7}}{4}=\frac{5 \pm \sqrt{7}}{2}
\end{aligned}
$$

We can then use a calculator to approximate

$$
x_{+} \approx 3.82 \quad x_{-} \approx 1.18
$$

So $x_{-}$is inside the domain, while $x_{+}$lies outside.
Alternatively ${ }^{a}$, we can bound $x_{ \pm}$by first noting that $2 \leq \sqrt{7} \leq 3$. From this we know that

$$
\begin{aligned}
1 & =\frac{5-3}{2} \leq x_{-}=\frac{5-\sqrt{7}}{2} \leq \frac{5-2}{2}=1.5 \\
3.5 & =\frac{5+2}{2} \leq x_{+}=\frac{5+\sqrt{7}}{2} \leq \frac{5+3}{2}=4
\end{aligned}
$$

- Since the volume is zero when $x=0,3$, it must be the case that the volume is maximised when $x=x_{-}=\frac{5-\sqrt{7}}{2}$.
- Notice that since $0<x_{-}<3$ we know that the other lengths are positive, so our answer makes sense. Further, the question only asks for the length $x$ and not the resulting volume so we have answered the question.
$a \quad$ Say if we do not have a calculator to hand, or your instructor insists that the problem be done without one.

Example 3.5.16
There is a new wrinkle in the next two examples. Each involves finding the minimum value of a function $f(x)$ with $x$ running over all real numbers, rather than just over a finite interval as in Corollary 3.5.13. Both in Example 3.5.18 and in Example 3.5.19 the function $f(x)$ tends to $+\infty$ as $x$ tends to either $+\infty$ or $-\infty$. So the minimum value of $f(x)$ will be achieved for some finite value of $x$, which will be a local minimum as well as a global minimum.

## Theorem 3.5.17

Let $f(x)$ be defined and continuous for all $-\infty<x<\infty$. Let $c$ be a finite real number.
a If $\lim _{x \rightarrow+\infty} f(x)=+\infty$ and $\lim _{x \rightarrow-\infty} f(x)=+\infty$ and if $f(x)$ has a global minimum at $x=c$, then there are 2 possibilities. Either

- $f^{\prime}(c)=0$, or
- $f^{\prime}(c)$ does not exist

That is, a global minimum must occur either at a critical point or at a singular point.
b If $\lim _{x \rightarrow+\infty} f(x)=-\infty$ and $\lim _{x \rightarrow-\infty} f(x)=-\infty$ and if $f(x)$ has a global maximum at $x=c$, then there are 2 possibilities. Either

- $f^{\prime}(c)=0$, or
- $f^{\prime}(c)$ does not exist

That is, a global maximum must occur either at a critical point or at a singular point.

Example 3.5.18 How far from a point to a line.
Find the point on the line $y=6-3 x$ that is closest to the point $(7,5)$.
Solution In this problem

- A simple picture

- Some notation is already given to us. Let a point on the line have coordinates $(x, y)$, and we do not need units. And let $\ell$ be the distance from the point $(x, y)$ to the point $(7,5)$.
- Since the points are on the line the coordinates $(x, y)$ must obey

$$
y=6-3 x
$$

Notice that $x$ and $y$ have no further constraints. The distance $\ell$ is given by

$$
\ell^{2}=(x-7)^{2}+(y-5)^{2}
$$

- We can now eliminate the variable $y$ :

$$
\begin{aligned}
\ell^{2} & =(x-7)^{2}+(y-5)^{2} \\
& =(x-7)^{2}+(6-3 x-5)^{2}=(x-7)^{2}+(1-3 x)^{2} \\
& =x^{2}-14 x+49+1-6 x+9 x^{2}=10 x^{2}-20 x+50 \\
& =10\left(x^{2}-2 x+5\right) \\
\ell & =\sqrt{10} \cdot \sqrt{x^{2}-2 x+5}
\end{aligned}
$$

Notice that as $x \rightarrow \pm \infty$ the distance $\ell \rightarrow+\infty$.

- We can now apply Theorem 3.5.17
- Since the distance is defined for all real $x$, we do not have to check the endpoints of the domain - there are none.
- Form the derivative:

$$
\frac{\mathrm{d} \ell}{\mathrm{~d} x}=\sqrt{10} \frac{2 x-2}{2 \sqrt{x^{2}-2 x+5}}
$$

It is zero when $x=1$, and undefined if $x^{2}-2 x+5<0$. However, since

$$
x^{2}-2 x+5=\left(x^{2}-2 x+1\right)+4=\underbrace{(x-1)^{2}}_{\geq 0}+4
$$

we know that $x^{2}-2 x+5 \geq 4$. Thus the function has no singular points and the only critical point occurs at $x=1$. The corresponding function value is then

$$
\ell(1)=\sqrt{10} \sqrt{1-2+5}=2 \sqrt{10}
$$

- Thus the minimum value of the distance is $\ell=2 \sqrt{10}$ and occurs at $x=1$.
- This answer makes sense - the distance is not negative.
- The question asks for the point that minimises the distance, not that minimum distance. Hence the answer is $x=1, y=6-3=3$. I.e.

The point that minimises the distance is $(1,3)$.
Notice that we can make the analysis easier by observing that the point that minimises the distance also minimises the squared-distance. So that instead of minimising the function $\ell$, we can just minimise $\ell^{2}$ :

$$
\ell^{2}=10\left(x^{2}-2 x+5\right)
$$

The resulting algebra is a bit easier and we don't have to hunt for singular points.

Example 3.5.19 How far from a point to a curve.
Find the minimum distance from $(2,0)$ to the curve $y^{2}=x^{2}+1$.
Solution This is very much like the previous question.

- After reading the problem carefully we can draw a picture
(
- In this problem we do not need units and the variables $x, y$ are supplied. We define the distance to be $\ell$ and it is given by

$$
\ell^{2}=(x-2)^{2}+y^{2} .
$$

As noted in the previous problem, we will minimise the squared-distance since that also minimises the distance.

- Since $x, y$ satisfy $y^{2}=x^{2}+1$, we can write the distance as a function of $x$ :

$$
\ell^{2}=(x-2)^{2}+y^{2}=(x-2)^{2}+\left(x^{2}+1\right)
$$

Notice that as $x \rightarrow \pm \infty$ the squared-distance $\ell^{2} \rightarrow+\infty$.

- Since the squared-distance is a polynomial it will not have any singular points, only critical points. The derivative is

$$
\frac{\mathrm{d}}{\mathrm{~d} x} \ell^{2}=2(x-2)+2 x=4 x-4
$$

so the only critical point occurs at $x=1$.

- When $x=1, y= \pm \sqrt{2}$ and the distance is

$$
\ell^{2}=(1-2)^{2}+(1+1)=3 \quad \ell=\sqrt{3}
$$

and thus the minimum distance from the curve to $(2,0)$ is $\sqrt{3}$.

Example 3.5.20 Constructing a trough.
A water trough is to be constructed from a metal sheet of width 45 cm by bending up one third of the sheet on each side through an angle $\theta$. Which $\theta$ will allow the trough to carry the maximum amount of water?
Solution Clearly $0 \leq \theta \leq \pi$, so we are back in the domain ${ }^{a}$ of Corollary 3.5.13.

- After reading the problem carefully we should realise that it is really asking us to maximise the cross-sectional area. A figure really helps.

- From this we are led to define the height $h \mathrm{~cm}$ and cross-sectional area $A \mathrm{~cm}^{2}$. Both are functions of $\theta$.

$$
h=15 \sin \theta
$$

while the area can be computed as the sum of the central $15 \times h$ rectangle, plus two triangles. Each triangle has height $h$ and base $15 \cos \theta$. Hence

$$
\begin{aligned}
A & =15 h+2 \cdot \frac{1}{2} \cdot h \cdot 15 \cos \theta \\
& =15 h(1+\cos \theta)
\end{aligned}
$$

- Since $h=15 \sin \theta$ we can rewrite the area as a function of just $\theta$ :

$$
A(\theta)=225 \sin \theta(1+\cos \theta)
$$

where $0 \leq \theta \leq \pi$.

- Now we use Corollary 3.5.13. The ends of the interval give

$$
\begin{aligned}
& A(0)=225 \sin 0(1+\cos 0)=0 \\
& A(\pi)=225 \sin \pi(1+\cos \pi)=0
\end{aligned}
$$

The derivative is

$$
\begin{aligned}
A^{\prime}(\theta) & =225 \cos \theta \cdot(1+\cos \theta)+225 \sin \theta \cdot(-\sin \theta) \\
& =225\left[\cos \theta+\cos ^{2} \theta-\sin ^{2} \theta\right] \quad \text { recall } \sin ^{2} \theta=1-\cos ^{2} \theta \\
& =225\left[\cos \theta+2 \cos ^{2} \theta-1\right]
\end{aligned}
$$

This is a continuous function, so there are no singular points. However we can still hunt for critical points by solving $A^{\prime}(\theta)=0$. That is

$$
\begin{array}{rlr}
2 \cos ^{2} \theta+\cos \theta-1 & =0 & \text { factor carefully } \\
(2 \cos \theta-1)(\cos \theta+1) & =0 &
\end{array}
$$

Hence we must have $\cos \theta=-1$ or $\cos \theta=\frac{1}{2}$. On the domain $0 \leq \theta \leq \pi$, this means $\theta=\pi / 3$ or $\theta=\pi$.

$$
\begin{aligned}
A(\pi) & =0 \\
A(\pi / 3) & =225 \sin (\pi / 3)(1+\cos (\pi / 3)) \\
& =225 \cdot \frac{\sqrt{3}}{2} \cdot\left(1+\frac{1}{2}\right) \\
& =225 \cdot \frac{3 \sqrt{3}}{4} \approx 292.28
\end{aligned}
$$

- Thus the cross-sectional area is maximised when $\theta=\frac{\pi}{3}$.


Example 3.5.21 Closest and farthest points on a curve to a given point.
Find the points on the ellipse $\frac{x^{2}}{4}+y^{2}=1$ that are nearest to and farthest from the point ( 1,0 ).
Solution While this is another distance problem, the possible values of $x, y$ are bounded, so we need Corollary 3.5.13 rather than Theorem 3.5.17.

- We start by drawing a picture:

- Let $\ell$ be the distance from the point $(x, y)$ on the ellipse to the point $(1,0)$. As
was the case above, we will maximise the squared-distance.

$$
\ell^{2}=(x-1)^{2}+y^{2} .
$$

- Since $(x, y)$ lie on the ellipse we have

$$
\frac{x^{2}}{4}+y^{2}=1
$$

Note that this also shows that $-2 \leq x \leq 2$ and $-1 \leq y \leq 1$.
Isolating $y^{2}$ and substituting this into our expression for $\ell^{2}$ gives

$$
\ell^{2}=(x-1)^{2}+\underbrace{1-x^{2} / 4}_{=y^{2}}
$$

- Now we can apply Corollary 3.5.13. The endpoints of the domain give

$$
\begin{aligned}
\ell^{2}(-2) & =(-2-1)^{2}+1-(-2)^{2} / 4=3^{2}+1-1=9 \\
\ell^{2}(2) & =(2-1)^{2}+1-2^{2} / 4=1+1-1=1
\end{aligned}
$$

The derivative is

$$
\frac{\mathrm{d}}{\mathrm{~d} x} \ell^{2}=2(x-1)-x / 2=\frac{3 x}{2}-2
$$

Thus there are no singular points, but there is a critical point at $x=4 / 3$. The corresponding squared-distance is

$$
\begin{aligned}
\ell^{2}(4 / 3) & =\left(\frac{4}{3}-1\right)^{2}+1-\frac{(4 / 3)^{2}}{4} \\
& =(1 / 3)^{2}+1-(4 / 9)=6 / 9=2 / 3
\end{aligned}
$$

- To summarise (and giving distances and coordinates of points):

| $x$ | $(x, y)$ | $\ell$ |
| :---: | :---: | :---: |
| -2 | $(-2,0)$ | 3 |
| $\frac{4}{3}$ | $\left(\frac{4}{3}, \pm \frac{\sqrt{5}}{3}\right)$ | $\sqrt{\frac{2}{3}}$ |
| 2 | $(2,0)$ | 1 |

The point of maximum distance is $(-2,0)$, and the point of minimum distance is $\left(\frac{4}{3}, \pm \frac{\sqrt{5}}{3}\right)$.

Example 3.5.21

Example 3.5.22 Largest rectangle inside a triangle.
Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side $a$ if one side of the rectangle lies on the base of the triangle.
Solution Since the rectangle must sit inside the triangle, its dimensions are bounded and we will end up using Corollary 3.5.13.

- Carefully draw a picture:


We have drawn (on the left) the triangle in the $x y$-plane with its base on the $x$-axis. The base has been drawn running from $(-a / 2,0)$ to $(a / 2,0)$ so its centre lies at the origin. A little Pythagoras (or a little trigonometry) tells us that the height of the triangle is

$$
\sqrt{a^{2}-(a / 2)^{2}}=\frac{\sqrt{3}}{2} \cdot a=a \cdot \sin \frac{\pi}{3}
$$

Thus the vertex at the top of the triangle lies at $\left(0, \frac{\sqrt{3}}{2} \cdot a\right)$.

- If we construct a rectangle that does not touch the sides of the triangle, then we can increase the dimensions of the rectangle until it touches the triangle and so make its area larger. Thus we can assume that the two top corners of the rectangle touch the triangle as drawn in the right-hand figure above.
- Now let the rectangle be $2 x$ wide and $y$ high. And let $A$ denote its area. Clearly

$$
A=2 x y .
$$

where $0 \leq x \leq a / 2$ and $0 \leq y \leq \frac{\sqrt{3}}{2} a$.

- Our construction means that the top-right corner of the rectangle will have coordinates $(x, y)$ and lie on the line joining the top vertex of the triangle at ( $0, \sqrt{3} a / 2$ ) to the bottom-right vertex at $(a / 2,0)$. In order to write the area as a function of $x$ alone, we need the equation for this line since it will tell us how to write $y$ as a function of $x$. The line has slope

$$
\text { slope }=\frac{\sqrt{3} a / 2-0}{0-a / 2}=-\sqrt{3} .
$$

and passes through the point $(0, \sqrt{3} a / 2)$, so any point $(x, y)$ on that line satisfies:

$$
y=-\sqrt{3} x+\frac{\sqrt{3}}{2} a
$$

- We can now write the area as a function of $x$ alone

$$
\begin{aligned}
A(x) & =2 x\left(-\sqrt{3} x+\frac{\sqrt{3}}{2} a\right) \\
& =\sqrt{3} x(a-2 x)
\end{aligned}
$$

with $0 \leq x \leq a / 2$.

- The ends of the domain give:

$$
A(0)=0 \quad A(a / 2)=0
$$

The derivative is

$$
A^{\prime}(x)=\sqrt{3}(x \cdot(-2)+1 \cdot(a-2 x))=\sqrt{3}(a-4 x)
$$

Since this is a polynomial there are no singular points, but there is a critical point at $x=a / 4$. There

$$
\begin{aligned}
A(a / 4) & =\sqrt{3} \cdot \frac{a}{4} \cdot(a-a / 2)=\sqrt{3} \cdot \frac{a^{2}}{8} \\
y & =-\sqrt{3} \cdot(a / 4)+\frac{\sqrt{3}}{2} a=\sqrt{3} \cdot \frac{a}{4}
\end{aligned}
$$

- Checking the question again, we see that we are asked for the dimensions rather than the area, so the answer is $2 x \times y$ :

The largest such rectangle has dimensions $\frac{a}{2} \times \frac{\sqrt{3} a}{4}$.
Example 3.5.22
This next one is a good physics example. In it we will derive Snell's Law ${ }^{2}$ from Fermat's principle ${ }^{3}$.

2 Snell's law is named after the Dutch astronomer Willebrord Snellius who derived it in around 1621, though it was first stated accurately in 984 by Ibn Sahl.
3 Named after Pierre de Fermat who described it in a letter in 1662. The beginnings of the idea, however, go back as far as Hero of Alexandria in around 60CE. Hero is credited with many inventions including the first vending machine, and a precursor of the steam engine called an aeolipile.

Example 3.5.23 Snell's law.
Consider the figure below which shows the trajectory of a ray of light as it passes through two different mediums (say air and water).


Let $c_{a}$ be the speed of light in air and $c_{w}$ be the speed of light in water. Fermat's principle states that a ray of light will always travel along a path that minimises the time taken. So if a ray of light travels from $P$ (in air) to $Q$ (in water) then it will "choose" the point $O$ (on the interface) so as to minimise the total time taken. Use this idea to show Snell's law,

$$
\frac{\sin \theta_{i}}{\sin \theta_{r}}=\frac{c_{a}}{c_{w}}
$$

where $\theta_{i}$ is the angle of incidence and $\theta_{r}$ is the angle of refraction (as illustrated in the figure above).
Solution This problem is a little more abstract than the others we have examined, but we can still apply Theorem 3.5.17.

- We are given a figure in the statement of the problem and it contains all the relevant points and angles. However it will simplify things if we decide on a coordinate system. Let's assume that the point $O$ lies on the $x$-axis, at coordinates $(x, 0)$. The point $P$ then lies above the axis at $\left(X_{P},+Y_{P}\right)$, while $Q$ lies below the axis at $\left(X_{Q},-Y_{Q}\right)$. This is drawn below.

- The statement of Snell's law contains terms $\sin \theta_{i}$ and $\sin \theta_{r}$, so it is a good idea for us to see how to express these in terms of the coordinates we have just introduced:

$$
\begin{aligned}
& \sin \theta_{i}=\frac{\text { opposite }}{\text { hypotenuse }}=\frac{\left(x-X_{P}\right)}{\sqrt{\left(X_{P}-x\right)^{2}+Y_{P}^{2}}} \\
& \sin \theta_{r}=\frac{\text { opposite }}{\text { hypotenuse }}=\frac{\left(X_{Q}-x\right)}{\sqrt{\left(X_{Q}-x\right)^{2}+Y_{Q}^{2}}}
\end{aligned}
$$

- Let $\ell_{P}$ denote the distance $P O$, and $\ell_{Q}$ denote the distance $O Q$. Then we have

$$
\begin{aligned}
\ell_{P} & =\sqrt{\left(X_{P}-x\right)^{2}+Y_{P}^{2}} \\
\ell_{Q} & =\sqrt{\left(X_{Q}-x\right)^{2}+Y_{Q}^{2}}
\end{aligned}
$$

If we then denote the total time taken by $T$, then

$$
T=\frac{\ell_{P}}{c_{a}}+\frac{\ell_{Q}}{c_{w}}=\frac{1}{c_{a}} \sqrt{\left(X_{P}-x\right)^{2}+Y_{P}^{2}}+\frac{1}{c_{w}} \sqrt{\left(X_{Q}-x\right)^{2}+Y_{Q}^{2}}
$$

which is written as a function of $x$ since all the other terms are constants.

- Notice that as $x \rightarrow+\infty$ or $x \rightarrow-\infty$ the total time $T \rightarrow \infty$ and so we can apply Theorem 3.5.17. The derivative is

$$
\frac{\mathrm{d} T}{\mathrm{~d} x}=\frac{1}{c_{a}} \frac{-2\left(X_{P}-x\right)}{2 \sqrt{\left(X_{P}-x\right)^{2}+Y_{P}^{2}}}+\frac{1}{c_{w}} \frac{-2\left(X_{Q}-x\right)}{2 \sqrt{\left(X_{Q}-x\right)^{2}+Y_{Q}^{2}}}
$$

Notice that the terms inside the square-roots cannot be zero or negative since they are both sums of squares and $Y_{P}, Y_{Q}>0$. So there are no singular points, but there is a critical point when $T^{\prime}(x)=0$, namely when

$$
\begin{aligned}
0 & =\frac{1}{c_{a}} \frac{X_{P}-x}{\sqrt{\left(X_{P}-x\right)^{2}+Y_{P}^{2}}}+\frac{1}{c_{w}} \frac{X_{Q}-x}{\sqrt{\left(X_{Q}-x\right)^{2}+Y_{Q}^{2}}} \\
& =\frac{-\sin \theta_{i}}{c_{a}}+\frac{\sin \theta_{r}}{c_{w}}
\end{aligned}
$$

Rearrange this to get

$$
\frac{\sin \theta_{i}}{c_{a}}=\frac{\sin \theta_{r}}{c_{w}}
$$

move sines to one side

$$
\frac{\sin \theta_{i}}{\sin \theta_{r}}=\frac{c_{a}}{c_{w}}
$$

which is exactly Snell's law.

Example 3.5.24 Finding the best viewing angle.
The Statue of Liberty has height 46 m and stands on a 47 m tall pedestal. How far from the statue should an observer stand to maximize the angle subtended by the statue at the observer's eye, which is 1.5 m above the base of the pedestal?
Solution Obviously if we stand too close then all the observer sees is the pedestal, while if they stand too far then everything is tiny. The best spot for taking a photograph is somewhere in between.

- Draw a careful picture ${ }^{a}$

and we can put in the relevant lengths and angles.
- The height of the statue is $h=46 \mathrm{~m}$, and the height of the pedestal (above the eye) is $p=47-1.5=45.5 \mathrm{~m}$. The horizontal distance from the statue to the eye is $x$. There are two relevant angles. First $\theta$ is the angle subtended by the statue, while $\varphi$ is the angle subtended by the portion of the pedestal above the eye.
- Some trigonometry gives us

$$
\begin{aligned}
\tan \varphi & =\frac{p}{x} \\
\tan (\varphi+\theta) & =\frac{p+h}{x}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\varphi & =\arctan \frac{p}{x} \\
\varphi+\theta & =\arctan \frac{p+h}{x}
\end{aligned}
$$

and so

$$
\theta=\arctan \frac{p+h}{x}-\arctan \frac{p}{x}
$$

- If we allow the viewer to stand at any point in front of the statue, then $0 \leq x<\infty$. Further observe that as $x \rightarrow \infty$ or $x \rightarrow 0$ the angle $\theta \rightarrow 0$, since

$$
\lim _{x \rightarrow \infty} \arctan \frac{p+h}{x}=\lim _{x \rightarrow \infty} \arctan \frac{p}{x}=0
$$

and

$$
\lim _{x \rightarrow 0^{+}} \arctan \frac{p+h}{x}=\lim _{x \rightarrow 0^{+}} \arctan \frac{p}{x}=\frac{\pi}{2}
$$

Clearly the largest value of $\theta$ will be strictly positive and so has to be taken for some $0<x<\infty$. (Note the strict inequalities.) This $x$ will be a local maximum as well as a global maximum. As $\theta$ is not singular at any $0<x<\infty$, we need only search for critical points.
A careful application of the chain rule shows that the derivative is

$$
\begin{aligned}
\frac{\mathrm{d} \theta}{\mathrm{~d} x} & =\frac{1}{1+\left(\frac{p+h}{x}\right)^{2}} \cdot\left(\frac{-(p+h)}{x^{2}}\right)-\frac{1}{1+\left(\frac{p}{x}\right)^{2}} \cdot\left(\frac{-p}{x^{2}}\right) \\
& =\frac{-(p+h)}{x^{2}+(p+h)^{2}}+\frac{p}{x^{2}+p^{2}}
\end{aligned}
$$

So a critical point occurs when

$$
\begin{array}{rlrl}
\frac{(p+h)}{x^{2}+(p+h)^{2}} & =\frac{p}{x^{2}+p^{2}} & & \text { cross multiply } \\
(p+h)\left(x^{2}+p^{2}\right) & =p\left(x^{2}+(p+h)^{2}\right) & & \text { collect } x \text { terms } \\
x^{2}(p+h-p) & =p(p+h)^{2}-p^{2}(p+h) & & \text { clean up } \\
h x^{2} & =p(p+h)(p+h-p)=p h(p+h) & & \\
x^{2} & =p(p+h) & & \\
x & = \pm \sqrt{p(p+h)} \approx \pm 64.9 m &
\end{array}
$$

- Thus the best place to stand approximately 64.9 m in front or behind the statue. At that point $\theta \approx 0.348$ radians or $19.9^{\circ}$.
a And make some healthy use of public domain clip art.

Example 3.5.25 Moving objects around corners.
Find the length of the longest rod that can be carried horizontally (no tilting allowed) from a corridor 3 m wide into a corridor 2 m wide. The two corridors are perpendicular to each other.

## Solution

- Suppose that we are carrying the rod around the corner, then if the rod is as long as possible it must touch the corner and the outside walls of both corridors. A picture of this is show below.


You can see that this gives rise to two similar triangles, one inside each corridor. Also the maximum length of the rod changes with the angle it makes with the walls of the corridor.

- Suppose that the angle between the rod and the inner wall of the 3 m corridor is $\theta$, as illustrated in the figure above. At the same time it will make an angle of $\frac{\pi}{2}-\theta$ with the outer wall of the 2 m corridor. Denote by $\ell_{1}(\theta)$ the length of the part of the rod forming the hypotenuse of the upper triangle in the figure above. Similarly, denote by $\ell_{2}(\theta)$ the length of the part of the rod forming the hypotenuse of the lower triangle in the figure above. Then

$$
\ell_{1}(\theta)=\frac{3}{\sin \theta} \quad \ell_{2}(\theta)=\frac{2}{\cos \theta}
$$

and the total length is

$$
\ell(\theta)=\ell_{1}(\theta)+\ell_{2}(\theta)=\frac{3}{\sin \theta}+\frac{2}{\cos \theta}
$$

where $0 \leq \theta \leq \frac{\pi}{2}$.

- The length of the longest rod we can move through the corridor in this way is the minimum of $\ell(\theta)$. Notice that $\ell(\theta)$ is not defined at $\theta=0, \frac{\pi}{2}$. Indeed we find that as $\theta \rightarrow 0^{+}$or $\theta \rightarrow \frac{\pi}{2}^{-}$, the length $\ell \rightarrow+\infty$. (You should be able to picture what happens to our rod in those two limits). Clearly the minimum allowed $\ell(\theta)$ is going to be finite and will be achieved for some $0<\theta<\frac{\pi}{2}$ (note the strict
inequalities) and so will be a local minimum as well as a global minimum. So we only need to find zeroes of $\ell^{\prime}(\theta)$.
Differentiating $\ell$ gives

$$
\frac{\mathrm{d} \ell}{\mathrm{~d} \theta}=-\frac{3 \cos \theta}{\sin ^{2} \theta}+\frac{2 \sin \theta}{\cos ^{2} \theta}=\frac{-3 \cos ^{3} \theta+2 \sin ^{3} \theta}{\sin ^{2} \theta \cos ^{2} \theta} .
$$

This does not exist at $\theta=0, \frac{\pi}{2}$ (which we have already analysed) but does exist at every $0<\theta<\frac{\pi}{2}$ and is equal to zero when the numerator is zero. Namely when

$$
\begin{aligned}
2 \sin ^{3} \theta & =3 \cos ^{3} \theta & \text { divide by } \cos ^{3} \theta \\
2 \tan ^{3} \theta & =3 & \\
\tan \theta & =\sqrt[3]{\frac{3}{2}} &
\end{aligned}
$$

- From this we can recover $\sin \theta$ and $\cos \theta$, without having to compute $\theta$ itself. We can, for example, construct a right-angle triangle with adjacent length $\sqrt[3]{2}$ and opposite length $\sqrt[3]{3}$ (so that $\tan \theta=\sqrt[3]{3 / 2}$ ):


It has hypotenuse $\sqrt{3^{2 / 3}+2^{2 / 3}}$, and so

$$
\begin{aligned}
\sin \theta & =\frac{3^{1 / 3}}{\sqrt{3^{2 / 3}+2^{2 / 3}}} \\
\cos \theta & =\frac{2^{1 / 3}}{\sqrt{3^{2 / 3}+2^{2 / 3}}}
\end{aligned}
$$

Alternatively could use the identities:

$$
1+\tan ^{2} \theta=\sec ^{2} \theta \quad 1+\cot ^{2} \theta=\csc ^{2} \theta
$$

to obtain expressions for $1 / \cos \theta$ and $1 / \sin \theta$.

- Using the above expressions for $\sin \theta, \cos \theta$ we find the minimum of $\ell$ (which is the longest rod that we can move):

$$
\begin{aligned}
\ell & =\frac{3}{\sin \theta}+\frac{2}{\cos \theta}=\frac{3}{\frac{\sqrt[3]{3}}{\sqrt{2^{\frac{2}{3}}+3^{\frac{2}{3}}}}}+\frac{2}{\frac{\sqrt[3]{2}}{\sqrt{2^{\frac{2}{3}}+3^{\frac{2}{3}}}}} \\
& =\sqrt{2^{\frac{2}{3}}+3^{\frac{2}{3}}\left[3^{\frac{2}{3}}+2^{\frac{2}{3}}\right]} \\
& =\left[2^{\frac{2}{3}}+3^{\frac{2}{3}}\right]^{\frac{3}{2}} \approx 7.02 \mathrm{~m}
\end{aligned}
$$

### 3.5.4 $\Perp$ Exercises

## Exercises for § 3.5.1

## Exercises - Stage 1

1. Identify every critical point and every singular point of $f(x)$ shown on the graph below. Which correspond to local extrema?

2. Identify every critical point and every singular point of $f(x)$ on the graph below. Which correspond to local extrema? Which correspond to global extrema over the interval shown?

3. Draw a graph $y=f(x)$ where $f(2)$ is a local maximum, but it is not a global maximum.

## Exercises - Stage 2

4. Suppose $f(x)=\frac{x-1}{x^{2}+3}$.
a Find all critical points.
b Find all singular points.
c What are the possible points where local extrema of $f(x)$ may exist?

## Exercises - Stage 3

5. Below are a number of curves, all of which have a singular point at $x=2$. For each, label whether $x=2$ is a local maximum, a local minimum, or neither.

6. Draw a graph $y=f(x)$ where $f(2)$ is a local maximum, but $x=2$ is not a critical point or an endpoint.
7. 

$$
f(x)=\sqrt{|(x-5)(x+7)|}
$$

Find all critical points and all singular points of $f(x)$. You do not have to specify whether a point is critical or singular.
8. Suppose $f(x)$ is the constant function $f(x)=4$. What are the critical points and singular points of $f(x)$ ? What are its local and global maxima and minima?

## Exercises for § 3.5.2

## Exercises - Stage 1

1. Sketch a function $f(x)$ such that:

- $f(x)$ is defined over all real numbers
- $f(x)$ has a global max but no global min.

2. Sketch a function $f(x)$ such that:

- $f(x)$ is defined over all real numbers
- $f(x)$ is always positive
- $f(x)$ has no global max and no global min.

3. Sketch a function $f(x)$ such that:

- $f(x)$ is defined over all real numbers
- $f(x)$ has a global minimum at $x=5$
- $f(x)$ has a global minimum at $x=-5$, too.


## Exercises - Stage 2

4. $f(x)=x^{2}+6 x-10$. Find all global extrema on the interval $[-5,5]$
5. $f(x)=\frac{2}{3} x^{3}-2 x^{2}-30 x+7$. Find all global extrema on the interval $[-4,0]$.

## $\leadsto$ Exercises for § 3.5.3

Exercises - Stage 1 For Questions 3.5.4.1 through 3.5.4.3, the quantity to optimize is already given to you as a function of a single variable.For Questions 3.5.4.4 and 3.5.4.5, you can decide whether a critical point is a local extremum by considering the derivative of the function.For Questions 3.5.4.6 through 3.5.4.13, you will have to find an expression for the quantity you want to optimize as a function of a single variable.

1. *. Find the global maximum and the global minimum for $f(x)=x^{5}-5 x+2$ on the interval $[-2,0]$.
2. *. Find the global maximum and the global minimum for $f(x)=x^{5}-5 x-10$ on the interval $[0,2]$.
3. *. Find the global maximum and the global minimum for $f(x)=2 x^{3}-6 x^{2}-2$ on the interval $[1,4]$.
4. *. Consider the function $h(x)=x^{3}-12 x+4$. What are the coordinates of the local maximum of $h(x)$ ? What are the coordinates of the local minimum of $h(x)$ ?
5. *. Consider the function $h(x)=2 x^{3}-24 x+1$. What are the coordinates of the local maximum of $h(x)$ ? What are the coordinates of the local minimum of $h(x)$ ?
6. *. You are in a dune buggy at a point $P$ in the desert, 12 km due south of the nearest point $A$ on a straight east-west road. You want to get to a town $B$ on the road 18 km east of $A$. If your dune buggy can travel at an average speed of $15 \mathrm{~km} / \mathrm{hr}$ through the desert and $30 \mathrm{~km} / \mathrm{hr}$ along the road, towards what point $Q$ on the road should you head to minimize your travel time from $P$ to $B$ ?

7. *. A closed three dimensional box is to be constructed in such a way that its volume is $4500 \mathrm{~cm}^{3}$. It is also specified that the length of the base is 3 times the width of the base. Find the dimensions of the box that satisfies these conditions and has the minimum possible surface area. Justify your answer.
8. *. A closed rectangular container with a square base is to be made from two different materials. The material for the base costs $\$ 5$ per square metre, while the material for the other five sides costs $\$ 1$ per square metre. Find the dimensions of the container which has the largest possible volume if the total cost of materials is $\$ 72$.
9. *. Find a point $X$ on the positive $x$-axis and a point $Y$ on the positive $y$-axis such that (taking $O=(0,0)$ )
i The triangle $X O Y$ contains the first quadrant portion of the unit circle $x^{2}+y^{2}=1$ and
ii the area of the triangle $X O Y$ is as small as possible.
A complete and careful mathematical justification of property 3.5.4.9.i is required.
10. *. A rectangle is inscribed in a semicircle of radius $R$ so that one side of the rectangle lies along a diameter of the semicircle. Find the largest possible perimeter of such a rectangle, if it exists, or explain why it does not. Do the same for the smallest possible perimeter.

11. *. Find the maximal possible volume of a cylinder with surface area $A .{ }^{a}$
a Food is often packaged in cylinders, and companies wouldn't want to waste the metal they are made out of. So, you might expect the dimensions you find in this problem to describe a tin of, say, cat food. Read here about why this isn't the case.
12. *. What is the largest possible area of a window, with perimeter $P$, in the shape of a rectangle with a semicircle on top (so the diameter of the semicircle equals the width of the rectangle)?
13. *. Consider an open-top rectangular baking pan with base dimensions $x$ centimetres by $y$ centimetres and height $z$ centimetres that is made from $A$ square centimetres of tin plate. Suppose $y=p x$ for some fixed constant $p$.
a Find the dimensions of the baking pan with the maximum capacity (i.e., maximum volume). Prove that your answer yields the baking pan with maximum capacity. Your answer will depend on the value of $p$.
b Find the value of the constant $p$ that yields the baking pan with maximum capacity and give the dimensions of the resulting baking pan. Prove that your answer yields the baking pan with maximum capacity.

## Exercises - Stage 3

14. *. Let $f(x)=x^{x}$ for $x>0$.
a Find $f^{\prime}(x)$.
b At what value of $x$ does the curve $y=f(x)$ have a horizontal tangent line?
c Does the function $f$ have a local maximum, a local minimum, or neither of these at the point $x$ found in part 3.5.4.14.b?
15. *. A length of wire is cut into two pieces, one of which is bent to form a circle, the other to form a square. How should the wire be cut if the area enclosed by the two curves is maximized? How should the wire be cut if the area enclosed by the two curves is minimized? Justify your answers.

## 3.6^ Sketching Graphs

One of the most obvious applications of derivatives is to help us understand the shape of the graph of a function. In this section we will use our accumulated knowledge of derivatives to identify the most important qualitative features of graphs $y=f(x)$. The goal of this section is to highlight features of the graph $y=f(x)$ that are easily

- determined from $f(x)$ itself, and
- deduced from $f^{\prime}(x)$, and
- read from $f^{\prime \prime}(x)$.

We will then use the ideas to sketch several examples.

### 3.6.1 Domain, Intercepts and Asymptotes

Given a function $f(x)$, there are several important features that we can determine from that expression before examining its derivatives.

- The domain of the function - take note of values where $f$ does not exist. If the function is rational, look for where the denominator is zero. Similarly be careful to look for roots of negative numbers or other possible sources of discontinuities.
- Intercepts - examine where the function crosses the $x$-axis and the $y$-axis by solving $f(x)=0$ and computing $f(0)$.
- Vertical asymptotes - look for values of $x$ at which $f(x)$ blows up. If $f(x)$ approaches either $+\infty$ or $-\infty$ as $x$ approaches $a$ (or possibly as $x$ approaches $a$ from one side) then $x=a$ is a vertical asymptote to $y=f(x)$. When $f(x)$ is a rational function (written so that common factors are cancelled), then $y=f(x)$ has vertical asymptotes at the zeroes of the denominator.
- Horizontal asymptotes - examine the limits of $f(x)$ as $x \rightarrow+\infty$ and $x \rightarrow-\infty$. Often $f(x)$ will tend to $+\infty$ or to $-\infty$ or to a finite limit $L$. If, for example, $\lim _{x \rightarrow+\infty} f(x)=L$, then $y=L$ is a horizontal asymptote to $y=f(x)$ as $x \rightarrow \infty$.

Example 3.6.1 Domain, intercepts and asymptotes of $\frac{x+1}{(x+3)(x-2)}$.
Consider the function

$$
f(x)=\frac{x+1}{(x+3)(x-2)}
$$

- We see that it is defined on all real numbers except $x=-3,+2$.
- Since $f(0)=-1 / 6$ and $f(x)=0$ only when $x=-1$, the graph has $y$-intercept $(0,-1 / 6)$ and $x$-intercept $(-1,0)$.
- Since the function is rational and its denominator is zero at $x=-3,+2$ it will have vertical asymptotes at $x=-3,+2$. To determine the shape around those asymptotes we need to examine the limits

$$
\lim _{x \rightarrow-3} f(x) \quad \lim _{x \rightarrow 2} f(x)
$$

Notice that when $x$ is close to -3 , the factors $(x+1)$ and $(x-2)$ are both negative, so the sign of $f(x)=\frac{x+1}{x-2} \cdot \frac{1}{x+3}$ is the same as the sign of $x+3$. Hence

$$
\lim _{x \rightarrow-3^{+}} f(x)=+\infty \quad \lim _{x \rightarrow-3^{-}} f(x)=-\infty
$$

A similar analysis when $x$ is near 2 gives

$$
\lim _{x \rightarrow 2^{+}} f(x)=+\infty \quad \lim _{x \rightarrow 2^{-}} f(x)=-\infty
$$

- Finally since the numerator has degree 1 and the denominator has degree 2, we see that as $x \rightarrow \pm \infty, f(x) \rightarrow 0$. So $y=0$ is a horizontal asymptote.
- Since we know the behaviour around the asymptotes and we know the locations of the intercepts (as shown in the left graph below), we can then join up the pieces and smooth them out to get the a good sketch of this function (below right).



### 3.6.2 First Derivative - Increasing or Decreasing

Now we move on to the first derivative, $f^{\prime}(x)$. This is a good time to revisit the mean-value theorem (Theorem 2.13.5) and some of its consequences (Corollary 2.13.12). There we considered any function $f(x)$ that is continuous on an interval $A \leq x \leq B$ and differentiable on $A<x<B$. Then

- if $f^{\prime}(x)>0$ for all $A<x<B$, then $f(x)$ is increasing on $(A, B)$
- that is, for all $A<a<b<B, f(a)<f(b)$.
- if $f^{\prime}(x)<0$ for all $A<x<B$, then $f(x)$ is decreasing on $(A, B)$
- that is, for all $A<a<b<B, f(a)>f(b)$.

Thus the sign of the derivative indicates to us whether the function is increasing or decreasing. Further, as we discussed in Section 3.5.1, we should also examine points at which the derivative is zero - critical points - and points where the derivative does not exist. These points may indicate a local maximum or minimum.

We will now consider a function $f(x)$ that is defined on an interval $I$, except possibly at finitely many points of $I$. If $f$ or its derivative $f^{\prime}$ is not defined at a point $a$ of $I$, then we call $a$ a singular point ${ }^{1}$ of $f$.

After studying the function $f(x)$ as described above, we should compute its derivative $f^{\prime}(x)$.

- Critical points - determine where $f^{\prime}(x)=0$. At a critical point, $f$ has a horizontal tangent.
- Singular points - determine where $f^{\prime}(x)$ is not defined. If $f^{\prime}(x)$ approaches $\pm \infty$ as $x$ approaches a singular point $a$, then $f$ has a vertical tangent there when $f$ approaches a finite value as $x$ approaches $a$ (or possibly approaches $a$ from one side) and a vertical asymptote when $f(x)$ approaches $\pm \infty$ as $x$ approaches $a$ (or possibly approaches $a$ from one side).
- Increasing and decreasing - where is the derivative positive and where is it negative. Notice that in order for the derivative to change sign, it must either pass through zero (a critical point) or have a singular point. Thus neighbouring regions of increase and decrease will be separated by critical and singular points.


## Example 3.6.2 A simple polynomial.

Consider the function

$$
f(x)=x^{4}-6 x^{3}
$$

1 This is the extension of the definition of "singular point" mentioned in the footnote in Definition 3.5.6.

- Before we move on to derivatives, let us first examine the function itself as we did above.
- As $f(x)$ is a polynomial its domain is all real numbers.
- Its $y$-intercept is at $(0,0)$. We find its $x$-intercepts by factoring

$$
f(x)=x^{4}-6 x^{3}=x^{3}(x-6)
$$

So it crosses the $x$-axis at $x=0,6$.

- Again, since the function is a polynomial it does not have any vertical asymptotes. And since

$$
\lim _{x \rightarrow \pm \infty} f(x)=\lim _{x \rightarrow \pm \infty} x^{4}(1-6 / x)=+\infty
$$

it does not have horizontal asymptotes - it blows up to $+\infty$ as $x$ goes to $\pm \infty$.

- We can also determine where the function is positive or negative since we know it is continuous everywhere and zero at $x=0,6$. Thus we must examine the intervals

$$
(-\infty, 0) \quad(0,6) \quad(6, \infty)
$$

When $x<0, x^{3}<0$ and $x-6<0$ so $f(x)=x^{3}(x-6)=$ (negative)(negative) $>0$. Similarly when $x>6, x^{3}>0, x-6>0$ we must have $f(x)>0$. Finally when $0<x<6, x^{3}>0$ but $x-6<0$ so $f(x)<0$. Thus

| interval | $(-\infty, 0)$ | 0 | $(0,6)$ | 6 | $(6, \infty)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $f(x)$ | positive | 0 | negative | 0 | positive |

- Based on this information we can already construct a rough sketch.

- Now we compute its derivative

$$
f^{\prime}(x)=4 x^{3}-18 x^{2}=2 x^{2}(2 x-9)
$$

- Since the function is a polynomial, it does not have any singular points, but it does have two critical points at $x=0,9 / 2$. These two critical points split the real line into 3 open intervals

$$
(-\infty, 0) \quad(0,9 / 2) \quad(9 / 2, \infty)
$$

We need to determine the sign of the derivative in each intervals.

- When $x<0, x^{2}>0$ but $(2 x-9)<0$, so $f^{\prime}(x)<0$ and the function is decreasing.
- When $0<x<9 / 2, x^{2}>0$ but $(2 x-9)<0$, so $f^{\prime}(x)<0$ and the function is still decreasing.
- When $x>9 / 2, x^{2}>0$ and $(2 x-9)>0$, so $f^{\prime}(x)>0$ and the function is increasing.

We can then summarise this in the following table

| interval | $(-\infty, 0)$ | 0 | $(0,9 / 2)$ | $9 / 2$ | $(9 / 2, \infty)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $f^{\prime}(x)$ | negative | 0 | negative | 0 | positive |
|  | decreasing | horizontal <br> tangent | decreasing | minimum | increasing |

Since the derivative changes sign from negative to positive at the critical point $x=9 / 2$, this point is a minimum. Its $y$-value is

$$
\begin{aligned}
y & =f(9 / 2)=\frac{9^{3}}{2^{3}}\left(\frac{9}{2}-6\right) \\
& =\frac{3^{6}}{2^{3}} \cdot\left(\frac{-3}{2}\right)=-\frac{3^{7}}{2^{4}}
\end{aligned}
$$

On the other hand, at $x=0$ the derivative does not change sign; while this point has a horizontal tangent line it is not a minimum or maximum.

- Putting this information together we arrive at a quite reasonable sketch.


To improve upon this further we will examine the second derivative.
Example 3.6.2

### 3.6.3 Second Derivative - Concavity

The second derivative $f^{\prime \prime}(x)$ tells us the rate at which the derivative changes. Perhaps the easiest way to understand how to interpret the sign of the second derivative is to think about what it implies about the slope of the tangent line to the graph of the function. Consider the following sketches of $y=1+x^{2}$ and $y=-1-x^{2}$.



- In the case of $y=f(x)=1+x^{2}, f^{\prime \prime}(x)=2>0$. Notice that this means the slope, $f^{\prime}(x)$, of the line tangent to the graph at $x$ increases as $x$ increases. Looking at the figure on the left above, we see that the graph always lies above the tangent lines.
- For $y=f(x)=-1-x^{2}, f^{\prime \prime}(x)=-2<0$. The slope, $f^{\prime}(x)$, of the line tangent to
the graph at $x$ decreases as $x$ increases. Looking at the figure on the right above, we see that the graph always lies below the tangent lines.

Similarly consider the following sketches of $y=x^{-1 / 2}$ and $y=\sqrt{4-x}$ :



Both of their derivatives, $-\frac{1}{2} x^{-3 / 2}$ and $-\frac{1}{2}(4-x)^{-1 / 2}$, are negative, so they are decreasing functions. Examining second derivatives shows some differences.

- For the first function, $y^{\prime \prime}(x)=\frac{3}{4} x^{-5 / 2}>0$, so the slopes of tangent lines are increasing with $x$ and the graph lies above its tangent lines.
- However, the second function has $y^{\prime \prime}(x)=-\frac{1}{4}(4-x)^{-3 / 2}<0$ so the slopes of the tangent lines are decreasing with $x$ and the graph lies below its tangent lines.

More generally

## Definition 3.6.3

Let $f(x)$ be a continuous function on the interval $[a, b]$ and suppose its first and second derivatives exist on that interval.

- If $f^{\prime \prime}(x)>0$ for all $a<x<b$, then the graph of $f$ lies above its tangent lines for $a<x<b$ and it is said to be concave up.

- If $f^{\prime \prime}(x)<0$ for all $a<x<b$, then the graph of $f$ lies below its tangent lines for $a<x<b$ and it is said to be concave down.

- If $f^{\prime \prime}(c)=0$ for some $a<c<b$, and the concavity of $f$ changes across $x=c$, then we call $(c, f(c))$ an inflection point.


Note that one might also see the terms

- "convex" or "convex up" used in place of "concave up", and
- "concave" or "convex down" used to mean "concave down".

To avoid confusion we recommend the reader stick with the terms "concave up" and "concave down".

Let's now continue Example 3.6.2 by discussing the concavity of the curve.

## Example 3.6.4 Continuation of 3.6.2.

Consider again the function

$$
f(x)=x^{4}-6 x^{3}
$$

- Its first derivative is $f^{\prime}(x)=4 x^{3}-18 x^{2}$, so

$$
f^{\prime \prime}(x)=12 x^{2}-36 x=12 x(x-3)
$$

- Thus the second derivative is zero (and potentially changes sign) at $x=0,3$. Thus we should consider the sign of the second derivative on the following intervals

$$
(-\infty, 0) \quad(0,3) \quad(3, \infty)
$$

A little algebra gives us

| interval | $(-\infty, 0)$ | 0 | $(0,3)$ | 3 | $(3, \infty)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $f^{\prime \prime}(x)$ | positive | 0 | negative | 0 | positive |
| concavity | up | inflection | down | inflection | up |

Since the concavity changes at both $x=0$ and $x=3$, the following are inflection points

$$
(0,0) \quad\left(3,3^{4}-6 \times 3^{3}\right)=\left(3,-3^{4}\right)
$$

- Putting this together with the information we obtained earlier gives us the following sketch
concave
up
concave
down

Example 3.6.5 Optional - $y=x^{1 / 3}$ and $y=x^{2 / 3}$.
In our Definition 3.6.3, concerning concavity and inflection points, we considered only functions having first and second derivatives on the entire interval of interest. In this example, we will consider the functions

$$
f(x)=x^{1 / 3} \quad g(x)=x^{2 / 3}
$$

We shall see that $x=0$ is a singular point for both of those functions. There is no universal agreement as to precisely when a singular point should also be called an inflection point. We choose to extend our definition of inflection point in Definition 3.6.3 as follows. If

- the function $f(x)$ is defined and continuous on an interval $a<x<b$ and if
- the first and second derivatives $f^{\prime}(x)$ and $f^{\prime \prime}(x)$ exist on $a<x<b$ except possibly at the single point $a<c<b$ and if
- $f$ is concave up on one side of $c$ and is concave down on the other side of $c$
then we say that $(c, f(c))$ is an inflection point of $y=f(x)$. Now let's check out $y=f(x)$ and $y=g(x)$ from this point of view.

1 Features of $y=f(x)$ and $y=g(x)$ that are read off of $f(x)$ and $g(x)$ :

- Since $f(0)=0^{1 / 3}=0$ and $g(0)=0^{2 / 3}=0$, the origin $(0,0)$ lies on both $y=f(x)$ and $y=g(x)$.
- For example, $1^{3}=1$ and $(-1)^{3}=-1$ so that the cube root of 1 is $1^{1 / 3}=1$ and the cube root of -1 is $(-1)^{1 / 3}=-1$. In general,

$$
x^{1 / 3} \begin{cases}<0 & \text { if } x<0 \\ =0 & \text { if } x=0 \\ >0 & \text { if } x>0\end{cases}
$$

Consequently the graph $y=f(x)=x^{1 / 3}$ lies below the $x$-axis when $x<0$ and lies above the $x$-axis when $x>0$. On the other hand, the graph $y=$ $g(x)=x^{2 / 3}=\left[x^{1 / 3}\right]^{2}$ lies on or above the $x$-axis for all $x$.

- As $x \rightarrow+\infty$, both $y=f(x)=x^{1 / 3}$ and $y=g(x)=x^{2 / 3}$ tend to $+\infty$.
- As $x \rightarrow-\infty, y=f(x)=x^{1 / 3}$ tends to $-\infty$ and $y=g(x)=x^{2 / 3}$ tends to $+\infty$.

2 Features of $y=f(x)$ and $y=g(x)$ that are read off of $f^{\prime}(x)$ and $g^{\prime}(x)$ :

$$
\begin{aligned}
& f^{\prime}(x)=\left\{\begin{array}{ll}
\frac{1}{3} x^{-2 / 3} & \text { if } x \neq 0 \\
\text { undefined } & \text { if } x=0
\end{array}\right\} \Longrightarrow f^{\prime}(x)>0 \text { for all } x \neq 0 \\
& g^{\prime}(x)=\left\{\begin{array}{ll}
\frac{2}{3} x^{-1 / 3} & \text { if } x \neq 0 \\
\text { undefined } & \text { if } x=0
\end{array}\right\} \Longrightarrow g^{\prime}(x) \begin{cases}<0 & \text { if } x<0 \\
>0 & \text { if } x>0\end{cases}
\end{aligned}
$$

So the graph $y=f(x)$ is increasing on both sides of the singular point $x=0$, while the graph $y=g(x)$ is decreasing to the left of $x=0$ and is increasing to the right of $x=0$. As $x \rightarrow 0, f^{\prime}(x)$ and $g^{\prime}(x)$ become infinite. That is, the slopes of the tangent lines at $(x, f(x))$ and $(x, g(x))$ become infinite and the tangent lines become vertical.

3 Features of $y=f(x)$ and $y=g(x)$ that are read off of $f^{\prime \prime}(x)$ and $g^{\prime \prime}(x)$ :

$$
f^{\prime \prime}(x)=\left\{\begin{array}{ll}
-\frac{2}{9} x^{-5 / 3}=-\frac{2}{9}\left[x^{-1 / 3}\right]^{5} & \text { if } x \neq 0 \\
\text { undefined } & \text { if } x=0
\end{array}\right\}
$$

$$
\begin{aligned}
& \Longrightarrow f^{\prime \prime}(x) \begin{cases}>0 & \text { if } x<0 \\
<0 & \text { if } x>0\end{cases} \\
g^{\prime \prime}(x) & =\left\{\begin{array}{ll}
-\frac{2}{9} x^{-4 / 3}=-\frac{2}{9}\left[x^{-1 / 3}\right]^{4} & \text { if } x \neq 0 \\
\text { undefined } & \text { if } x=0
\end{array}\right\} \\
& \Longrightarrow g^{\prime \prime}(x)<0 \text { for all } x \neq 0
\end{aligned}
$$

So the graph $y=g(x)$ is concave down on both sides of the singular point $x=0$, while the graph $y=f(x)$ is concave up to the left of $x=0$ and is concave down to the right of $x=0$.

By way of summary, we have, for $f(x)$,

|  | $(-\infty, 0)$ | 0 | $(0, \infty)$ |
| :--- | :--- | :--- | :--- |
| $f(x)$ | negative | 0 | positive |
| $f^{\prime}(x)$ | positive | undefined | positive |
|  | increasing |  | increasing |
| $f^{\prime \prime}(x)$ | positive | undefined | negative |
|  | concave up | inflection | concave down |

and for $g(x)$,

|  | $(-\infty, 0)$ | 0 | $(0, \infty)$ |
| :--- | :--- | :--- | :--- |
| $g(x)$ | positive | 0 | positive |
| $g^{\prime}(x)$ | negative | undefined | positive |
|  | decreasing |  | increasing |
| $g^{\prime \prime}(x)$ | negative | undefined | negative |
|  | concave down | inflection | concave down |

Since the concavity changes at $x=0$ for $y=f(x)$, but not for $y=g(x),(0,0)$ is an inflection point for $y=f(x)$, but not for $y=g(x)$. We have the following sketch for $y=f(x)=x^{1 / 3}$

and the following sketch for $y=g(x)=x^{2 / 3}$.


Note that the curve $y=f(x)=x^{1 / 3}$ looks perfectly smooth, even though $f^{\prime}(x) \rightarrow \infty$ as $x \rightarrow 0$. There is no kink or discontinuity at $(0,0)$. The singularity at $x=0$ has caused the $y$-axis to be a vertical tangent to the curve, but has not prevented the curve from looking smooth.

Example 3.6.5

### 3.6.4 Symmetries

Before we proceed to some examples, we should examine some simple symmetries possessed by some functions. We'll look at three symmetries - evenness, oddness and periodicity. If a function possesses one of these symmetries then it can be exploited to reduce the amount of work required to sketch the graph of the function.

Let us start with even and odd functions.

## Definition 3.6.6

A function $f(x)$ is said to be even if $f(-x)=f(x)$ for all $x$.

## Definition 3.6.7

A function $f(x)$ is said to be odd if $f(-x)=-f(x)$ for all $x$.

Example 3.6.8 An even function and an odd funtion.
Let $f(x)=x^{2}$ and $g(x)=x^{3}$. Then

$$
\begin{aligned}
& f(-x)=(-x)^{2}=x^{2}=f(x) \\
& g(-x)=(-x)^{3}=-x^{3}=-g(x)
\end{aligned}
$$

Hence $f(x)$ is even and $g(x)$ is odd.
Notice any polynomial involving only even powers of $x$ will be even

$$
\begin{array}{rlrl}
f(x) & =7 x^{6}+2 x^{4}-3 x^{2}+5 & \text { remember that } 5=5 x^{0} \\
f(-x) & =7(-x)^{6}+2(-x)^{4}-3(-x)^{2}+5 & & \\
& =7 x^{6}+2 x^{4}-3 x^{2}+5=f(x) & &
\end{array}
$$

Similarly any polynomial involving only odd powers of $x$ will be odd

$$
\begin{aligned}
g(x) & =2 x^{5}-8 x^{3}-3 x \\
g(-x) & =2(-x)^{5}-8(-x)^{3}-3(-x) \\
& =-2 x^{5}+8 x^{3}+3 x=-g(x)
\end{aligned}
$$

Not all even and odd functions are polynomials. For example

$$
|x| \quad \cos x \quad \text { and }\left(e^{x}+e^{-x}\right)
$$

are all even, while

$$
\sin x \quad \tan x \quad \text { and }\left(e^{x}-e^{-x}\right)
$$

are all odd. Indeed, given any function $f(x)$, the function

$$
\begin{array}{lr}
g(x)=f(x)+f(-x) & \text { will be even, and } \\
h(x)=f(x)-f(-x) & \text { will be odd. }
\end{array}
$$

Now let us see how we can make use of these symmetries to make graph sketching easier. Let $f(x)$ be an even function. Then
the point $\left(x_{0}, y_{0}\right)$ lies on the graph of $y=f(x)$
if and only if $y_{0}=f\left(x_{0}\right)=f\left(-x_{0}\right)$ which is the case if and only if
the point $\left(-x_{0}, y_{0}\right)$ lies on the graph of $y=f(x)$.


Notice that the points $\left(x_{0}, y_{0}\right)$ and $\left(-x_{0}, y_{0}\right)$ are just reflections of each other across the $y$-axis. Consequently, to draw the graph $y=f(x)$, it suffices to draw the part of the graph with $x \geq 0$ and then reflect it in the $y$-axis. Here is an example. The part with $x \geq 0$ is on the left and the full graph is on the right.


Very similarly, when $f(x)$ is an odd function then

$$
\left(x_{0}, y_{0}\right) \text { lies on the graph of } y=f(x)
$$

if and only if
$\left(-x_{0},-y_{0}\right)$ lies on the graph of $y=f(x)$


Now the symmetry is a little harder to interpret pictorially. To get from $\left(x_{0}, y_{0}\right)$ to $\left(-x_{0},-y_{0}\right)$ one can first reflect $\left(x_{0}, y_{0}\right)$ in the $y$-axis to get to $\left(-x_{0}, y_{0}\right)$ and then reflect the result in the $x$-axis to get to $\left(-x_{0},-y_{0}\right)$. Consequently, to draw the graph $y=f(x)$, it suffices to draw the part of the graph with $x \geq 0$ and then reflect it first in the $y$-axis and then in the $x$-axis. Here is an example. First, here is the part of the graph with $x \geq 0$.


Next, as an intermediate step (usually done in our heads rather than on paper), we add in the reflection in the $y$-axis.


Finally to get the full graph, we reflect the dashed line in the $x$-axis

and then remove the dashed line.


Let's do a more substantial example of an even function
Example 3.6.9 An even rational function.
Consider the function

$$
g(x)=\frac{x^{2}-9}{x^{2}+3}
$$

- The function is even since

$$
g(-x)=\frac{(-x)^{2}-9}{(-x)^{2}+3}=\frac{x^{2}-9}{x^{2}+3}=g(x)
$$

Thus it suffices to study the function for $x \geq 0$ because we can then use the even symmetry to understand what happens for $x<0$.

- The function is defined on all real numbers since its denominator $x^{2}+3$ is never zero. Hence it has no vertical asymptotes.
- The $y$-intercept is $g(0)=\frac{-9}{3}=-3$. And $x$-intercepts are given by the solution of $x^{2}-9=0$, namely $x= \pm 3$. Note that we only need to establish $x=3$ as an intercept. Then since $g$ is even, we know that $x=-3$ is also an intercept.
- To find the horizontal asymptotes we compute the limit as $x \rightarrow+\infty$

$$
\begin{aligned}
\lim _{x \rightarrow \infty} g(x) & =\lim _{x \rightarrow \infty} \frac{x^{2}-9}{x^{2}+3} \\
& =\lim _{x \rightarrow \infty} \frac{x^{2}\left(1-9 / x^{2}\right)}{x^{2}\left(1+3 / x^{2}\right)} \\
& =\lim _{x \rightarrow \infty} \frac{1-9 / x^{2}}{1+3 / x^{2}}=1
\end{aligned}
$$

Thus $y=1$ is a horizontal asymptote. Indeed, this is also the asymptote as $x \rightarrow-\infty$ since by the even symmetry

$$
\lim _{x \rightarrow-\infty} g(x)=\lim _{x \rightarrow \infty} g(-x)=\lim _{x \rightarrow \infty} g(x) .
$$

- We can already produce a quite reasonable sketch just by putting in the horizontal asymptote and the intercepts and drawing a smooth curve between them.


Note that we have drawn the function as never crossing the asymptote $y=1$, however we have not yet proved that. We could by trying to solve $g(x)=1$.

$$
\begin{aligned}
& \frac{x^{2}-9}{x^{2}+3}=1 \\
& x^{2}-9=x^{2}+3 \\
& -9=3 \text { so no solutions. }
\end{aligned}
$$

Alternatively we could analyse the first derivative to see how the function approaches the asymptote.

- Now we turn to the first derivative:

$$
\begin{aligned}
g^{\prime}(x) & =\frac{\left(x^{2}+3\right)(2 x)-\left(x^{2}-9\right)(2 x)}{\left(x^{2}+3\right)^{2}} \\
& =\frac{24 x}{\left(x^{2}+3\right)^{2}}
\end{aligned}
$$

There are no singular points since the denominator is nowhere zero. The only critical point is at $x=0$. Thus we must find the sign of $g^{\prime}(x)$ on the intervals

$$
(-\infty, 0) \quad(0, \infty)
$$

- When $x>0,24 x>0$ and $\left(x^{2}+3\right)>0$, so $g^{\prime}(x)>0$ and the function is increasing. By even symmetry we know that when $x<0$ the function must be decreasing. Hence the critical point $x=0$ is a local minimum of the function.
- Notice that since the function is increasing for $x>0$ and the function must approach the horizontal asymptote $y=1$ from below. Thus the sketch above is quite accurate.
- Now consider the second derivative:

$$
\begin{aligned}
g^{\prime \prime}(x) & =\frac{\mathrm{d}}{\mathrm{~d} x} \frac{24 x}{\left(x^{2}+3\right)^{2}} \\
& =\frac{\left(x^{2}+3\right)^{2} \cdot 24-24 x \cdot 2\left(x^{2}+3\right) \cdot 2 x}{\left(x^{2}+3\right)^{4}}
\end{aligned}
$$

cancel a factor of $\left(x^{2}+3\right)$

$$
\begin{aligned}
& =\frac{\left(x^{2}+3\right) \cdot 24-96 x^{2}}{\left(x^{2}+3\right)^{3}} \\
& =\frac{72\left(1-x^{2}\right)}{\left(x^{2}+3\right)^{3}}
\end{aligned}
$$

- It is clear that $g^{\prime \prime}(x)=0$ when $x= \pm 1$. Note that, again, we can infer the zero at $x=-1$ from the zero at $x=1$ by the even symmetry. Thus we need to examine the sign of $g^{\prime \prime}(x)$ the intervals

$$
(-\infty,-1) \quad(-1,1) \quad(1, \infty)
$$

- When $|x|<1$ we have $\left(1-x^{2}\right)>0$ so that $g^{\prime \prime}(x)>0$ and the function is concave up. When $|x|>1$ we have $\left(1-x^{2}\right)<0$ so that $g^{\prime \prime}(x)<0$ and the function is concave down. Thus the points $x= \pm 1$ are inflection points. Their coordinates are $( \pm 1, g( \pm 1))=( \pm 1,-2)$.
- Putting this together gives the following sketch:
$(-1,-2) \cdot(1,-2)$
concave
down
$\qquad$ Example 3.6.9
Another symmetry we should consider is periodicity.


## Definition 3.6.10

A function $f(x)$ is said to be periodic, with period $P>0$, if $f(x+P)=f(x)$ for all $x$.

Note that if $f(x+P)=f(x)$ for all $x$, then replacing $x$ by $x+P$, we have

$$
f(x+2 P)=f(x+P+P)=f(x+P)=f(x)
$$

More generally $f(x+k P)=f(x)$ for all integers $k$. Thus if $f$ has period $P$, then it also has period $n P$ for all natural numbers $n$. The smallest period is called the fundamental period.

Example 3.6.11 $\sin x$ is periodic.
The classic example of a periodic function is $f(x)=\sin x$, which has period $2 \pi$ since $f(x+2 \pi)=\sin (x+2 \pi)=\sin x=f(x)$.

If $f(x)$ has period $P$ then

$$
\left(x_{0}, y_{0}\right) \text { lies on the graph of } y=f(x)
$$

if and only if $y_{0}=f\left(x_{0}\right)=f\left(x_{0}+P\right)$ which is the case if and only if

$$
\left(x_{0}+P, y_{0}\right) \text { lies on the graph of } y=f(x)
$$

and, more generally,

$$
\left(x_{0}, y_{0}\right) \text { lies on the graph of } y=f(x)
$$

if and only if

$$
\left(x_{0}+n P, y_{0}\right) \text { lies on the graph of } y=f(x)
$$

for all integers $n$.
Note that the point $\left(x_{0}+P, y_{0}\right)$ can be obtained by translating $\left(x_{0}, y_{0}\right)$ horizontally by $P$. Similarly the point $\left(x_{0}+n P, y_{0}\right)$ can be found by repeatedly translating $\left(x_{0}, y_{0}\right)$ horizontally by $P$.
$\left.\begin{array}{l|lll}\begin{array}{l}\left(x_{0}-P, y_{0}\right) \\ \bullet\end{array} & y_{0}- & \left(x_{0}, y_{0}\right) & \left(x_{0}+P, y_{0}\right)\end{array}\right)\left(x_{0}+2 P, y_{0}\right)$

Consequently, to draw the graph $y=f(x)$, it suffices to draw one period of the graph, say the part with $0 \leq x \leq P$, and then translate it repeatedly. Here is an example. Here is a sketch of one period

and here is the full sketch.
(

### 3.6.5 A Checklist for Sketching

Above we have described how we can use our accumulated knowledge of derivatives to quickly identify the most important qualitative features of graphs $y=f(x)$. Here we give the reader a quick checklist of things to examine in order to produce an accurate sketch based on properties that are easily read off from $f(x), f^{\prime}(x)$ and $f^{\prime \prime}(x)$.

### 3.6.5.1 A Sketching Checklist

1 Features of $y=f(x)$ that are read off of $f(x)$ :

- First check where $f(x)$ is defined. Then
- $y=f(x)$ is plotted only for $x$ 's in the domain of $f(x)$, i.e. where $f(x)$ is defined.
- $y=f(x)$ has vertical asymptotes at the points where $f(x)$ blows up to $\pm \infty$.
- Next determine whether the function is even, odd, or periodic.
- $y=f(x)$ is first plotted for $x \geq 0$ if the function is even or odd. The rest of the sketch is then created by reflections.
- $y=f(x)$ is first plotted for a single period if the function is periodic. The rest of the sketch is then created by translations.
- Next compute $f(0), \lim _{x \rightarrow \infty} f(x)$ and $\lim _{x \rightarrow-\infty} f(x)$ and look for solutions to $f(x)=0$ that you can easily find. Then
- $y=f(x)$ has $y$-intercept $(0, f(0))$.
- $y=f(x)$ has $x$-intercept $(a, 0)$ whenever $f(a)=0$
- $y=f(x)$ has horizontal asymptote $y=Y$ if $\lim _{x \rightarrow \infty} f(x)=L$ or $\lim _{x \rightarrow-\infty} f(x)=$ $L$.

2 Features of $y=f(x)$ that are read off of $f^{\prime}(x)$ :

- Compute $f^{\prime}(x)$ and determine its critical points and singular points, then
- $y=f(x)$ has a horizontal tangent at the points where $f^{\prime}(x)=0$.
- $y=f(x)$ is increasing at points where $f^{\prime}(x)>0$.
- $y=f(x)$ is decreasing at points where $f^{\prime}(x)<0$.
- $y=f(x)$ has vertical tangents or vertical asymptotes at the points where $f^{\prime}(x)= \pm \infty$.

3 Features of $y=f(x)$ that are read off of $f^{\prime \prime}(x)$ :

- Compute $f^{\prime \prime}(x)$ and determine where $f^{\prime \prime}(x)=0$ or does not exist, then
- $y=f(x)$ is concave up at points where $f^{\prime \prime}(x)>0$.
- $y=f(x)$ is concave down at points where $f^{\prime \prime}(x)<0$.
- $y=f(x)$ may or may not have inflection points where $f^{\prime \prime}(x)=0$.


### 3.6.6 ~ Sketching Examples

Example 3.6.12 Sketch $f(x)=x^{3}-3 x+1$.

1 Reading from $f(x)$ :

- The function is a polynomial so it is defined everywhere.
- Since $f(-x)=-x^{3}+3 x+1 \neq \pm f(x)$, it is not even or odd. Nor is it periodic.
- The $y$-intercept is $y=1$. The $x$-intercepts are not easily computed since it is a cubic polynomial that does not factor nicely ${ }^{a}$. So for this example we don't worry about finding them.
- Since it is a polynomial it has no vertical asymptotes.
- For very large $x$, both positive and negative, the $x^{3}$ term in $f(x)$ dominates the other two terms so that

$$
f(x) \rightarrow \begin{cases}+\infty & \text { as } x \rightarrow+\infty \\ -\infty & \text { as } x \rightarrow-\infty\end{cases}
$$

and there are no horizontal asymptotes.

2 We now compute the derivative:

$$
f^{\prime}(x)=3 x^{2}-3=3\left(x^{2}-1\right)=3(x+1)(x-1)
$$

- The critical points (where $f^{\prime}(x)=0$ ) are at $x= \pm 1$. Further since the derivative is a polynomial it is defined everywhere and there are no singular points. The critical points split the real line into the intervals $(-\infty,-1),(-1,1)$ and $(1, \infty)$.
- When $x<-1$, both factors $(x+1),(x-1)<0$ so $f^{\prime}(x)>0$.
- Similarly when $x>1$, both factors $(x+1),(x-1)>0$ so $f^{\prime}(x)>0$.
- When $-1<x<1,(x-1)<0$ but $(x+1)>0$ so $f^{\prime}(x)<0$.
- Summarising all this

|  | $(-\infty,-1)$ | -1 | $(-1,1)$ | 1 | $(1, \infty)$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | positive | 0 | negative | 0 | positive |
|  | increasing | maximum | decreasing | minimum | increasing |

So $(-1, f(-1))=(-1,3)$ is a local maximum and $(1, f(1))=(1,-1)$ is a local minimum.

3 Compute the second derivative:

$$
f^{\prime \prime}(x)=6 x
$$

- The second derivative is zero when $x=0$, and the problem is quite easy to analyse. Clearly, $f^{\prime \prime}(x)<0$ when $x<0$ and $f^{\prime \prime}(x)>0$ when $x>0$.
- Thus $f$ is concave down for $x<0$, concave up for $x>0$ and has an inflection point at $x=0$.

Putting this all together gives:

$a$ With the aid of a computer we can find the $x$-intercepts numerically: $x \approx$ $-1.879385242,0.3472963553$, and 1.532088886 . If you are interested in more details check out Appendix C.

Example 3.6.12

Example 3.6.13 Sketch $f(x)=x^{4}-4 x^{3}$.

1 Reading from $f(x)$ :

- The function is a polynomial so it is defined everywhere.
- Since $f(-x)=x^{4}+4 x^{3} \neq \pm f(x)$, it is not even or odd. Nor is it periodic.
- The $y$-intercept is $y=f(0)=0$, while the $x$-intercepts are given by the solution of

$$
\begin{array}{r}
f(x)=x^{4}-4 x^{3}=0 \\
x^{3}(x-4)=0
\end{array}
$$

Hence the $x$-intercepts are 0,4 .

- Since $f$ is a polynomial it does not have any vertical asymptotes.
- For very large $x$, both positive and negative, the $x^{4}$ term in $f(x)$ dominates
the other term so that

$$
f(x) \rightarrow \begin{cases}+\infty & \text { as } x \rightarrow+\infty \\ +\infty & \text { as } x \rightarrow-\infty\end{cases}
$$

and the function has no horizontal asymptotes.
2 Now compute the derivative $f^{\prime}(x)$ :

$$
f^{\prime}(x)=4 x^{3}-12 x^{2}=4(x-3) x^{2}
$$

- The critical points are at $x=0,3$. Since the function is a polynomial there are no singular points. The critical points split the real line into the intervals $(-\infty, 0),(0,3)$ and $(3, \infty)$.
- When $x<0, x^{2}>0$ and $x-3<0$, so $f^{\prime}(x)<0$.
- When $0<x<3, x^{2}>0$ and $x-3<0$, so $f^{\prime}(x)<0$.
- When $3<x, x^{2}>0$ and $x-3>0$, so $f^{\prime}(x)>0$.
- Summarising all this

|  | $(-\infty, 0)$ | 0 | $(0,3)$ | 3 | $(3, \infty)$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | negative | 0 | negative | 0 | positive |
|  | decreasing | horizontal <br> tangent | decreasing | minimum | increasing |

So the point $(3, f(3))=(3,-27)$ is a local minimum. The point $(0, f(0))=$ $(0,0)$ is neither a minimum nor a maximum, even though $f^{\prime}(0)=0$.

3 Now examine $f^{\prime \prime}(x)$ :

$$
f^{\prime \prime}(x)=12 x^{2}-24 x=12 x(x-2)
$$

- So $f^{\prime \prime}(x)=0$ when $x=0,2$. This splits the real line into the intervals $(-\infty, 0),(0,2)$ and $(2, \infty)$.
- When $x<0, x-2<0$ and so $f^{\prime \prime}(x)>0$.
- When $0<x<2, x>0$ and $x-2<0$ and so $f^{\prime \prime}(x)<0$.
- When $2<x, x>0$ and $x-2>0$ and so $f^{\prime \prime}(x)>0$.
- Thus the function is convex up for $x<0$, then convex down for $0<x<$ 2 , and finally convex up again for $x>2$. Hence $(0, f(0))=(0,0)$ and $(2, f(2))=(2,-16)$ are inflection points.

Putting all this information together gives us the following sketch.

$\uparrow$ Example 3.6.13

Example 3.6.14 $f(x)=x^{3}-6 x^{2}+9 x-54$.

1 Reading from $f(x)$ :

- The function is a polynomial so it is defined everywhere.
- Since $f(-x)=-x^{3}-6 x^{2}-9 x-54 \neq \pm f(x)$, it is not even or odd. Nor is it periodic.
- The $y$-intercept is $y=f(0)=-54$, while the $x$-intercepts are given by the solution of

$$
\begin{aligned}
f(x)=x^{3}-6 x^{2}+9 x-54 & =0 \\
x^{2}(x-6)+9(x-6) & =0 \\
\left(x^{2}+9\right)(x-6) & =0
\end{aligned}
$$

Hence the only $x$-intercept is 6 .

- Since $f$ is a polynomial it does not have any vertical asymptotes.
- For very large $x$, both positive and negative, the $x^{3}$ term in $f(x)$ dominates the other term so that

$$
f(x) \rightarrow \begin{cases}+\infty & \text { as } x \rightarrow+\infty \\ -\infty & \text { as } x \rightarrow-\infty\end{cases}
$$

and the function has no horizontal asymptotes.

2 Now compute the derivative $f^{\prime}(x)$ :

$$
\begin{aligned}
f^{\prime}(x) & =3 x^{2}-12 x+9 \\
& =3\left(x^{2}-4 x+3\right)=3(x-3)(x-1)
\end{aligned}
$$

- The critical points are at $x=1,3$. Since the function is a polynomial there are no singular points. The critical points split the real line into the intervals $(-\infty, 1),(1,3)$ and $(3, \infty)$.
- When $x<1,(x-1)<0$ and $(x-3)<0$, so $f^{\prime}(x)>0$.
- When $1<x<3,(x-1)>0$ and $(x-3)<0$, so $f^{\prime}(x)<0$.
- When $3<x,(x-1)>0$ and $(x-3)>0$, so $f^{\prime}(x)>0$.
- Summarising all this

|  | $(-\infty, 1)$ | 1 | $(1,3)$ | 3 | $(3, \infty)$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | positive | 0 | negative | 0 | positive |
|  | increasing | maximum | decreasing | minimum | increasing |

So the point $(1, f(1))=(1,-50)$ is a local maximum. The point $(3, f(3))=$ $(3,-54)$ is a local minimum.

3 Now examine $f^{\prime \prime}(x)$ :

$$
f^{\prime \prime}(x)=6 x-12
$$

- So $f^{\prime \prime}(x)=0$ when $x=2$. This splits the real line into the intervals $(-\infty, 2)$ and $(2, \infty)$.
- When $x<2, f^{\prime \prime}(x)<0$.
- When $x>2, f^{\prime \prime}(x)>0$.
- Thus the function is convex down for $x<2$, then convex up for $x>2$. Hence $(2, f(2))=(2,-52)$ is an inflection point.

Putting all this information together gives us the following sketch.

and if we zoom in around the interesting points (minimum, maximum and inflection point), we have


An example of sketching a simple rational function.
Example 3.6.15 $f(x)=\frac{x}{x^{2}-4}$.

1 Reading from $f(x)$ :

- The function is rational so it is defined except where its denominator is zero
- namely at $x= \pm 2$.
- Since $f(-x)=\frac{-x}{x^{2}-4}=-f(x)$, it is odd. Indeed this means that we only need to examine what happens to the function for $x \geq 0$ and we can then infer what happens for $x \leq 0$ using $f(-x)=-f(x)$. In practice we will sketch the graph for $x \geq 0$ and then infer the rest from this symmetry.
- The $y$-intercept is $y=f(0)=0$, while the $x$-intercepts are given by the solution of $f(x)=0$. So the only $x$-intercept is 0 .
- Since $f$ is rational, it may have vertical asymptotes where its denominator is zero - at $x= \pm 2$. Since the function is odd, we only have to analyse the asymptote at $x=2$ and we can then infer what happens at $x=-2$ by symmetry.

$$
\begin{aligned}
& \lim _{x \rightarrow 2^{+}} f(x)=\lim _{x \rightarrow 2^{+}} \frac{x}{(x-2)(x+2)}=+\infty \\
& \lim _{x \rightarrow 2^{-}} f(x)=\lim _{x \rightarrow 2^{-}} \frac{x}{(x-2)(x+2)}=-\infty
\end{aligned}
$$

- We now check for horizontal asymptotes:

$$
\begin{aligned}
\lim _{x \rightarrow+\infty} f(x) & =\lim _{x \rightarrow+\infty} \frac{x}{x^{2}-4} \\
& =\lim _{x \rightarrow+\infty} \frac{1}{x-4 / x}=0
\end{aligned}
$$

2 Now compute the derivative $f^{\prime}(x)$ :

$$
\begin{aligned}
f^{\prime}(x) & =\frac{\left(x^{2}-4\right) \cdot 1-x \cdot 2 x}{\left(x^{2}-4\right)^{2}} \\
& =\frac{-\left(x^{2}+4\right)}{\left(x^{2}-4\right)^{2}}
\end{aligned}
$$

- Hence there are no critical points. There are singular points where the denominator is zero, namely $x= \pm 2$. Before we proceed, notice that the numerator is always negative and the denominator is always positive. Hence $f^{\prime}(x)<0$ except at $x= \pm 2$ where it is undefined.
- The function is decreasing except at $x= \pm 2$.
- We already know that at $x=2$ we have a vertical asymptote and that $f^{\prime}(x)<0$ for all $x$. So

$$
\lim _{x \rightarrow 2} f^{\prime}(x)=-\infty
$$

- Summarising all this

|  | $[0,2)$ | 2 | $(2, \infty)$ |
| :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | negative | DNE | negative |
|  | decreasing | vertical <br> asymptote | decreasing |

Remember - we will draw the graph for $x \geq 0$ and then use the odd symmetry to infer the graph for $x<0$.

3 Now examine $f^{\prime \prime}(x)$ :

$$
\begin{aligned}
f^{\prime \prime}(x) & =-\frac{\left(x^{2}-4\right)^{2} \cdot(2 x)-\left(x^{2}+4\right) \cdot 2 \cdot 2 x \cdot\left(x^{2}-4\right)}{\left(x^{2}-4\right)^{4}} \\
& =-\frac{\left(x^{2}-4\right) \cdot(2 x)-\left(x^{2}+4\right) \cdot 4 x}{\left(x^{2}-4\right)^{3}} \\
& =-\frac{2 x^{3}-8 x-4 x^{3}-16 x}{\left(x^{2}-4\right)^{3}} \\
& =\frac{2 x\left(x^{2}+12\right)}{\left(x^{2}-4\right)^{3}}
\end{aligned}
$$

- So $f^{\prime \prime}(x)=0$ when $x=0$ and does not exist when $x= \pm 2$. This splits the real line into the intervals $(-\infty,-2),(-2,0),(0,2)$ and $(2, \infty)$. However we only need to consider $x \geq 0$ (because of the odd symmetry).
- When $0<x<2, x>0,\left(x^{2}+12\right)>0$ and $\left(x^{2}-4\right)<0$ so $f^{\prime \prime}(x)<0$.
- When $x>2, x>0,\left(x^{2}+12\right)>0$ and $\left(x^{2}-4\right)>0$ so $f^{\prime \prime}(x)>0$.

Putting all this information together gives the following sketch for $x \geq 0$ :


We can then draw in the graph for $x<0$ using $f(-x)=-f(x)$ :


Notice that this means that the concavity changes at $x=0$, so the point $(0, f(0))=$ $(0,0)$ is a point of inflection (as indicated).
$\begin{array}{ll}\text { 亿 } & \text { Example 3.6.15 }\end{array}$
This final example is more substantial since the function has singular points (points where the derivative is undefined). The analysis is more involved.

Example 3.6.16 $f(x)=\sqrt[3]{\frac{x^{2}}{(x-6)^{2}}}$.

1 Reading from $f(x)$ :

- First notice that we can rewrite

$$
f(x)=\sqrt[3]{\frac{x^{2}}{(x-6)^{2}}}=\sqrt[3]{\frac{x^{2}}{x^{2} \cdot(1-6 / x)^{2}}}=\sqrt[3]{\frac{1}{(1-6 / x)^{2}}}
$$

- The function is the cube root of a rational function. The rational function is defined except at $x=6$, so the domain of $f$ is all reals except $x=6$.
- Clearly the function is not periodic, and examining

$$
\begin{aligned}
f(-x) & =\sqrt[3]{\frac{1}{(1-6 /(-x))^{2}}} \\
& =\sqrt[3]{\frac{1}{(1+6 / x)^{2}}} \neq \pm f(x)
\end{aligned}
$$

shows the function is neither even nor odd.

- To compute horizontal asymptotes we examine the limit of the portion of the function inside the cube-root

$$
\lim _{x \rightarrow \pm \infty} \frac{1}{\left(1-\frac{6}{x}\right)^{2}}=1
$$

This means we have

$$
\lim _{x \rightarrow \pm \infty} f(x)=1
$$

That is, the line $y=1$ will be a horizontal asymptote to the graph $y=f(x)$ both for $x \rightarrow+\infty$ and for $x \rightarrow-\infty$.

- Our function $f(x) \rightarrow+\infty$ as $x \rightarrow 6$, because of the $(1-6 / x)^{2}$ in its denominator. So $y=f(x)$ has $x=6$ as a vertical asymptote.

2 Now compute $f^{\prime}(x)$. Since we rewrote

$$
f(x)=\sqrt[3]{\frac{1}{(1-6 / x)^{2}}}=\left(1-\frac{6}{x}\right)^{-\frac{2}{3}}
$$

we can use the chain rule

$$
\begin{aligned}
f^{\prime}(x) & =-\frac{2}{3}\left(1-\frac{6}{x}\right)^{-\frac{5}{3}} \frac{6}{x^{2}} \\
& =-4\left(\frac{x-6}{x}\right)^{-\frac{5}{3}} \frac{1}{x^{2}} \\
& =-4\left(\frac{1}{x-6}\right)^{\frac{5}{3}} \frac{1}{x^{\frac{1}{3}}}
\end{aligned}
$$

- Notice that the derivative is nowhere equal to zero, so the function has no critical points. However there are two places the derivative is undefined. The terms

$$
\left(\frac{1}{x-6}\right)^{\frac{5}{3}} \quad \frac{1}{x^{\frac{1}{3}}}
$$

are undefined at $x=6,0$ respectively. Hence $x=0,6$ are singular points. These split the real line into the intervals $(-\infty, 0),(0,6)$ and $(6, \infty)$.

- When $x<0,(x-6)<0$, we have that $(x-6)^{-\frac{5}{3}}<0$ and $x^{-\frac{1}{3}}<0$ and so $f^{\prime}(x)=-4 \cdot($ negative $) \cdot($ negative $)<0$.
- When $0<x<6,(x-6)<0$, we have that $(x-6)^{-\frac{5}{3}}<0$ and $x^{-\frac{1}{3}}>0$ and so $f^{\prime}(x)>0$.
- When $x>6,(x-6)>0$, we have that $(x-6)^{-\frac{5}{3}}>0$ and $x^{-\frac{1}{3}}>0$ and so $f^{\prime}(x)<0$.
- We should also examine the behaviour of the derivative as $x \rightarrow 0$ and $x \rightarrow 6$.

$$
\begin{aligned}
& \lim _{x \rightarrow 0^{-}} f^{\prime}(x)=-4\left(\lim _{x \rightarrow 0^{-}}(x-6)^{-\frac{5}{3}}\right)\left(\lim _{x \rightarrow 0^{-}} x^{-\frac{1}{3}}\right)=-\infty \\
& \lim _{x \rightarrow 0^{+}} f^{\prime}(x)=-4\left(\lim _{x \rightarrow 0^{+}}(x-6)^{-\frac{5}{3}}\right)\left(\lim _{x \rightarrow 0^{+}} x^{-\frac{1}{3}}\right)=+\infty \\
& \lim _{x \rightarrow 6^{-}} f^{\prime}(x)=-4\left(\lim _{x \rightarrow 6^{-}}(x-6)^{-\frac{5}{3}}\right)\left(\lim _{x \rightarrow 6^{-}} x^{-\frac{1}{3}}\right)=+\infty \\
& \lim _{x \rightarrow 6^{+}} f^{\prime}(x)=-4\left(\lim _{x \rightarrow 6^{+}}(x-6)^{-\frac{5}{3}}\right)\left(\lim _{x \rightarrow 6^{+}} x^{-\frac{1}{3}}\right)=-\infty
\end{aligned}
$$

We already know that $x=6$ is a vertical asymptote of the function, so it is not surprising that the lines tangent to the graph become vertical as we approach 6 . The behavior around $x=0$ is less standard, since the lines tangent to the graph become vertical, but $x=0$ is not a vertical asymptote of the function. Indeed the function takes a finite value $y=f(0)=0$.

- Summarising all this

|  | $(-\infty, 0)$ | 0 | $(0,6)$ | 6 | $(6, \infty)$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | negative | DNE | positive | DNE | negative |
|  | decreasing | vertical <br> tangents | increasing | vertical <br> asymptote | decreasing |

3 Now look at $f^{\prime \prime}(x)$ :

$$
\begin{aligned}
f^{\prime \prime}(x) & =-4 \frac{\mathrm{~d}}{\mathrm{~d} x}\left[\left(\frac{1}{x-6}\right)^{\frac{5}{3}} \frac{1}{x^{\frac{1}{3}}}\right] \\
& =-4\left[-\frac{5}{3}\left(\frac{1}{x-6}\right)^{\frac{8}{3}} \frac{1}{x^{\frac{1}{3}}}-\frac{1}{3}\left(\frac{1}{x-6}\right)^{\frac{5}{3}} \frac{1}{x^{\frac{4}{3}}}\right] \\
& =\frac{4}{3}\left(\frac{1}{x-6}\right)^{\frac{8}{3}} \frac{1}{x^{\frac{4}{3}}}[5 x+(x-6)] \\
& =8\left(\frac{1}{x-6}\right)^{\frac{8}{3}} \frac{1}{x^{\frac{4}{3}}}[x-1]
\end{aligned}
$$

Oof!

- Both of the factors $\left(\frac{1}{x-6}\right)^{\frac{8}{3}}=\left(\frac{1}{\sqrt[3]{x-6}}\right)^{8}$ and $\frac{1}{x^{\frac{4}{3}}}=\left(\frac{1}{\sqrt[3]{x}}\right)^{4}$ are even powers and so are positive (though possibly infinite). So the sign of $f^{\prime \prime}(x)$ is the same as the sign of the factor $x-1$. Thus

|  | $(-\infty, 1)$ | 1 | $(1, \infty)$ |
| :--- | :--- | :--- | :--- |
| $f^{\prime \prime}(x)$ | negative | 0 | positive |
|  | concave down | inflection <br> point | concave up |

Here is a sketch of the graph $y=f(x)$.


It is hard to see the inflection point at $x=1, y=f(1)=\frac{1}{\sqrt[3]{25}}$ in the above sketch. So here is a blow up of the part of the sketch around $x=1$.


And if we zoom in even more we have


### 3.6.7 $\leadsto$ Exercises

$\rightarrow$ Exercises for § 3.6.1

## Exercises - Stage 1

1. Suppose $f(x)$ is a function given by

$$
f(x)=\frac{g(x)}{x^{2}-9}
$$

where $g(x)$ is also a function. True or false: $f(x)$ has a vertical asymptote at $x=-3$.

## Exercises - Stage 2

2. Match the functions $f(x), g(x), h(x)$, and $k(x)$ to the curves $y=A(x)$ through $y=D(x)$.

$$
\begin{array}{ll}
f(x)=\sqrt{x^{2}+1} & g(x)=\sqrt{x^{2}-1} \\
h(x)=\sqrt{x^{2}+4} & k(x)=\sqrt{x^{2}-4}
\end{array}
$$


3. Below is the graph of

$$
y=f(x)=\sqrt{\log ^{2}(x+p)}
$$

a What is $p$ ?
b What is $b$ (marked on the graph)?
c What is the $x$-intercept of $f(x)$ ?
Remember $\log (x+p)$ is the natural logarithm of $x+p, \log _{e}(x+p)$.

4. Find all asymptotes of $f(x)=\frac{x(2 x+1)(x-7)}{3 x^{3}-81}$.
5. Find all asymptotes of $f(x)=10^{3 x-7}$.

## Exercises for § 3.6.2

## Exercises - Stage 1

1. Match each function graphed below to its derivative from the list. (For example, which function on the list corresponds to $A^{\prime}(x)$ ?)
The $y$-axes have been scaled to make the curve's behaviour clear, so the vertical scales differ from graph to graph.
$l(x)=(x-2)^{4}$
$m(x)=(x-2)^{4}(x+2)$
$n(x)=(x-2)^{2}(x+2)^{2}$
$o(x)=(x-2)(x+2)^{3}$
$p(x)=(x+2)^{4}$




## Exercises - Stage 2

2. *. Find the largest open interval on which $f(x)=\frac{e^{x}}{x+3}$ is increasing.
3. *. Find the largest open interval on which $f(x)=\frac{\sqrt{x-1}}{2 x+4}$ is increasing.
4. *. Find the largest open interval on which $f(x)=2 \arctan (x)-\log \left(1+x^{2}\right)$ is increasing.

## Exercises for § 3.6.3

## Exercises - Stage 1

1. On the graph below, mark the intervals where $f^{\prime \prime}(x)>0$ (i.e. $f(x)$ is concave up) and where $f^{\prime \prime}(x)<0$ (i.e. $f(x)$ is concave down).

2. Sketch a curve that is:

- concave up when $|x|>5$,
- concave down when $|x|<5$,
- increasing when $x<0$, and
- decreasing when $x>0$.

3. Suppose $f(x)$ is a function whose second derivative exists and is continuous for all real numbers.
True or false: if $f^{\prime \prime}(3)=0$, then $x=3$ is an inflection point of $f(x)$.
Remark: compare to Question 3.6.7.7

## Exercises - Stage 2

4. *. Find all inflection points for the graph of $f(x)=3 x^{5}-5 x^{4}+13 x$.

Exercises - Stage 3 Questions 3.6.7.5 through 3.6.7.7 ask you to show that certain things are true. Give a clear explanation using concepts and theorems from this textbook.
5. *. Let

$$
f(x)=\frac{x^{5}}{20}+\frac{5 x^{3}}{6}-10 x^{2}+500 x+1000
$$

Show that $f(x)$ has exactly one inflection point.
6. *. Let $f(x)$ be a function whose first two derivatives exist everywhere, and $f^{\prime \prime}(x)>0$ for all $x$.
a Show that $f(x)$ has at most one critical point and that any critical point is an absolute minimum for $f(x)$.
b Show that the maximum value of $f(x)$ on any finite interval occurs at one of the endpoints of the interval.
7. Suppose $f(x)$ is a function whose second derivative exists and is continuous for all real numbers, and $x=3$ is an inflection point of $f(x)$. Use the Intermediate Value Theorem to show that $f^{\prime \prime}(3)=0$. Remark: compare to Question 3.6.7.3.

Exercises for § 3.6.4

## Exercises - Stage 1

1. What symmetries (even, odd, periodic) does the function graphed below have?

2. What symmetries (even, odd, periodic) does the function graphed below have?

3. Suppose $f(x)$ is an even function defined for all real numbers. Below is the curve $y=f(x)$ when $x>0$. Complete the sketch of the curve.

4. Suppose $f(x)$ is an odd function defined for all real numbers. Below is the curve $y=f(x)$ when $x>0$. Complete the sketch of the curve.


Exercises - Stage 2 In Questions 3.6.7.7 through 3.6.7.10, find the symmetries of a function from its equation.
5.

$$
f(x)=\frac{x^{4}-x^{6}}{e^{x^{2}}}
$$

Show that $f(x)$ is even.
6.

$$
f(x)=\sin (x)+\cos \left(\frac{x}{2}\right)
$$

Show that $f(x)$ is periodic.
7.

$$
f(x)=x^{4}+5 x^{2}+\cos \left(x^{3}\right)
$$

What symmetries (even, odd, periodic) does $f(x)$ have?
8.

$$
f(x)=x^{5}+5 x^{4}
$$

What symmetries (even, odd, periodic) does $f(x)$ have?
9.

$$
f(x)=\tan (\pi x)
$$

What is the period of $f(x)$ ?
Exercises - Stage 3
10.

$$
f(x)=\tan (3 x)+\sin (4 x)
$$

What is the period of $f(x)$ ?

## $\rightarrow$ Exercises for § 3.6.6

Exercises - Stage 1 In Questions 3.6.7.2 through 3.6.7.4, you will sketch the graphs of rational functions.In Questions 3.6.7.6 and 3.6.7.7, you will sketch the graphs of functions with an exponential component. In the next section, you will learn how to find their horizontal asymptotes, but for now these are given to you.In Questions 3.6.7.8 and 3.6.7.9, you will sketch the graphs of functions that have a trigonometric component.

1. *. Let $f(x)=x \sqrt{3-x}$.
a Find the domain of $f(x)$.
b Determine the $x$-coordinates of the local maxima and minima (if any) and intervals where $f(x)$ is increasing or decreasing.
c Determine intervals where $f(x)$ is concave upwards or downwards, and the $x$ coordinates of inflection points (if any). You may use, without verifying it, the formula $f^{\prime \prime}(x)=(3 x-12)(3-x)^{-3 / 2} / 4$.
d There is a point at which the tangent line to the curve $y=f(x)$ is vertical. Find this point.
e Sketch the graph $y=f(x)$, showing the features given in items (a) to (d) above and giving the $(x, y)$ coordinates for all points occurring above.
2. *. Sketch the graph of

$$
f(x)=\frac{x^{3}-2}{x^{4}}
$$

Indicate the critical points, local and absolute maxima and minima, vertical and horizontal asymptotes, inflection points and regions where the curve is concave upward or downward.
3. *. The first and second derivatives of the function $f(x)=\frac{x^{4}}{1+x^{3}}$ are:

$$
f^{\prime}(x)=\frac{4 x^{3}+x^{6}}{\left(1+x^{3}\right)^{2}} \quad \text { and } \quad f^{\prime \prime}(x)=\frac{12 x^{2}-6 x^{5}}{\left(1+x^{3}\right)^{3}}
$$

Graph $f(x)$. Include local and absolute maxima and minima, regions where $f(x)$ is increasing or decreasing, regions where the curve is concave upward or downward, and any asymptotes.
4. *. The first and second derivatives of the function $f(x)=\frac{x^{3}}{1-x^{2}}$ are:

$$
f^{\prime}(x)=\frac{3 x^{2}-x^{4}}{\left(1-x^{2}\right)^{2}} \quad \text { and } \quad f^{\prime \prime}(x)=\frac{6 x+2 x^{3}}{\left(1-x^{2}\right)^{3}}
$$

Graph $f(x)$. Include local and absolute maxima and minima, regions where the curve is concave upward or downward, and any asymptotes.
5. *. The function $f(x)$ is defined by

$$
f(x)= \begin{cases}e^{x} & x<0 \\ \frac{x^{2}+3}{3(x+1)} & x \geq 0\end{cases}
$$

a Explain why $f(x)$ is continuous everywhere.
b Determine all of the following if they are present:
i $x$-coordinates of local maxima and minima, intervals where $f(x)$ is increasing or decreasing;
ii intervals where $f(x)$ is concave upwards or downwards;
iii equations of any horizontal or vertical asymptotes.
c Sketch the graph of $y=f(x)$, giving the $(x, y)$ coordinates for all points of interest above.
6. *. The function $f(x)$ and its derivative are given below:

$$
f(x)=(1+2 x) e^{-x^{2}} \quad \text { and } \quad f^{\prime}(x)=2\left(1-x-2 x^{2}\right) e^{-x^{2}}
$$

Sketch the graph of $f(x)$. Indicate the critical points, local and/or absolute maxima and minima, and asymptotes. Without actually calculating the inflection points, indicate on the graph their approximate location.
Note: $\lim _{x \rightarrow \pm \infty} f(x)=0$.
7. *. Consider the function $f(x)=x e^{-x^{2} / 2}$.

Note: $\lim _{x \rightarrow \pm \infty} f(x)=0$.
a Find all inflection points and intervals of increase, decrease, convexity up, and convexity down. You may use without proof the formula $f^{\prime \prime}(x)=\left(x^{3}-3 x\right) e^{-x^{2} / 2}$.
b Find local and global minima and maxima.
c Use all the above to draw a graph for $f$. Indicate all special points on the graph.
8. Use the techniques from this section to sketch the graph of $f(x)=x+2 \sin x$.
9. *.

$$
f(x)=4 \sin x-2 \cos 2 x
$$

Graph the equation $y=f(x)$, including all important features. (In particular, find all local maxima and minima and all inflection points.) Additionally, find the maximum and minimum values of $f(x)$ on the interval $[0, \pi]$.
10. Sketch the curve $y=\sqrt[3]{\frac{x+1}{x^{2}}}$.

You may use the facts $y^{\prime}(x)=\frac{-(x+2)}{3 x^{5 / 3}(x+1)^{2 / 3}}$ and $y^{\prime \prime}(x)=\frac{4 x^{2}+16 x+10}{9 x^{8 / 3}(x+1)^{5 / 3}}$.

## Exercises - Stage 3

11. *. A function $f(x)$ defined on the whole real number line satisfies the following conditions

$$
f(0)=0 \quad f(2)=2 \quad \lim _{x \rightarrow+\infty} f(x)=0 \quad f^{\prime}(x)=K\left(2 x-x^{2}\right) e^{-x}
$$

for some positive constant $K$. (Read carefully: you are given the derivative of $f(x)$, not $f(x)$ itself.)
a Determine the intervals on which $f$ is increasing and decreasing and the location of any local maximum and minimum values of $f$.
b Determine the intervals on which $f$ is concave up or down and the $x-$ coordinates of any inflection points of $f$.
c Determine $\lim _{x \rightarrow-\infty} f(x)$.
d Sketch the graph of $y=f(x)$, showing any asymptotes and the information determined in parts 3.6.7.11.a and 3.6.7.11.b.
12. *. Let $f(x)=e^{-x}, x \geq 0$.
a Sketch the graph of the equation $y=f(x)$. Indicate any local extrema and inflection points.
b Sketch the graph of the inverse function $y=g(x)=f^{-1}(x)$.
c Find the domain and range of the inverse function $g(x)=f^{-1}(x)$.
d Evaluate $g^{\prime}\left(\frac{1}{2}\right)$.
13. *.
a Sketch the graph of $y=f(x)=x^{5}-x$, indicating asymptotes, local maxima and minima, inflection points, and where the graph is concave up/concave down.
b Consider the function $f(x)=x^{5}-x+k$, where $k$ is a constant, $-\infty<$ $k<\infty$. How many roots does the function have? (Your answer might depend on the value of $k$.)
14. *. The hyperbolic trigonometric functions $\sinh (x)$ and $\cosh (x)$ are defined by

$$
\sinh (x)=\frac{e^{x}-e^{-x}}{2} \quad \cosh (x)=\frac{e^{x}+e^{-x}}{2}
$$

They have many properties that are similar to corresponding properties of $\sin (x)$ and $\cos (x)$. In particular, it is easy to see that

$$
\frac{\mathrm{d}}{\mathrm{~d} x} \sinh (x)=\cosh (x) \quad \frac{\mathrm{d}}{\mathrm{~d} x} \cosh (x)=\sinh (x) \quad \cosh ^{2}(x)-\sinh ^{2}(x)=1
$$

You may use these properties in your solution to this question.
a Sketch the graphs of $\sinh (x)$ and $\cosh (x)$.
b Define inverse hyperbolic trigonometric functions $\sinh ^{-1}(x)$ and $\cosh ^{-1}(x)$, carefully specifing their domains of definition. Sketch the graphs of $\sinh ^{-1}(x)$ and $\cosh ^{-1}(x)$.
c Find $\frac{\mathrm{d}}{\mathrm{d} x}\left\{\cosh ^{-1}(x)\right\}$.

## 3.7^ L'Hôpital's Rule, Indeterminate Forms

### 3.7.1 L'Hôpital's Rule and Indeterminate Forms

Let us return to limits (Chapter 1) and see how we can use derivatives to simplify certain families of limits called indeterminate forms. We know, from Theorem 1.4.3 on the arithmetic of limits, that if

$$
\lim _{x \rightarrow a} f(x)=F \quad \lim _{x \rightarrow a} g(x)=G
$$

and $G \neq 0$, then

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)}=\frac{F}{G}
$$

The requirement that $G \neq 0$ is critical - we explored this in Example 1.4.7. Please reread that example.

Of course ${ }^{1}$ it is not surprising that if $F \neq 0$ and $G=0$, then

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)}=D N E
$$

and if $F=0$ but $G \neq 0$ then

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)}=0
$$

However when both $F, G=0$ then, as we saw in Example 1.4.7, almost anything can happen

$$
\begin{array}{lll}
f(x)=x & g(x)=x^{2} & \lim _{x \rightarrow 0} \frac{x}{x^{2}}=\lim _{x \rightarrow 0} \frac{1}{x}=D N E \\
f(x)=x^{2} & g(x)=x & \lim _{x \rightarrow 0} \frac{x^{2}}{x}=\lim _{x \rightarrow 0} x=0 \\
f(x)=x & g(x)=x & \lim _{x \rightarrow 0} \frac{x}{x}=\lim _{x \rightarrow 0} 1=1 \\
f(x)=7 x^{2} & g(x)=3 x^{2} & \lim _{x \rightarrow 0} \frac{7 x^{2}}{3 x^{2}}=\lim _{x \rightarrow 0} \frac{7}{3}=\frac{7}{3}
\end{array}
$$

Indeed after exploring Example 1.4.12 and 1.4.14 we gave ourselves the rule of thumb that if we found $0 / 0$, then there must be something that cancels.

Because the limit that results from these $0 / 0$ situations is not immediately obvious, but also leads to some interesting mathematics, we should give it a name.

## Definition 3.7.1 First indeterminate forms.

Let $a \in \mathbb{R}$ and let $f(x)$ and $g(x)$ be functions. If

$$
\lim _{x \rightarrow a} f(x)=0 \quad \text { and } \quad \lim _{x \rightarrow a} g(x)=0
$$

then the limit

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)}
$$

is called a $\frac{0}{0}$ indeterminate form.

There are quite a number of mathematical tools for evaluating such indeterminate forms - Taylor series for example. A simpler method, which works in quite a few cases, is L'Hôpital's rule ${ }^{2}$.

1 Now it is not so surprising, but perhaps back when we started limits, this was not so obvious.
2 Named for the 17th century mathematician, Guillaume de l'Hôpital, who published the first textbook on differential calculus. The eponymous rule appears in that text, but is believed to have been developed by Johann Bernoulli. The book was the source of some controversy since it

## Theorem 3.7.2 L'Hôpital's Rule.

Let $a \in \mathbb{R}$ and assume that

$$
\lim _{x \rightarrow a} f(x)=\lim _{x \rightarrow a} g(x)=0
$$

Then
a if $f^{\prime}(a)$ and $g^{\prime}(a)$ exist and $g^{\prime}(a) \neq 0$, then

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)}=\frac{f^{\prime}(a)}{g^{\prime}(a)}
$$

b while, if $f^{\prime}(x)$ and $g^{\prime}(x)$ exist, with $g^{\prime}(x)$ nonzero, on an open interval that contains $a$, except possibly at $a$ itself, and if the limit

$$
\lim _{x \rightarrow a} \frac{f^{\prime}(x)}{g^{\prime}(x)} \text { exists or is }+\infty \text { or is }-\infty
$$

then

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)}=\lim _{x \rightarrow a} \frac{f^{\prime}(x)}{g^{\prime}(x)}
$$

Proof. We only give the proof for part (a). The proof of part (b) is not very difficult, but uses the Generalised Mean-Value Theorem (Theorem 3.4.38), which is optional and most readers have not seen it.

- First note that we must have $f(a)=g(a)=0$. To see this note that since derivative $f^{\prime}(a)$ exists, we know that the limit

$$
\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a} \text { exists }
$$

Since we know that the denominator goes to zero, we must also have that the numerator goes to zero (otherwise the limit would be undefined). Hence we must have

$$
\lim _{x \rightarrow a}(f(x)-f(a))=\left(\lim _{x \rightarrow a} f(x)\right)-f(a)=0
$$

contained many results by Bernoulli, which l'Hôpital acknowledged in the preface, but Bernoulli felt that l'Hôpital got undue credit.

Note that around that time l'Hôpital's name was commonly spelled l'Hospital, but the spelling of silent s in French was changed subsequently; many texts spell his name l'Hospital. If you find yourself in Paris, you can hunt along Boulevard de l'Hôpital for older street signs carved into the sides of buildings which spell it "l'Hospital" - though arguably there are better things to do there.

We are told that $\lim _{x \rightarrow a} f(x)=0$ so we must have $f(a)=0$. Similarly we know that $g(a)=0$.

- Now consider the indeterminate form

$$
\begin{aligned}
\lim _{x \rightarrow a} \frac{f(x)}{g(x)} & =\lim _{x \rightarrow a} \frac{f(x)-0}{g(x)-0} \\
& =\lim _{x \rightarrow a} \frac{f(x)-f(a)}{g(x)-g(a)} \\
& =\lim _{x \rightarrow a} \frac{f(x)-f(a)}{g(x)-g(a)} \cdot \frac{(x-a)^{-1}}{(x-a)^{-1}} \\
& =\lim _{x \rightarrow a}\left[\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}\right] \\
& =\frac{\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a}}{\lim _{x \rightarrow a} \frac{g(x)-g(a)}{x-a}}=\frac{f^{\prime}(a)}{g^{\prime}(a)}
\end{aligned}
$$

We can justify this step and apply Theorem 1.4.3, since the limits in the numerator and denominator exist, because they are just $f^{\prime}(a)$ and $g^{\prime}(a)$.

### 3.7.1.1 Optional — Proof of Part (b) of l'Hôpital's Rule

To prove part (b) we must work around the possibility that $f^{\prime}(a)$ and $g^{\prime}(a)$ do not exist or that $f^{\prime}(x)$ and $g^{\prime}(x)$ are not continuous at $x=a$. To do this, we make use of the Generalised Mean-Value Theorem (Theorem 3.4.38) that was used to prove Equation 3.4.33. We recommend you review the GMVT before proceeding.

For simplicity we consider the limit

$$
\lim _{x \rightarrow a^{+}} \frac{f(x)}{g(x)}
$$

By assumption, we know that

$$
\lim _{x \rightarrow a^{+}} f(x)=\lim _{x \rightarrow a^{+}} g(x)=0
$$

For simplicity, we also assume that $f(a)=g(a)=0$. This allows us to write

$$
\frac{f(x)}{g(x)}=\frac{f(x)-f(a)}{g(x)-g(a)}
$$

which is the right form for an application of the GMVT.
By assumption $f^{\prime}(x)$ and $g^{\prime}(x)$ exist, with $g^{\prime}(x)$ nonzero, in some open interval around $a$, except possibly at $a$ itself. So we know that they exist, with $g^{\prime}(x) \neq 0$, in some interval $(a, b]$ with $b>a$. Then the GMVT (Theorem 3.4.38) tells us that for $x \in(a, b]$

$$
\frac{f(x)}{g(x)}=\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f^{\prime}(c)}{g^{\prime}(c)}
$$

where $c \in(a, x)$. As we take the limit as $x \rightarrow a$, we also have that $c \rightarrow a$, and so

$$
\lim _{x \rightarrow a^{+}} \frac{f(x)}{g(x)}=\lim _{x \rightarrow a^{+}} \frac{f^{\prime}(c)}{g^{\prime}(c)}=\lim _{c \rightarrow a^{+}} \frac{f^{\prime}(c)}{g^{\prime}(c)}
$$

as required.

### 3.7.2 $\leadsto$ Standard Examples

Here are some simple examples using L'Hôpital's rule.
Example 3.7.3 Find $\lim _{x \rightarrow 0} \frac{\sin x}{x}$.
Consider the limit

$$
\lim _{x \rightarrow 0} \frac{\sin x}{x}
$$

- Notice that

$$
\begin{array}{r}
\lim _{x \rightarrow 0} \sin x=0 \\
\lim _{x \rightarrow 0} x=0
\end{array}
$$

so this is a $\frac{0}{0}$ indeterminate form, and suggests we try l'Hôpital's rule.

- To apply the rule we must first check the limits of the derivatives.

$$
\begin{array}{llll}
f(x)=\sin x & f^{\prime}(x)=\cos x & & \text { and } \\
g(x)=x & g^{\prime}(x)=1 & & f^{\prime}(0)=1 \\
& \text { and } & g^{\prime}(0)=1
\end{array}
$$

- So by l'Hôpital's rule

$$
\lim _{x \rightarrow 0} \frac{\sin x}{x}=\frac{f^{\prime}(0)}{g^{\prime}(0)}=\frac{1}{1}=1 .
$$

Example 3.7.4 Compute $\lim _{x \rightarrow 0} \frac{\sin (x)}{\sin (2 x)}$.
Consider the limit

$$
\lim _{x \rightarrow 0} \frac{\sin (x)}{\sin (2 x)}
$$

- First check

$$
\begin{array}{r}
\lim _{x \rightarrow 0} \sin 2 x=0 \\
\lim _{x \rightarrow 0} \sin x=0
\end{array}
$$

so we again have a $\frac{0}{0}$ indeterminate form.

- Set $f(x)=\sin x$ and $g(x)=\sin 2 x$, then

$$
\begin{array}{ll}
f^{\prime}(x)=\cos x & f^{\prime}(0)=1 \\
g^{\prime}(x)=2 \cos 2 x & g^{\prime}(0)=2
\end{array}
$$

- And by l'Hôpital's rule

$$
\lim _{x \rightarrow 0} \frac{\sin x}{\sin 2 x}=\frac{f^{\prime}(0)}{g^{\prime}(0)}=\frac{1}{2}
$$

Example 3.7.5 $\lim _{x \rightarrow 0} \frac{q^{x}-1}{x}$.
Let $q>1$ and compute the limit

$$
\lim _{x \rightarrow 0} \frac{q^{x}-1}{x}
$$

This limit arose in our discussion of exponential functions in Section 2.7.

- First check

$$
\begin{aligned}
\lim _{x \rightarrow 0}\left(q^{x}-1\right) & =1-1=0 \\
\lim _{x \rightarrow 0} x & =0
\end{aligned}
$$

so we have a $\frac{0}{0}$ indeterminate form.

- Set $f(x)=q^{x}-1$ and $g(x)=x$, then (maybe after a quick review of Section 2.7)

$$
f^{\prime}(x)=\frac{\mathrm{d}}{\mathrm{~d} x}\left(q^{x}-1\right)=q^{x} \cdot \log q \quad f^{\prime}(0)=\log q
$$

$$
g^{\prime}(x)=1
$$

$$
g^{\prime}(0)=1
$$

- And by l'Hôpital's rule ${ }^{a}$

$$
\lim _{h \rightarrow 0} \frac{q^{h}-1}{h}=\log q .
$$

$a \quad$ While it might not be immediately obvious, this example relies on circular reasoning. In order to apply l'Hôpital's rule, we need to compute the derivative of $q^{x}$. However in order to compute that limit (see Section 2.7) we needed to evaluate this limit.

A more obvious example of this sort of circular reasoning can be seen if we use l'Hôpital's rule to compute the derivative of $f(x)=x^{n}$ at $x=a$ using the limit

$$
f^{\prime}(a)=\lim _{x \rightarrow a} \frac{x^{n}-a^{n}}{x-a}=\lim _{x \rightarrow a} \frac{n x^{n-1}-0}{1-0}=n a^{n-1}
$$

We have used the result $\frac{\mathrm{d}}{\mathrm{d} x} x^{n}=n x^{n-1}$ to prove itself!
Example 3.7.5
In this example, we shall apply L'Hôpital's rule twice before getting the answer.
Example 3.7.6 Double L'Hôpital.
Compute the limit

$$
\lim _{x \rightarrow 0} \frac{\sin \left(x^{2}\right)}{1-\cos x}
$$

- Again we should check

$$
\begin{aligned}
\lim _{x \rightarrow 0} \sin \left(x^{2}\right) & =\sin 0=0 \\
\lim _{x \rightarrow 0}(1-\cos x) & =1-\cos 0=0
\end{aligned}
$$

and we have a $\frac{0}{0}$ indeterminate form.

- Let $f(x)=\sin \left(x^{2}\right)$ and $g(x)=1-\cos x$ then

$$
\begin{aligned}
f^{\prime}(x) & =2 x \cos \left(x^{2}\right) & f^{\prime}(0) & =0 \\
g^{\prime}(x) & =\sin x & g^{\prime}(0) & =0
\end{aligned}
$$

So if we try to apply l'Hôpital's rule naively we will get

$$
\lim _{x \rightarrow 0} \frac{\sin \left(x^{2}\right)}{1-\cos x}=\frac{f^{\prime}(0)}{g^{\prime}(0)}=\frac{0}{0}
$$

which is another $\frac{0}{0}$ indeterminate form.

- It appears that we are stuck until we remember that l'Hôpital's rule (as stated in Theorem 3.7.2) has a part (b) - now is a good time to reread it.
- It says that

$$
\lim _{x \rightarrow 0} \frac{f(x)}{g(x)}=\lim _{x \rightarrow 0} \frac{f^{\prime}(x)}{g^{\prime}(x)}
$$

provided this second limit exists. In our case this requires us to compute

$$
\lim _{x \rightarrow 0} \frac{2 x \cos \left(x^{2}\right)}{\sin (x)}
$$

which we can do using l'Hôpital's rule again. Now

$$
\begin{aligned}
& h(x)=2 x \cos \left(x^{2}\right) \quad h^{\prime}(x)=2 \cos \left(x^{2}\right)-4 x^{2} \sin \left(x^{2}\right) \quad h^{\prime}(0)=2 \\
& \ell(x)=\sin (x) \quad \ell^{\prime}(x)=\cos (x) \quad \ell^{\prime}(0)=1
\end{aligned}
$$

By l'Hôpital's rule

$$
\lim _{x \rightarrow 0} \frac{2 x \cos \left(x^{2}\right)}{\sin (x)}=\frac{h^{\prime}(0)}{\ell^{\prime}(0)}=2
$$

- Thus our original limit is

$$
\lim _{x \rightarrow 0} \frac{\sin \left(x^{2}\right)}{1-\cos x}=\lim _{x \rightarrow 0} \frac{2 x \cos \left(x^{2}\right)}{\sin (x)}=2 .
$$

- We can succinctly summarise the two applications of L'Hôpital's rule in this example by

$$
\lim _{x \rightarrow 0} \frac{\sin \left(x^{2}\right)}{\underbrace{1-\cos x}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow 0}}}=\lim _{x \rightarrow 0} \underbrace{\frac{2 x \cos \left(x^{2}\right)}{\sin x}}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow 0}}=\lim _{x \rightarrow 0} \underbrace{\frac{2 \cos \left(x^{2}\right)-4 x^{2} \sin \left(x^{2}\right)}{\cos x}}_{\substack{\text { num } \rightarrow 2 \\ \text { den } \rightarrow 1}}=2
$$

Here "num" and "den" are used as abbreviations of "numerator" and "denominator" respectively."

Example 3.7.6
One must be careful to ensure that the hypotheses of l'Hôpital's rule are satisfied before applying it. The following "warnings" show the sorts of things that can go wrong.

Warning 3.7.7 Denominator limit nonzero.
If

$$
\lim _{x \rightarrow a} f(x)=0 \quad \text { but } \quad \lim _{x \rightarrow a} g(x) \neq 0
$$

then

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)} \quad \text { need not be the same as } \quad \frac{f^{\prime}(a)}{g^{\prime}(a)} \text { or } \lim _{x \rightarrow a} \frac{f^{\prime}(x)}{g^{\prime}(x)} .
$$

Here is an example. Take

$$
a=0 \quad f(x)=3 x \quad g(x)=4+5 x
$$

Then

$$
\begin{aligned}
\lim _{x \rightarrow 0} \frac{f(x)}{g(x)} & =\lim _{x \rightarrow 0} \frac{3 x}{4+5 x} \\
\lim _{x \rightarrow 0} \frac{f^{\prime}(x)}{g^{\prime}(x)} & =\frac{f^{\prime}(0)}{g^{\prime}(0)}=\frac{3}{5}
\end{aligned}
$$

Warning 3.7.8 Numerator limit nonzero.
If

$$
\lim _{x \rightarrow a} g(x)=0 \quad \text { but } \quad \lim _{x \rightarrow a} f(x) \neq 0
$$

then

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)} \quad \text { need not be the same as } \quad \lim _{x \rightarrow a} \frac{f^{\prime}(x)}{g^{\prime}(x)}
$$

Here is an example. Take

$$
a=0 \quad f(x)=4+5 x \quad g(x)=3 x
$$

Then

$$
\begin{aligned}
\lim _{x \rightarrow 0} \frac{f(x)}{g(x)} & =\lim _{x \rightarrow 0} \frac{4+5 x}{3 x} \\
\lim _{x \rightarrow 0} \frac{f^{\prime}(x)}{g^{\prime}(x)} & =\lim _{x \rightarrow 0} \frac{5}{3}=\frac{5}{3}
\end{aligned} \quad=\mathrm{DNE}
$$

This next one is more subtle; the limits of the original numerator and denominator functions both go to zero, but the limit of the ratio their derivatives does not exist.

Warning 3.7.9 Limit of ratio of derivatives DNE.
If

$$
\lim _{x \rightarrow a} f(x)=0 \quad \text { and } \quad \lim _{x \rightarrow a} g(x)=0
$$

but

$$
\lim _{x \rightarrow a} \frac{f^{\prime}(x)}{g^{\prime}(x)} \text { does not exist }
$$

then it is still possible that

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)} \text { exists. }
$$

Here is an example. Take

$$
a=0 \quad f(x)=x^{2} \sin \frac{1}{x} \quad g(x)=x
$$

Then (with an application of the squeeze theorem)

$$
\lim _{x \rightarrow 0} f(x)=0 \quad \text { and } \quad \lim _{x \rightarrow 0} g(x)=0
$$

If we attempt to apply l'Hôptial's rule then we have $g^{\prime}(x)=1$ and

$$
f^{\prime}(x)=2 x \sin \frac{1}{x}-\cos \frac{1}{x}
$$

and we then try to compute the limit

$$
\lim _{x \rightarrow 0} \frac{f^{\prime}(x)}{g^{\prime}(x)}=\lim _{x \rightarrow 0}\left(2 x \sin \frac{1}{x}-\cos \frac{1}{x}\right)
$$

However, this limit does not exist. The first term converges to 0 (by the squeeze theorem), but the second term $\cos (1 / x)$ just oscillates wildly between $\pm 1$. All we can conclude from this is

Since the limit of the ratio of derivatives does not exist, we cannot apply l'Hôpital's rule.

Instead we should go back to the original limit and apply the squeeze theorem:

$$
\lim _{x \rightarrow 0} \frac{f(x)}{g(x)}=\lim _{x \rightarrow 0} \frac{x^{2} \sin \frac{1}{x}}{x}=\lim _{x \rightarrow 0} x \sin \frac{1}{x}=0
$$

since $|x \sin (1 / x)|<|x|$ and $|x| \rightarrow 0$ as $x \rightarrow 0$.

It is also easy to construct an example in which the limits of numerator and denominator are both zero, but the limit of the ratio and the limit of the ratio of the derivatives do not exist. A slight change of the previous example shows that it is possible that

$$
\lim _{x \rightarrow a} f(x)=0 \quad \text { and } \quad \lim _{x \rightarrow a} g(x)=0
$$

but neither of the limits

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)} \quad \text { or } \quad \lim _{x \rightarrow a} \frac{f^{\prime}(x)}{g^{\prime}(x)}
$$

exist. Take

$$
a=0 \quad f(x)=x \sin \frac{1}{x} \quad g(x)=x
$$

Then (with a quick application of the squeeze theorem)

$$
\lim _{x \rightarrow 0} f(x)=0 \quad \text { and } \quad \lim _{x \rightarrow 0} g(x)=0
$$

However,

$$
\lim _{x \rightarrow 0} \frac{f(x)}{g(x)}=\lim _{x \rightarrow 0} \frac{x \sin \frac{1}{x}}{x}=\lim _{x \rightarrow 0} \sin \frac{1}{x}
$$

does not exist. And similarly

$$
\lim _{x \rightarrow 0} \frac{f^{\prime}(x)}{g^{\prime}(x)}=\lim _{x \rightarrow 0} \frac{\sin \frac{1}{x}-\frac{1}{x} \cos \frac{1}{x}}{x^{2}}
$$

does not exist.

### 3.7.3 $\leadsto$ Variations

Theorem 3.7.2 is the basic form of L'Hôpital's rule, but there are also many variations. Here are a bunch of them.

### 3.7.3.1 Limits at $\pm \infty$

L'Hôpital's rule also applies when the limit of $x \rightarrow a$ is replaced by $\lim _{x \rightarrow a+}$ or by $\lim _{x \rightarrow a-}$ or by $\lim _{x \rightarrow+\infty}$ or by $\lim _{x \rightarrow-\infty}$.

We can justify adapting the rule to the limits to $\pm \infty$ via the following reasoning

$$
\begin{array}{rlr}
\lim _{x \rightarrow \infty} \frac{f(x)}{g(x)} & =\lim _{y \rightarrow 0^{+}} \frac{f(1 / y)}{g(1 / y)} & \text { substitute } x=1 / y \\
& =\lim _{y \rightarrow 0^{+}} \frac{-\frac{1}{y^{2}} f^{\prime}(1 / y)}{-\frac{1}{y^{2}} g^{\prime}(1 / y)} &
\end{array}
$$

where we have used l'Hôpital's rule (assuming this limit exists) and the fact that $\frac{\mathrm{d}}{\mathrm{d} y} f(1 / y)=-\frac{1}{y^{2}} f^{\prime}(1 / y)$ (and similarly for $g$ ). Cleaning this up and substituting $y=1 / x$ gives the required result:

$$
\lim _{x \rightarrow \infty} \frac{f(x)}{g(x)}=\lim _{y \rightarrow 0^{+}} \frac{f^{\prime}(1 / y)}{g^{\prime}(1 / y)}=\lim _{x \rightarrow \infty} \frac{f^{\prime}(x)}{g^{\prime}(x)}
$$

Example 3.7.10 L'Hôpital at infinity.
Consider the limit

$$
\lim _{x \rightarrow \infty} \frac{\arctan x-\frac{\pi}{2}}{\frac{1}{x}}
$$

Both numerator and denominator go to 0 as $x \rightarrow \infty$, so this is an $\frac{0}{0}$ indeterminate form.
We find

$$
\lim _{x \rightarrow+\infty} \underbrace{\frac{\operatorname{rrctan} x-\frac{\pi}{2}}{x}}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow 0}}=\lim _{x \rightarrow+\infty} \frac{\frac{1}{1+x^{2}}}{-\frac{1}{x^{2}}}=-\lim _{x \rightarrow+\infty} \underbrace{\frac{1}{1+\frac{1}{x^{2}}}}_{\substack{\text { num } \rightarrow 1 \\ \text { den } \rightarrow 1}}=-1
$$

We have applied L'Hôpital's rule with

$$
\begin{aligned}
f(x) & =\arctan x-\frac{\pi}{2} & g(x) & =\frac{1}{x} \\
f^{\prime}(x) & =\frac{1}{1+x^{2}} & g^{\prime}(x) & =-\frac{1}{x^{2}}
\end{aligned}
$$

### 3.7.3.2 $\leadsto \frac{\infty}{\infty}$ indeterminate form

L'Hôpital's rule also applies when $\lim _{x \rightarrow a} f(x)=0, \lim _{x \rightarrow a} g(x)=0$ is replaced by $\lim _{x \rightarrow a} f(x)=$ $\pm \infty, \lim _{x \rightarrow a} g(x)= \pm \infty$.

Example 3.7.11 Compute $\lim _{x \rightarrow \infty} \frac{\log x}{x}$.
Consider the limit

$$
\lim _{x \rightarrow \infty} \frac{\log x}{x}
$$

The numerator and denominator both blow up towards infinity so this is an $\frac{\infty}{\infty}$ indeterminate form. An application of l'Hôpital's rule gives

$$
\begin{aligned}
\lim _{x \rightarrow \infty} \underbrace{\frac{\log x}{x}}_{\substack{\text { num } \rightarrow \infty \\
\text { den } \rightarrow \infty}} & =\lim _{x \rightarrow \infty} \frac{1 / x}{1} \\
& =\lim _{x \rightarrow \infty} \frac{1}{x}=0
\end{aligned}
$$

Example 3.7.11

Example 3.7.12 Find $\lim _{x \rightarrow \infty} \frac{5 x^{2}+3 x-3}{x^{2}+1}$.
Consider the limit

$$
\lim _{x \rightarrow \infty} \frac{5 x^{2}+3 x-3}{x^{2}+1}
$$

Then by two applications of l'Hôpital's rule we get

$$
\lim _{x \rightarrow \infty} \underbrace{\frac{5 x^{2}+3 x-3}{x^{2}+1}}_{\substack{\text { num } \rightarrow \infty \\ \text { den } \rightarrow \infty}}=\lim _{x \rightarrow \infty} \underbrace{\frac{10 x+3}{2 x}}_{\substack{\text { num } \rightarrow \infty \\ \text { den } \rightarrow \infty}}=\lim _{x \rightarrow \infty} \frac{10}{2}=5
$$

Example 3.7.13 A messier double l'Hôpital.
Compute the limit

$$
\lim _{x \rightarrow 0+} \frac{\log x}{\tan \left(\frac{\pi}{2}-x\right)}
$$

We can compute this using l'Hôpital's rule twice:

$$
\begin{aligned}
\lim _{x \rightarrow 0+}+\underbrace{\frac{\log x}{\tan \left(\frac{\pi}{2}-x\right)}}_{\substack{\text { num } \rightarrow-\infty \\
\text { den } \rightarrow+\infty}} & =\lim _{x \rightarrow 0+} \frac{\frac{1}{x}}{-\sec ^{2}\left(\frac{\pi}{2}-x\right)}=-\lim _{x \rightarrow 0+} \underbrace{\frac{\cos ^{2}\left(\frac{\pi}{2}-x\right)}{x}}_{\substack{\text { num } \rightarrow 0 \\
\text { den } \rightarrow 0}} \\
& =-\lim _{x \rightarrow 0+} \underbrace{\frac{2 \cos \left(\frac{\pi}{2}-x\right) \sin \left(\frac{\pi}{2}-x\right)}{1}}_{\substack{\text { num } \rightarrow 0 \\
\text { den } \rightarrow 1}}=0
\end{aligned}
$$

The first application of L'Hôpital's was with

$$
f(x)=\log x \quad g(x)=\tan \left(\frac{\pi}{2}-x\right)
$$

$$
f^{\prime}(x)=\frac{1}{x} \quad g^{\prime}(x)=-\sec ^{2}\left(\frac{\pi}{2}-x\right)
$$

and the second time with

$$
\begin{aligned}
f(x) & =\cos ^{2}\left(\frac{\pi}{2}-x\right) & g(x) & =x \\
f^{\prime}(x) & =2 \cos \left(\frac{\pi}{2}-x\right)\left[-\sin \left(\frac{\pi}{2}-x\right)\right](-1) & g^{\prime}(x) & =1
\end{aligned}
$$

Example 3.7.13
Sometimes things don't quite work out as we would like and l'Hôpital's rule can get stuck in a loop. Remember to think about the problem before you apply any rule.

Example 3.7.14 Stuck in a loop.
Consider the limit

$$
\lim _{x \rightarrow \infty} \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}
$$

Clearly both numerator and denominator go to $\infty$, so we have a $\frac{\infty}{\infty}$ indeterminate form. Naively applying l'Hôpital's rule gives

$$
\lim _{x \rightarrow \infty} \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}=\lim _{x \rightarrow \infty} \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}
$$

which is again a $\frac{\infty}{\infty}$ indeterminate form. So apply l'Hôpital's rule again:

$$
\lim _{x \rightarrow \infty} \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}=\lim _{x \rightarrow \infty} \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}
$$

which is right back where we started!
The correct approach to such a limit is to apply the methods we learned in Chapter 1 and rewrite

$$
\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}=\frac{e^{x}\left(1+e^{-2 x}\right)}{e^{x}\left(1-e^{-2 x}\right)}=\frac{1+e^{-2 x}}{1-e^{-2 x}}
$$

and then take the limit.
A similar sort of l'Hôpital-rule-loop will occur if you naively apply l'Hôpital's rule to the limit

$$
\lim _{x \rightarrow \infty} \frac{\sqrt{4 x^{2}+1}}{5 x-1}
$$

which appeared in Example 1.5.6.

Applications of derivatives 3.7 L'Hôpital's Rule, Indeterminate Forms

### 3.7.3.3 Optional - Proof of l'Hôpital's Rule for $\frac{\infty}{\infty}$

We can justify this generalisation of l'Hôpital's rule with some careful manipulations. Since the derivatives $f^{\prime}, g^{\prime}$ exist in some interval around $a$, we know that $f, g$ are continuous in some interval around $a$; let $x, t$ be points inside that interval. Now rewrite 3

$$
\begin{aligned}
\frac{f(x)}{g(x)} & =\frac{f(x)}{g(x)}+\underbrace{\left(\frac{f(t)}{g(x)}-\frac{f(t)}{g(x)}\right)}_{=0}+\underbrace{\left(\frac{f(x)-f(t)}{g(x)-g(t)}-\frac{f(x)-f(t)}{g(x)-g(t)}\right)}_{\text {we can clean it up }} \\
& =\underbrace{\frac{f(x)-f(t)}{g(x)-g(t)}}_{\text {ready for GMVT }}+\frac{f(t)}{g(x)}+\underbrace{\left(\frac{f(x)}{g(x)}-\frac{f(t)}{g(x)}-\frac{f(x)-f(t)}{g(x)-g(t)}\right)}_{=0} \\
& =\frac{f(x)-f(t)}{g(x)-g(t)}+\frac{f(t)}{g(x)}+\left(\frac{f(x)-f(t)}{g(x)}-\frac{f(x)-f(t)}{g(x)-g(t)}\right) \\
& =\frac{f(x)-f(t)}{g(x)-g(t)}+\frac{f(t)}{g(x)}+\left(\frac{1}{g(x)}-\frac{1}{g(x)-g(t)}\right) \cdot(f(x)-f(t)) \\
& =\frac{f(x)-f(t)}{g(x)-g(t)}+\frac{f(t)}{g(x)}+\left(\frac{g(x)-g(t)-g(x)}{g(x)(g(x)-g(t))}\right) \cdot(f(x)-f(t)) \\
& =\underbrace{\frac{f(x)-f(t)}{g(x)-g(t)}+\frac{f(t)}{g(x)}-\frac{g(t)}{g(x)} \cdot \underbrace{\frac{f(x)-f(t)}{g(x)-g(t)}}_{\text {ready for GMVT }}}_{\text {ready for GMVT }}
\end{aligned}
$$

Oof! Now the generalised mean-value theorem (Theorem 3.4.38) tells us there is a $c$ between $x$ and $t$ so that

$$
\frac{f(x)-f(t)}{g(x)-g(t)}=\frac{f^{\prime}(c)}{g^{\prime}(c)}
$$

Now substitute this into the large expression we derived above:

$$
\frac{f(x)}{g(x)}=\frac{f^{\prime}(c)}{g^{\prime}(c)}+\frac{1}{g(x)}\left(f(t)-\frac{f^{\prime}(c)}{g^{\prime}(c)} \cdot g(t)\right)
$$

At first glance this does not appear so useful, however if we fix $t$ and take the limit as $x \rightarrow a$, then it becomes

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)}=\lim _{x \rightarrow a} \frac{f^{\prime}(c)}{g^{\prime}(c)}+\lim _{x \rightarrow a} \frac{1}{g(x)}\left(f(t)-\frac{f^{\prime}(c)}{g^{\prime}(c)} \cdot g(t)\right)
$$

Since $g(x) \rightarrow \infty$ as $x \rightarrow a$, this last term goes to zero

$$
=\lim _{x \rightarrow a} \frac{f^{\prime}(c)}{g^{\prime}(c)}+0
$$

3 This is quite a clever argument, but it is not immediately obvious why one rewrites things this way. After the fact it becomes clear that it is done to massage the expression into the form where we can apply the generalised mean-value theorem (Theorem 3.4.38).

Now take the limit as $t \rightarrow a$. The left-hand side is unchanged since it is independent of $t$. The right-hand side, however, does change; the number $c$ is trapped between $x$ and $t$. Since we have already taken the limit $x \rightarrow a$, so when we take the limit $t \rightarrow a$, we are effectively taking the limit $c \rightarrow a$. Hence

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)}=\lim _{c \rightarrow a} \frac{f^{\prime}(c)}{g^{\prime}(c)}
$$

which is the desired result.

### 3.7.3.4 $0 \cdot \infty$ indeterminate form

When $\lim _{x \rightarrow a} f(x)=0$ and $\lim _{x \rightarrow a} g(x)=\infty$. We can use a little algebra to manipulate this into either a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form:

$$
\lim _{x \rightarrow a} \frac{f(x)}{1 / g(x)} \quad \lim _{x \rightarrow a} \frac{g(x)}{1 / f(x)}
$$

Example 3.7.15 $\lim _{x \rightarrow 0^{+}} x \cdot \log x$.
Consider the limit

$$
\lim _{x \rightarrow 0^{+}} x \cdot \log x
$$

Here the function $f(x)=x$ goes to zero, while $g(x)=\log x$ goes to $-\infty$. If we rewrite this as the fraction

$$
x \cdot \log x=\frac{\log x}{1 / x}
$$

then the $0 \cdot \infty$ form has become an $\frac{\infty}{\infty}$ form.
The result is then

$$
\lim _{x \rightarrow 0+} \underbrace{x}_{\rightarrow 0} \underbrace{\log x}_{\rightarrow-\infty}=\lim _{x \rightarrow 0+} \frac{\log }{\substack{\text { num } \rightarrow-\infty \\ \text { den } \rightarrow \infty}} \frac{\frac{1}{x}}{\frac{1}{x}}=\lim _{x \rightarrow 0+} \frac{\frac{1}{x}}{-\frac{1}{x^{2}}}=-\lim _{x \rightarrow 0+} x=0
$$

Example 3.7.16 Computing $\lim _{x \rightarrow+\infty} x^{n} e^{-x}$.
In this example we'll evaluate $\lim _{x \rightarrow+\infty} x^{n} e^{-x}$, for all natural numbers $n$. We'll start with $n=1$ and $n=2$ and then, using what we have learned from those cases, move on to
general $n$.

$$
\lim _{x \rightarrow+\infty} \underbrace{x}_{\rightarrow \infty} \underbrace{e^{-x}}_{\rightarrow 0}=\lim _{x \rightarrow+\infty} \underbrace{\frac{x}{e^{x}}}_{\substack{\text { num } \rightarrow+\infty \\ \text { den } \rightarrow+\infty}}=\lim _{x \rightarrow+\infty} \underbrace{\frac{1}{e^{x}}}_{\substack{\text { num } \rightarrow 1 \\ \text { den } \rightarrow+\infty}}=\lim _{x \rightarrow+\infty} e^{-x}=0
$$

Applying l'Hôpital twice,

$$
\begin{aligned}
\lim _{x \rightarrow+\infty} \underbrace{x^{2}}_{\rightarrow \infty} \underbrace{e^{-x}}_{\rightarrow 0} & =\lim _{x \rightarrow+\infty} \underbrace{\frac{x^{2}}{e^{x}}}_{\substack{\text { num } \rightarrow+\infty \\
\text { den } \rightarrow+\infty}}=\lim _{x \rightarrow+\infty} \underbrace{\frac{2 x}{e^{x}}}_{\substack{\text { num } \rightarrow \infty \\
\text { den } \rightarrow+\infty}}=\lim _{x \rightarrow+\infty} \underbrace{\frac{2}{e^{x}}}_{\substack{\text { num } \rightarrow 2 \\
\text { den } \rightarrow+\infty}}=\lim _{x \rightarrow+\infty} 2 e^{-x} \\
& -0
\end{aligned}
$$

Indeed, for any natural number $n$, applying l'Hôpital $n$ times gives

$$
\begin{aligned}
\lim _{x \rightarrow+\infty} \underbrace{x^{n}}_{\rightarrow \infty} \underbrace{e^{-x}}_{\rightarrow 0} & =\lim _{x \rightarrow+\infty} \underbrace{\frac{x^{n}}{e^{x}}}_{\substack{\text { num } \rightarrow+\infty \\
\text { den } \rightarrow+\infty}} \\
& =\lim _{x \rightarrow+\infty} \underbrace{\frac{n x^{n-1}}{e^{x}}}_{\substack{\text { num } \rightarrow \infty \\
\text { den } \rightarrow+\infty}} \\
& =\lim _{x \rightarrow+\infty} \underbrace{\lim _{x \rightarrow+\infty} \underbrace{\frac{n!}{d^{x}}}_{\substack{n u m \\
\text { num } \rightarrow n!}}=0}_{\substack{\text { num } \rightarrow \infty \\
\text { den } \rightarrow+\infty \\
\frac{n(n-1) x^{n-2}}{e^{x}}}}=0
\end{aligned}
$$

### 3.7.3.5 $\infty-\infty$ indeterminate form

When $\lim _{x \rightarrow a} f(x)=\infty$ and $\lim _{x \rightarrow a} g(x)=\infty$. We rewrite the difference as a fraction using a common denominator

$$
f(x)-g(x)=\frac{h(x)}{\ell(x)}
$$

which is then a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

Example 3.7.17 Compute $\lim _{x \rightarrow \frac{\pi}{2}^{-}}(\sec x-\tan x)$.
Consider the limit

$$
\lim _{x \rightarrow \frac{\pi}{2}^{-}}(\sec x-\tan x)
$$

Since the limit of both $\sec x$ and $\tan x$ is $+\infty$ as $x \rightarrow \frac{\pi^{-}}{2}$, this is an $\infty-\infty$ indeterminate form. However we can rewrite this as

$$
\sec x-\tan x=\frac{1}{\cos x}-\frac{\sin x}{\cos x}=\frac{1-\sin x}{\cos x}
$$

which is then a $\frac{0}{0}$ indeterminate form. This then gives

$$
\lim _{x \rightarrow \frac{\pi}{2}-}(\underbrace{\sec x}_{\rightarrow+\infty}-\underbrace{\tan x}_{\rightarrow+\infty})=\lim _{x \rightarrow \frac{\pi}{2}-} \underbrace{\frac{1-\sin x}{\cos x}}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow 0}}=\lim _{x \rightarrow \frac{\pi}{2}-} \frac{-\cos x}{\underbrace{-\sin x}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow-1}}}=0
$$

In the last example, Example 3.7.17, we converted an $\infty-\infty$ indeterminate form into a $\frac{0}{0}$ indeterminate form by exploiting the fact that the two terms, $\sec x$ and $\tan x$, in the $\infty-\infty$ indeterminate form shared a common denominator, namely $\cos x$. In the "real world" that will, of course, almost never happen. However as the next couple of examples show, you can often massage these expressions into suitable forms.

Here is another, much more complicated, example, where it doesn't happen.
Example 3.7.18 A complicated $\infty-\infty$ example.
In this example, we evaluate the $\infty-\infty$ indeterminate form

$$
\lim _{x \rightarrow 0}(\underbrace{\frac{1}{x}}_{\rightarrow \pm \infty}-\underbrace{\frac{1}{\log (1+x)}}_{\rightarrow \pm \infty})
$$

We convert it into a $\frac{0}{0}$ indeterminate form simply by putting the two fractions, $\frac{1}{x}$ and $\frac{1}{\log (1+x)}$ over a common denominator.

$$
\begin{equation*}
\lim _{x \rightarrow 0}(\underbrace{\frac{1}{x}}_{\rightarrow \pm \infty}-\underbrace{\frac{1}{\log (1+x)}}_{\rightarrow \pm \infty})=\lim _{x \rightarrow 0} \underbrace{\frac{\log (1+x)-x}{x \log (1+x)}}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow 0}} \tag{E1}
\end{equation*}
$$

Now we apply L'Hôpital's rule, and simplify

$$
\lim _{x \rightarrow 0} \underbrace{\frac{\log (1+x)-x}{x \log (1+x)}}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow 0}}=\lim _{x \rightarrow 0} \frac{\frac{1}{1+x}-1}{\log (1+x)+\frac{x}{1+x}}
$$

$$
\begin{align*}
& =\lim _{x \rightarrow 0} \frac{1-(1+x)}{(1+x) \log (1+x)+x} \\
& =-\lim _{x \rightarrow 0} \frac{x}{\underbrace{(1+x) \log (1+x)+x}_{\substack{\text { num } \rightarrow 0 \\
\text { den } \rightarrow 1 \times 0+0=0}}} \tag{E2}
\end{align*}
$$

Then we apply L'Hôpital's rule a second time

$$
\begin{align*}
-\lim _{x \rightarrow 0} \underbrace{\frac{x}{(1+x) \log (1+x)+x}}_{\substack{\text { num } \rightarrow 0 \\
\text { den } \rightarrow 1 \times 0+0=0}} & =-\lim _{x \rightarrow 0} \underbrace{\frac{1}{\log (1+x)+\frac{1+x}{1+x}+1}}_{\substack{\text { num } \rightarrow 1 \\
\text { den } \rightarrow 0+1+1=2}} \\
& =-\frac{1}{2} \tag{E3}
\end{align*}
$$

Combining (E1), (E2) and (E3) gives our final answer

$$
\lim _{x \rightarrow 0}\left(\frac{1}{x}-\frac{1}{\log (1+x)}\right)=-\frac{1}{2}
$$

Example 3.7.18
The following example can be done by l'Hôpital's rule, but it is actually far simpler to multiply by the conjugate and take the limit using the tools of Chapter 1.

$$
\text { Example 3.7.19 Compute } \lim _{x \rightarrow \infty} \sqrt{x^{2}+4 x}-\sqrt{x^{2}-3 x} \text {. }
$$

Consider the limit

$$
\lim _{x \rightarrow \infty} \sqrt{x^{2}+4 x}-\sqrt{x^{2}-3 x}
$$

Neither term is a fraction, but we can write

$$
\begin{aligned}
\sqrt{x^{2}+4 x}-\sqrt{x^{2}-3 x} & =x \sqrt{1+4 / x}-x \sqrt{1-3 / x} \quad \text { assuming } x>0 \\
& =x(\sqrt{1+4 / x}-\sqrt{1-3 / x}) \\
& =\frac{\sqrt{1+4 / x}-\sqrt{1-3 / x}}{1 / x}
\end{aligned}
$$

which is now a $\frac{0}{0}$ form with $f(x)=\sqrt{1+4 / x}-\sqrt{1-3 / x}$ and $g(x)=1 / x$. Then

$$
f^{\prime}(x)=\frac{-4 / x^{2}}{2 \sqrt{1+4 / x}}-\frac{3 / x^{2}}{2 \sqrt{1-3 / x}} \quad g^{\prime}(x)=-\frac{1}{x^{2}}
$$

Hence

$$
\frac{f^{\prime}(x)}{g^{\prime}(x)}=\frac{4}{2 \sqrt{1+4 / x}}+\frac{3}{\sqrt{1-3 / x}}
$$

And so in the limit as $x \rightarrow \infty$

$$
\lim _{x \rightarrow \infty} \frac{f^{\prime}(x)}{g^{\prime}(x)}=\frac{4}{2}+\frac{3}{2}=\frac{7}{2}
$$

and so our original limit is also $7 / 2$.
By comparison, if we multiply by the conjugate we have

$$
\begin{aligned}
\sqrt{x^{2}+4 x}-\sqrt{x^{2}-3 x} & =\left(\sqrt{x^{2}+4 x}-\sqrt{x^{2}-3 x}\right) \cdot \frac{\sqrt{x^{2}+4 x}+\sqrt{x^{2}-3 x}}{\sqrt{x^{2}+4 x}+\sqrt{x^{2}-3 x}} \\
& =\frac{x^{2}+4 x-\left(x^{2}-3 x\right)}{\sqrt{x^{2}+4 x}+\sqrt{x^{2}-3 x}} \\
& =\frac{7 x}{\sqrt{x^{2}+4 x}+\sqrt{x^{2}-3 x}} \\
& =\frac{7}{\sqrt{1+4 / x}+\sqrt{1-3 / x}} \quad \text { assuming } x>0
\end{aligned}
$$

Now taking the limit as $x \rightarrow \infty$ gives $7 / 2$ as required. Just because we know l'Hôpital's rule, it does not mean we should use it everywhere it might be applied.


### 3.7.3.6 ${ }^{\Perp} 1^{\infty}$ indeterminate form

We can use l'Hôpital's rule on limits of the form

$$
\begin{array}{ll}
\lim _{x \rightarrow a} f(x)^{g(x)} \text { with } & \\
\lim _{x \rightarrow a} f(x)=1 \quad \text { and } \quad \lim _{x \rightarrow a} g(x)=\infty
\end{array}
$$

by considering the logarithm of the limit ${ }^{4}$ :

$$
\log \left(\lim _{x \rightarrow a} f(x)^{g(x)}\right)=\lim _{x \rightarrow a} \log \left(f(x)^{g(x)}\right)=\lim _{x \rightarrow a} \log (f(x)) \cdot g(x)
$$

which is now an $0 \cdot \infty$ form. This can be further transformed into a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form:

$$
\begin{aligned}
\log \left(\lim _{x \rightarrow a} f(x)^{g(x)}\right) & =\lim _{x \rightarrow a} \log (f(x)) \cdot g(x) \\
& =\lim _{x \rightarrow a} \frac{\log (f(x))}{1 / g(x)}
\end{aligned}
$$

[^9]Example 3.7.20 Find $\lim _{x \rightarrow 0}(1+x)^{\frac{a}{x}}$.
The following limit appears quite naturally when considering systems which display exponential growth or decay.

$$
\lim _{x \rightarrow 0}(1+x)^{\frac{a}{x}} \quad \text { with the constant } a \neq 0
$$

Since $(1+x) \rightarrow 1$ and $a / x \rightarrow \infty$ this is an $1^{\infty}$ indeterminate form.
By considering its logarithm we have

$$
\begin{aligned}
\log \left(\lim _{x \rightarrow 0}(1+x)^{\frac{a}{x}}\right) & =\lim _{x \rightarrow 0} \log \left((1+x)^{\frac{a}{x}}\right) \\
& =\lim _{x \rightarrow 0} \frac{a}{x} \log (1+x) \\
& =\lim _{x \rightarrow 0} \frac{a \log (1+x)}{x}
\end{aligned}
$$

which is now a $\frac{0}{0}$ form. Applying l'Hôpital's rule gives

$$
\lim _{x \rightarrow 0} \underbrace{\frac{a \log (1+x)}{x}}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow 0}}=\lim _{x \rightarrow 0} \underbrace{\frac{\frac{a}{1+x}}{1}}_{\substack{\text { num } \rightarrow a \\ \text { den } \rightarrow 1}}=a
$$

Since $(1+x)^{a / x}=\exp \left[\log \left((1+x)^{a / x}\right)\right]$ and the exponential function is continuous, $\uparrow$ our original limit is $e^{a}$.

Here is a more complicated example of a $1^{\infty}$ indeterminate form.
Example 3.7.21 A more complicated example.
In the limit

$$
\lim _{x \rightarrow 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^{2}}}
$$

the base, $\frac{\sin x}{x}$, converges to 1 (see Example 3.7.3) and the exponent, $\frac{1}{x^{2}}$, goes to $\infty$. But if we take logarithms then

$$
\log \left(\frac{\sin x}{x}\right)^{\frac{1}{x^{2}}}=\frac{\log \frac{\sin x}{x}}{x^{2}}
$$

then, in the limit $x \rightarrow 0$, we have a $\frac{0}{0}$ indeterminate form. One application of l'Hôpital's rule gives

$$
\lim _{x \rightarrow 0} \underbrace{\frac{\log \frac{\sin x}{x}}{x^{2}}}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow 0}}=\lim _{x \rightarrow 0} \frac{\frac{x}{\sin x} \frac{x \cos x-\sin x}{x^{2}}}{2 x}=\lim _{x \rightarrow 0} \frac{\frac{x \cos x-\sin x}{x \sin x}}{2 x}=\lim _{x \rightarrow 0} \frac{x \cos x-\sin x}{2 x^{2} \sin x}
$$

which is another $\frac{0}{0}$ form. Applying l'Hôpital's rule again gives:

$$
\begin{aligned}
\lim _{x \rightarrow 0} \underbrace{\frac{x \cos x-\sin x}{2 x^{2} \sin x}}_{\substack{\text { num } \rightarrow 0 \\
\text { den } \rightarrow 0}} & =\lim _{x \rightarrow 0} \frac{\cos x-x \sin x-\cos x}{4 x \sin x+2 x^{2} \cos x} \\
& =-\lim _{x \rightarrow 0} \frac{x \sin x}{4 x \sin x+2 x^{2} \cos x}=-\lim _{x \rightarrow 0} \frac{\sin x}{4 \sin x+2 x \cos x}
\end{aligned}
$$

which is yet another $\frac{0}{0}$ form. Once more with l'Hôpital's rule:

$$
\begin{aligned}
-\lim _{x \rightarrow 0} \underbrace{\frac{\sin x}{4 \sin x+2 x \cos x}}_{\substack{\text { num } \rightarrow 0 \\
\text { den } \rightarrow 0}} & =-\lim _{x \rightarrow 0} \underbrace{\frac{\cos x}{4 \cos x+2 \cos x-2 x \sin x}}_{\substack{\text { num } \rightarrow 1 \\
\text { den } \rightarrow 6}} \\
& =-\frac{1}{6}
\end{aligned}
$$

Oof! We have just shown that the logarithm of our original limit is $-\frac{1}{6}$. Hence the original limit itself is $e^{-1 / 6}$.
This was quite a complicated example. However it does illustrate the importance of cleaning up your algebraic expressions. This will both reduce the amount of work you have to do and will also reduce the number of errors you make.

Example 3.7.21

### 3.7.3.7 $\leftrightarrows 0^{0}$ indeterminate form

Like the $1^{\infty}$ form, this can be treated by considering its logarithm.
Example 3.7.22 Compute $\lim _{x \rightarrow 0+} x^{x}$.
For example, in the limit

$$
\lim _{x \rightarrow 0+} x^{x}
$$

both the base, $x$, and the exponent, also $x$, go to zero. But if we consider the logarithm then we have

$$
\log x^{x}=x \log x
$$

which is a $0 \cdot \infty$ indeterminate form, which we already know how to treat. In fact, we already found, in Example 3.7.15, that

$$
\lim _{x \rightarrow 0+} x \log x=0
$$

Since the exponential is a continuous function

$$
\lim _{x \rightarrow 0+} x^{x}=\lim _{x \rightarrow 0+} \exp (x \log x)=\exp \left(\lim _{x \rightarrow 0+} x \log x\right)=e^{0}=1
$$

$\begin{array}{ll} \\ & \text { Example 3.7.22 }\end{array}$

### 3.7.3.8 $\leadsto \infty^{0}$ indeterminate form

Again, we can treat this form by considering its logarithm.
Example 3.7.23 Find $\lim _{x \rightarrow+\infty} x^{\frac{1}{x}}$.
For example, in the limit

$$
\lim _{x \rightarrow+\infty} x^{\frac{1}{x}}
$$

the base, $x$, goes to infinity and the exponent, $\frac{1}{x}$, goes to zero. But if we take logarithms

$$
\log x^{\frac{1}{x}}=\frac{\log x}{x}
$$

which is an $\frac{\infty}{\infty}$ form, which we know how to treat.

$$
\lim _{x \rightarrow+\infty} \underbrace{\frac{\log x}{x}}_{\substack{\text { num } \rightarrow \infty \\ \text { den } \rightarrow \infty}}=\lim _{x \rightarrow+\infty} \underbrace{\frac{\frac{1}{x}}{1}}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow 1}}=0
$$

Since the exponential is a continuous function

$$
\lim _{x \rightarrow+\infty} x^{\frac{1}{x}}=\lim _{x \rightarrow+\infty} \exp \left(\frac{\log x}{x}\right)=\exp \left(\lim _{x \rightarrow \infty} \frac{\log x}{x}\right)=e^{0}=1
$$

### 3.7.4 $\Perp$ Exercises

Exercises - Stage 1 In Questions 3.7.4.1 to 3.7.4.3, you are asked to give pairs of functions that combine to make indeterminate forms. Remember that an indeterminate form is indeterminate precisely because its limit can take on a number of values.

1. Give two functions $f(x)$ and $g(x)$ with the following properties:
i $\lim _{x \rightarrow \infty} f(x)=\infty$
ii $\lim _{x \rightarrow \infty} g(x)=\infty$
iii $\lim _{x \rightarrow \infty} \frac{f(x)}{g(x)}=2.5$
2. Give two functions $f(x)$ and $g(x)$ with the following properties:
i $\lim _{x \rightarrow \infty} f(x)=\infty$
ii $\lim _{x \rightarrow \infty} g(x)=\infty$
iii $\lim _{x \rightarrow \infty} \frac{f(x)}{g(x)}=0$
3. Give two functions $f(x)$ and $g(x)$ with the following properties:
i $\lim _{x \rightarrow \infty} f(x)=1$
ii $\lim _{x \rightarrow \infty} g(x)=\infty$
iii $\lim _{x \rightarrow \infty}[f(x)]^{g(x)}=5$

## Exercises - Stage 2

4. *. Evaluate $\lim _{x \rightarrow 1} \frac{x^{3}-e^{x-1}}{\sin (\pi x)}$.
5. *. Evaluate $\lim _{x \rightarrow 0+} \frac{\log x}{x}$. (Remember: in these notes, log means logarithm base e.)
6. *. Evaluate $\lim _{x \rightarrow \infty}(\log x)^{2} e^{-x}$.
7. *. Evaluate $\lim _{x \rightarrow \infty} x^{2} e^{-x}$.
8. *. Evaluate $\lim _{x \rightarrow 0} \frac{x-x \cos x}{x-\sin x}$.
9. Evaluate $\lim _{x \rightarrow 0} \frac{\sqrt{x^{6}+4 x^{4}}}{x^{2} \cos x}$.
10. *. Evaluate $\lim _{x \rightarrow \infty} \frac{(\log x)^{2}}{x}$.
11. *. Evaluate $\lim _{x \rightarrow 0} \frac{1-\cos x}{\sin ^{2} x}$.
12. Evaluate $\lim _{x \rightarrow 0} \frac{x}{\sec x}$.
13. Evaluate $\lim _{x \rightarrow 0} \frac{\csc x \cdot \tan x \cdot\left(x^{2}+5\right)}{e^{x}}$.
14. Evaluate $\lim _{x \rightarrow \infty} \sqrt{2 x^{2}+1}-\sqrt{x^{2}+x}$.
15. *. Evaluate $\lim _{x \rightarrow 0} \frac{\sin \left(x^{3}+3 x^{2}\right)}{\sin ^{2} x}$.
16. *. Evaluate $\lim _{x \rightarrow 1} \frac{\log \left(x^{3}\right)}{x^{2}-1}$.
17. *. Evaluate $\lim _{x \rightarrow 0} \frac{e^{-1 / x^{2}}}{x^{4}}$.
18. *. Evaluate $\lim _{x \rightarrow 0} \frac{x e^{x}}{\tan (3 x)}$.
19. Evaluate $\lim _{x \rightarrow 0} \sqrt[x^{2}]{\sin ^{2} x}$.
20. Evaluate $\lim _{x \rightarrow 0} \sqrt[x^{2}]{\cos x}$.
21. Evaluate $\lim _{x \rightarrow 0^{+}} e^{x \log x}$.
22. Evaluate $\lim _{x \rightarrow 0}\left[-\log \left(x^{2}\right)\right]^{x}$.
23. *. Find $c$ so that $\lim _{x \rightarrow 0} \frac{1+c x-\cos x}{e^{x^{2}}-1}$ exists.
24. *. Evaluate $\lim _{x \rightarrow 0} \frac{e^{k \sin \left(x^{2}\right)}-\left(1+2 x^{2}\right)}{x^{4}}$, where $k$ is a constant.

## Exercises - Stage 3

25. Suppose an algorithm, given an input with with $n$ variables, will terminate in at most $S(n)=5 n^{4}-13 n^{3}-4 n+\log (n)$ steps. A researcher writes that the algorithm will terminate in roughly at most $A(n)=5 n^{4}$ steps. Show that the percentage error involved in using $A(n)$ instead of $S(n)$ tends to zero as $n$ gets very large. What happens to the absolute error?
Remark: this is a very common kind of approximation. When people deal with
functions that give very large numbers, often they don't care about the exact large number-they only want a ballpark. So, a complicated function might be replaced by an easier function that doesn't give a large relative error. If you would like to know more about the ways people describe functions that give very large numbers, you can read about "Big O Notation" in Section 3.6.3 of the CLP2 textbook.

## ToWARDS INTEGRAL CALCULUS

We have now come to the final topic of the course - antiderivatives. This is only a short section since it is really just to give a taste of the next calculus subject: integral calculus.

## 4.1^ Introduction to Antiderivatives

### 4.1.1 Introduction to Antiderivatives

So far in the course we have learned how to determine the rate of change (i.e. the derivative) of a given function. That is

$$
\text { given a function } f(x) \text { find } \frac{\mathrm{d} f}{\mathrm{~d} x} \text {. }
$$

Along the way we developed an understanding of limits, which allowed us to define instantaneous rates of change - the derivative. We then went on to develop a number of applications of derivatives to modelling and approximation. In this last section we want to just introduce the idea of antiderivatives. That is

$$
\text { given a derivative } \frac{\mathrm{d} f}{\mathrm{~d} x} \text { find the original function } f(x) \text {. }
$$

For example - say we know that

$$
\frac{\mathrm{d} f}{\mathrm{~d} x}=x^{2}
$$

and we want to find $f(x)$. From our previous experience differentiating we know that derivatives of polynomials are again polynomials. So we guess that our unknown function $f(x)$ is a polynomial. Further we know that when we differentiate $x^{n}$ we get $n x^{n-1}$

- multiply by the exponent and reduce the exponent by 1 . So to end up with a derivative of $x^{2}$ we need to have differentiated an $x^{3}$. But $\frac{\mathrm{d}}{\mathrm{d} x} x^{3}=3 x^{2}$, so we need to divide both sides by 3 to get the answer we want. That is

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left(\frac{1}{3} x^{3}\right)=x^{2}
$$

However we know that the derivative of a constant is zero, so we also have

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left(\frac{1}{3} x^{3}+1\right)=x^{2}
$$

and

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left(\frac{1}{3} x^{3}-\pi\right)=x^{2}
$$

At this point it will really help the discussion to give a name to what we are doing.

## Definition 4.1.1

A function $F(x)$ that satisfies

$$
\frac{\mathrm{d}}{\mathrm{~d} x} F(x)=f(x)
$$

is called an antiderivative of $f(x)$.

Notice the use of the indefinite article there - an antiderivative. This is precisely because we can always add or subtract a constant to an antiderivative and when we differentiate we'll get the same answer. We can write this as a lemma, but it is actually just Corollary 2.13.13 (from back in the section on the mean-value theorem) in disguise.

## Lemma 4.1.2

Let $F(x)$ be an antiderivative of $f(x)$, then for any constant $c$, the function $F(x)+c$ is also an antiderivative of $f(x)$.

Because of this lemma we typically write antiderivatives with " $+c$ " tacked on the end. That is, if we know that $F^{\prime}(x)=f(x)$, then we would state that the antiderivative of $f(x)$ is

$$
F(x)+c
$$

where this " $+c$ " is there to remind us that we can always add or subtract some constant and it will still be an antiderivative of $f(x)$. Hence the antiderivative of $x^{2}$ is

$$
\frac{1}{3} x^{3}+c
$$

Similarly, the antiderivative of $x^{4}$ is

$$
\frac{1}{5} x^{5}+c
$$

and for $\sqrt{x}=x^{1 / 2}$ it is

$$
\frac{2}{3} x^{3 / 2}+c
$$

This last one is tricky (at first glance) - but we can always check our answer by differentiating.

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left(\frac{2}{3} x^{3 / 2}+c\right)=\frac{2}{3} \cdot \frac{3}{2} x^{1 / 2}+0
$$

Now in order to determine the value of $c$ we need more information. For example, we might be asked

Given that $g^{\prime}(t)=t^{2}$ and $g(3)=7$ find $g(t)$.
We are given the derivative and one piece of additional information and from these two facts we need to find the original function. From our work above we know that

$$
g(t)=\frac{1}{3} t^{3}+c
$$

and we can find $c$ from the other piece of information

$$
7=g(3)=\frac{1}{3} \cdot 27+c=9+c
$$

Hence $c=-2$ and so

$$
g(t)=\frac{1}{3} t^{3}-2
$$

We can then very easily check our answer by recomputing $g(3)$ and $g^{\prime}(t)$. This is a good habit to get into.

Finding antiderivatives of polynomials is generally not too hard. We just need to use the rule

$$
\text { if } f(x)=x^{n} \text { then } F(x)=\frac{1}{n+1} x^{n+1}+c
$$

Of course this breaks down when $n=-1$. In order to find an antiderivative for $f(x)=\frac{1}{x}$ we need to remember that $\frac{\mathrm{d}}{\mathrm{d} x} \log x=\frac{1}{x}$, and more generally that $\frac{\mathrm{d}}{\mathrm{d} x} \log |x|=\frac{1}{x}$. See Example 2.10.4. So

$$
\text { if } f(x)=\frac{1}{x} \text { then } F(x)=\log |x|+c
$$

Example 4.1.3 Antiderivative of $3 x^{5}-7 x^{2}+2 x+3+x^{-1}-x^{-2}$.
Let $f(x)=3 x^{5}-7 x^{2}+2 x+3+x^{-1}-x^{-2}$. Then the antiderivative of $f(x)$ is

$$
\begin{aligned}
F(x) & =\frac{3}{6} x^{6}-\frac{7}{3} x^{3}+\frac{2}{2} x^{2}+3 x+\log |x|-\frac{1}{-1} x^{-1}+c \quad \text { clean it up } \\
& =\frac{1}{2} x^{6}-\frac{7}{3} x^{3}+x^{2}+3 x+\log |x|+x^{-1}+c
\end{aligned}
$$

Now to check we should differentiate and hopefully we get back to where we started

$$
\begin{aligned}
F^{\prime}(x) & =\frac{6}{2} x^{5}-\frac{7}{3} \cdot 3 x^{2}+2 x+3+\frac{1}{x}-x^{-2} \\
& =3 x^{5}-7 x^{2}+2 x+3+\frac{1}{x}-x^{-2}
\end{aligned}
$$

In your next calculus course you will develop a lot of machinery to help you find antiderivatives. At this stage about all that we can do is continue the sort of thing we have done. Think about the derivatives we know and work backwards. So, for example, we can take a list of derivatives

| $F(x)$ | 1 | $x^{n}$ | $\sin x$ | $\cos x$ | $\tan x$ | $e^{x}$ | $\ln \|x\|$ | $\arcsin x$ | $\arctan x$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $f(x)=\frac{\mathrm{d}}{\mathrm{d} x} F(x)$ | 0 | $n x^{n-1}$ | $\cos x$ | $-\sin x$ | $\sec ^{2} x$ | $e^{x}$ | $\frac{1}{x}$ | $\frac{1}{\sqrt{1-x^{2}}}$ | $\frac{1}{1+x^{2}}$ |

and flip it upside down to give the tables of antiderivatives.

| $f(x)=\frac{\mathrm{d}}{\mathrm{d} x} F(x)$ | 0 | $n x^{n-1}$ | $\cos x$ | $-\sin x$ | $\sec ^{2} x$ | $e^{x}$ | $\frac{1}{x}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $F(x)$ | $c$ | $x^{n}+c$ | $\sin x+c$ | $\cos x+c$ | $\tan x+c$ | $e^{x}+c$ | $\ln \|x\|+c$ |


| $f(x)=\frac{\mathrm{d}}{\mathrm{d} x} F(x)$ | $\frac{1}{\sqrt{1-x^{2}}}$ | $\frac{1}{1+x^{2}}$ |
| :--- | :--- | :--- |
| $F(x)$ | $\arcsin x+c$ | $\arctan x+c$ |

Here $c$ is just a constant - any constant. But we can do a little more; clean up $x^{n}$ by dividing by $n$ and then replacing $n$ by $n+1$. Similarly we can tweak $\sin x$ by multiplying by -1 :

| $f(x)=\frac{\mathrm{d}}{\mathrm{d} x} F(x)$ | 0 | $x^{n}$ | $\cos x$ | $\sin x$ | $\sec ^{2} x$ | $e^{x}$ | $\frac{1}{x}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $F(x)$ | $c$ | $\frac{1}{n+1} x^{n+1}+c$ | $\sin x+c$ | $-\cos x+c$ | $\tan x+c$ | $e^{x}+c$ | $\ln \|x\|+c$ |


| $f(x)=\frac{\mathrm{d}}{\mathrm{d} x} F(x)$ | $\frac{1}{\sqrt{1-x^{2}}}$ | $\frac{1}{1+x^{2}}$ |
| :--- | :--- | :--- |
| $F(x)$ | $\arcsin x+c$ | $\arctan x+c$ |

Here are a couple more examples.

Example 4.1.4 Antiderivatives of $\sin x, \cos 2 x$ and $\frac{1}{1+4 x^{2}}$.
Consider the functions

$$
f(x)=\sin x+\cos 2 x \quad g(x)=\frac{1}{1+4 x^{2}}
$$

Find their antiderivatives.
Solution The first one we can almost just look up our table. Let $F$ be the antiderivative of $f$, then

$$
F(x)=-\cos x+\sin 2 x+c \quad \text { is not quite right. }
$$

When we differentiate to check things, we get a factor of two coming from the chain rule. Hence to compensate for that we multiply $\sin 2 x$ by $\frac{1}{2}$ :

$$
F(x)=-\cos x+\frac{1}{2} \sin 2 x+c
$$

Differentiating this shows that we have the right answer.
Similarly, if we use $G$ to denote the antiderivative of $g$, then it appears that $G$ is nearly $\arctan x$. To get this extra factor of 4 we need to substitute $x \mapsto 2 x$. So we try

$$
G(x)=\arctan (2 x)+c \quad \text { which is nearly correct. }
$$

Differentiating this gives us

$$
G^{\prime}(x)=\frac{2}{1+(2 x)^{2}}=2 g(x)
$$

Hence we should multiply by $\frac{1}{2}$. This gives us

$$
G(x)=\frac{1}{2} \arctan (2 x)+c
$$

$\uparrow$ We can then check that this is, in fact, correct just by differentiating.

Now let's do a more substantial example.
Example 4.1.5 Position as antiderivative of velocity.
Suppose that we are driving to class. We start at $x=0$ at time $t=0$. Our velocity is $v(t)=50 \sin (t)$. The class is at $x=25$. When do we get there?
Solution Let's denote by $x(t)$ our position at time $t$. We are told that

- $x(0)=0$
- $x^{\prime}(t)=50 \sin t$

We have to determine $x(t)$ and then find the time $T$ that obeys $x(T)=25$. Now armed with our table above we know that the antiderivative of $\sin t$ is just $-\cos t$. We can check this:

$$
\frac{\mathrm{d}}{\mathrm{~d} t}(-\cos t)=\sin t
$$

We can then get the factor of 50 by multiplying both sides of the above equation by 50 :

$$
\frac{\mathrm{d}}{\mathrm{~d} t}(-50 \cos t)=50 \sin t
$$

And of course, this is just an antiderivative of $50 \sin t$; to write down the general antiderivative we just add a constant $c$ :

$$
\frac{\mathrm{d}}{\mathrm{~d} t}(-50 \cos t+c)=50 \sin t
$$

Since $v(t)=\frac{\mathrm{d}}{\mathrm{d} t} x(t)$, this antiderivative is $x(t)$ :

$$
x(t)=-50 \cos t+c
$$

To determine $c$ we make use of the other piece of information we are given, namely

$$
x(0)=0 .
$$

Substituting this in gives us

$$
x(0)=-50 \cos 0+c=-50+c
$$

Hence we must have $c=50$ and so

$$
x(t)=-50 \cos t+50=50(1-\cos t) .
$$

Now that we have our position as a function of time, we can determine how long it takes us to arrive there. That is, we can find the time $T$ so that $x(T)=25$.

$$
\begin{aligned}
25=x(T) & =50(1-\cos T) \\
\frac{1}{2} & =1-\cos T \\
-\frac{1}{2} & =-\cos T \\
\frac{1}{2} & =\cos T
\end{aligned}
$$

Recalling our special triangles, we see that $T=\frac{\pi}{3}$.
Example 4.1.5
The example below shows how antiderivatives arise naturally when studying differ-
ential equations.
Example 4.1.6 Theorem 3.3.2 revisited..
Back in Section 3.3 we encountered a simple differential equation, namely equation 3.3.1. We were able to solve this equation by guessing the answer and then checking it carefully. We can derive the solution more systematically by using antiderivatives.
Recall equation 3.3.1:

$$
\frac{\mathrm{d} Q}{\mathrm{~d} t}=-k Q
$$

where $Q(t)$ is the amount of radioactive material at time $t$ and we assume $Q(t)>0$. Take this equation and divide both sides by $Q(t)$ to get

$$
\frac{1}{Q(t)} \frac{\mathrm{d} Q}{\mathrm{~d} t}=-k
$$

At this point we should ${ }^{a}$ think that the left-hand side is familiar. Now is a good moment to look back at logarithmic differentiation in Section 2.10.
The left-hand side is just the derivative of $\log Q(t)$ :

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t}(\log Q(t)) & =\frac{1}{Q(t)} \frac{\mathrm{d} Q}{\mathrm{~d} t} \\
& =-k
\end{aligned}
$$

So to solve this equation, we are really being asked to find all functions $\log Q(t)$ having derivative $-k$. That is, we need to find all antiderivatives of $-k$. Of course that is just $-k t+c$. Hence we must have

$$
\log Q(t)=-k t+c
$$

and then taking the exponential of both sides gives

$$
Q(t)=e^{-k t+c}=e^{c} \cdot e^{-k t}=C e^{-k t}
$$

where $C=e^{c}$. This is precisely Theorem 3.3.2.
$a$ Well - perhaps it is better to say "notice that". Let's not make this a moral point.

The above is a small example of the interplay between antiderivatives and differential equations.

Here is another example of how we might use antidifferentiation to compute areas or volumes.

Example 4.1.7 Volume of a cone.
We know (especially if one has revised the material in the appendix and Appendix B.5.2 in particular) that the volume of a right-circular cone is

$$
V=\frac{\pi}{3} r^{2} h
$$

where $h$ is the height of the cone and $r$ is the radius of its base. Now, the derivation of this formula given in Appendix B.5.2 is not too simple. We present an alternate proof here that uses antiderivatives.


Consider cutting off a portion of the cone so that its new height is $x$ (rather than $h$ ). Call the volume of the resulting smaller cone $V(x)$. We are going to determine $V(x)$ for all $x \geq 0$, including $x=h$, by first evaluating $V^{\prime}(x)$ and $V(0)$ (which is obviously $0)$.
Call the radius of the base of the new smaller cone $y$ (rather than $r$ ). By similar triangles we know that

$$
\frac{r}{h}=\frac{y}{x} .
$$

Now keep $x$ and $y$ fixed and consider cutting off a little more of the cone so its height is $X$. When we do so, the radius of the base changes from $y$ to $Y$ and again by similar triangles we know that

$$
\frac{Y}{X}=\frac{y}{x}=\frac{r}{h}
$$

The change in volume is then

$$
V(x)-V(X)
$$

Of course if we knew the formula for the volume of a cone, then we could compute the above exactly. However, even without knowing the volume of a cone, it is easy to derive upper and lower bounds on this quantity. The piece removed has bottom radius $y$ and top radius $Y$. Hence its volume is bounded above and below by the cylinders of height $x-X$ and with radius $y$ and $Y$ respectively. Hence

$$
\pi Y^{2}(x-X) \leq V(x)-V(X) \leq \pi y^{2}(x-X)
$$

since the volume of a cylinder is just the area of its base times its height. Now massage this expression a little

$$
\pi Y^{2} \leq \frac{V(x)-V(X)}{x-X} \leq \pi y^{2}
$$

The middle term now looks like a derivative; all we need to do is take the limit as $X \rightarrow x$ :

$$
\lim _{X \rightarrow x} \pi Y^{2} \leq \lim _{X \rightarrow x} \frac{V(x)-V(X)}{x-X} \leq \lim _{X \rightarrow x} \pi y^{2}
$$

The rightmost term is independent of $X$ and so is just $\pi y^{2}$. In the leftmost term, as $X \rightarrow x$, we must have that $Y \rightarrow y$. Hence the leftmost term is just $\pi y^{2}$. Then by the squeeze theorem (Theorem 1.4.18) we know that

$$
\frac{\mathrm{d} V}{\mathrm{~d} x}=\lim _{X \rightarrow x} \frac{V(x)-V(X)}{x-X}=\pi y^{2}
$$

But we know that

$$
y=\frac{r}{h} \cdot x
$$

so

$$
\frac{\mathrm{d} V}{\mathrm{~d} x}=\pi\left(\frac{r}{h}\right)^{2} x^{2}
$$

Now we can antidifferentiate to get back to $V$ :

$$
V(x)=\frac{\pi}{3}\left(\frac{r}{h}\right)^{2} x^{3}+c
$$

To determine $c$ notice that when $x=0$ the volume of the cone is just zero and so $c=0$. Thus

$$
V(x)=\frac{\pi}{3}\left(\frac{r}{h}\right)^{2} x^{3}
$$

and so when $x=h$ we are left with

$$
V(h)=\frac{\pi}{3}\left(\frac{r}{h}\right)^{2} h^{3}=\frac{\pi}{3} r^{2} h
$$

as required.
Example 4.1.7

### 4.1.2 $\leadsto$ Exercises

## Exercises - Stage 1

1. Let $f(x)$ be a function with derivative $f^{\prime}(x)$. What is the most general antiderivative of $f^{\prime}(x)$ ?
2. On the graph below, the black curve is $y=f(x)$. Which of the coloured curves is an antiderivative of $f(x)$ ?


Exercises - Stage 2 In Questions 4.1.2.3 through 4.1.2.12, you are asked to find the antiderivative of a function. Phrased like this, we mean the most general antiderivative. These will all include some added constant. The table after Example 4.1.3 might be of help.In Questions 4.1.2.13 through 4.1.2.16, you are asked to find a specific antiderivative of a function. In this case, you should be able to solve for the entire function-no unknown constants floating about.In Questions 4.1.2.17 through 4.1.2.19, we will explore some simple situations where antiderivatives might arise.
3. Find the antiderivative of $f(x)=3 x^{2}+5 x^{4}+10 x-9$.
4. Find the antiderivative of $f(x)=\frac{3}{5} x^{7}-18 x^{4}+x$.
5. Find the antiderivative of $f(x)=4 \sqrt[3]{x}-\frac{9}{2 x^{2.7}}$.
6. Find the antiderivative of $f(x)=\frac{1}{7 \sqrt{x}}$.
7. Find the antiderivative of $f(x)=e^{5 x+11}$.
8. Find the antiderivative of $f(x)=3 \sin (5 x)+7 \cos (13 x)$.
9. Find the antiderivative of $f(x)=\sec ^{2}(x+1)$.
10. Find the antiderivative of $f(x)=\frac{1}{x+2}$.
11. Find the antiderivative of $f(x)=\frac{7}{\sqrt{3-3 x^{2}}}$.
12. Find the antiderivative of $f(x)=\frac{1}{1+25 x^{2}}$
13. Find the function $f(x)$ with $f^{\prime}(x)=3 x^{2}-9 x+4$ and $f(1)=10$.
14. Find the function $f(x)$ with $f^{\prime}(x)=\cos (2 x)$ and $f(\pi)=\pi$.
15. Find the function $f(x)$ with $f^{\prime}(x)=\frac{1}{x}$ and $f(-1)=0$.
16. Find the function $f(x)$ with $f^{\prime}(x)=\frac{1}{\sqrt{1-x^{2}}}+1$ and $f(1)=-\frac{\pi}{2}$.
17. Suppose a population of bacteria at time $t$ (measured in hours) is growing at a rate of $100 e^{2 t}$ individuals per hour. Starting at time $t=0$, how long will it take the initial colony to increase by 300 individuals?
18. Your bank account at time $t$ (measured in years) is growing at a rate of

$$
1500 e^{\frac{t}{50}}
$$

dollars per year. How much money is in your account at time $t$ ?
19. At time $t$ during a particular day, $0 \leq t \leq 24$, your house consumes energy at a rate of

$$
0.5 \sin \left(\frac{\pi}{24} t\right)+0.25
$$

kW . (Your consumption was smallest in the middle of the night, and peaked at noon.) How much energy did your house consume in that day?

Exercises - Stage 3 For Questions 4.1.2.21 through 4.1.2.26, you are again asked to find the antiderivatives of certain functions. In general, finding antiderivatives can
be extremely difficult-indeed, it will form the main topic of next semester's calculus course. However, you can work out the antiderivatives of the functions below using what you've learned so far about derivatives.
20. *. Let $f(x)=2 \sin ^{-1} \sqrt{x}$ and $g(x)=\sin ^{-1}(2 x-1)$. Find the derivative of $f(x)-g(x)$ and simplify your answer. What does the answer imply about the relation between $f(x)$ and $g(x)$ ?
21. Find the antiderivative of $f(x)=2 \cos (2 x) \cos (3 x)-3 \sin (2 x) \sin (3 x)$.
22. Find the antiderivative of $f(x)=\frac{\left(x^{2}+1\right) e^{x}-e^{x}(2 x)}{\left(x^{2}+1\right)^{2}}$.
23. Find the antiderivative of $f(x)=3 x^{2} e^{x^{3}}$.
24. Find the antiderivative of $f(x)=5 x \sin \left(x^{2}\right)$.
25. Find the antiderivative of $f(x)=e^{\log x}$.
26. Find the antiderivative of $f(x)=\frac{7}{\sqrt{3-x^{2}}}$.
27. Imagine forming a solid by revolving the parabola $y=x^{2}+1$ around the $x$-axis, as in the picture below.


Use the method of Example 4.1.7 to find the volume of such an object if the segment of the parabola that we rotate runs from $x=-H$ to $x=H$.

```
Appendix A
```


## High School Material

This chapter is really split into two parts.

- Sections A. 1 to A. 13 we expect you to understand and know.
- The very last section, Section A.14, contains results that we don't expect you to memorise, but that we think you should be able to quickly derive from other results you know.


## A.1^ Similar Triangles



Two triangles $T_{1}, T_{2}$ are similar when

- (AAA - angle angle angle) The angles of $T_{1}$ are the same as the angles of $T_{2}$.
- (SSS - side side side) The ratios of the side lengths are the same. That is

$$
\frac{A}{a}=\frac{B}{b}=\frac{C}{c}
$$

- (SAS - side angle side) Two sides have lengths in the same ratio and the angle between them is the same. For example

$$
\frac{A}{a}=\frac{C}{c} \text { and angle } \beta \text { is same }
$$

## A. $2 \times$ Pythagoras

For a right-angled triangle the length of the hypotenuse is related to the lengths of the other two sides by

$(\text { adjacent })^{2}+(\text { opposite })^{2}=(\text { hypotenuse })^{2}$

## A.3ム Trigonometry - Definitions



$$
\begin{aligned}
\sin \theta & =\frac{\text { opposite }}{\text { hypotenuse }} \\
\cos \theta & =\frac{\text { adjacent }}{\text { hypotenuse }} \\
\csc \theta & =\frac{1}{\sin \theta} \\
\tan \theta & =\frac{\text { opposite }}{\text { oppacent }}
\end{aligned} \quad \frac{1}{\cos \theta}
$$

## A.4^Radians, Arcs and Sectors



For a circle of radius $r$ and angle of $\theta$ radians:

- Arc length $L(\theta)=r \theta$.
- Area of sector $A(\theta)=\frac{\theta}{2} r^{2}$.


## A.5』 Trigonometry - Graphs





## A. $6 \pm$ Trigonometry - Special Triangles



From the above pair of special triangles we have

$$
\begin{array}{llrl}
\sin \frac{\pi}{4}=\frac{1}{\sqrt{2}} & \sin \frac{\pi}{6}=\frac{1}{2} & \sin \frac{\pi}{3}=\frac{\sqrt{3}}{2} \\
\cos \frac{\pi}{4}=\frac{1}{\sqrt{2}} & \cos \frac{\pi}{6}=\frac{\sqrt{3}}{2} & \cos \frac{\pi}{3}=\frac{1}{2} \\
\tan \frac{\pi}{4}=1 & \tan \frac{\pi}{6}=\frac{1}{\sqrt{3}} & \tan \frac{\pi}{3}=\sqrt{3}
\end{array}
$$

## A.7』 Trigonometry - Simple Identities

- Periodicity

$$
\sin (\theta+2 \pi)=\sin (\theta) \quad \cos (\theta+2 \pi)=\cos (\theta)
$$

- Reflection

$$
\sin (-\theta)=-\sin (\theta) \quad \cos (-\theta)=\cos (\theta)
$$

- Reflection around $\pi / 4$

$$
\sin \left(\frac{\pi}{2}-\theta\right)=\cos \theta \quad \cos \left(\frac{\pi}{2}-\theta\right)=\sin \theta
$$

- Reflection around $\pi / 2$

$$
\sin (\pi-\theta)=\sin \theta \quad \cos (\pi-\theta)=-\cos \theta
$$

- Rotation by $\pi$

$$
\sin (\theta+\pi)=-\sin \theta \quad \cos (\theta+\pi)=-\cos \theta
$$

- Pythagoras

$$
\frac{\sin ^{2} \theta+\cos ^{2} \theta=1}{491}
$$

## A.8』 Trigonometry - Add and Subtract Angles

- Sine

$$
\sin (\alpha \pm \beta)=\sin (\alpha) \cos (\beta) \pm \cos (\alpha) \sin (\beta)
$$

- Cosine

$$
\cos (\alpha \pm \beta)=\cos (\alpha) \cos (\beta) \mp \sin (\alpha) \sin (\beta)
$$

## A.94 Inverse Trigonometric Functions

Some of you may not have studied inverse trigonometric functions in highschool, however we still expect you to know them by the end of the course.


Since these functions are inverses of each other we have

$$
\begin{aligned}
\arcsin (\sin \theta) & =\theta & -\frac{\pi}{2} & \leq \theta \leq \frac{\pi}{2} \\
\arccos (\cos \theta) & =\theta & 0 & \leq \theta \leq \pi \\
\arctan (\tan \theta) & =\theta & -\frac{\pi}{2} & \leq \theta \leq \frac{\pi}{2}
\end{aligned}
$$

and also

$$
\begin{aligned}
\sin (\arcsin x) & =x & & -1 \leq x \leq 1 \\
\cos (\arccos x) & =x & & -1 \leq x \leq 1 \\
\tan (\arctan x) & =x & & \text { any real } x
\end{aligned}
$$



Again

$$
\begin{array}{rr}
\operatorname{arccsc}(\csc \theta)=\theta & -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \theta \neq 0 \\
\operatorname{arcsec}(\sec \theta)=\theta & 0 \leq \theta \leq \pi, \theta \neq \frac{\pi}{2} \\
\operatorname{arccot}(\cot \theta)=\theta & 0<\theta<\pi
\end{array}
$$

and

$$
\begin{array}{rr}
\csc (\operatorname{arccsc} x)=x & |x| \geq 1 \\
\sec (\operatorname{arcsec} x)=x & |x| \geq 1 \\
\cot (\operatorname{arccot} x)=x & \text { any real } x
\end{array}
$$

## A.10^ Areas



- Area of a rectangle

$$
A=b h
$$

- Area of a triangle

$$
A=\frac{1}{2} b h=\frac{1}{2} a b \sin \theta
$$

- Area of a circle

$$
A=\pi r^{2}
$$

- Area of an ellipse

$$
A=\pi a b
$$

## A.11^ Volumes



- Volume of a rectangular prism

$$
V=l w h
$$

- Volume of a cylinder

$$
V=\pi r^{2} h
$$

- Volume of a cone

$$
V=\frac{1}{3} \pi r^{2} h
$$

- Volume of a sphere

$$
V=\frac{4}{3} \pi r^{3}
$$

## A.12^ Powers

In the following, $x$ and $y$ are arbitrary real numbers, and $q$ is an arbitrary constant that is strictly bigger than zero.

- $q^{0}=1$
- $q^{x+y}=q^{x} q^{y}, q^{x-y}=\frac{q^{x}}{q^{y}}$
- $q^{-x}=\frac{1}{q^{x}}$
- $\left(q^{x}\right)^{y}=q^{x y}$
- $\lim _{x \rightarrow \infty} q^{x}=\infty, \lim _{x \rightarrow-\infty} q^{x}=0$ if $q>1$
- $\lim _{x \rightarrow \infty} q^{x}=0, \lim _{x \rightarrow-\infty} q^{x}=\infty$ if $0<q<1$
- The graph of $2^{x}$ is given below. The graph of $q^{x}$, for any $q>1$, is similar.



## A.13ム Logarithms

In the following, $x$ and $y$ are arbitrary real numbers that are strictly bigger than 0 , and $p$ and $q$ are arbitrary constants that are strictly bigger than one.

- $q^{\log _{q} x}=x, \quad \log _{q}\left(q^{x}\right)=x$
- $\log _{q} x=\frac{\log _{p} x}{\log _{p} q}$
- $\log _{q} 1=0, \quad \log _{q} q=1$
- $\log _{q}(x y)=\log _{q} x+\log _{q} y$
- $\log _{q}\left(\frac{x}{y}\right)=\log _{q} x-\log _{q} y$
- $\log _{q}\left(\frac{1}{y}\right)=-\log _{q} y$,
- $\log _{q}\left(x^{y}\right)=y \log _{q} x$
- $\lim _{x \rightarrow \infty} \log _{q} x=\infty, \quad \lim _{x \rightarrow 0+} \log _{q} x=-\infty$
- The graph of $\log _{10} x$ is given below. The graph of $\log _{q} x$, for any $q>1$, is similar.



## A.14』 Highschool Material You Should be Able to Derive

- Graphs of $\csc \theta, \sec \theta$ and $\cot \theta$ :



- More Pythagoras

$$
\begin{array}{lll}
\sin ^{2} \theta+\cos ^{2} \theta=1 & \stackrel{\text { divide by } \cos ^{2} \theta}{\longmapsto} & \tan ^{2} \theta+1=\sec ^{2} \theta \\
\sin ^{2} \theta+\cos ^{2} \theta=1 & \stackrel{\text { divide by } \sin ^{2} \theta}{\longmapsto} & 1+\cot ^{2} \theta=\csc ^{2} \theta
\end{array}
$$

- Sine - double angle (set $\beta=\alpha$ in sine angle addition formula)

$$
\sin (2 \alpha)=2 \sin (\alpha) \cos (\alpha)
$$

- Cosine - double angle (set $\beta=\alpha$ in cosine angle addition formula)

$$
\begin{aligned}
\cos (2 \alpha) & =\cos ^{2}(\alpha)-\sin ^{2}(\alpha) & & \\
& =2 \cos ^{2}(\alpha)-1 & & \left(\text { use } \sin ^{2}(\alpha)=1-\cos ^{2}(\alpha)\right) \\
& =1-2 \sin ^{2}(\alpha) & & \text { (use } \left.\cos ^{2}(\alpha)=1-\sin ^{2}(\alpha)\right)
\end{aligned}
$$

- Composition of trigonometric and inverse trigonometric functions:

$$
\cos (\arcsin x)=\sqrt{1-x^{2}} \quad \sec (\arctan x)=\sqrt{1+x^{2}}
$$

and similar expressions.

# Origin of Trig, Area And Volume Formulas 

## B. $1 \wedge$ Theorems about Triangles

## B.1.1 $\leadsto$ Thales' Theorem

We want to get at right-angled triangles. A classic construction for this is to draw a triangle inside a circle, so that all three corners lie on the circle and the longest side forms the diameter of the circle. See the figure below in which we have scaled the circle to have radius 1 and the triangle has longest side 2 .


Thales theorem states that the angle at $C$ is always a right-angle. The proof is quite straight-forward and relies on two facts:

- the angles of a triangle add to $\pi$, and
- the angles at the base of an isosceles triangle are equal.

So we split the triangle $A B C$ by drawing a line from the centre of the circle to $C$. This creates two isosceles triangles $O A C$ and $O B C$. Since they are isosceles, the angles at their bases $\alpha$ and $\beta$ must be equal (as shown). Adding the angles of the original triangle now gives

$$
\pi=\alpha+(\alpha+\beta)+\beta=2(\alpha+\beta)
$$

So the angle at $C=\pi-(\alpha+\beta)=\pi / 2$.

## B.1.2 $\leadsto$ Pythagoras

Since trigonometry, at its core, is the study of lengths and angles in right-angled triangles, we must include a result you all know well, but likely do not know how to prove.


The lengths of the sides of any right-angled triangle are related by the famous result due to Pythagoras

$$
c^{2}=a^{2}+b^{2}
$$

There are many ways to prove this, but we can do so quite simply by studying the following diagram:


We start with a right-angled triangle with sides labeled $a, b$ and $c$. Then we construct a square of side-length $a+b$ and draw inside it 4 copies of the triangle arranged as shown in the centre of the above figure. The area in white is then $a^{2}+b^{2}$. Now move the triangles around to create the arrangement shown on the right of the above figure. The
area in white is bounded by a square of side-length $c$ and so its area is $c^{2}$. The area of the outer square didn't change when the triangles were moved, nor did the area of the triangles, so the white area cannot have changed either. This proves $a^{2}+b^{2}=c^{2}$.

## B.2^ Trigonometry

## B.2.1 Angles - Radians vs Degrees

For mathematics, and especially in calculus, it is much better to measure angles in units called radians rather than degrees. By definition, an arc of length $\theta$ on a circle of radius one subtends an angle of $\theta$ radians at the centre of the circle.


The circle on the left has radius 1 , and the arc swept out by an angle of $\theta$ radians has length $\theta$. Because a circle of radius one has circumference $2 \pi$ we have

$$
\begin{aligned}
2 \pi \text { radians } & =360^{\circ} & \pi \text { radians } & =180^{\circ}
\end{aligned} \frac{\pi}{2} \text { radians }=90^{\circ}
$$

More generally, consider a circle of radius $r$. Let $L(\theta)$ denote the length of the arc swept out by an angle of $\theta$ radians and let $A(\theta)$ denote the area of the sector (or wedge) swept out by the same angle. Since the angle sweeps out the fraction $\frac{\theta}{2 \pi}$ of a whole circle, we have

$$
\begin{aligned}
& L(\theta)=2 \pi r \cdot \frac{\theta}{2 \pi}=\theta r \\
& A(\theta)=\pi r^{2} \cdot \frac{\theta}{2 \pi}=\frac{\theta}{2} r^{2}
\end{aligned}
$$

and

## B.2.2 $\leadsto$ Trig Function Definitions

The trigonometric functions are defined as ratios of the lengths of the sides of a rightangle triangle as shown in the left of the diagram below. These ratios depend only on the angle $\theta$.


The trigonometric functions sine, cosine and tangent are defined as ratios of the lengths of the sides

$$
\sin \theta=\frac{\text { opposite }}{\text { hypotenuse }} \quad \cos \theta=\frac{\text { adjacent }}{\text { hypotenuse }} \quad \tan \theta=\frac{\text { opposite }}{\text { adjacent }}=\frac{\sin \theta}{\cos \theta} .
$$

These are frequently abbreviated as

$$
\sin \theta=\frac{\mathrm{o}}{\mathrm{~h}} \quad \cos \theta=\frac{\mathrm{a}}{\mathrm{~h}} \quad \tan \theta=\frac{\mathrm{o}}{\mathrm{a}}
$$

which gives rise to the mnemonic

$$
\begin{array}{lll}
\mathrm{SOH} & \mathrm{CAH} & \text { TOA }
\end{array}
$$

If we scale the triangle so that they hypotenuse has length 1 then we obtain the diagram on the right. In that case, $\sin \theta$ is the height of the triangle, $\cos \theta$ the length of its base and $\tan \theta$ is the length of the line tangent to the circle of radius 1 as shown.

Since the angle $2 \pi$ sweeps out a full circle, the angles $\theta$ and $\theta+2 \pi$ are really the same.


Hence all the trigonometric functions are periodic with period $2 \pi$. That is

$$
\sin (\theta+2 \pi)=\sin (\theta) \quad \cos (\theta+2 \pi)=\cos (\theta) \quad \tan (\theta+2 \pi)=\tan (\theta)
$$

The plots of these functions are shown below




The reciprocals (cosecant, secant and cotangent) of these functions also play important roles in trigonometry and calculus:

$$
\csc \theta=\frac{1}{\sin \theta}=\frac{\mathrm{h}}{\mathrm{o}} \quad \sec \theta=\frac{1}{\cos \theta}=\frac{\mathrm{h}}{\mathrm{a}} \quad \cot \theta=\frac{1}{\tan \theta}=\frac{\cos \theta}{\sin \theta}=\frac{\mathrm{a}}{\mathrm{o}}
$$

The plots of these functions are shown below


These reciprocal functions also have geometric interpretations:


Since these are all right-angled triangles we can use Pythagoras to obtain the following identities:

$$
\sin ^{2} \theta+\cos ^{2} \theta=1 \quad \tan ^{2} \theta+1=\sec ^{2} \theta \quad 1+\cot ^{2} \theta=\csc ^{2} \theta
$$

Of these it is only necessary to remember the first

$$
\sin ^{2} \theta+\cos ^{2} \theta=1
$$

The second can then be obtained by dividing this by $\cos ^{2} \theta$ and the third by dividing by $\sin ^{2} \theta$.

## B.2.3 $\leadsto$ Important Triangles

Computing sine and cosine is non-trivial for general angles - we need Taylor series (or similar tools) to do this. However there are some special angles (usually small integer fractions of $\pi$ ) for which we can use a little geometry to help. Consider the following two triangles.


The first results from cutting a square along its diagonal, while the second is obtained by cutting an equilateral triangle from one corner to the middle of the opposite side. These, together with the angles $0, \frac{\pi}{2}$ and $\pi$ give the following table of values

| $\theta$ | $\sin \theta$ | $\cos \theta$ | $\tan \theta$ | $\csc \theta$ | $\sec \theta$ | $\cot \theta$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 rad | 0 | 1 | 0 | DNE | 1 | DNE |
| $\frac{\pi}{2} \mathrm{rad}$ | 1 | 0 | DNE | 1 | DNE | 0 |
| $\pi \mathrm{rad}$ | 0 | -1 | 0 | DNE | -1 | DNE |
| $\frac{\pi}{4} \mathrm{rad}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | 1 | $\sqrt{2}$ | $\sqrt{2}$ | 1 |
| $\frac{\pi}{6} \mathrm{rad}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ | 2 | $\frac{2}{\sqrt{3}}$ | $\sqrt{3}$ |
| $\frac{\pi}{3} \mathrm{rad}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ | $\frac{2}{\sqrt{3}}$ | 2 | $\frac{1}{\sqrt{3}}$ |

## B.2.4 $\leadsto$ Some More Simple Identities

Consider the figure below


The pair triangles on the left shows that there is a simple relationship between trigonometric functions evaluated at $\theta$ and at $-\theta$ :

$$
\sin (-\theta)=-\sin (\theta) \quad \cos (-\theta)=\cos (\theta)
$$

That is - sine is an odd function, while cosine is even. Since the other trigonometric functions can be expressed in terms of sine and cosine we obtain

$$
\tan (-\theta)=-\tan (\theta) \quad \csc (-\theta)=-\csc (\theta) \quad \sec (-\theta)=\sec (\theta) \quad \cot (-\theta)=-\cot (\theta)
$$

Now consider the triangle on the right - if we consider the angle $\frac{\pi}{2}-\theta$ the side-lengths of the triangle remain unchanged, but the roles of "opposite" and "adjacent" are swapped. Hence we have

$$
\sin \left(\frac{\pi}{2}-\theta\right)=\cos \theta \quad \cos \left(\frac{\pi}{2}-\theta\right)=\sin \theta
$$

Again these imply that
$\tan \left(\frac{\pi}{2}-\theta\right)=\cot \theta \quad \csc \left(\frac{\pi}{2}-\theta\right)=\sec \theta \quad \sec \left(\frac{\pi}{2}-\theta\right)=\csc \theta \quad \cot \left(\frac{\pi}{2}-\theta\right)=\tan \theta$
We can go further. Consider the following diagram:


This implies that

$$
\begin{array}{ll}
\sin (\pi-\theta)=\sin (\theta) & \cos (\pi-\theta)=-\cos (\theta) \\
\sin (\pi+\theta)=-\sin (\theta) & \cos (\pi+\theta)=-\cos (\theta)
\end{array}
$$

From which we can get the rules for the other four trigonometric functions.

## B.2.5 - Identities - Adding Angles

We wish to explain the origins of the identity

$$
\sin (\alpha+\beta)=\sin (\alpha) \cos (\beta)+\cos (\alpha) \sin (\beta) .
$$

A very geometric demonstration uses the figure below and an observation about areas.


- The left-most figure shows two right-angled triangles with angles $\alpha$ and $\beta$ and both with hypotenuse length 1 .
- The next figure simply rearranges the triangles - translating and rotating the lower triangle so that it lies adjacent to the top of the upper triangle.
- Now scale the lower triangle by a factor of $q$ so that edges opposite the angles $\alpha$ and $\beta$ are flush. This means that $q \cos \beta=\cos \alpha$. ie

$$
q=\frac{\cos \alpha}{\cos \beta}
$$

Now compute the areas of these (blue and red) triangles

$$
\begin{aligned}
A_{\mathrm{red}} & =\frac{1}{2} q^{2} \sin \beta \cos \beta \\
A_{\mathrm{blue}} & =\frac{1}{2} \sin \alpha \cos \alpha
\end{aligned}
$$

So twice the total area is

$$
2 A_{\text {total }}=\sin \alpha \cos \alpha+q^{2} \sin \beta \cos \beta
$$

- But we can also compute the total area using the rightmost triangle:

$$
2 A_{\text {total }}=q \sin (\alpha+\beta)
$$

Since the total area must be the same no matter how we compute it we have

$$
\begin{aligned}
q \sin (\alpha+\beta) & =\sin \alpha \cos \alpha+q^{2} \sin \beta \cos \beta \\
\sin (\alpha+\beta) & =\frac{1}{q} \sin \alpha \cos \alpha+q \sin \beta \cos \beta \\
& =\frac{\cos \beta}{\cos \alpha} \sin \alpha \cos \alpha+\frac{\cos \alpha}{\cos \beta} \sin \beta \cos \beta \\
& =\sin \alpha \cos \beta+\cos \alpha \sin \beta
\end{aligned}
$$

as required.
We can obtain the angle addition formula for cosine by substituting $\alpha \mapsto \pi / 2-\alpha$ and $\beta \mapsto-\beta$ into our sine formula:

$$
\begin{aligned}
\sin (\alpha+\beta) & =\sin (\alpha) \cos (\beta)+\cos (\alpha) \sin (\beta) \\
\underbrace{\sin (\pi / 2-\alpha-\beta)}_{\cos (\alpha+\beta)} & =\underbrace{\sin (\pi / 2-\alpha)}_{\cos (\alpha)} \cos (-\beta)+\underbrace{\cos (\pi / 2-\alpha)}_{\sin (\alpha)} \sin (-\beta) \\
\cos (\alpha+\beta) & =\cos (\alpha) \cos (\beta)-\sin (\alpha) \sin (\beta)
\end{aligned}
$$

where we have used $\sin (\pi / 2-\theta)=\cos (\theta)$ and $\cos (\pi / 2-\theta)=\sin (\theta)$.
It is then a small step to the formulas for the difference of angles. From the relation

$$
\sin (\alpha+\beta)=\sin (\alpha) \cos (\beta)+\cos (\alpha) \sin (\beta)
$$

we can substitute $\beta \mapsto-\beta$ and so obtain

$$
\begin{aligned}
\sin (\alpha-\beta) & =\sin (\alpha) \cos (-\beta)+\cos (\alpha) \sin (-\beta) \\
& =\sin (\alpha) \cos (\beta)-\cos (\alpha) \sin (\beta)
\end{aligned}
$$

The formula for cosine can be obtained in a similar manner. To summarise

$$
\begin{aligned}
& \sin (\alpha \pm \beta)=\sin (\alpha) \cos (\beta) \pm \cos (\alpha) \sin (\beta) \\
& \cos (\alpha \pm \beta)=\cos (\alpha) \cos (\beta) \mp \sin (\alpha) \sin (\beta)
\end{aligned}
$$

The formulas for tangent are a bit more work, but

$$
\begin{aligned}
\tan (\alpha+\beta) & =\frac{\sin (\alpha+\beta)}{\cos (\alpha+\beta)} \\
& =\frac{\sin (\alpha) \cos (\beta)+\cos (\alpha) \sin (\beta)}{\cos (\alpha) \cos (\beta)-\sin (\alpha) \sin (\beta)} \\
& =\frac{\sin (\alpha) \cos (\beta)+\cos (\alpha) \sin (\beta)}{\cos (\alpha) \cos (\beta)-\sin (\alpha) \sin (\beta)} \cdot \frac{\sec (\alpha) \sec (\beta)}{\sec (\alpha) \sec (\beta)} \\
& =\frac{\sin (\alpha) \sec (\alpha)+\sin (\beta) \sec (\beta)}{1-\sin (\alpha) \sec (\alpha) \sin (\beta) \sec (\beta)} \\
& =\frac{\tan (\alpha)+\tan (\beta)}{1-\tan (\alpha) \tan (\beta)}
\end{aligned}
$$

and similarly we get

$$
\tan (\alpha-\beta)=\frac{\tan (\alpha)-\tan (\beta)}{1+\tan (\alpha) \tan (\beta)}
$$

## B.2.6 $\leadsto$ Identities - Double-angle Formulas

If we set $\beta=\alpha$ in the angle-addition formulas we get

$$
\begin{array}{rlr}
\sin (2 \alpha) & =2 \sin (\alpha) \cos (\alpha) & \\
\cos (2 \alpha) & =\cos ^{2}(\alpha)-\sin ^{2}(\alpha) & \\
& =2 \cos ^{2}(\alpha)-1 & \\
& =1-2 \sin ^{2}(\alpha) & \\
\tan (2 \alpha) & =\frac{2 \tan (\alpha)}{1-\tan ^{2}(\alpha)} & \\
& =\frac{2}{\cot (\alpha)-\tan (\alpha)} \cos ^{2} \theta=1-\cos ^{2} \theta \\
& \text { divide top and bottom by } \tan (\alpha)
\end{array}
$$

## B.2.7 $\leadsto$ Identities — Extras

## B.2.7.1 $\leadsto$ Sums to Products

Consider the identities

$$
\sin (\alpha+\beta)=\sin (\alpha) \cos (\beta)+\cos (\alpha) \sin (\beta) \quad \sin (\alpha-\beta)=\sin (\alpha) \cos (\beta)-\cos (\alpha) \sin (\beta)
$$

If we add them together some terms on the right-hand side cancel:

$$
\sin (\alpha+\beta)+\sin (\alpha-\beta)=2 \sin (\alpha) \cos (\beta)
$$

If we now set $u=\alpha+\beta$ and $v=\alpha-\beta$ (i.e. $\alpha=\frac{u+v}{2}, \beta=\frac{u-v}{2}$ ) then

$$
\sin (u)+\sin (v)=2 \sin \left(\frac{u+v}{2}\right) \cos \left(\frac{u-v}{2}\right)
$$

This transforms a sum into a product. Similarly:

$$
\begin{aligned}
\sin (u)-\sin (v) & =2 \sin \left(\frac{u-v}{2}\right) \cos \left(\frac{u+v}{2}\right) \\
\cos (u)+\cos (v) & =2 \cos \left(\frac{u+v}{2}\right) \cos \left(\frac{u-v}{2}\right) \\
\cos (u)-\cos (v) & =-2 \sin \left(\frac{u+v}{2}\right) \sin \left(\frac{u-v}{2}\right)
\end{aligned}
$$

## B.2.7.2 $\leadsto$ Products to sums

Again consider the identities

$$
\sin (\alpha+\beta)=\sin (\alpha) \cos (\beta)+\cos (\alpha) \sin (\beta) \quad \sin (\alpha-\beta)=\sin (\alpha) \cos (\beta)-\cos (\alpha) \sin (\beta)
$$ and add them together:

$$
\sin (\alpha+\beta)+\sin (\alpha-\beta)=2 \sin (\alpha) \cos (\beta)
$$

Then rearrange:

$$
\sin (\alpha) \cos (\beta)=\frac{\sin (\alpha+\beta)+\sin (\alpha-\beta)}{2}
$$

In a similar way, start with the identities
$\cos (\alpha+\beta)=\cos (\alpha) \cos (\beta)-\sin (\alpha) \sin (\beta) \quad \cos (\alpha-\beta)=\cos (\alpha) \cos (\beta)+\sin (\alpha) \sin (\beta)$ If we add these together we get

$$
2 \cos (\alpha) \cos (\beta)=\cos (\alpha+\beta)+\cos (\alpha-\beta)
$$

while taking their difference gives

$$
2 \sin (\alpha) \sin (\beta)=\cos (\alpha-\beta)-\cos (\alpha+\beta)
$$

Hence

$$
\begin{aligned}
\sin (\alpha) \sin (\beta) & =\frac{\cos (\alpha-\beta)-\cos (\alpha+\beta)}{2} \\
\cos (\alpha) \cos (\beta) & =\frac{\cos (\alpha-\beta)+\cos (\alpha+\beta)}{2}
\end{aligned}
$$

## B.34 Inverse Trigonometric Functions

In order to construct inverse trigonometric functions we first have to restrict their domains so as to make them one-to-one (or injective). We do this as shown below


Since these functions are inverses of each other we have

$$
\begin{aligned}
& \arcsin (\sin \theta)=\theta \\
& -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\
& \arccos (\cos \theta)=\theta \quad 0 \leq \theta \leq \pi \\
& \arctan (\tan \theta)=\theta \\
& -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}
\end{aligned}
$$

and also

$$
\begin{aligned}
\sin (\arcsin x) & =x & & -1 \leq x \leq 1 \\
\cos (\arccos x) & =x & & -1 \leq x \leq 1 \\
\tan (\arctan x) & =x & & \text { any real } x
\end{aligned}
$$

We can read other combinations of trig functions and their inverses, like, for example, $\cos (\arcsin x)$, off of triangles like


We have chosen the hypotenuse and opposite sides of the triangle to be of length 1 and $x$, respectively, so that $\sin (\theta)=x$. That is, $\theta=\arcsin x$. We can then read off of the triangle that

$$
\cos (\arcsin x)=\cos (\theta)=\sqrt{1-x^{2}}
$$

We can reach the same conclusion using trig identities, as follows.

- Write $\arcsin x=\theta$. We know that $\sin (\theta)=x$ and we wish to compute $\cos (\theta)$. So we just need to express $\cos (\theta)$ in terms of $\sin (\theta)$.
- To do this we make use of one of the Pythagorean identities

$$
\begin{aligned}
\sin ^{2} \theta+\cos ^{2} \theta & =1 \\
\cos \theta & = \pm \sqrt{1-\sin ^{2} \theta}
\end{aligned}
$$

- Thus

$$
\cos (\arcsin x)=\cos \theta= \pm \sqrt{1-\sin ^{2} \theta}
$$

- To determine which branch we should use we need to consider the domain and range of $\arcsin x$ :

$$
\text { Domain: }-1 \leq x \leq 1 \quad \text { Range: }-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}
$$

Thus we are applying cosine to an angle that always lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Cosine is non-negative on this range. Hence we should take the positive branch and

$$
\begin{aligned}
\cos (\arcsin x) & =\sqrt{1-\sin ^{2} \theta}=\sqrt{1-\sin ^{2}(\arcsin x)} \\
& =\sqrt{1-x^{2}}
\end{aligned}
$$

In a very similar way we can simplify $\tan (\arccos x)$.

- Write $\arccos x=\theta$, and then

$$
\tan (\arccos x)=\tan \theta=\frac{\sin \theta}{\cos \theta}
$$

- Now the denominator is easy since $\cos \theta=\cos \arccos x=x$.
- The numerator is almost the same as the previous computation.

$$
\begin{aligned}
\sin \theta & = \pm \sqrt{1-\cos ^{2} \theta} \\
& = \pm \sqrt{1-x^{2}}
\end{aligned}
$$

- To determine which branch we again consider domains and and ranges:

$$
\text { Domain: }-1 \leq x \leq 1 \quad \text { Range: } 0 \leq \arccos x \leq \pi
$$

Thus we are applying sine to an angle that always lies between 0 and $\pi$. Sine is non-negative on this range and so we take the positive branch.

- Putting everything back together gives

$$
\tan (\arccos x)=\frac{\sqrt{1-x^{2}}}{x}
$$

Completing the 9 possibilities gives:

$$
\begin{array}{lll}
\sin (\arcsin x)=x & \sin (\arccos x)=\sqrt{1-x^{2}} & \sin (\arctan x)=\frac{x}{\sqrt{1+x^{2}}} \\
\cos (\arcsin x)=\sqrt{1-x^{2}} & \cos (\arccos x)=x & \cos (\arctan x)=\frac{1}{\sqrt{1+x^{2}}} \\
\tan (\arcsin x)=\frac{x}{\sqrt{1-x^{2}}} & \tan (\arccos x)=\frac{\sqrt{1-x^{2}}}{x} & \tan (\arctan x)=x
\end{array}
$$

## B.4^ Cosine and Sine Laws

## B.4.1 Cosine Law or Law of Cosines

The cosine law says that, if a triangle has sides of length $a, b$ and $c$ and the angle opposite the side of length $c$ is $\gamma$, then

$$
c^{2}=a^{2}+b^{2}-2 a b \cos \gamma
$$

Observe that, when $\gamma=\frac{\pi}{2}$, this reduces to, (surpise!) Pythagoras' theorem $c^{2}=a^{2}+b^{2}$. Let's derive the cosine law.


Consider the triangle on the left. Now draw a perpendicular line from the side of length $c$ to the opposite corner as shown. This demonstrates that

$$
c=a \cos \beta+b \cos \alpha
$$

Multiply this by $c$ to get an expression for $c^{2}$ :

$$
c^{2}=a c \cos \beta+b c \cos \alpha
$$

Doing similarly for the other corners gives

$$
\begin{aligned}
& a^{2}=a c \cos \beta+a b \cos \gamma \\
& b^{2}=b c \cos \alpha+a b \cos \gamma
\end{aligned}
$$

Now combining these:

$$
\begin{aligned}
a^{2}+b^{2}-c^{2} & =(b c-b c) \cos \alpha+(a c-a c) \cos \beta+2 a b \cos \gamma \\
& =2 a b \cos \gamma
\end{aligned}
$$

as required.

## B.4.2 $\leadsto$ Sine Law or Law of Sines

The sine law says that, if a triangle has sides of length $a, b$ and $c$ and the angles opposite those sides are $\alpha, \beta$ and $\gamma$, then

$$
\frac{a}{\sin \alpha}=\frac{b}{\sin \beta}=\frac{c}{\sin \gamma}
$$



This rule is best understood by computing the area of the triangle using the formula $A=\frac{1}{2} a b \sin \theta$ of Appendix A.10. Doing this three ways gives

$$
\begin{aligned}
& 2 A=b c \sin \alpha \\
& 2 A=a c \sin \beta \\
& 2 A=a b \sin \gamma
\end{aligned}
$$

Dividing these expressions by $a b c$ gives

$$
\frac{2 A}{a b c}=\frac{\sin \alpha}{a}=\frac{\sin \beta}{b}=\frac{\sin \gamma}{c}
$$

as required.

## B.5 A Circles, cones and spheres

## B.5.1 Where Does the Formula for the Area of a Circle Come From?

Typically when we come across $\pi$ for the first time it is as the ratio of the circumference of a circle to its diameter

$$
\pi=\frac{C}{d}=\frac{C}{2 r}
$$

Indeed this is typically the first definition we see of $\pi$. It is easy to build an intuition that the area of the circle should be propotional to the square of its radius. For example we can draw the largest possible square inside the circle (an inscribed square) and the smallest possible square outside the circle (a circumscribed square):


The smaller square has side-length $\sqrt{2} r$ and the longer has side-length $2 r$. Hence

$$
2 r^{2} \leq A \leq 4 r^{2} \quad \text { or } 2 \leq \frac{A}{r^{2}} \leq 4
$$

That is, the area of the circle is between 2 and 4 times the square of the radius. What is perhaps less obvious (if we had not been told this in school) is that the constant of propotionality for area is also $\pi$ :

$$
\pi=\frac{A}{r^{2}}
$$

We will show this using Archimedes' proof. He makes use of these inscribed and circumscribed polygons to make better and better approximations of the circle. The steps of the proof are somewhat involved and the starting point is to rewrite the area of a circle as

$$
A=\frac{1}{2} C r
$$

where $C$ is (still) the circumference of the circle. This suggests that this area is the same as that of a triangle of height $r$ and base length $C$

$$
T=\frac{1}{2} C r
$$



Archimedes' proof then demonstrates that indeed this triangle and the circle have the same area. It relies on a "proof by contradiction" - showing that $T<A$ and $T>A$ cannot be true and so the only possibility is that $A=T$.

We will first show that $T<A$ cannot happen. Construct an $n$-sided "inscribed" polygon as shown below:


Let $p_{n}$ be the inscribed polygon as shown.


We need 4 steps.

- The area of $p_{n}$ is smaller than that of the circle - this follows since we can construct $p_{n}$ by cutting slices from the circle.
- Let $E_{n}$ be the difference between the area of the circle and $p_{n}: E_{n}=A-A\left(p_{n}\right)$ (see the left of the previous figure). By the previous point we know $E_{n}>0$. Now as we increase the number of sides, this difference becomes smaller. To be more precise

$$
E_{2 n} \leq \frac{1}{2} E_{n}
$$

The error $E_{n}$ is made up of $n$ "lobes". In the centre-left of the previous figure we draw one such lobe and surround it by a rectangle of dimensions $a \times 2 b-$
we could determine these more precisely using a little trigonometry, but it is not necessary.
This diagram shows the lobe is smaller than the rectangle of base $2 b$ and height $a$ Since there are $n$ copies of the lobe, we have

$$
E_{n} \leq n \times 2 a b \quad \text { rewrite as } \frac{E_{n}}{2} \leq n a b
$$

Now draw in the polygon $p_{2 n}$ and consider the associated "error" $E_{2 n}$. If we focus on the two lobes shown then we see that the area of these two new lobes is equal to that of the old lobe (shown in centre-left) minus the area of the triangle with base $2 b$ and height $a$ (drawn in purple). Since there are $n$ copies of this picture we have

$$
\begin{array}{rlr}
E_{2 n} & =E_{n}-n a b & \text { now use that } n a b \geq E_{n} / 2 \\
& \leq E_{n}-\frac{E_{n}}{2}=\frac{E_{n}}{2} &
\end{array}
$$

- The area of $p_{n}$ is smaller than $T$. To see this decompose $p_{n}$ into $n$ isosceles triangles. Each of these has base shorter than $C / n$; the straight line is shorter than the corresponding arc - though strictly speaking we should prove this. The height of each triangle is shorter than $r$. Thus

$$
\begin{aligned}
A\left(p_{n}\right) & =n \times \frac{1}{2}(\text { base }) \times(\text { height }) \\
& \leq n \times \frac{C r}{2 n}=T
\end{aligned}
$$

- If we assume that $T<A$, then $A-T=d$ where $d$ is some positive number. However we know from point 2 that we can make $n$ large enough so that $E_{n}<d$ (each time we double $n$ we halve the error). But now we have a contradiction to step 3, since we have just shown that

$$
\begin{gathered}
E_{n}=A-A\left(p_{n}\right)<A-T \quad \text { which implies that } \\
A\left(p_{n}\right)>T .
\end{gathered}
$$

Thus we cannot have $T<A$.
If we now assume that $T>A$ we will get a similar contradiction by a similar construction. Now we use regular $n$-sided circumscribed polygons, $P_{n}$.


The proof can be broken into 4 similar steps.


1. The area of $P_{n}$ is greater than that of the circle - this follows since we can construct the circle by trimming the polygon $P_{n}$.
2. Let $E_{n}$ be the difference between the area of the polygon and the circle: $E_{n}=$ $A\left(P_{n}\right)-A$ (see the left of the previous figure). By the previous point we know $E_{n}>0$. Now as we increase the number of sides, this difference becomes smaller. To be more precise we will show

$$
E_{2 n} \leq \frac{1}{2} E_{n}
$$

The error $E_{n}$ is made up of $n$ "lobes". In the centre-left of the previous figure we draw one such lobe. Let $L_{n}$ denote the area of one of these lobes, so $E_{n}=n L_{n}$. In the centre of the previous figure we have labelled this lobe carefully and also shown how it changes when we create the polygon $P_{2 n}$. In particular, the original lobe is bounded by the straight lines $\overrightarrow{a d}, \overrightarrow{a f}$ and the arc $\widehat{f b d}$. We create $P_{2 n}$ from $P_{n}$ by cutting away the corner triangle $\triangle a e c$. Accordingly the lines $\overrightarrow{e c}$ and $\overrightarrow{b a}$ are orthogonal and the segments $|b c|=|c d|$.
By the construction of $P_{2 n}$ from $P_{n}$, we have

$$
2 L_{2 n}=L_{n}-A(\triangle a e c) \quad \text { or equivalently } L_{2 n}=\frac{1}{2} L_{n}-A(\triangle a b c)
$$

And additionally

$$
L_{2 n} \leq A(\triangle b c d)
$$

Now consider the triangle $\triangle a b d$ (centre-right of the previous figure) and the two triangles within it $\triangle a b c$ and $\triangle b c d$. We know that $\overrightarrow{a b}$ and $\overrightarrow{c b}$ form a right-angle. Consequently $\overrightarrow{a c}$ is the hypotenuse of a right-angled triangle, so $|a c|>|b c|=|c d|$. So now, the triangles $\triangle a b c$ and $\triangle b c d$ have the same heights, but the base of $\overrightarrow{a c}$ is longer than $\overrightarrow{c d}$. Hence the area of $\triangle a b c$ is strictly larger than that of $\triangle b c d$.
Thus we have

$$
L_{2 n} \leq A(\triangle b c d)<A(\triangle a b c)
$$

But now we can write

$$
\begin{array}{rlrl}
L_{2 n} & =\frac{1}{2} L_{n}-A(\triangle a b c)<\frac{1}{2} L_{n}-L_{2 n} & \text { rearrange } \\
2 L_{2 n} & <\frac{1}{2} L_{n} & & \text { there are } n \text { such lobes, so } \\
2 n L_{2 n} & <\frac{n}{2} L_{n} & & \text { since } E_{n}=n L_{n}, \text { we have } \\
E_{2 n} & <\frac{1}{2} E_{n} & \text { which is what we wanted to show. }
\end{array}
$$

3. The area of $P_{n}$ is greater than $T$. To see this decompose $P_{n}$ into $n$ isosceles triangles. The height of each triangle is $r$, while the base of each is longer than $C / n$ (this is a subtle point and its proof is equivalent to showing that $\tan \theta>\theta$ ). Thus

$$
\begin{aligned}
A\left(P_{n}\right) & =n \times \frac{1}{2}(\text { base }) \times(\text { height }) \\
& \geq n \times \frac{C r}{2 n}=T
\end{aligned}
$$

4. If we assume that $T>A$, then $T-A=d$ where $d$ is some positive number. However we know from point 2 that we can make $n$ large enough so that $E_{n}<d$ (each time we double $n$ we halve the error). But now we have a contradiction since we have just shown that

$$
\begin{array}{cl}
E_{n}=A\left(P_{n}\right)-A & <T-A \quad \text { which implies that } \\
A\left(p_{n}\right) & >T .
\end{array}
$$

Thus we cannot have $T>A$. The only possibility that remains is that $T=A$.

## B.5.2 Where Do These Volume Formulas Come From?

We can establish the volumes of cones and spheres from the formula for the volume of a cylinder and a little work with limits and some careful summations. We first need a few facts.

- Every square number can be written as a sum of consecutive odd numbers. More precisely

$$
n^{2}=1+3+\cdots+(2 n-1)
$$

- The sum of the first $n$ positive integers is $\frac{1}{2} n(n+1)$. That is

$$
1+2+3+\cdots+n=\frac{1}{2} n(n+1)
$$

- The sum of the squares of the first $n$ positive integers is $\frac{1}{6} n(n+1)(2 n+1)$.

$$
1^{2}+2^{2}+3^{2}+\cdots+n^{2}=\frac{1}{6} n(n+1)(2 n+1)
$$

We will not give completely rigorous proofs of the above identities (since we are not going to assume that the reader knows mathematical induction), rather we will explain them using pictorial arguments. The first two of these we can explain by some quite simple pictures:


We see that we can decompose any square of unit-squares into a sequence of strips, each of which consists of an odd number of unit-squares. This is really just from the fact that

$$
n^{2}-(n-1)^{2}=2 n-1
$$

Similarly, we can represent the sum of the first $n$ integers as a triangle of unit squares as shown. If we make a second copy of that triangle and arrange it as shown, it gives a rectangle of dimensions $n$ by $n+1$. Hence the rectangle, being exactly twice the size of the original triangle, contains $n(n+1)$ unit squares.

The explanation of the last formula takes a little more work and a carefully constructed picture:


Let us break these pictures down step by step

- Leftmost represents the sum of the squares of the first $n$ integers.
- Centre - We recall from above that each square number can be written as a sum of consecutive odd numbers, which have been represented as coloured bands of unit-squares.
- Make three copies of the sum and arrange them carefully as shown. The first and third copies are obvious, but the central copy is rearranged considerably; all bands of the same colour have the same length and have been arranged into rectangles as shown.
Putting everything from the three copies together creates a rectangle of dimensions $(2 n+1) \times(1+2+3+\cdots+n)$.

We know (from above) that $1+2+3+\cdots+n=\frac{1}{2} n(n+1)$ and so

$$
\left(1^{2}+2^{2}+\cdots+n^{2}\right)=\frac{1}{3} \times \frac{1}{2} n(n+1)(2 n+1)
$$

as required.
Now we can start to look at volumes. Let us start with the volume of a cone; consider the figure below. We bound the volume of the cone above and below by stacks of cylinders. The cross-sections of the cylinders and cone are also shown.


To obtain the bounds we will construct two stacks of $n$ cylinders, $C_{1}, C_{2}, \ldots, C_{n}$. Each cylinder has height $h / n$ and radius that varies with height. In particular, we define cylinder $C_{k}$ to have height $h / n$ and radius $k \times r / n$. This radius was determined using similar triangles so that cylinder $C_{n}$ has radius $r$. Now cylinder $C_{k}$ has volume

$$
\begin{aligned}
V_{k} & =\pi \times \text { radius }^{2} \times \text { height }=\pi\left(\frac{k r}{n}\right)^{2} \cdot \frac{h}{n} \\
& =\frac{\pi r^{2} h}{n^{3}} k^{2}
\end{aligned}
$$

We obtain an upper bound by stacking cylinders $C_{1}, C_{2}, \ldots, C_{n}$ as shown. This object has volume

$$
\begin{aligned}
V & =V_{1}+V_{2}+\cdots+V_{n} \\
& =\frac{\pi r^{2} h}{n^{3}}\left(1^{2}+2^{2}+3^{2}+\cdots+n^{2}\right) \\
& =\frac{\pi r^{2} h}{n^{3}} \cdot \frac{n(n+1)(2 n+1)}{6}
\end{aligned}
$$

A similar lower bound is obtained by stacking cylinders $C_{1}, \ldots, C_{n-1}$ which gives a volume of

$$
\begin{aligned}
V & =V_{1}+V_{2}+\cdots+V_{n-1} \\
& =\frac{\pi r^{2} h}{n^{3}}\left(1^{2}+2^{2}+3^{2}+\cdots+(n-1)^{2}\right) \\
& =\frac{\pi r^{2} h}{n^{3}} \cdot \frac{(n-1)(n)(2 n-1)}{6}
\end{aligned}
$$

Thus the true volume of the cylinder is bounded between

$$
\frac{\pi r^{2} h}{n^{3}} \cdot \frac{(n-1)(n)(2 n-1)}{6} \leq \text { correct volume } \leq \frac{\pi r^{2} h}{n^{3}} \cdot \frac{n(n+1)(2 n+1)}{6}
$$

We can now take the limit as the number of cylinders, $n$, goes to infinity. The upper bound becomes

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \frac{\pi r^{2} h}{n^{3}} \frac{n(n+1)(2 n+1)}{6} & =\frac{\pi r^{2} h}{6} \lim _{n \rightarrow \infty} \frac{n(n+1)(2 n+1)}{n^{3}} \\
& =\frac{\pi r^{2} h}{6} \lim _{n \rightarrow \infty} \frac{(1+1 / n)(2+1 / n)}{1} \\
& =\frac{\pi r^{2} h}{6} \times 2 \\
& =\frac{\pi r^{2} h}{3}
\end{aligned}
$$

The other limit is identical, so by the squeeze theorem we have

$$
\text { Volume of cone }=\frac{1}{3} \pi r^{2} h
$$

Now the sphere - though we will do the analysis for a hemisphere of radius $R$. Again we bound the volume above and below by stacks of cylinders. The cross-sections of the cylinders and cone are also shown.


To obtain the bounds we will construct two stacks of $n$ cylinders, $C_{1}, C_{2}, \ldots, C_{n}$. Each cylinder has height $R / n$ and radius that varies with its position in the stack. To describe the position, define

$$
y_{k}=k \times \frac{R}{n}
$$

That is, $y_{k}$, is $k$ steps of distance $\frac{R}{n}$ from the top of the hemisphere. Then we set the $k^{\text {th }}$ cylinder, $C_{k}$ to have height $R / n$ and radius $r_{k}$ given by

$$
\begin{aligned}
r_{k}^{2} & =R^{2}-\left(R-y_{k}\right)^{2}=R^{2}-R^{2}(1-k / n)^{2} \\
& =R^{2}\left(2 k / n-k^{2} / n^{2}\right)
\end{aligned}
$$

as shown in the top-right and bottom-left illustrations. The volume of $C_{k}$ is then

$$
\begin{aligned}
V_{k} & =\pi \times \text { radius }^{2} \times \text { height }=\pi \times R^{2}\left(2 k / n-k^{2} / n^{2}\right) \times \frac{R}{n} \\
& =\pi R^{3} \cdot\left(\frac{2 k}{n^{2}}-\frac{k^{2}}{n^{3}}\right)
\end{aligned}
$$

We obtain an upper bound by stacking cylinders $C_{1}, C_{2}, \ldots, C_{n}$ as shown. This object has volume

$$
\begin{aligned}
V & =V_{1}+V_{2}+\cdots+V_{n} \\
& =\pi R^{3} \cdot\left(\frac{2}{n^{2}}(1+2+3+\cdots+n)-\frac{1}{n^{3}}\left(1^{2}+2^{2}+3^{2}+\cdots+n^{2}\right)\right)
\end{aligned}
$$

Now recall from above that

$$
1+2+3+\cdots+n=\frac{1}{2} n(n+1) \quad 1^{2}+2^{2}+3^{2}+\cdots+n^{2}=\frac{1}{6} n(n+1)(2 n+1)
$$

so

$$
V=\pi R^{3} \cdot\left(\frac{n(n+1)}{n^{2}}-\frac{n(n+1)(2 n+1)}{6 n^{3}}\right)
$$

Again, a lower bound is obtained by stacking cylinders $C_{1}, \ldots, C_{n-1}$ and a similar analysis gives

$$
V=\pi R^{3} \cdot\left(\frac{n(n-1)}{(n-1)^{2}}-\frac{n(n-1)(2 n-1)}{6(n-1)^{3}}\right)
$$

Thus the true volume of the hemisphere is bounded between
$\pi R^{3} \cdot\left(\frac{n(n+1)}{n^{2}}-\frac{n(n+1)(2 n+1)}{6 n^{3}}\right) \leq$ correct volume $\leq \pi R^{3} \cdot\left(\frac{n(n+1)}{n^{2}}-\frac{n(n+1)(2 n+1)}{6 n^{3}}\right)$
We can now take the limit as the number of cylinders, $n$, goes to infinity. The upper bound becomes

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \pi R^{3} \cdot\left(\frac{n(n+1)}{n^{2}}-\frac{n(n+1)(2 n+1)}{6 n^{3}}\right) & =\pi R^{3}\left(\lim _{n \rightarrow \infty} \frac{n(n+1)}{n^{2}}-\frac{n(n+1)(2 n+1)}{6 n^{3}}\right) \\
& =\pi R^{3}\left(1-\frac{2}{6}\right)=\frac{2}{3} \pi R^{3} .
\end{aligned}
$$

The other limit is identical, so by the squeeze theorem we have

$$
\begin{aligned}
\text { Volume of hemisphere } & =\frac{2}{3} \pi R^{3} & \text { and so } \\
\text { Volume of sphere } & =\frac{4}{3} \pi R^{3} &
\end{aligned}
$$



To this point you have found solutions to equations almost exclusively by algebraic manipulation. This is possible only for the artificially simple equations of problem sets and tests. In the "real world" it is very common to encounter equations that cannot be solved by algebraic manipulation. For example, you found, by completing a square, that the solutions to the quadratic equation $a x^{2}+b x+c=0$ are $x=\left(-b \pm \sqrt{b^{2}-4 a c}\right) / 2 a$. But it is known that there simply does not exist a corresponding formula for the roots of a general polynomial of degree five or more. Fortunately, encountering such an equation is not the end of the world, because usually one does not need to know the solutions exactly. One only needs to know them to within some specified degree of accuracy. For example, one rarely needs to know $\pi$ to more than a few decimal places. There is a whole subject, called numerical analysis, that concerns using algorithms to solve equations (and perform other tasks) approximately, to any desired degree of accuracy.

We have already had, in Examples 1.6.14 and 1.6.15, and the lead up to them, a really quick introduction to the bisection method, which is a crude, but effective, algorithm for finding approximate solutions to equations of the form $f(x)=0$. We shall shortly use a little calculus to derive a very efficient algorithm for finding approximate solutions to such equations. But first here is a simple example which provides a review of some of the basic ideas of root finding and the bisection method.

## Example C.0.1 Bisection method.

Suppose that we are given some function $f(x)$ and we have to find solutions to the equation $f(x)=0$. To be concrete, suppose that $f(x)=8 x^{3}+12 x^{2}+6 x-15$. How do we go about solving $f(x)=0$ ? To get a rough idea of the lay of the land, sketch the graph of $f(x)$. First observe that

- when $x$ is very large and negative, $f(x)$ is very large and negative
- when $x$ is very large and positive, $f(x)$ is very large and positive
- when $x=0, f(x)=f(0)=-15<0$
- when $x=1, f(x)=f(1)=11>0$
- $f^{\prime}(x)=24 x^{2}+24 x+6=24\left(x^{2}+x+\frac{1}{4}\right)=24\left(x+\frac{1}{2}\right)^{2} \geq 0$ for all $x$. So $f(x)$ increases monotonically with $x$. The graph has a tangent of slope 0 at $x=-\frac{1}{2}$ and tangents of strictly positive slope everywhere else.
This tells us that the graph of $f(x)$ looks like


Since $f(x)$ strictly increases ${ }^{a}$ as $x$ increases, $f(x)$ can take the value zero for at most one value of $x$.

- Since $f(0)<0$ and $f(1)>0$ and $f$ is continuous, $f(x)$ must pass through 0 as $x$ travels from $x=0$ to $x=1$, by Theorem 1.6.12 (the intermediate value theorem). So $f(x)$ takes the value zero for some $x$ between 0 and 1 . We will often write this as "the root is $x=0.5 \pm 0.5$ " to indicate the uncertainty.
- To get closer to the root, we evaluate $f(x)$ halfway between 0 and 1 .

$$
f\left(\frac{1}{2}\right)=8\left(\frac{1}{2}\right)^{3}+12\left(\frac{1}{2}\right)^{2}+6\left(\frac{1}{2}\right)-15=-8
$$

Since $f\left(\frac{1}{2}\right)<0$ and $f(1)>0$ and $f$ is continuous, $f(x)$ must take the value zero for some $x$ between $\frac{1}{2}$ and 1 . The root is $0.75 \pm 0.25$.

- To get still closer to the root, we evaluate $f(x)$ halfway between $\frac{1}{2}$ and 1 .

$$
f\left(\frac{3}{4}\right)=8\left(\frac{3}{4}\right)^{3}+12\left(\frac{3}{4}\right)^{2}+6\left(\frac{3}{4}\right)-15=-\frac{3}{8}
$$

Since $f\left(\frac{3}{4}\right)<0$ and $f(1)>0$ and $f$ is continuous, $f(x)$ must take the value zero for some $x$ between $\frac{3}{4}$ and 1 . The root is $0.875 \pm 0.125$.

- And so on.
$a$ By " $f(x)$ is strictly increasing" we mean that $f(a)<f(b)$ whenever $a<b$. As $f^{\prime}(x)>0$ for all $x \neq-\frac{1}{2}, f(x)$ is strictly increasing even as $x$ passes through $-\frac{1}{2}$. For example, for any $x>-\frac{1}{2}$, the mean value theorem (Theorem 2.13.5) tells us that there is a $c$ strictly between $-\frac{1}{2}$ and $x$ such that $f(x)-f\left(-\frac{1}{2}\right)=f^{\prime}(c)\left(x+\frac{1}{2}\right)>0$.

The root finding strategy used in Example C.0.1 is called the bisection method. The bisection method will home in on a root of the function $f(x)$ whenever

- $f(x)$ is continuous ( $f(x)$ need not have a derivative) and
- you can find two numbers $a_{1}<b_{1}$ with $f\left(a_{1}\right)$ and $f\left(b_{1}\right)$ being of opposite sign.

Denote by $I_{1}$ the interval $\left[a_{1}, b_{1}\right]=\left\{x \mid a_{1} \leq x \leq b_{1}\right\}$. Once you have found the interval $I_{1}$, the bisection method generates a sequence $I_{1}, I_{2}, I_{3}, \cdots$ of intervals by the following rule.

## Equation C.0.2 (bisection method).

Denote by $c_{n}=\frac{a_{n}+b_{n}}{2}$ the midpoint of the interval $I_{n}=\left[a_{n}, b_{n}\right]$. If $f\left(c_{n}\right)$ has the same sign as $f\left(a_{n}\right)$, then

$$
I_{n+1}=\left[a_{n+1}, b_{n+1}\right] \quad \text { with } \quad a_{n+1}=c_{n}, b_{n+1}=b_{n}
$$

and if $f\left(c_{n}\right)$ and $f\left(a_{n}\right)$ have opposite signs, then

$$
I_{n+1}=\left[a_{n+1}, b_{n+1}\right] \quad \text { with } \quad a_{n+1}=a_{n}, b_{n+1}=c_{n}
$$

This rule was chosen so that $f\left(a_{n}\right)$ and $f\left(b_{n}\right)$ have opposite sign for every $n$. Since $f(x)$ is continuous, $f(x)$ has a zero in each interval $I_{n}$. Thus each step reduces the error bars by a factor of 2 . That isn't too bad, but we can come up with something that is much more efficient. We just need a little calculus.

## C.1』 Newton's Method

Newton's method ${ }^{1}$, also known as the Newton-Raphson method, is another technique for generating numerical approximate solutions to equations of the form $f(x)=0$. For example, one can easily get a good approximation to $\sqrt{2}$ by applying Newton's method to the equation $x^{2}-2=0$. This will be done in Example C.1.2, below.

Here is the derivation of Newton's method. We start by simply making a guess for the solution. For example, we could base the guess on a sketch of the graph of $f(x)$. Call the initial guess $x_{1}$. Next recall, from Theorem 2.3.4, that the tangent line to $y=f(x)$ at $x=x_{1}$ is $y=F(x)$, where

$$
F(x)=f\left(x_{1}\right)+f^{\prime}\left(x_{1}\right)\left(x-x_{1}\right)
$$

1 The algorithm that we are about to describe grew out of a method that Newton wrote about in 1669. But the modern method incorporates substantial changes introduced by Raphson in 1690 and Simpson in 1740.

Usually $F(x)$ is a pretty good approximation to $f(x)$ for $x$ near $x_{1}$. So, instead of trying to solve $f(x)=0$, we solve the linear equation $F(x)=0$ and call the solution $x_{2}$.

$$
\begin{aligned}
0=F(x)=f\left(x_{1}\right)+f^{\prime}\left(x_{1}\right)\left(x-x_{1}\right) & \Longleftrightarrow x-x_{1}=-\frac{f\left(x_{1}\right)}{f^{\prime}\left(x_{1}\right)} \\
& \Longleftrightarrow x=x_{2}=x_{1}-\frac{f\left(x_{1}\right)}{f^{\prime}\left(x_{1}\right)}
\end{aligned}
$$

Note that if $f(x)$ were a linear function, then $F(x)$ would be exactly $f(x)$ and $x_{2}$ would solve $f(x)=0$ exactly.


Now we repeat, but starting with the (second) guess $x_{2}$ rather than $x_{1}$. This gives the (third) guess $x_{3}=x_{2}-\frac{f\left(x_{2}\right)}{f^{\prime}\left(x_{2}\right)}$. And so on. By way of summary, Newton's method is

1 Make a preliminary guess $x_{1}$.
2 Define $x_{2}=x_{1}-\frac{f\left(x_{1}\right)}{f^{\prime}\left(x_{1}\right)}$.
3 Iterate. That is, for each natural number $n$, once you have computed $x_{n}$, define

Equation C.1.1 (Newton's method).

$$
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}
$$

Example C.1.2 (Approximating $\sqrt{2}$ ).
In this example we compute, approximately, the square root of two. We will of course pretend that we do not already know that $\sqrt{2}=1.41421 \cdots$. So we cannot find it by solving, approximately, the equation $f(x)=x-\sqrt{2}=0$. Instead we apply Newton's method to the equation

$$
f(x)=x^{2}-2=0
$$

Since $f^{\prime}(x)=2 x$, Newton's method says that we should generate approximate solutions
by iteratively applying

$$
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}=x_{n}-\frac{x_{n}^{2}-2}{2 x_{n}}=\frac{x_{n}}{2}+\frac{1}{x_{n}}
$$

We need a starting point. Since $1^{2}=1<2$ and $2^{2}=4>2$, the square root of two must be between 1 and 2, so let's start Newton's method with the initial guess $x_{1}=1.5$. Here goes ${ }^{a}$ :

$$
\begin{aligned}
x_{1} & =1.5 \\
x_{2} & =\frac{1}{2} x_{1}+\frac{1}{x_{1}}=\frac{1}{2}(1.5)+\frac{1}{1.5} \\
& =1.416666667 \\
x_{3} & =\frac{1}{2} x_{2}+\frac{1}{x_{2}}=\frac{1}{2}(1.416666667)+\frac{1}{1.416666667} \\
& =1.414215686 \\
x_{4} & =\frac{1}{2} x_{3}+\frac{1}{x_{3}}=\frac{1}{2}(1.414215686)+\frac{1}{1.414215686} \\
& =1.414213562 \\
x_{5} & =\frac{1}{2} x_{4}+\frac{1}{x_{4}}=\frac{1}{2}(1.414213562)+\frac{1}{1.414213562} \\
& =1.414213562
\end{aligned}
$$

It looks like the $x_{n}$ 's, rounded to nine decimal places, have stabilized to 1.414213562 . So it is reasonable to guess that $\sqrt{2}$, rounded to nine decimal places, is exactly 1.414213562 . Recalling that all numbers $1.4142135615 \leq y<1.4142135625$ round to 1.414213562 , we can check our guess by evaluating $f(1.4142135615)$ and $f(1.4142135625)$. Since $f(1.4142135615)=-2.5 \times 10^{-9}<0$ and $f(1.4142135625)=3.6 \times 10^{-10}>0$ the square root of two must indeed be between 1.4142135615 and 1.4142135625 .
a The following computations have been carried out in double precision, which is computer speak for about 15 significant digits. We are displaying each $x_{n}$ rounded to 10 significant digits ( 9 decimal places). So each displayed $x_{n}$ has not been impacted by roundoff error, and still contains more decimal places than are usually needed.

Example C.1.2

Example C.1.3 (Approximating $\pi$ ).
In this example we compute, approximately, $\pi$ by applying Newton's method to the equation

$$
f(x)=\sin x=0
$$

starting with $x_{1}=3$. Since $f^{\prime}(x)=\cos x$, Newton's method says that we should
generate approximate solutions by iteratively applying

$$
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}=x_{n}-\frac{\sin x_{n}}{\cos x_{n}}=x_{n}-\tan x_{n}
$$

Here goes

$$
\begin{aligned}
x_{1} & =3 \\
x_{2} & =x_{1}-\tan x_{1}=3-\tan 3 \\
& =3.142546543 \\
x_{3} & =3.142546543-\tan 3.142546543 \\
& =3.141592653 \\
x_{4} & =3.141592653-\tan 3.141592653 \\
& =3.141592654 \\
x_{5} & =3.141592654-\tan 3.141592654 \\
& =3.141592654
\end{aligned}
$$

Since $f(3.1415926535)=9.0 \times 10^{-11}>0$ and $f(3.1415926545)=-9.1 \times 10^{-11}<0, \pi$ must be between 3.1415926535 and 3.1415926545 . Of course to compute $\pi$ in this way, we (or at least our computers) have to be able to evaluate $\tan x$ for various values of $x$. Taylor expansions can help us do that. See Example 3.4.22.

Example C.1.3

## Example C.1. 4 wild instability.

This example illustrates how Newton's method can go badly wrong if your initial guess is not good enough. We'll try to solve the equation

$$
f(x)=\arctan x=0
$$

starting with $x_{1}=1.5$. (Of course the solution to $f(x)=0$ is just $x=0$; we chose $x_{1}=1.5$ for demonstration purposes.) Since the derivative $f^{\prime}(x)=\frac{1}{1+x^{2}}$, Newton's method gives

$$
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}=x_{n}-\left(1+x_{n}^{2}\right) \arctan x_{n}
$$

So ${ }^{a}$

$$
\begin{aligned}
& x_{1}=1.5 \\
& x_{2}=1.5-\left(1+1.5^{2}\right) \arctan 1.5=-1.69 \\
& x_{3}=-1.69-\left(1+1.69^{2}\right) \arctan (-1.69)=2.32 \\
& x_{4}=2.32-\left(1+2.32^{2}\right) \arctan (2.32)=-5.11 \\
& x_{5}=-5.11-\left(1+5.11^{2}\right) \arctan (-5.11)=32.3
\end{aligned}
$$

$$
\begin{aligned}
& x_{6}=32.3-\left(1+32.3^{2}\right) \arctan (32.3)=-1575 \\
& x_{7}=3,894,976
\end{aligned}
$$

Looks pretty bad! Our $x_{n}$ 's are not settling down at all!
The figure below shows what went wrong. In this figure, $y=F_{1}(x)$ is the tangent line to $y=\arctan x$ at $x=x_{1}$. Under Newton's method, this tangent line crosses the $x$-axis at $x=x_{2}$. Then $y=F_{2}(x)$ is the tangent to $y=\arctan x$ at $x=x_{2}$. Under Newton's method, this tangent line crosses the $x$-axis at $x=x_{3}$. And so on.
The problem arose because the $x_{n}$ 's were far enough from the solution, $x=0$, that the tangent line approximations, while good approximations to $f(x)$ for $x \approx x_{n}$, were very poor approximations to $f(x)$ for $x \approx 0$. In particular, $y=F_{1}(x)$ (i.e. the tangent line at $x=x_{1}$ ) was a bad enough approximation to $y=\arctan x$ for $x \approx 0$ that $x=x_{2}$ (i.e. the value of $x$ where $y=F_{1}(x)$ crosses the $x$-axis) is farther from the solution $x=0$ than our original guess $x=x_{1}$.


If we had started with $x_{1}=0.5$ instead of $x_{1}=1.5$, Newton's method would have succeeded very nicely:

$$
x_{1}=0.5 \quad x_{2}=-0.0796 \quad x_{3}=0.000335 \quad x_{4}=-2.51 \times 10^{-11}
$$

$a$ Once again, the following computations have been carried out in double precision. This time, it is clear that the $x_{n}$ 's are growing madly as $n$ increases. So there is not much point to displaying many decimal places and we have not done so.

Example C.1.4

Example C.1.5 interest rate.
A car dealer sells a new car for $\$ 23,520$. He also offers to finance the same car for payments of $\$ 420$ per month for five years. What interest rate is this dealer charging? Solution. By way of preparation, we'll start with a simpler problem. Suppose that you will have to make a single $\$ 420$ payment $n$ months in the future. The simpler problem is to determine how much money you have to deposit now in an account that pays an interest rate of $100 r \%$ per month, compounded monthly ${ }^{a}$, in order to be able to make the $\$ 420$ payment in $n$ months.
Let's denote by $P$ the initial deposit. Because the interest rate is $100 r \%$ per month, compounded monthly,

- the first month's interest is $P \times r$. So at the end of month $\# 1$, the account balance is $P+P r=P(1+r)$.
- The second month's interest is $[P(1+r)] \times r$. So at the end of month $\# 2$, the account balance is $P(1+r)+P(1+r) r=P(1+r)^{2}$.
- And so on.
- So at the end of $n$ months, the account balance is $P(1+r)^{n}$.

In order for the balance at the end of $n$ months, $P(1+r)^{n}$, to be $\$ 420$, the initial deposit has to be $P=420(1+r)^{-n}$. That is what is meant by the statement "The present value ${ }^{b}$ of a $\$ 420$ payment made $n$ months in the future, when the interest rate is $100 r \%$ per month, compounded monthly, is $420(1+r)^{-n}$."
Now back to the original problem. We will be making 60 monthly payments of $\$ 420$. The present value of all 60 payments is ${ }^{c}$

$$
\begin{aligned}
420(1+r)^{-1}+ & 420(1+r)^{-2}+\cdots+420(1+r)^{-60} \\
& =420 \frac{(1+r)^{-1}-(1+r)^{-61}}{1-(1+r)^{-1}} \\
& =420 \frac{1-(1+r)^{-60}}{(1+r)-1} \\
& =420 \frac{1-(1+r)^{-60}}{r}
\end{aligned}
$$

The interest rate $100 r \%$ being charged by the car dealer is such that the present value of 60 monthly payments of $\$ 420$ is $\$ 23520$. That is, the monthly interest rate being charged by the car dealer is the solution of

$$
\begin{array}{lll}
23520=420 \frac{1-(1+r)^{-60}}{r} \quad & \text { or } & 56=\frac{1-(1+r)^{-60}}{r} \\
& \text { or } & 56 r=1-(1+r)^{-60} \\
& \text { or } & 56 r(1+r)^{60}=(1+r)^{60}-1 \\
& \text { or } & (1-56 r)(1+r)^{60}=1
\end{array}
$$

Set $f(r)=(1-56 r)(1+r)^{60}-1$. Then

$$
f^{\prime}(r)=-56(1+r)^{60}+60(1-56 r)(1+r)^{59}
$$

or

$$
f^{\prime}(r)=[-56(1+r)+60(1-56 r)](1+r)^{59}=(4-3416 r)(1+r)^{59}
$$

Apply Newton's method with an initial guess of $r_{1}=.002$. (That's $0.2 \%$ per month or $2.4 \%$ per year.) Then

$$
\begin{aligned}
& r_{2}=r_{1}-\frac{\left(1-56 r_{1}\right)\left(1+r_{1}\right)^{60}-1}{\left(4-3416 r_{1}\right)\left(1+r_{1}\right)^{59}}=0.002344 \\
& r_{3}=r_{2}-\frac{\left(1-56 r_{2}\right)\left(1+r_{2}\right)^{60}-1}{\left(4-3416 r_{2}\right)\left(1+r_{2}\right)^{59}}=0.002292 \\
& r_{4}=r_{3}-\frac{\left(1-56 r_{3}\right)\left(1+r_{3}\right)^{60}-1}{\left(4-3416 r_{3}\right)\left(1+r_{3}\right)^{59}}=0.002290 \\
& r_{5}=r_{4}-\frac{\left(1-56 r_{4}\right)\left(1+r_{4}\right)^{60}-1}{\left(4-3416 r_{4}\right)\left(1+r_{4}\right)^{59}}=0.002290
\end{aligned}
$$

So the interest rate is $0.229 \%$ per month or $2.75 \%$ per year.
$a$ "Compounded monthly", means that, each month, interest is paid on the accumulated interest that was paid in all previous months.
$b$ Inflation means that prices of goods (typically) increase with time, and hence $\$ 100$ now is worth more than $\$ 100$ in 10 years time. The term "present value" is widely used in economics and finance to mean "the current amount of money that will have a specified value at a specified time in the future". It takes inflation into account. If the money is invested, it takes into account the rate of return of the investment. We recommend that the interested reader do some search-engining to find out more.
$c$ Don't worry if you don't know how to evaluate such sums. They are called geometric sums, and will be covered in the CLP-2 text. (See (1.1.3) in the CLP-2 text. In any event, you can check that this is correct, by multiplying the whole equation by $1-(1+r)^{-1}$. When you simplify the left hand side, you should get the right hand side.


## C.2』 The Error Behaviour of Newton's Method

Newton's method usually works spectacularly well, provided your initial guess is reasonably close to a solution of $f(x)=0$. A good way to select this initial guess is to sketch the graph of $y=f(x)$. We now explain why "Newton's method usually works spectacularly well, provided your initial guess is reasonably close to a solution of $f(x)=0$ ".

Let $r$ be any solution of $f(x)=0$. Then $f(r)=0$. Suppose that we have already
computed $x_{n}$. The error in $x_{n}$ is $\left|x_{n}-r\right|$. We now derive a formula that relates the error after the next step, $\left|x_{n+1}-r\right|$, to $\left|x_{n}-r\right|$. We have seen in (3.4.32) that

$$
f(x)=f\left(x_{n}\right)+f^{\prime}\left(x_{n}\right)\left(x-x_{n}\right)+\frac{1}{2} f^{\prime \prime}(c)\left(x-x_{n}\right)^{2}
$$

for some $c$ between $x_{n}$ and $x$. In particular, choosing $x=r$,

$$
\begin{equation*}
0=f(r)=f\left(x_{n}\right)+f^{\prime}\left(x_{n}\right)\left(r-x_{n}\right)+\frac{1}{2} f^{\prime \prime}(c)\left(r-x_{n}\right)^{2} \tag{E1}
\end{equation*}
$$

Recall that $x_{n+1}$ is the solution of $0=f\left(x_{n}\right)+f^{\prime}\left(x_{n}\right)\left(x-x_{n}\right)$. So

$$
\begin{equation*}
0=f\left(x_{n}\right)+f^{\prime}\left(x_{n}\right)\left(x_{n+1}-x_{n}\right) \tag{E2}
\end{equation*}
$$

We need to get an expression for $x_{n+1}-r$. Subtracting (E2) from (E1) gives

$$
\begin{aligned}
0=f^{\prime}\left(x_{n}\right)\left(r-x_{n+1}\right)+\frac{1}{2} f^{\prime \prime}(c)\left(r-x_{n}\right)^{2} & \Longrightarrow x_{n+1}-r=\frac{f^{\prime \prime}(c)}{2 f^{\prime}\left(x_{n}\right)}\left(x_{n}-r\right)^{2} \\
& \Longrightarrow\left|x_{n+1}-r\right|=\frac{\left|f^{\prime \prime}(c)\right|}{2\left|f^{\prime}\left(x_{n}\right)\right|}\left|x_{n}-r\right|^{2}
\end{aligned}
$$

If the guess $x_{n}$ is close to $r$, then $c$, which must be between $x_{n}$ and $r$, is also close to $r$ and we will have $f^{\prime \prime}(c) \approx f^{\prime \prime}(r)$ and $f^{\prime}\left(x_{n}\right) \approx f^{\prime}(r)$ and

$$
\begin{equation*}
\left|x_{n+1}-r\right| \approx \frac{\left|f^{\prime \prime}(r)\right|}{2\left|f^{\prime}(r)\right|}\left|x_{n}-r\right|^{2} \tag{E3}
\end{equation*}
$$

Even when $x_{n}$ is not close to $r$, if we know that there are two numbers $L, M>0$ such that $f$ obeys:
(H1) $\left|f^{\prime}\left(x_{n}\right)\right| \geq L$
(H2) $\left|f^{\prime \prime}(c)\right| \leq M$
(we'll see examples of this below) then we will have

$$
\begin{equation*}
\left|x_{n+1}-r\right| \leq \frac{M}{2 L}\left|x_{n}-r\right|^{2} \tag{E4}
\end{equation*}
$$

Let's denote by $\varepsilon_{1}$ the error, $\left|x_{1}-r\right|$, of our initial guess. In fact, let's denote by $\varepsilon_{n}$ the error, $\left|x_{n}-r\right|$, in $x_{n}$. Then (E4) says

$$
\varepsilon_{n+1} \leq \frac{M}{2 L} \varepsilon_{n}^{2}
$$

In particular

$$
\begin{aligned}
& \varepsilon_{2} \leq \frac{M}{2 L} \varepsilon_{1}^{2} \\
& \varepsilon_{3} \leq \frac{M}{2 L} \varepsilon_{2}^{2} \leq \frac{M}{2 L}\left(\frac{M}{2 L} \varepsilon_{1}^{2}\right)^{2}=\left(\frac{M}{2 L}\right)^{3} \varepsilon_{1}^{4}
\end{aligned}
$$

$$
\begin{aligned}
& \varepsilon_{4} \leq \frac{M}{2 L} \varepsilon_{3}^{2} \leq \frac{M}{2 L}\left[\left(\frac{M}{2 L}\right)^{3} \varepsilon_{1}^{4}\right]^{2}=\left(\frac{M}{2 L}\right)^{7} \varepsilon_{1}^{8} \\
& \varepsilon_{5} \leq \frac{M}{2 L} \varepsilon_{4}^{2} \leq \frac{M}{2 L}\left[\left(\frac{M}{2 L}\right)^{7} \varepsilon_{1}^{8}\right]^{2}=\left(\frac{M}{2 L}\right)^{15} \varepsilon_{1}^{16}
\end{aligned}
$$

By now we can see a pattern forming, that is easily verified by induction ${ }^{1}$.

$$
\begin{equation*}
\varepsilon_{n} \leq\left(\frac{M}{2 L}\right)^{2^{n-1}-1} \varepsilon_{1}^{2^{n-1}}=\frac{2 L}{M}\left(\frac{M}{2 L} \varepsilon_{1}\right)^{2^{n-1}} \tag{E5}
\end{equation*}
$$

As long as $\frac{M}{2 L} \varepsilon_{1}<1$ (which gives us a quantitative idea as to how good our first guess has to be in order for Newton's method to work), this goes to zero extremely quickly as $n$ increases. For example, suppose that $\frac{M}{2 L} \varepsilon_{1} \leq \frac{1}{2}$. Then

$$
\varepsilon_{n} \leq \frac{2 L}{M}\left(\frac{1}{2}\right)^{2^{n-1}} \leq \frac{2 L}{M} \cdot \begin{cases}0.25 & \text { if } n=2 \\ 0.0625 & \text { if } n=3 \\ 0.0039=3.9 \times 10^{-3} & \text { if } n=4 \\ 0.000015=1.5 \times 10^{-5} & \text { if } n=5 \\ 0.00000000023=2.3 \times 10^{-10} & \text { if } n=6 \\ 0.000000000000000000054=5.4 \times 10^{-20} & \text { if } n=7\end{cases}
$$

Each time you increase $n$ by one, the number of zeroes after the decimal place roughly doubles. You can see why from (E5). Since

$$
\left(\frac{M}{2 L} \varepsilon_{1}\right)^{2^{(n+1)-1}}=\left(\frac{M}{2 L} \varepsilon_{1}\right)^{2^{n-1} \times 2}=\left[\left(\frac{M}{2 L} \varepsilon_{1}\right)^{2^{n-1}}\right]^{2}
$$

we have, very roughly speaking, $\varepsilon_{n+1} \approx \varepsilon_{n}^{2}$. This quadratic behaviour is the reason that Newton's method is so useful.

Example C.2.1 (Example C.1.2, continued).
Let's consider, as we did in Example C.1.2, $f(x)=x^{2}-2$, starting with $x_{1}=\frac{3}{2}$. Then

$$
f^{\prime}(x)=2 x \quad f^{\prime \prime}(x)=2
$$

Recalling, from (H1) and (H2), that $L$ is a lower bound on $\left|f^{\prime}\right|$ and $M$ is an upper bound on $\left|f^{\prime \prime}\right|$, we may certainly take $M=2$ and if, for example, $x_{n} \geq 1$ for all $n$ (as happened in Example C.1.2), we may take $L=2$ too. While we do not know what $r$ is, we do know that $1 \leq r \leq 2$ (since $f(1)=1^{1}-2<0$ and $f(2)=2^{2}-2>0$ ). As we

1 Mathematical induction is a technique for proving a sequence $S_{1}, S_{2}, S_{3}, \cdots$ of statements. That technique consists of first proving that $S_{1}$ is true, and then proving that, for any natural number $n$, if $S_{n}$ is true then $S_{n+1}$ is true.
took $x_{1}=\frac{3}{2}$, we have $\varepsilon_{1}=\left|x_{1}-r\right| \leq \frac{1}{2}$, so that $\frac{M}{2 L} \varepsilon_{1} \leq \frac{1}{4}$ and

$$
\begin{equation*}
\varepsilon_{n+1} \leq \frac{2 L}{M}\left(\frac{M}{2 L} \varepsilon_{1}\right)^{2^{n-1}} \leq 2\left(\frac{1}{4}\right)^{2^{n-1}} \tag{E6}
\end{equation*}
$$

This tends to zero very quickly as $n$ increases. Furthermore this is an upper bound on the error and not the actual error. In fact (E6) is a very crude upper bound. For example, setting $n=3$ gives the bound

$$
\varepsilon_{4} \leq 2\left(\frac{1}{4}\right)^{2^{2}}=7 \times 10^{-3}
$$

and we saw in Example C.1.2 that the actual error in $x_{4}$ was smaller than $5 \times 10^{-10}$.
Example C.2.1

## Example C.2.2 (Example C.1.3, continued).

Let's consider, as we did in Example C.1.3, $f(x)=\sin x$, starting with $x_{1}=3$. Then

$$
f^{\prime}(x)=\cos x \quad f^{\prime \prime}(x)=-\sin x
$$

As $|-\sin x| \leq 1$, we may certainly take $M=1$. In Example C.1.3, all $x_{n}$ 's were between 3 and 3.2. Since (to three decimal places)

$$
\sin (3)=0.141>0 \quad \sin (3.2)=-0.058<0
$$

the IVT (intermediate value theorem) tells us that $3<r<3.2$ and $\varepsilon_{1}=\left|x_{1}-r\right|<0.2$. So $r$ and all $x_{n}$ 's and hence all $c$ 's lie in the interval $(3,3.2)$. Since

$$
-0.9990=\cos (3)<\cos c<\cos (3.2)=-0.9983
$$

we necessarily have $\left|f^{\prime}(c)\right|=|\cos c| \geq 0.9$ and we may take $L=0.9$. So

$$
\varepsilon_{n+1} \leq \frac{2 L}{M}\left(\frac{M}{2 L} \varepsilon_{1}\right)^{2^{n-1}} \leq \frac{2 \times 0.9}{1}\left(\frac{1}{2 \times 0.9} 0.2\right)^{2^{n-1}} \leq 2\left(\frac{1}{9}\right)^{2^{n-1}}
$$

$\uparrow$ This tends to zero very quickly as $n$ increases.

We have now seen two procedures for finding roots of a function $f(x)$ - the bisection method (which does not use the derivative of $f(x)$, but which is not very efficient) and Newton's method (which does use the derivative of $f(x)$, and which is very efficient). In fact, there is a whole constellation of other methods ${ }^{2}$ and the interested reader should search engine their way to, for example, Wikipedia's article on root finding algorithms.

2 What does it say about mathematicians that they have developed so many ways of finding zero?

Here, we will just mention two other methods, one being a variant of the bisection method and the other being a variant of Newton's method.

## C.3』 The false position (regula falsi) method

Let $f(x)$ be a continuous function and let $a_{1}<b_{1}$ with $f\left(a_{1}\right)$ and $f\left(b_{1}\right)$ being of opposite sign.

As we have seen, the bisection method generates a sequence of intervals $I_{n}=\left[a_{n}, b_{n}\right]$, $n=1,2,3, \cdots$ with, for each $n, f\left(a_{n}\right)$ and $f\left(b_{n}\right)$ having opposite sign (so that, by continuity, $f$ has a root in $I_{n}$ ). Once we have $I_{n}$, we choose $I_{n+1}$ based on the sign of $f$ at the midpoint, $\frac{a_{n}+b_{n}}{2}$, of $I_{n}$. Since we always test the midpoint, the possible error decreases by a factor of 2 each step.

The false position method tries to make the whole procedure more efficient by testing the sign of $f$ at a point that is closer to the end of $I_{n}$ where the magnitude of $f$ is smaller. To be precise, we approximate $y=f(x)$ by the equation of the straight line through $\left(a_{n}, f\left(a_{n}\right)\right)$ and $\left(b_{n}, f\left(b_{n}\right)\right)$.


The equation of that straight line is

$$
y=F(x)=f\left(a_{n}\right)+\frac{f\left(b_{n}\right)-f\left(a_{n}\right)}{b_{n}-a_{n}}\left(x-a_{n}\right)
$$

Then the false position method tests the sign of $f(x)$ at the value of $x$ where $F(x)=0$.

$$
\begin{aligned}
& F(x)=f\left(a_{n}\right)+\frac{f\left(b_{n}\right)-f\left(a_{n}\right)}{b_{n}-a_{n}}\left(x-a_{n}\right)=0 \\
& \quad \Longleftrightarrow x=a_{n}-\frac{b_{n}-a_{n}}{f\left(b_{n}\right)-f\left(a_{n}\right)} f\left(a_{n}\right)=\frac{a_{n} f\left(b_{n}\right)-b_{n} f\left(a_{n}\right)}{f\left(b_{n}\right)-f\left(a_{n}\right)}
\end{aligned}
$$

So once we have the interval $I_{n}$, the false position method generates the interval $I_{n+1}$ by the following rule. ${ }^{1}$

Equation C.3.1 false position method.
Set $c_{n}=\frac{a_{n} f\left(b_{n}\right)-b_{n} f\left(a_{n}\right)}{f\left(b_{n}\right)-f\left(a_{n}\right)}$. If $f\left(c_{n}\right)$ has the same sign as $f\left(a_{n}\right)$, then

$$
I_{n+1}=\left[a_{n+1}, b_{n+1}\right] \quad \text { with } \quad a_{n+1}=c_{n}, b_{n+1}=b_{n}
$$

and if $f\left(c_{n}\right)$ and $f\left(a_{n}\right)$ have opposite signs, then

$$
I_{n+1}=\left[a_{n+1}, b_{n+1}\right] \quad \text { with } \quad a_{n+1}=a_{n}, b_{n+1}=c_{n}
$$

## C.4^ The secant method

Let $f(x)$ be a continuous function. The secant method is a variant of Newton's method that avoids the use of the derivative of $f(x)$ - which can be very helpful when dealing with the derivative is not easy. It avoids the use of the derivative by approximating $f^{\prime}(x)$ by $\frac{f(x+h)-f(x)}{h}$ for some $h$. That is, it approximates the tangent line to $f$ at $x$ by a secant line for $f$ that passes through $x$. To limit the number of evaluations of $f(x)$ required, it uses $x=x_{n-1}$ and $x+h=x_{n}$. Here is how it works.

Suppose that we have already found $x_{n}$. Then we denote by $y=F(x)$ the equation of the (secant) line that passes through $\left(x_{n-1}, f\left(x_{n-1}\right)\right)$ and $\left(x_{n}, f\left(x_{n}\right)\right)$ and we choose $x_{n+1}$ to be the value of $x$ where $F(x)=0$.


The equation of the secant line is

$$
y=F(x)=f\left(x_{n-1}\right)+\frac{f\left(x_{n}\right)-f\left(x_{n-1}\right)}{x_{n}-x_{n-1}}\left(x-x_{n-1}\right)
$$

1 The convergence behaviour of the false position method is relatively complicated. So we do not discuss it here. As always, we invite the interested reader to visit their favourite search engine.
so that $x_{n+1}$ is determined by

$$
\begin{aligned}
0 & =F\left(x_{n+1}\right)=f\left(x_{n-1}\right)+\frac{f\left(x_{n}\right)-f\left(x_{n-1}\right)}{x_{n}-x_{n-1}}\left(x_{n+1}-x_{n-1}\right) \\
& \Longleftrightarrow x_{n+1}=x_{n-1}-\frac{x_{n}-x_{n-1}}{f\left(x_{n}\right)-f\left(x_{n-1}\right)} f\left(x_{n-1}\right)
\end{aligned}
$$

or, simplifying,

## Equation C.4.1 secant method.

$$
x_{n+1}=\frac{x_{n-1} f\left(x_{n}\right)-x_{n} f\left(x_{n-1}\right)}{f\left(x_{n}\right)-f\left(x_{n-1}\right)}
$$

Of course, to get started with $n=1$, we need two initial guesses, $x_{0}$ and $x_{1}$, for the root.

Example C.4.2 Approximating $\sqrt{2}$, again.
In this example we compute, approximately, the square root of two by applying the secant method to the equation

$$
f(x)=x^{2}-2=0
$$

and we'll compare the secant method results with the corresponding Newton's method results. (See Example C.1.2.)
Since $f^{\prime}(x)=2 x$, (C.1.1) says that, under Newton's method, we should iteratively apply

$$
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}=x_{n}-\frac{x_{n}^{2}-2}{2 x_{n}}=\frac{x_{n}}{2}+\frac{1}{x_{n}}
$$

while (C.4.1) says that, under the secant method, we should iteratively apply (after a little simplifying algebra)

$$
\begin{aligned}
x_{n+1} & =\frac{x_{n-1} f\left(x_{n}\right)-x_{n} f\left(x_{n-1}\right)}{f\left(x_{n}\right)-f\left(x_{n-1}\right)}=\frac{x_{n-1}\left[x_{n}^{2}-2\right]-x_{n}\left[x_{n-1}^{2}-2\right]}{x_{n}^{2}-x_{n-1}^{2}} \\
& =\frac{x_{n-1} x_{n}\left[x_{n}-x_{n-1}\right]+2\left[x_{n}-x_{n-1}\right]}{x_{n}^{2}-x_{n-1}^{2}} \\
& =\frac{x_{n-1} x_{n}+2}{x_{n-1}+x_{n}}
\end{aligned}
$$

Here are the results, starting Newton's method with $x_{1}=4$ and starting the secant method with $x_{0}=4, x_{1}=3$. (So we are giving the secant method a bit of a head start.)
secant method Newton's method
$x_{0} \quad 4$

| $x_{1}$ | 3 | 4 |
| :--- | :--- | :--- |
| $x_{2}$ | 2 | 2.25 |
| $x_{3}$ | 1.6 | 1.57 |
| $x_{4}$ | 1.444 | 1.422 |
| $x_{5}$ | 1.4161 | 1.414234 |
| $x_{6}$ | 1.414233 | 1.414213562525 |
| $x_{7}$ | 1.414213575 | 1.414213562373095 |

For comparison purposes, the square root of 2 , to 15 decimal places, is 1.414213562373095. So the secant method $x_{7}$ is accurate to 7 decimal places and the Newton's method $x_{7}$ is accurate to at least 15 decimal places.

```
Example C.4.2
```

The advantage that the secant method has over Newton's method is that it does not use the derivative of $f$. This can be a substantial advantage, for example when evaluation of the derivative is computationally difficult or expensive. On the other hand, the above example suggests that the secant method is not as fast as Newton's method. The following subsection shows that this is indeed the case.

## C.5^ The Error Behaviour of the Secant Method

Let $f(x)$ have two continuous derivatives, and let $r$ be any solution of $f(x)=0$. We will now get a pretty good handle on the error behaviour of the secant method near $r$.

Denote by $\tilde{\varepsilon}_{n}=x_{n}-r$ the (signed) error in $x_{n}$ and by $\varepsilon_{n}=\left|x_{n}-r\right|$ the (absolute) error in $x_{n}$. Then, $x_{n}=r+\tilde{\varepsilon}_{n}$, and, by (C.4.1),

$$
\begin{aligned}
\tilde{\varepsilon}_{n+1} & =\frac{x_{n-1} f\left(x_{n}\right)-x_{n} f\left(x_{n-1}\right)}{f\left(x_{n}\right)-f\left(x_{n-1}\right)}-r \\
& =\frac{\left[r+\tilde{\varepsilon}_{n-1}\right] f\left(x_{n}\right)-\left[r+\tilde{\varepsilon}_{n}\right] f\left(x_{n-1}\right)}{f\left(x_{n}\right)-f\left(x_{n-1}\right)}-r \\
& =\frac{\tilde{\varepsilon}_{n-1} f\left(x_{n}\right)-\tilde{\varepsilon}_{n} f\left(x_{n-1}\right)}{f\left(x_{n}\right)-f\left(x_{n-1}\right)}
\end{aligned}
$$

By the Taylor expansion (3.4.32) and the mean value theorem (Theorem 2.13.5),

$$
\begin{aligned}
f\left(x_{n}\right) & =f(r)+f^{\prime}(r) \tilde{\varepsilon}_{n}+\frac{1}{2} f^{\prime \prime}\left(c_{1}\right) \tilde{\varepsilon}_{n}^{2} \\
& =f^{\prime}(r) \tilde{\varepsilon}_{n}+\frac{1}{2} f^{\prime \prime}\left(c_{1}\right) \tilde{\varepsilon}_{n}^{2} \\
f\left(x_{n}\right)-f\left(x_{n-1}\right) & =f^{\prime}\left(c_{2}\right)\left[x_{n}-x_{n-1}\right] \\
& =f^{\prime}\left(c_{2}\right)\left[\tilde{\varepsilon}_{n}-\tilde{\varepsilon}_{n-1}\right]
\end{aligned}
$$

for some $c_{1}$ between $r$ and $x_{n}$ and some $c_{2}$ between $x_{n-1}$ and $x_{n}$. So, for $x_{n-1}$ and $x_{n}$ near $r, c_{1}$ and $c_{2}$ also have to be near $r$ and

$$
\begin{aligned}
f\left(x_{n}\right) & \approx f^{\prime}(r) \tilde{\varepsilon}_{n}+\frac{1}{2} f^{\prime \prime}(r) \tilde{\varepsilon}_{n}^{2} \\
f\left(x_{n-1}\right) & \approx f^{\prime}(r) \tilde{\varepsilon}_{n-1}+\frac{1}{2} f^{\prime \prime}(r) \tilde{\varepsilon}_{n-1}^{2} \\
f\left(x_{n}\right)-f\left(x_{n-1}\right) & \approx f^{\prime}(r)\left[\tilde{\varepsilon}_{n}-\tilde{\varepsilon}_{n-1}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\varepsilon}_{n+1} & =\frac{\tilde{\varepsilon}_{n-1} f\left(x_{n}\right)-\tilde{\varepsilon}_{n} f\left(x_{n-1}\right)}{f\left(x_{n}\right)-f\left(x_{n-1}\right)} \\
& \approx \frac{\tilde{\varepsilon}_{n-1}\left[f^{\prime}(r) \tilde{\varepsilon}_{n}+\frac{1}{2} f^{\prime \prime}(r) \tilde{\varepsilon}_{n}^{2}\right]-\tilde{\varepsilon}_{n}\left[f^{\prime}(r) \tilde{\varepsilon}_{n-1}+\frac{1}{2} f^{\prime \prime}(r) \tilde{\varepsilon}_{n-1}^{2}\right]}{f^{\prime}(r)\left[\tilde{\varepsilon}_{n}-\tilde{\varepsilon}_{n-1}\right]} \\
& =\frac{\frac{1}{2} \tilde{\varepsilon}_{n-1} \tilde{\varepsilon}_{n} f^{\prime \prime}(r)\left[\tilde{\varepsilon}_{n}-\tilde{\varepsilon}_{n-1}\right]}{f^{\prime}(r)\left[\tilde{\varepsilon}_{n}-\tilde{\varepsilon}_{n-1}\right]} \\
& =\frac{f^{\prime \prime}(r)}{2 f^{\prime}(r)} \tilde{\varepsilon}_{n-1} \tilde{\varepsilon}_{n}
\end{aligned}
$$

Taking absolute values, we have

$$
\begin{equation*}
\varepsilon_{n+1} \approx K \varepsilon_{n-1} \varepsilon_{n} \quad \text { with } K=\left|\frac{f^{\prime \prime}(r)}{2 f^{\prime}(r)}\right| \tag{E7}
\end{equation*}
$$

We have seen that Newton's method obeys a similar formula - (E3) says that, when $x_{n}$ is near $r$, Newton's method obeys $\varepsilon_{n+1} \approx K \varepsilon_{n}^{2}$, also with $K=\left|\frac{f^{\prime \prime}(r)}{2 f^{\prime}(r)}\right|$. As we shall now see, the change from $\varepsilon_{n}^{2}$, in $\varepsilon_{n+1} \approx K \varepsilon_{n}^{2}$, to $\varepsilon_{n-1} \varepsilon_{n}$, in $\varepsilon_{n+1} \approx K \varepsilon_{n-1} \varepsilon_{n}$, does have a substantial impact on the behaviour of $\varepsilon_{n}$ for large $n$.

To see the large $n$ behaviour, we now iterate (E7). The formulae will look simpler if we multiply (E7) by $K$ and write $\delta_{n}=K \varepsilon_{n}$. Then (E7) becomes $\delta_{n+1} \approx \delta_{n-1} \delta_{n}$ (and we have eliminated $K$ ). The first iterations are

$$
\begin{aligned}
& \delta_{2} \quad \approx \delta_{0} \delta_{1} \\
& \delta_{3} \approx \delta_{1} \delta_{2} \approx \delta_{0} \delta_{1}^{2} \\
& \delta_{4} \approx \delta_{2} \delta_{3} \approx \delta_{0}^{2} \delta_{1}^{3} \\
& \delta_{5} \approx \delta_{3} \delta_{4} \approx \delta_{0}^{3} \delta_{1}^{5} \\
& \delta_{6} \approx \delta_{4} \delta_{5} \approx \delta_{0}^{5} \delta_{1}^{8} \\
& \delta_{7} \approx \delta_{5} \delta_{6} \approx \delta_{0}^{8} \delta_{1}^{13}
\end{aligned}
$$

Notice that every $\delta_{n}$ is of the form $\delta_{0}^{\alpha_{n}} \delta_{1}^{\beta_{n}}$. Substituting $\delta_{n}=\delta_{0}^{\alpha_{n}} \delta_{1}^{\beta_{n}}$ into $\delta_{n+1} \approx \delta_{n-1} \delta_{n}$ gives

$$
\delta_{0}^{\alpha_{n+1}} \delta_{1}^{\beta_{n+1}} \approx \delta_{0}^{\alpha_{n-1}} \delta_{1}^{\beta_{n-1}} \delta_{0}^{\alpha_{n}} \delta_{1}^{\beta_{n}}
$$

and we have

$$
\begin{equation*}
\alpha_{n+1}=\alpha_{n-1}+\alpha_{n} \quad \beta_{n+1}=\beta_{n-1}+\beta_{n} \tag{E8}
\end{equation*}
$$

The recursion rule in (E8) is famous ${ }^{1}$. The Fibonacci ${ }^{2}$ sequence (which is $0,1,1,2$, $3,5,8,13, \cdots)$, is defined by

$$
\begin{aligned}
& F_{0}=0 \\
& F_{1}=1 \\
& F_{n}=F_{n-1}+F_{n-2} \quad \text { for } n>1
\end{aligned}
$$

So, for $n \geq 2, \alpha_{n}=F_{n-1}$ and $\beta_{n}=F_{n}$ and

$$
\delta_{n} \approx \delta_{0}^{\alpha_{n}} \delta_{1}^{\beta_{n}}=\delta_{0}^{F_{n-1}} \delta_{1}^{F_{n}}
$$

One of the known properties of the Fibonacci sequence is that, for large $n$,

$$
F_{n} \approx \frac{\varphi^{n}}{\sqrt{5}} \quad \text { where } \varphi=\frac{1+\sqrt{5}}{2} \approx 1.61803
$$

This $\varphi$ is the golden ratio ${ }^{3}$. So, for large $n$,

$$
\begin{aligned}
K \varepsilon_{n} & =\delta_{n} \approx \delta_{0}^{F_{n-1}} \delta_{1}^{F_{n}} \approx \delta_{0}^{\frac{\varphi^{n-1}}{\sqrt{5}}} \delta_{1}^{\varphi^{n}}=\delta_{0}^{\frac{1}{\sqrt{5 \varphi}} \times \varphi^{n}} \delta_{1}^{\frac{1}{\sqrt{5}} \times \varphi^{n}} \\
& =d^{\varphi^{n}} \quad \text { where } \quad d=\delta_{0}^{\frac{1}{\sqrt{5 \varphi}}} \delta_{1}^{\frac{1}{\sqrt{5}}} \\
& \approx d^{1.6^{n}}
\end{aligned}
$$

Assuming that $0<\delta_{0}=K \varepsilon_{0}<1$ and $0<\delta_{1}=K \varepsilon_{1}<1$, we will have $0<d<1$.
By way of contrast, for Newton's method, for large $n$,

$$
K \varepsilon_{n} \approx d^{2^{n}} \quad \text { where } \quad d=\left(K \varepsilon_{1}\right)^{1 / 2}
$$

As $2^{n}$ grows quite a bit more quickly than $1.6^{n}$ (for example, when $\mathrm{n}=5,2^{n}=32$ and $1.6^{n}=10.5$, and when $n=10,2^{n}=1024$ and $1.6^{n}=110$ ) Newton's method homes in on the root quite a bit faster than the secant method, assuming that you start reasonably close to the root.

[^10]2 Fibonacci (1170-1250) was an Italian mathematician who was also known as Leonardo of Pisa, Leonardo Bonacci and Leonardo Biglio Pisano.
3 Also worth a quick trip to your search engine.

## Hints for Exercises

## 1 . Limits

## 1.1 • Drawing Tangents and a First Limit

### 1.1.2 • Exercises

## Exercises - Stage 1

1.1.2.2. Hint. The tangent line to a curve at point $P$ passes through $P$.
1.1.2.3. Hint. Try drawing tangent lines to the following curves, at the given points $P$ :


## 1.2 • Another Limit and Computing Velocity

### 1.2.2 • Exercises

## Exercises - Stage 1

1.2.2.3. Hint. Where did you start, and where did you end?
1.2.2.4. Hint. Is the object falling faster and faster, slower and slower, or at a constant rate?
1.2.2.5. Hint. Slope is change in vertical component over change in horizontal component.
1.2.2.6. Hint. Sign of velocity gives direction of motion: the velocity is positive at time $t$ if $s(t)$ is increasing at time $t$.

## Exercises - Stage 2

1.2.2.7. Hint. Velocity is distance over time.
1.2.2.8. Hint. Use that $\frac{\sqrt{a}-b}{c}=\frac{\sqrt{a}-b}{c} \cdot\left(\frac{\sqrt{a}+b}{\sqrt{a}+b}\right)=\frac{a-b^{2}}{c(\sqrt{a}+b)}$.

## 1.3 - The Limit of a Function

1.3.2 • Exercises

## Exercises - Stage 1

1.3.2.2. Hint. Consider the difference between a limit and a one-sided limit.
1.3.2.3. Hint. Pay careful attention to which limits are one-sided and which are not.
1.3.2.5. Hint. The function doesn't have to be continuous.
1.3.2.6. Hint. See Question 1.3.2.5
1.3.2.7. Hint. See Question 1.3.2.5
1.3.2.8. Hint. What is the relationship between the limit and the two one-sided limits?
1.3.2.9. Hint. What is the relationship between the limit and the two one-sided limits?

## Exercises - Stage 2

1.3.2.14. Hint. What are the one-sided limits?
1.3.2.16. Hint. Think about what it means that $x$ does not appear in the function $f(x)=\frac{1}{10}$.
1.3.2.17. Hint. We only care about what happens really, really close to $x=3$.

## 1.4 - Calculating Limits with Limit Laws 1.4.2 • Exercises

Exercises - Stage 1
1.4.2.2. Hint. Try to make two functions with factors that will cancel.
1.4.2.3. Hint. Try to make $g(x)$ cancel out.
1.4.2.5. Hint. See Questions 1.4.2.2, 1.4.2.3, and 1.4.2.4.

## Exercises - Stage 2

1.4.2.6. Hint. Find the limit of the numerator and denominator separately.
1.4.2.7. Hint. Break it up into smaller pieces, evaluate the limits of the pieces.
1.4.2.8. Hint. First find the limit of the "inside" function, $\frac{4 x-2}{x+2}$.
1.4.2.9. *. Hint. Is $\cos (-3)$ zero?
1.4.2.10. *. Hint. Expand, then simplify.
1.4.2.14. *. Hint. Try the simplest method first.
1.4.2.15. *. Hint. Factor the denominator.
1.4.2.16. *. Hint. Factor the numerator and the denominator.
1.4.2.17. *. Hint. Factor the numerator.
1.4.2.18. *. Hint. Simplify first by factoring the numerator.
1.4.2.19. Hint. The function is a polynomial.
1.4.2.20. *. Hint. Multiply both the numerator and the denominator by the conjugate of the numerator, $\sqrt{x^{2}+8}+3$.
1.4.2.21. *. Hint. Multiply both the numerator and the denominator by the conjugate of the numerator, $\sqrt{x+7}+\sqrt{11-x}$.
1.4.2.22. *. Hint. Multiply both the numerator and the denominator by the conjugate of the numerator, $\sqrt{x+2}+\sqrt{4-x}$.
1.4.2.23. *. Hint. Multiply both the numerator and the denominator by the conjugate of the numerator, $\sqrt{x-2}+\sqrt{4-x}$.
1.4.2.24. *. Hint. Multiply both the numerator and the denominator by the conjugate of the denominator, $2+\sqrt{5-t}$.
1.4.2.25. Hint. Consider the factors $x^{2}$ and $\cos \left(\frac{3}{x}\right)$ separately. Review the squeeze theorem.
1.4.2.26. Hint. Look for a reason to ignore the trig. Review the squeeze theorem.
1.4.2.27. *. Hint. As in the previous questions, we want to use the Squeeze Theorem. If $x<0$, then $-x$ is positive, so $x<-x$. Use this fact when you bound your expressions.
1.4.2.28. Hint. Factor the numerator.
1.4.2.29. Hint. Factor the denominator; pay attention to signs.
1.4.2.30. Hint. First find the limit of the "inside" function.
1.4.2.31. Hint. Factor; pay attention to signs.
1.4.2.32. Hint. Look for perfect squares
1.4.2.33. Hint. Think about what effect changing $d$ has on the function $x^{5}-$ $32 x+15$.
1.4.2.34. Hint. There's an easy way.
1.4.2.35. *. Hint. What can you do to safely ignore the sine function?
1.4.2.36. *. Hint. Factor
1.4.2.37. Hint. If you're looking at the hints for this one, it's probably easier than you think.
1.4.2.38. Hint. You'll want to simplify this, since $t=\frac{1}{2}$ is not in the domain of the function. One way to start your simplification is to add the fractions in the numerator by finding a common denominator.
1.4.2.39. Hint. If you're not sure how $\frac{|x|}{x}$ behaves, try plugging in a few values of $x$, like $x= \pm 1$ and $x= \pm 2$.
1.4.2.40. Hint. Look to Question 1.4.2.39 to see how a function of the form $\frac{|X|}{X}$ behaves.
1.4.2.41. Hint. Is anything weird happening to this function at $x=0$ ?
1.4.2.42. Hint. Use the limit laws.
1.4.2.43. *. Hint. The denominator goes to zero; what must the numerator go to?

## Exercises - Stage 3

1.4.2.45. Hint. Try plotting points. If you can't divide by $f(x)$, take a limit.
1.4.2.46. Hint. There is a close relationship between $f$ and $g$. Fill in the following table:

| $x$ | $f(x)$ | $g(x)$ | $\frac{f(x)}{g(x)}$ |
| :--- | :--- | :--- | :--- |
| -3 |  |  |  |
| -2 |  |  |  |
| -1 |  |  |  |
| -0 |  |  |  |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |

1.4.2.47. Hint. Velocity of white ball when $t=1$ is $\lim _{h \rightarrow 0} \frac{s(1+h)-s(1)}{h}$.
1.4.2.49. Hint. When you're evaluating $\lim _{x \rightarrow 0^{-}} f(x)$, you're only considering values of $x$ that are less than 0 .
1.4.2.50. Hint. When you're considering $\lim _{x \rightarrow-4^{-}} f(x)$, you're only considering values of $x$ that are less than -4 . When you're considering $\lim _{x \rightarrow-4^{+}} f(x)$, think about the domain of the rational function in the top line.

## 1.5 - Limits at Infinity

### 1.5.2 • Exercises

Exercises - Stage 1
1.5.2.1. Hint. It might not look like a traditional polynomial.
1.5.2.2. Hint. The degree of the polynomial matters.

## Exercises - Stage 2

1.5.2.3. Hint. What does a negative exponent do?
1.5.2.4. Hint. You can think about the behaviour of this function by remembering how you first learned to describe exponentiation.
1.5.2.5. Hint. The exponent will be a negative number.
1.5.2.6. Hint. What single number is the function approaching?
1.5.2.7. Hint. The highest-order term dominates when $x$ is large.
1.5.2.8. Hint. Factor the highest power of $x$ out of both the numerator and the denominator. You can factor through square roots (carefully).
1.5.2.9. *. Hint. Multiply and divide by the conjugate, $\sqrt{x^{2}+5 x}+\sqrt{x^{2}-x}$.
1.5.2.10. *. Hint. Divide both the numerator and the denominator by the highest power of $x$ that is in the denominator.
Remember that $\sqrt{ }$ is defined to be the positive square root. Consequently, if $x<0$, then $\sqrt{x^{2}}$, which is positive, is not the same as $x$, which is negative.
1.5.2.11. *. Hint. Factor out the highest power of the denominator.
1.5.2.12. *. Hint. The conjugate of $\left(\sqrt{x^{2}+x}-x\right)$ is $\left(\sqrt{x^{2}+x}+x\right)$.

Multiply by $1=\frac{\sqrt{x^{2}+x}+x}{\sqrt{x^{2}+x}+x}$ to coax your function into a fraction.
1.5.2.13. *. Hint. Divide both the numerator and the denominator by the highest power of $x$ that is in the denominator.
1.5.2.14. *. Hint. Divide both the numerator and the denominator by the highest power of $x$ that is in the denominator.
1.5.2.15. *. Hint. Divide both the numerator and the denominator by the highest power of $x$ that is in the denominator.
1.5.2.16. Hint. Divide both the numerator and the denominator by $x$ (which is the largest power of $x$ in the denominator). In the numerator, move the resulting factor of $1 / x$ inside the two roots. Be careful about the signs when you do so. Even and odd roots behave differently- see Question 1.5.2.10.
1.5.2.17. *. Hint. Divide both the numerator and the denominator by the highest power of $x$ that is in the denominator.
1.5.2.18. Hint. Divide both the numerator and the denominator by the highest power of $x$ that is in the denominator. It is not always true that $\sqrt{x^{2}}=x$.
1.5.2.19. Hint. Simplify.
1.5.2.20. *. Hint. What is a simpler version of $|x|$ when you know $x<0$ ?
1.5.2.22. *. Hint. Divide both the numerator and the denominator by the highest power of $x$ that is in the denominator. When is $\sqrt{x}=x$, and when is $\sqrt{x}=-x$ ?
1.5.2.23. Hint. Divide both the numerator and the denominator by the highest power of $x$ that is in the denominator. Pay careful attention to signs.
1.5.2.24. *. Hint. Multiply and divide the expression by its conjugate, $\left(\sqrt{n^{2}+5 n}+n\right)$.
1.5.2.25. Hint. Consider what happens to the function as $a$ becomes very, very small. You shouldn't need to do much calculation.
1.5.2.26. Hint. Since $x=3$ is not in the domain of the function, we need to be a little creative. Try simplifying the function.

## Exercises - Stage 3

1.5.2.27. Hint. This is a bit of a trick question. Consider what happens to a rational function as $x \rightarrow \pm \infty$ in each of these three cases:

- the degree of the numerator is smaller than the degree of the denominator,
- the degree of the numerator is the same as the degree of the denominator, and
- the degree of the numerator is larger than the degree of the denominator.
1.5.2.28. Hint. We tend to conflate "infinity" with "some really large number."


## 1.6 • Continuity

### 1.6.4 • Exercises

## Exercises - Stage 1

1.6.4.1. Hint. Try a repeating pattern.
1.6.4.2. Hint. $f$ is my height.
1.6.4.3. Hint. The intermediate value theorem only works for a certain kind of function.
1.6.4.7. Hint. Compare what is given to you to the definition of continuity.
1.6.4.8. Hint. Compare what is given to you to the definition of continuity.
1.6.4.9. Hint. What if the function is discontinuous?
1.6.4.10. Hint. What is $h(0)$ ?

## Exercises - Stage 2

1.6.4.11. Hint. Use the definition of continuity.
1.6.4.12. Hint. If this is your password, you might want to change it.
1.6.4.13. *. Hint. Find the domain: when is the denominator zero?
1.6.4.14. *. Hint. When is the denominator zero? When is the argument of the square root negative?
1.6.4.15. *. Hint. When is the denominator zero? When is the argument of the square root negative?
1.6.4.16. *. Hint. There are infinitely many points where it is not continuous.
1.6.4.17. *. Hint. $x=c$ is the important point.
1.6.4.18. *. Hint. The important place is $x=0$.
1.6.4.19. *. Hint. The important point is $x=c$.
1.6.4.20. *. Hint. The important point is $x=2 c$.

## Exercises - Stage 3

1.6.4.21. Hint. Consider the function $f(x)=\sin x-x+1$.
1.6.4.22. *. Hint. Consider the function $f(x)=3^{x}-x^{2}$, and how it relates to the problem and the IVT.
1.6.4.23. *. Hint. Consider the function $2 \tan x-x-1$ and its roots.
1.6.4.24. *. Hint. Consider the function $f(x)=\sqrt{\cos (\pi x)}-\sin (2 \pi x)-1 / 2$, and be careful about where it is continuous.
1.6.4.25. *. Hint. Consider the function $f(x)=1 / \cos ^{2}(\pi x)-x-\frac{3}{2}$, paying attention to where it is continuous.
1.6.4.26. Hint. We want $f(x)$ to be $0 ; 0$ is between a positive number and a negative number. Try evaluating $f(x)$ for some integer values of $x$.
1.6.4.27. Hint. $\quad \sqrt[3]{7}$ is the value where $x^{3}=7$.
1.6.4.28. Hint. You need to consider separately the cases where $f(a)<g(a)$ and $f(a)=g(a)$. Let $h(x)=f(x)-g(x)$. What is $h(c)$ ?

## 2 - Derivatives

## 2.1 $\cdot$ Revisiting Tangent Lines

### 2.1.2 • Exercises

## Exercises - Stage 1

2.1.2.2. Hint. You can use 2.1.2.2.a to explain 2.1.2.2.b.
2.1.2.3. Hint. Your calculations for slope of the secant lines will all have the same denominators; to save yourself some time, you can focus on the numerators.

## Exercises - Stage 2

2.1.2.4. Hint. You can do this by calculating several secant lines. You can also do this by getting out a ruler and trying to draw the tangent line very carefully.
2.1.2.5. Hint. There are many possible values for $Q$ and $R$.
2.1.2.6. Hint. A line with slope 0 is horizontal.

## 2.2 • Definition of the Derivative

### 2.2.4 • Exercises

## Exercises - Stage 1

2.2.4.1. Hint. What are the properties of $f^{\prime}$ when $f$ is a line?
2.2.4.2. Hint. Be very careful not to confuse $f$ and $f^{\prime}$.
2.2.4.3. Hint. Be very careful not to confuse $f$ and $f^{\prime}$.
2.2.4.5. Hint. The slope has to look "the same" from the left and the right.
2.2.4.6. Hint. Use the definition of the derivative, and what you know about limits.
2.2.4.7. Hint. Consider continuity.
2.2.4.8. Hint. Look at the definition of the derivative. Your answer will be a fraction.

## Exercises - Stage 2

2.2.4.9. Hint. You need a point (given), and a slope (derivative).
2.2.4.10. Hint. You'll need to add some fractions.
2.2.4.11. *. Hint. You don't have to take the limit from the left and right separately-things will cancel nicely.
2.2.4.12. *. Hint. You might have to add fractions.
2.2.4.14. Hint. Your limit should be easy.
2.2.4.15. *. Hint. Add fractions.
2.2.4.16. *. Hint. For $f$ to be differentiable at $x=2$, two things must be true: it must be continuous at $x=2$, and the derivative from the right must equal the derivative from the left.
2.2.4.17. *. Hint. After you plug in $f(x)$ to the definition of a derivative, you'll want to multiply and divide by the conjugate $\sqrt{1+x+h}+\sqrt{1+x}$.

## Exercises - Stage 3

2.2.4.18. Hint. From Section 1.2, compare the definition of velocity to the definition of a derivative. When you're finding the derivative, you'll need to cancel a lot on the numerator, which you can do by expanding the polynomials.
2.2.4.19. *. Hint. You'll need to look at limits from the left and right. The fact that $f(0)=0$ is useful for your computation. Recall that if $x<0$ then $\sqrt{x^{2}}=|x|=-x$.
2.2.4.20. *. Hint. You'll need to look at limits from the left and right. The fact that $f(0)=0$ is useful for your computation.
2.2.4.21. *. Hint. You'll need to look at limits from the left and right. The fact that $f(0)=0$ is useful for your computation.
2.2.4.22. *. Hint. You'll need to look at limits from the left and right. The fact that $f(1)=0$ is useful for your computation.
2.2.4.23. . Hint. There's lots of room between 0 and $\frac{1}{8}$; see what you can do with it.
2.2.4.24. Hint. Set up your usual limit, then split it into two pieces
2.2.4.25. Hint. You don't need the definition of the derivative for a line.
2.2.4.26. *. Hint. A generic point on the curve has coordinates ( $\alpha, \alpha^{2}$ ). In terms of $\alpha$, what is the equation of the tangent line to the curve at the point $\left(\alpha, \alpha^{2}\right)$ ? What does it mean for $(1,-3)$ to be on that line?
2.2.4.27. *. Hint. Remember for a constant $n$,

$$
\lim _{h \rightarrow 0} h^{n}= \begin{cases}0 & n>0 \\ 1 & n=0 \\ D N E & n<0\end{cases}
$$

## 2.3 • Interpretations of the Derivative

### 2.3.3 • Exercises

## Exercises - Stage 2

2.3.3.1. Hint. Think about units.

## Exercises - Stage 3

2.3.3.8. Hint. There are 360 degrees in one rotation.
2.3.3.9. Hint. $\quad P^{\prime}(t)$ was discussed in Question 2.3.3.7.

## 2.4 • Arithmetic of Derivatives - a Differentiation Toolbox

### 2.4.2 • Exercises

Exercises - Stage 1
2.4.2.1. Hint. Look at the Sum rule.
2.4.2.2. Hint. Try an example, like $f(x)=g(x)=x$.
2.4.2.3. Hint. Simplify.
2.4.2.4. Hint. $g(x)=f(x)+f(x)+f(x)$

## Exercises - Stage 2

2.4.2.5. Hint. Use linearity and the known derivatives of $x^{2}$ and $x^{1 / 2}$.
2.4.2.6. Hint. You have already seen $\frac{\mathrm{d}}{\mathrm{d} x}\{\sqrt{x}\}$.
2.4.2.7. *. Hint. The equation of a line can be determined using a point, and the slope. The derivative of $x^{3}$ can be found by writing $x^{3}=(x)\left(x^{2}\right)$.
2.4.2.8. *. Hint. Be careful to distinguish between speed and velocity.
2.4.2.10. Hint. How do you take care of that power?
2.4.2.11. Hint. You know how to take the derivative of a reciprocal; this might be faster than using the quotient rule.

## Exercises - Stage 3

2.4.2.12. Hint. Population growth is rate of change of population.
2.4.2.14. *. Hint. Interpret it as a derivative that you know how to compute.
2.4.2.15. Hint. The answer is not 10 square metres per second.
2.4.2.16. Hint. You don't need to know $g(0)$ or $g^{\prime}(0)$.

## 2.6 • Using the Arithmetic of Derivatives - Examples

### 2.6.2 • Exercises

## Exercises - Stage 1

2.6.2.1. Hint. Check signs.
2.6.2.2. Hint. Read Lemma 2.6.9 carefully.

## Exercises - Stage 2

2.6.2.3. Hint. First, factor an $x$ out of the derivative. What's left over looks like a quadratic equation, if you take $x^{2}$ to be your variable, instead of $x$.
2.6.2.4. Hint. $\frac{1}{t}=t^{-1}$
2.6.2.5. Hint. First simplify. Don't be confused by the role reversal of $x$ and $y$ : $x$ is just the name of the function $\left(2 y+\frac{1}{y}\right) \cdot y^{3}$, which is a function of the variable $y$. You are to differentiate with respect to $y$.
2.6.2.6. Hint. $\sqrt{x}=x^{1 / 2}$
2.6.2.8. Hint. You don't need to multiply through.
2.6.2.9. Hint. You can use the quotient rule.
2.6.2.13. *. Hint. There are two pieces of the given function that could cause problems.
2.6.2.14. Hint. $\quad \sqrt[3]{x}=x^{1 / 3}$
2.6.2.15. Hint. Simplify first.

## Exercises - Stage 3

2.6.2.17. *. Hint. Let $m$ be the slope of such a tangent line, and let $P_{1}$ and $P_{2}$ be the points where the tangent line is tangent to the two curves, respectively. There are three equations $m$ fulfils: it has the same slope as the curves at the given points, and it is the slope of the line passing through the points $P_{1}$ and $P_{2}$.
2.6.2.18. Hint. A line has equation $y=m x+b$, for some constants $m$ and $b$. What has to be true for $y=m b+x$ to be tangent to the first curve at the point $x=\alpha$, and to the second at the point $x=\beta$ ?
2.6.2.19. *. Hint. Compare this to one of the forms given in the text for the definition of the derivative.

## 2.7 • Derivatives of Exponential Functions

### 2.7.3 • Exercises

Exercises - Stage 1
2.7.3.1. Hint. Two of the functions are the same.
2.7.3.3. Hint. When can you use the power rule?
2.7.3.4. Hint. What is the shape of the curve $e^{a x}$, when $a$ is a positive consant?

## Exercises - Stage 2

2.7.3.5. Hint. Quotient rule
2.7.3.6. Hint. $e^{2 x}=\left(e^{x}\right)^{2}$
2.7.3.7. Hint. $e^{a+x}=e^{a} e^{x}$
2.7.3.8. Hint. Figure out where the derivative is positive.
2.7.3.9. Hint. $e^{-x}=\frac{1}{e^{x}}$
2.7.3.10. Hint. Product rule will work nicely here. Alternately, review the result of Question 2.7.3.6.
2.7.3.11. Hint. To find the sign of a product, compare the signs of each factor. The function $e^{t}$ is always positive.

## Exercises - Stage 3

2.7.3.12. Hint. After you differentiate, factor out $e^{x}$.
2.7.3.13. Hint. Simplify.
2.7.3.14. *. Hint. In order to be differentiable, a function should be continuous. To determine the differentiability of the function at $x=1$, use the definition of the derivative.

## 2.8 • Derivatives of Trigonometric Functions

### 2.8.8 • Exercises

## Exercises - Stage 1

2.8.8.1. Hint. A horizontal tangent line is where the graph appears to "level off."
2.8.8.2. Hint. You are going to mark there points on the sine graph where the graph is the steepest, going up.

## Exercises - Stage 2

2.8.8.3. Hint. You need to memorize the derivatives of sine, cosine, and tangent.
2.8.8.4. Hint. There are infinitely many values. You need to describe them all.
2.8.8.5. Hint. Simplify first.
2.8.8.6. Hint. The identity won't help you.
2.8.8.8. Hint. Quotient rule
2.8.8.11. Hint. Use an identity.
2.8.8.12. Hint. How can you move the negative signs to a location that you can more easily deal with?
2.8.8.13. Hint. Apply the quotient rule.
2.8.8.14. *. Hint. The only spot to worry about is when $x=0$. For $f(x)$ to be differentiable, it must be continuous, so first find the value of $b$ that makes $f$ continuous at $x=0$. Then, find the value of $a$ that makes the derivatives from the left and right of $x=0$ equal to each other.

## Exercises - Stage 3

2.8.8.16. *. Hint. Compare this to one of the forms given in the text for the definition of the derivative.
2.8.8.17. *. Hint. Compare this to one of the forms given in the text for the definition of the derivative.
2.8.8.18. *. Hint. Compare this to one of the forms given in the text for the definition of the derivative.
2.8.8.19. Hint. $\tan \theta=\frac{\sin \theta}{\cos \theta}$
2.8.8.20. *. Hint. In order for a derivative to exist, the function must be continuous, and the derivative from the left must equal the derivative from the right.
2.8.8.21. *. Hint. There are infinitely many places where it does not exist.
2.8.8.27. Hint. You can set up the derivative using the limit definition: $f^{\prime}(0)=$ $\lim _{h \rightarrow 0} \frac{f(h)-f(0)}{h}$. If the limit exists, it gives you $f^{\prime}(0)$; if the limit does not exist, you conclude $f^{\prime}(0)$ does not exist.
To evaluate the limit, recall that when we differentiated sine, we learned that for $h$ near 0,

$$
\cos h \leq \frac{\sin h}{h} \leq 1
$$

2.8.8.28. *. Hint. Recall $|x|=\left\{\begin{array}{rl}x & x \geq 0 \\ -x & x<0\end{array}\right.$. To determine whether $h(x)$ is differentiable at $x=0$, use the definition of the derivative.
2.8.8.29. *. Hint. To decide whether the function is differentiable, use the definition of the derivative.
2.8.8.30. *. Hint. In this chapter, we learned $\lim _{x \rightarrow 0} \frac{\sin x}{x}=1$. If you divide the numerator and denominator by $x^{5}$, you can make use of this knowledge.

## 2.9 - One More Tool - the Chain Rule

### 2.9.4 • Exercises

## Exercises - Stage 1

2.9.4.1. Hint. For parts 2.9.4.1.a and 2.9.4.1.b, remember the definition of a derivative:

$$
\frac{\mathrm{d} K}{\mathrm{~d} U}=\lim _{h \rightarrow 0} \frac{K(U+h)-K(U)}{h} .
$$

When $h$ is positive, $U+h$ is an increased urchin population; what is the sign of $K(U+h)-K(U)$ ?
For part 2.9.4.1.c, use the chain rule!
2.9.4.2. Hint. Remember that Leibniz notation suggests fractional cancellation.

## Exercises - Stage 2

2.9.4.3. Hint. If $g(x)=\cos x$ and $h(x)=5 x+3$, then $f(x)=g(h(x))$. So we apply the chain rule, with "outside" function $\cos x$ and "inside" function $5 x+3$.
2.9.4.4. Hint. You can expand this into a polynomial, but it's easier to use the chain rule. If $g(x)=x^{5}$, and $h(x)=x^{2}+2$, then $f(x)=g(h(x))$.
2.9.4.5. Hint. You can expand this into a polynomial, but it's easier to use the chain rule. If $g(k)=k^{17}$, and $h(k)=4 k^{4}+2 k^{2}+1$, then $T(k)=g(h(k))$.
2.9.4.6. Hint. If we define $g(x)=\sqrt{x}$ and $h(x)=\frac{x^{2}+1}{x^{2}-1}$, then $f(x)=g(h(x))$. To differentiate the square root function: $\frac{\mathrm{d}}{\mathrm{d} x}\{\sqrt{x}\}=\frac{\mathrm{d}}{\mathrm{d} x}\left\{x^{1 / 2}\right\}=\frac{1}{2} x^{-1 / 2}=\frac{1}{2 \sqrt{x}}$.
2.9.4.7. Hint. You'll need to use the chain rule twice.
2.9.4.8. *. Hint. Use the chain rule.
2.9.4.9. *. Hint. Use the chain rule.
2.9.4.10. *. Hint. Use the chain rule.
2.9.4.11. *. Hint. Use the chain rule.
2.9.4.12. *. Hint. Recall $\frac{1}{x^{2}}=x^{-2}$ and $\sqrt{x^{2}-1}=\left(x^{2}-1\right)^{1 / 2}$.
2.9.4.14. Hint. If we let $g(x)=\sec x$ and $h(x)=e^{2 x+7}$, then $f(x)=g(h(x))$, so by the chain rule, $f^{\prime}(x)=g^{\prime}(h(x)) \cdot h^{\prime}(x)$. However, in order to evaluate $h^{\prime}(x)$, we'll need to use the chain rule again.
2.9.4.15. Hint. What trig identity can you use to simplify the first factor in the equation?
2.9.4.16. Hint. Velocity is the derivative of position with respect to time. In this case, the velocity of the particle is given by $s^{\prime}(t)$.
2.9.4.17. Hint. The slope of the tangent line is the derivative. You'll need to use the chain rule twice.
2.9.4.18. *. Hint. Start with the product rule, then use the chain rule to differentiate $e^{4 x}$.
2.9.4.19. *. Hint. Start with the quotient rule; you'll need the chain rule only to differentiate $e^{3 x}$.
2.9.4.20. *. Hint. More than one chain rule needed here.
2.9.4.21. *. Hint. More than one chain rule application is needed here.
2.9.4.22. *. Hint. More than one chain rule application is needed here.
2.9.4.23. *. Hint. More than one chain rule application is needed here.
2.9.4.24. *. Hint. What rule do you need, besides chain? Also, remember that $\cos ^{2} x=[\cos x]^{2}$.
2.9.4.27. *. Hint. The product of two functions is zero exactly when at least one of the functions is zero.
2.9.4.28. Hint. If $t \geq 1$, then $0<\frac{1}{t} \leq 1$.
2.9.4.29. Hint. The notation $\cos ^{3}(5 x-7)$ means $[\cos (5 x-7)]^{3}$. So, if $g(x)=x^{3}$ and $h(x)=\cos (5 x-7)$, then $g(h(x))=[\cos (5 x+7)]^{3}=\cos ^{3}(5 x+7)$.
2.9.4.30. *. Hint. In Example 2.6.6, we generalized the product rule to three factors:

$$
\begin{array}{r}
\frac{\mathrm{d}}{\mathrm{~d} x}\{f(x) g(x) h(x)\}=f^{\prime}(x) g(x) h(x)+f(x) g^{\prime}(x) h(x) \\
+f(x) g(x) h^{\prime}(x)
\end{array}
$$

This isn't strictly necessary, but it will simplify your computations.

## Exercises - Stage 3

2.9.4.31. Hint. At time $t$, the particle is at the point $(x(t), y(t))$, with $x(t)=\cos t$ and $y(t)=\sin t$. Over time, the particle traces out a curve; let's call that curve $y=f(x)$. Then $y(t)=f(x(t))$, so the slope of the curve at the point $(x(t), y(t))$ is $f^{\prime}(x(t))$. You are to determine the values of $t$ for which $f^{\prime}(x(t))=-1$.
2.9.4.32. *. Hint. Set $f(x)=e^{x+x^{2}}$ and $g(x)=1+x$. Compare $f(0)$ and $g(0)$, and compare $f^{\prime}(x)$ and $g^{\prime}(x)$.
2.9.4.33. Hint. If $\sin 2 x$ and $2 \sin x \cos x$ are the same, then they also have the same derivatives.
2.9.4.34. Hint. This is a long, nasty problem, but it doesn't use anything you haven't seen before. Be methodical, and break the question into as many parts as you have to. At the end, be proud of yourself for your problem-solving abilities and tenaciousness!
2.9.4.35. Hint. To sketch the curve, you can start by plotting points. Alternately, consider $x^{2}+y$.

### 2.10 • The Natural Logarithm

### 2.10.3 • Exercises

## Exercises - Stage 1

2.10.3.1. Hint. Each speaker produces 3 dB of noise, so if $P$ is the power of one speaker, $3=V(P)=10 \log _{10}\left(\frac{P}{S}\right)$. Use this to find $V(10 P)$ and $V(100 P)$.
2.10.3.2. Hint. The question asks you when $A(t)=2000$. So, solve $2000=$ $1000 e^{t / 20}$ for $t$.
2.10.3.3. Hint. What happens when $\cos x$ is a negative number?

## Exercises - Stage 2

2.10.3.4. Hint. There are two easy ways: use the chain rule, or simplify first.
2.10.3.5. Hint. There are two easy ways: use the chain rule, or simplify first.
2.10.3.6. Hint. Don't be fooled by a common mistake: $\log \left(x^{2}+x\right)$ is not the same as $\log \left(x^{2}\right)+\log x$.
2.10.3.7. Hint. Use the base-change formula to convert this to natural logarithm (base $e$ ).
2.10.3.9. Hint. Use the chain rule.
2.10.3.10. Hint. Use the chain rule twice.
2.10.3.11. *. Hint. You'll need to use the chain rule twice.
2.10.3.12. *. Hint. Use the chain rule.
2.10.3.13. *. Hint. Use the chain rule to differentiate.
2.10.3.14. *. Hint. You can differentiate this by using the chain rule several times.
2.10.3.15. *. Hint. Using logarithm rules before you differentiate will make this easier.
2.10.3.16. Hint. Using logarithm rules before you differentiate will make this easier.
2.10.3.17. Hint. First, differentiate using the chain rule and any other necessary rules. Then, plug in $x=2$.
2.10.3.18. *. Hint. In the text, you are given the derivative $\frac{\mathrm{d}}{\mathrm{d} x} a^{x}$, where $a$ is a constant.
2.10.3.19. Hint. You'll need to use logarithmic differentiation. Set $g(x)=$ $\log (f(x))$, and find $g^{\prime}(x)$. Then, use that to find $f^{\prime}(x)$. This is the method used in the text to find $\frac{\mathrm{d}}{\mathrm{d} x} a^{x}$.
2.10.3.20. *. Hint. Use Question 2.10.3.19 and the base-change formula, $\log _{b}(a)=\frac{\log a}{\log b}$.
2.10.3.21. Hint. To make this easier, use logarithmic differentiation. Set $g(x)=$ $\log (f(x))$, and find $g^{\prime}(x)$. Then, use that to find $f^{\prime}(x)$. This is the method used in the text to find $\frac{\mathrm{d}}{\mathrm{d} x} a^{x}$, and again in Question 2.10.3.19.
2.10.3.22. Hint. To make this easier, use logarithmic differentiation. Set $g(x)=$ $\log (f(x))$, and find $g^{\prime}(x)$. Then, use that to find $f^{\prime}(x)$. This is the method used in the text to find $\frac{\mathrm{d}}{\mathrm{d} x} a^{x}$, and again in Question 2.10.3.19.
2.10.3.23. Hint. It's not going to come out nicely, but there's a better way than blindly applying quotient and product rules, or expanding giant polynomials.
2.10.3.24. *. Hint. You'll need to use logarithmic differentiation. Set $g(x)=$ $\log (f(x))$, and find $g^{\prime}(x)$. Then, use that to find $f^{\prime}(x)$. This is the method used in the text to find $\frac{\mathrm{d}}{\mathrm{d} x} a^{x}$, and again in Question 2.10.3.19.
2.10.3.25. *. Hint. You'll need to use logarithmic differentiation. Set $g(x)=$ $\log (f(x))$, and find $g^{\prime}(x)$. Then, use that to find $f^{\prime}(x)$. This is the method used in the text to find $\frac{\mathrm{d}}{\mathrm{d} x} a^{x}$, and again in Question 2.10.3.19.
2.10.3.26. *. Hint. You'll need to use logarithmic differentiation. Set $g(x)=$ $\log (f(x))$, and find $g^{\prime}(x)$. Then, use that to find $f^{\prime}(x)$. This is the method used in the text to find $\frac{\mathrm{d}}{\mathrm{d} x} a^{x}$, and again in Question 2.10.3.19.
2.10.3.27. *. Hint. You'll need to use logarithmic differentiation. Differentiate $\log (f(x))$, then solve for $f^{\prime}(x)$. This is the method used in the text to find $\frac{\mathrm{d}}{\mathrm{d} x} a^{x}$.
2.10.3.28. *. Hint. You'll need to use logarithmic differentiation. Differentiate
$\log (f(x))$, then solve for $f^{\prime}(x)$. This is the method used in the text to find $\frac{\mathrm{d}}{\mathrm{d} x} a^{x}$.
2.10.3.29. *. Hint. You'll need to use logarithmic differentiation. Differentiate $\log (f(x))$, then solve for $f^{\prime}(x)$. This is the method used in the text to find $\frac{\mathrm{d}}{\mathrm{d} x} a^{x}$.

## Exercises - Stage 3

2.10.3.30. Hint. Evaluate $\frac{\mathrm{d}}{\mathrm{d} x}\left\{\log \left([f(x)]^{g(x)}\right)\right\}$.
2.10.3.31. Hint. Differentiate $y=\log (f(x))$. When is the derivative equal to zero?

### 2.11 • Implicit Differentiation

### 2.11.2 • Exercises

## Exercises - Stage 1

2.11.2.1. Hint. Where did the $y^{\prime}$ come from?
2.11.2.2. Hint. The three points to look at are $(0,-4),(0,0)$, and $(0,4)$. What does the slope of the tangent line look like there?
2.11.2.3. Hint. A function must pass the vertical line test: one input cannot result in two different outputs.

## Exercises - Stage 2

2.11.2.4. *. Hint. Remember that $y$ is a function of $x$. Use implicit differentiation, then collect all the terms containing $\frac{\mathrm{d} y}{\mathrm{~d} x}$ on one side of the equation to solve for $\frac{\mathrm{d} y}{\mathrm{~d} x}$.
2.11.2.5. *. Hint. Differentiate implicitly, then solve for $y^{\prime}$.
2.11.2.6. *. Hint. Remember that $y$ is a function of $x$. You can determine explicitly the values of $x$ for which $y(x)=1$.
2.11.2.8. *. Hint. Plug in $y=0$ at a strategic point in your work to simplify your computation.
2.11.2.10. *. Hint. Plug in $y=0$ at a strategic point in your work to simplify your computation.
2.11.2.11. Hint. If the tangent line has slope $y^{\prime}$, and it is parallel to $y=x$, then $y^{\prime}=1$.
2.11.2.12. *. Hint. You don't need to solve for $y^{\prime}$ in general: only at a single point.
2.11.2.13. *. Hint. After you differentiate implicitly, get all the terms containing $y^{\prime}$ onto one side so you can solve for $y^{\prime}$.

## Exercises - Stage 3

2.11.2.14. *. Hint. You don't need to solve for $\frac{\mathrm{d} y}{\mathrm{~d} x}$ for all values of $x$-only when $y=0$.

### 2.12 • Inverse Trigonometric Functions

### 2.12.2 • Exercises

## Exercises - Stage 1

2.12.2.1. Hint. Remember that only certain numbers can come out of sine and cosine, but any numbers can go in.
2.12.2.2. Hint. What is the range of the arccosine function?
2.12.2.3. Hint. A one-to-one function passes the horizontal line test. To graph the inverse of a function, reflect it across the line $y=x$.
2.12.2.4. Hint. Your answer will depend on $a$. The arcsine function alone won't give you every value.
2.12.2.5. Hint. In order for $x$ to be in the domain of $f$, you must be able to plug $x$ into both arcsine and arccosecant.

## Exercises - Stage 2

2.12.2.6. Hint. For the domain of $f$, remember the domain of arcsine is $[-1,1]$.
2.12.2.7. Hint. The domain of $\arccos (t)$ is $[-1,1]$, but you also have to make sure you aren't dividing by zero.
2.12.2.8. Hint. $\quad \frac{\mathrm{d}}{\mathrm{d} x}\{\operatorname{arcsec} x\}=\frac{1}{|x| \sqrt{x^{2}-1}}$, and the domain of $\operatorname{arcsec} x$ is $|x| \geq$ 1.
2.12.2.9. Hint. The domain of $\arctan (x)$ is all real numbers.
2.12.2.10. Hint. The domain of $\arcsin x$ is $[-1,1]$, and the domain of $\sqrt{x}$ is $x \geq 0$.
2.12.2.11. Hint. This occurs only once.
2.12.2.12. Hint. The answer is a very simple expression.
2.12.2.13. *. Hint. chain rule
2.12.2.16. Hint. You can simplify the expression before you differentiate to remove the trigonometric functions. If $\arctan x=\theta$, then fill in the sides of the triangle below using the definition of arctangent and the Pythagorean theorem:


With the sides labeled, you can figure out $\sin (\arctan x)=\sin (\theta)$.
2.12.2.17. Hint. You can simplify the expression before you differentiate to remove the trigonometric functions. If $\arcsin x=\theta$, then fill in the sides of the triangle below using the definition of arctangent and the Pythagorean theorem:


With the sides labeled, you can figure out $\cot (\arcsin x)=\cot (\theta)$.
2.12.2.18. *. Hint. What is the slope of the line $y=2 x+9$ ?
2.12.2.19. Hint. Differentiate using the chain rule.

## Exercises - Stage 3

2.12.2.20. *. Hint. If $g(y)=f^{-1}(y)$, then $f(g(y))=f\left(f^{-1}(y)\right)=y$. Differentiate this last equality using the chain rule.
2.12.2.21. *. Hint. To simplify notation, let $g(y)=f^{-1}(y)$. Simplify and differentiate $g(f(x))$.
2.12.2.22. *. Hint. To simplify notation, let $g(y)=f^{-1}(y)$. Simplify and differentiate $g(f(x))$.
2.12.2.23. Hint. Use logarithmic differentiation.
2.12.2.24. Hint. Where are those functions defined?
2.12.2.25. Hint. Compare this to one of the forms given in the text for the definition of the derivative.
2.12.2.26. Hint. $f^{-1}(7)$ is the number $y$ that satisfies $f(y)=7$.
2.12.2.27. Hint. If $f^{-1}(y)=0$, that means $f(0)=y$. So, we're looking for the number that we plug into $f^{-1}$ to get 0 .
2.12.2.28. Hint. As usual, after you differentiate implicitly, get all the terms containing $y^{\prime}$ onto one side of the equation, so you can factor out $y^{\prime}$.

### 2.13 . The Mean Value Theorem

### 2.13.5 • Exercises

## Exercises - Stage 1

2.13.5.1. Hint. How long would it take the caribou to travel 5000 km , travelling at its top speed?
2.13.5.2. Hint. Let $f(x)$ be the position of the crane, where $x$ is the hour of the day.
2.13.5.3. Hint. For an example, look at Figure 2.13.4.
2.13.5.4. Hint. How does this question differ from the statement of the mean value theorem?
2.13.5.6. Hint. Where is $f(x)$ differentiable?

## Exercises - Stage 2

2.13.5.7. *. Hint. To use Rolle's Theorem, you will want two values where the function is zero. If you're stuck finding one of them, think about when $x^{2}-2 \pi x$ is equal to zero.
2.13.5.11. Hint. To show that there are exactly $n$ roots, you need to not only show that $n$ exist, but also that there are not more than $n$.
2.13.5.12. Hint. To show that there are exactly $n$ roots, you need to not only show that $n$ exist, but also that there are not more than $n$. If you can't explicitly find the root(s), you can use the intermediate value theorem to show they exist.
2.13.5.13. Hint. If $f(x)=0$, then $\left|x^{3}\right|=\left|\sin \left(x^{5}\right)\right| \leq 1$. When $|x|<1$, is $\cos \left(x^{5}\right)$ positive or negative?
2.13.5.14. Hint. Let $f(x)=e^{x}-4 \cos (2 x)$, and use Rolle's Theorem. What is the interval where $f(x)$ can have a positive root?
2.13.5.15. *. Hint. For 2.13.5.15.b, what does Rolle's Theorem tell you has to happen in order for $f(x)$ to have more than one root in $[-1,1]$ ?
2.13.5.16. *. Hint. Since $f(x)=e^{x}$ is a continuous and differentiable function, the MVT promises that there exists some number $c$ such that

$$
f^{\prime}(c)=\frac{f(T)-f(0)}{T}
$$

Find that $c$, in terms of $T$.
2.13.5.17. Hint. Let $f(x)=\operatorname{arcsec} x+\operatorname{arccsc} c-C$. What is $f^{\prime}(x)$ ?

## Exercises - Stage 3

2.13.5.18. *. Hint. Show that $f$ is differentiable by showing that $f^{\prime}(x)$ exists for every $x$. Then, the Mean Value Theorem applies. What is the largest $f^{\prime}(x)$ can
be, for any $x$ ? If $f(100)<100$, what does the MVT tell you must be true of $f^{\prime}(c)$ for some $c$ ?
2.13.5.19. Hint. In order for $f^{-1}(x)$ to be defined over an interval, $f(x)$ must be one-to-one over that interval.
2.13.5.20. Hint. In order for $f^{-1}(x)$ to be defined over an interval, $f(x)$ must be one-to-one over that interval.
2.13.5.21. Hint. Let $h(x)=f(x)-g(x)$. What does the Mean Value Theorem tell you about the derivative of $h$ ?
2.13.5.22. Hint. Rolle's Theorem relates the roots of a function to the roots of its derivative.
2.13.5.23. Hint. To show that there are exactly $n$ distinct roots, you need to not only show that $n$ exist, but also that there are not more than $n$.

## $2.14 \cdot$ Higher Order Derivatives

### 2.14.2 • Exercises

## Exercises - Stage 1

2.14.2.1. Hint. If you know the first derivative, this should be easy.
2.14.2.2. Hint. Exactly two of the statements must be true.
2.14.2.3. Hint. Use factorials, as in Example 2.14.2.
2.14.2.4. Hint. The problem isn't with any of the algebra.

## Exercises - Stage 2

2.14.2.5. Hint. Recall $\frac{\mathrm{d}}{\mathrm{d} x} \log x=\frac{1}{x}$.
2.14.2.6. Hint. Recall $\frac{\mathrm{d}}{\mathrm{d} x}\{\arctan x\}=\frac{1}{1+x^{2}}=\left(1+x^{2}\right)^{-1}$.
2.14.2.7. Hint. Use implicit differentiation.
2.14.2.8. Hint. The acceleration is given by $s^{\prime \prime}(t)$.
2.14.2.9. Hint. Remember to use the chain rule.
2.14.2.10. Hint. $h^{\prime}(t)$ gives the velocity of the particle, and $h^{\prime \prime}(t)$ gives its acceleration-the rate the velocity is changing.
2.14.2.11. Hint. $h^{\prime}(t)$ gives the velocity of the particle, and $h^{\prime \prime}(t)$ gives its acceleration-the rate the velocity is changing. Be wary of signs-as in legends, they may be misleading.
2.14.2.12. Hint. You don't need to solve for $y^{\prime \prime}$ in general-only when $x=y=0$. To do this, you also need to find $y^{\prime}$ at the point $(0,0)$.
2.14.2.13. Hint. To show that two functions are unequal, you can show that one input results in different outputs.

## Exercises - Stage 3

2.14.2.14. Hint. Only one of the curves could possibly represent $y=f(x)$.
2.14.2.15. Hint. Remember $\frac{\mathrm{d}}{\mathrm{d} x}\left\{2^{x}\right\}=2^{x} \log 2$.
2.14.2.16. Hint. Differentiate a few times until you get zero, remembering that $a, b, c$, and $d$ are all constants.
2.14.2.17. *. Hint. Use a similar method to Question 2.9.4.32, Section 2.9.
2.14.2.18. *. Hint. For 2.14.2.18.b, you know a point where the curve and tangent line intersect, and you know what the tangent line looks like. What do the derivatives tell you about the shape of the curve?
2.14.2.19. Hint. Review Pascal's Triangle.
2.14.2.20. Hint. Rolle's Theorem relates the roots of a function to the roots of its derivative. So, the fifth derivative tells us something about the fourth, the fourth derivative tells us something about the third, and so on.
2.14.2.21. Hint. You'll want to use Rolle's Theorem, but the first derivative won't be very tractable-use the idea behind Question 2.14.2.20.
2.14.2.22. *. Hint. You can re-write this function as a piecewise function, with branches $x \geq 0$ and $x<0$. To figure out the derivatives at $x=0$, use the definition of a derivative.

## 3 - Applications of derivatives 3.1 • Velocity and Acceleration <br> 3.1.2 • Exercises

## Exercises - Stage 1

3.1.2.1. Hint. Is the velocity changing at $t=2$ ?
3.1.2.2. Hint. The acceleration (rate of change of velocity) is constant.
3.1.2.3. Hint. Remember the difference between speed and velocity.
3.1.2.4. Hint. How is this different from the wording of Question 3.1.2.3?

## Exercises - Stage 2

3.1.2.5. Hint. The equation of an object falling from rest on the earth is derived in Example 3.1.2. It would be difficult to use exactly the version given for $s(t)$, but using the same logic, you can find an equation for the height of the flower pot at time $t$.
3.1.2.6. Hint. Remember that a falling object has an acceleration of $9.8 \frac{\mathrm{~m}}{\mathrm{~s}^{2}}$.
3.1.2.7. Hint. Acceleration is constant, so finding a formula for the distance your keys have travelled is a similar problem to finding a formula for something falling.
3.1.2.8. Hint. See Example 3.1.3.
3.1.2.9. Hint. Be careful to match up the units.
3.1.2.10. Hint. Think about what it means for the car to decelerate at a constant rate. You might also review Question 3.1.2.2.
3.1.2.11. Hint. Let $a$ be the acceleration of the shuttle. Start by finding $a$, then find the position function of the shuttle.
3.1.2.12. Hint. Review Example 3.1.2, but account for the fact that your initial velocity is not zero.
3.1.2.13. Hint. Be very careful with units. The acceleration of gravity you're used to is 9.8 metres per second squared, so you might want to convert 325 kpm to metres per second.
3.1.2.14. Hint. Since gravity alone brings it down, its acceleration is a constant $-9.8 \frac{\mathrm{~m}}{\mathrm{~s}^{2}}$.
3.1.2.15. Hint. First, find an equation for $a(t)$, the acceleration of the car, noting that $a^{\prime}(t)$ is constant. Then, use this to find an equation for the velocity of the car. Be careful about seconds versus hours.

## Exercises - Stage 3

3.1.2.16. Hint. We recommend using two different functions to describe your height: $h_{1}(t)$ while you are in the air, not yet touching the trampoline, and $h_{2}(t)$ while you are in the trampoline, going down.
Both $h_{1}(t)$ and $h_{2}(t)$ are quadratic equations, since your acceleration is constant over both intervals, but be very careful about signs.
3.1.2.17. Hint. First, find an expression for the speed of the object. You can let $v_{0}$ be its velocity at time $t=0$.

## 3.2 - Related Rates <br> 3.2.2 • Exercises

## Exercises - Stage 1

3.2.2.1. Hint. If you know $P$, you can figure out $Q$.

## Exercises - Stage 2

3.2.2.2. *. Hint. Since the point moves along the unit circle, we know that $x^{2}+y^{2}=1$, where $x$ and $y$ are functions of time.
3.2.2.3. *. Hint. You'll need some implicit differentiation: what should your variable be? Example 3.2 .3 shows how to work with percentage rate of change.
3.2.2.4. *. Hint. For 3.2.2.4.b, refer to Example 3.2.3 for percentage rate of change.
3.2.2.5. *. Hint. Pay attention to direction, and what it means for the sign (plus/minus) of the velocities of the particles.
3.2.2.6. *. Hint. You'll want to think about the difference in the $y$-coordinates of the two particles.
3.2.2.7. *. Hint. Draw a picture, and be careful about signs.
3.2.2.8. *. Hint. You'll want to think about the difference in height of the two snails.
3.2.2.9. *. Hint. The length of the ladder is changing.
3.2.2.10. Hint. If a trapezoid has height $h$ and (parallel) bases $b_{1}$ and $b_{2}$, then its area is $h\left(\frac{b_{1}+b_{2}}{2}\right)$. To figure out how wide the top of the water is when the water is at height $h$, you can cut the trapezoid up into a rectangle and two triangles, and make use of similar triangles.
3.2.2.11. Hint. Be careful with units. One litre is $1000 \mathrm{~cm}^{3}$, which is not the same as $10 \mathrm{~m}^{3}$.
3.2.2.12. Hint. You, the rocket, and the rocket's original position form a right triangle.
3.2.2.13. *. Hint. Your picture should be a triangle.
3.2.2.14. Hint. Let $\theta$ be the angle between the two hands. Using the Law of Cosines, you can get an expression for $D$ in terms of $\theta$. To find $\frac{\mathrm{d} \theta}{\mathrm{d} t}$, use what you know about how fast clock hands move.
3.2.2.15. *. Hint. The area in the annulus is the area of the outer circle minus the area of the inner circle.
3.2.2.16. Hint. The volume of a sphere with radius $R$ is $\frac{4}{3} \pi r^{3}$.
3.2.2.17. Hint. The area of a triangle is half its base times its height. To find the base, split the triangle into two right triangles.
3.2.2.18. Hint. The easiest way to figure out the area of the sector of an annulus (or a circle) is to figure out the area of the entire annulus, then multiply by what proportion of the entire annulus the sector is. For example, if your sector is $\frac{1}{10}$ of the entire annulus, then its area is $\frac{1}{10}$ of the area of the entire annulus. (See Section A. 4 to see how this works out for circles.)
3.2.2.19. Hint. Think about the ways in which this problem is similar to and different from Example 3.2.6 and Question 3.2.2.18.
3.2.2.20. Hint. The volume of a cone with height $h$ and radius $r$ is $\frac{1}{3} \pi r^{2} h$. Also, one millilitre is the same as one cubic centimetre.

## Exercises - Stage 3

3.2 .2 .21 . Hint. If you were to install the buoy, how would you choose the length of rope? For which values of $\theta$ do $\sin \theta$ and $\cos \theta$ have different signs? How would those values of $\theta$ look on the diagram?
3.2.2.22. Hint. At both points of interest, the point is moving along a straight line. From the diagram, you can figure out the equation of that line.
For the question "How fast is the point moving?" in part (b), remember that the velocity of an object can be found by differentiating (with respect to time) the equation that gives the position of the object. The complicating factors in this case are that (1) the position of our object is not given as a function of time, and (2) the position of our object is given in two dimensions, not one.
3.2.2.23. Hint. (a) Since the perimeter of the cross section of the bottle does not change, $p$ (the perimeter of the ellipse) is the same as the perimeter of the circle of radius 5 .
(b) The volume of the bottle will be the area of its cross section times its height. This is always the case when you have some two-dimensional shape, and turn it into a three-dimensional object by "pulling" the shape straight up. (For example, you can think of a cylinder as a circle that has been "pulled" straight up. To understand why this formula works, think about what is means to measure the area of a shape in square centimetres, and the volume of an object in cubic centimetres.)
(c) You can use what you know about $a$ and the formula from (a) to find $b$ and $\frac{\mathrm{d} b}{\mathrm{~d} t}$. Then use the formula from (b).
3.2.2.24. Hint. If $A=0$, you can figure out $C$ and $D$ from the relationship given.

## 3.3 - Exponential Growth and Decay - a First Look at Differential Equations

### 3.3.4 • Exercises

- Exercises for § 3.3.1


## Exercises - Stage 1

3.3.4.1. Hint. Review the definition of a differential equation at the beginning of this section.
3.3.4.2. Hint. You can test whether a given function solves a differential equation by substituting the function into the equation.
3.3.4.3. Hint. Solve $0=C e^{-k t}$ for $t$.

Exercises - Stage 2
3.3.4.4. *. Hint. No calculus here-just a review of the algebra of exponentials.
3.3.4.5. *. Hint. Use Theorem 3.3.2.
3.3.4.6. Hint. From the text, we see the half-life of Carbon-14 is 5730 years. A microgram $(\mu \mathrm{g})$ is one-millionth of a gram, but you don't need to know that to solve this problem.
3.3.4.7. Hint. The quantity of Radium- 226 in the sample at time $t$ will be $Q(t)=$ $C e^{-k t}$ for some positive constants $C$ and $k$. You can use the given information to
find $C$ and $e^{-k}$.
In the following work, remember we use $\log$ to mean natural logarithm, $\log _{e}$.
3.3.4.8. *. Hint. The fact that the mass of the sample decreases at a rate proportional to its mass tells us that, if $Q(t)$ is the mass of Polonium-201, the following differential equation holds:

$$
\frac{\mathrm{d} Q}{\mathrm{~d} t}=-k Q(t)
$$

where $k$ is some positive constant. Compare this to Theorem 3.3.2.
3.3.4.9. Hint. The amount of Radium-221 in a sample at time $t$ will be $Q(t)=$ $C e^{-k t}$ for some positive constants $C$ and $k$. You can leave $C$ as a variable-it's the original amount in the sample, which isn't specified. What you want to find is the value of $t$ such that $Q(t)=0.0001 Q(0)=0.0001 C$.

## Exercises - Stage 3

3.3.4.10. Hint. You don't need to know the original amount of Polonium-210 in order to answer this question: you can leave it as some constant $C$, or you can call it $100 \%$.
3.3.4.11. Hint. Try to find the most possible and least possible remaining Uranium-232, given the bounds in the problem.

## - Exercises for § 3.3.2

## Exercises - Stage 1

3.3.4.1. Hint. You can refer to Corollary 3.3.8, but you can also just differentiate the various proposed functions and see whether, in fact, $\frac{\mathrm{d} T}{\mathrm{~d} t}$ is the same as $5[T-20]$.
3.3.4.2. Hint. From Newton's Law of Cooling and Corollary 3.3.8, the temperature of the object will be

$$
T(t)=[T(0)-A] e^{K t}+A
$$

where $A$ is the ambient temperature, $T(0)$ is the initial temperature of the copper, and $K$ is some constant.
3.3.4.3. Hint. What is $\lim _{t \rightarrow \infty} e^{K t}$ when $K$ is positive, negative, or zero?
3.3.4.4. Hint. Solve $A=[T(0)-A] e^{k t}$ for $t$.

## Exercises - Stage 2

3.3.4.5. Hint. From Newton's Law of Cooling and Corollary 3.3.8, we know the temperature of the copper will be

$$
T(t)=[T(0)-A] e^{K t}+A
$$

where $A$ is the ambient temperature, $T(0)$ is the initial temperature of the copper, and $K$ is some constant. Use the given information to find an expression for $T(t)$ not involving any unknown constants.
3.3.4.6. Hint. From Newton's Law of Cooling and Corollary 3.3.8, we know the temperature of the stone $t$ minutes after it leaves the fire is

$$
T(t)=[T(0)-A] e^{K t}+A
$$

where $A$ is the ambient temperature, $T(0)$ is the temperature of the stone the instant it left the fire, and $K$ is some constant.

## Exercises - Stage 3

3.3.4.8. *. Hint. Newton's Law of Cooling models the temperature of the tea after $t$ minutes as

$$
T(t)=[T(0)-A] e^{K t}+A
$$

where $A$ is the ambient temperature, $T(0)$ is the initial temperature of the tea, and $K$ is some constant.
3.3.4.9. Hint. What is $\lim _{t \rightarrow \infty} T(t)$ ?

## - Exercises for § 3.3.3

## Exercises - Stage 1

3.3.4.1. Hint. $P(0)$ is also (probably) a positive constant.

## Exercises - Stage 2

3.3.4.2. Hint. The assumption that the animals grow according to the Malthusian model tells us that their population $t$ years after 2015 is given by $P(t)=121 e^{b t}$ for some constant $b$.
3.3.4.3. Hint. The Malthusian model says that the population of bacteria $t$ hours after being placed in the dish will be $P(t)=1000 e^{b t}$ for some constant $b$.
3.3.4.4. Hint. If 1928 is $a$ years after the shipwreck, you might want to make use of the fact that $e^{b(a+1)}=e^{b a} e^{b}$.
3.3.4.5. Hint. If the population has a net birthrate per individual per unit time of $b$, then the Malthusian model predicts that the number of individuals at time $t$ will be $P(t)=P(0) e^{b t}$. You can use the test population to find $e^{b}$.

## Exercises - Stage 3

3.3.4.6. Hint. One way to investigate the sign of $k$ is to think about $f^{\prime}(t)$ : is it positive or negative?

## - Further problems for § 3.3

3.3.4.1. *. Hint. Use Theorem 3.3.2 to figure out what $f(x)$ looks like.
3.3.4.2. Hint. To use Corollary 3.3.8, you need to re-write the differential equation as

$$
\frac{\mathrm{d} T}{\mathrm{~d} t}=7\left[T-\left(-\frac{9}{7}\right)\right]
$$

3.3.4.3. *. Hint. The amount of the material at time $t$ will be $Q(t)=C e^{-k t}$ for some constants $C$ and $k$.
3.3.4.4. Hint. In your calculations, it might come in handy that $e^{30 K}=\left(e^{15 K}\right)^{2}$.
3.3.4.5. *. Hint. The differential equation in the problem has the same form as the differential equation from Newton's Law of Cooling.
3.3.4.6. *. Hint. We know the form of the solution $A(t)$ from Corollary 3.3.8.
3.3.4.7. *. Hint. If a function's rate of change is proportional to the function itself, what does the function looks like?
3.3.4.8. *. Hint. The equation from Newton's Law of Cooling, in Corollary 3.3.8, has a similar form to the differential equation in this question.

### 3.4 Approximating Functions Near a Specified Point - Taylor Polynomials

### 3.4.11 • Exercises

- Exercises for § 3.4.1

Exercises - Stage 1
3.4.11.1. Hint. An approximation should be something you can actually figure out-otherwise it's no use.

## Exercises - Stage 2

3.4.11.2. Hint. You'll need some constant $a$ to approximation $\log (0.93) \approx \log (a)$. This $a$ should have two properties: it should be close to 0.93 , and you should be able to easily evaluate $\log (a)$.
3.4.11.3. Hint. You'll need some constant $a$ to approximate $\arcsin (0.1) \approx$ $\arcsin (a)$. This $a$ should have two properties: it should be close to 0.1 , and you should be able to easily evaluate $\arcsin (a)$.
3.4.11.4. Hint. You'll need some constant $a$ to approximate $\sqrt{3} \tan (1) \approx$ $\sqrt{3} \tan (a)$. This $a$ should have two properties: it should be close to 1 , and you should be able to easily evaluate $\sqrt{3} \tan (a)$.

## Exercises - Stage 3

3.4.11.5. Hint. We could figure out $10.1^{3}$ exactly, if we wanted, with pen and paper. Since we're asking for an approximation, we aren't after perfect accuracy. Rather, we're after ease of calculation.

## - Exercises for § 3.4.2

## Exercises - Stage 1

3.4.11.1. Hint. The linear approximation $L(x)$ is chosen so that $f(5)=L(5)$ and $f^{\prime}(5)=L^{\prime}(5)$.
3.4.11.2. Hint. The graph of the linear approximation is a line, passing through $(2, f(2))$, with slope $f^{\prime}(2)$.
3.4.11.3. Hint. It's an extremely accurate approximation.

## Exercises - Stage 2

3.4.11.4. Hint. You'll need to centre your approximation about some $x=a$, which should have two properties: you can easily compute $\log (a)$, and $a$ is close to 0.93 .
3.4.11.5. Hint. Approximate the function $f(x)=\sqrt{x}$.
3.4.11.6. Hint. Approximate the function $f(x)=\sqrt[5]{x}$.

## Exercises - Stage 3

3.4.11.7. Hint. Approximate the function $f(x)=x^{3}$.
3.4.11.8. Hint. One possible choice of $f(x)$ is $f(x)=\sin x$.
3.4.11.9. Hint. Compare the derivatives.

## - Exercises for § 3.4.3

## Exercises - Stage 1

3.4.11.1. Hint. If $Q(x)$ is the quadratic approximation of $f$ about 3 , then $Q(3)=$ $f(3), Q^{\prime}(3)=f^{\prime}(3)$, and $Q^{\prime \prime}(3)=f^{\prime \prime}(3)$.
3.4.11.2. Hint. It is a very good approximation.

## Exercises - Stage 2

3.4.11.3. Hint. Approximate $f(x)=\log x$.
3.4.11.4. Hint. You'll probably want to centre your approximation about $x=0$.
3.4.11.5. Hint. The quadratic approximation of a function $f(x)$ about $x=a$ is

$$
f(x) \approx f(a)+f^{\prime}(a)(x-a)+\frac{1}{2} f^{\prime \prime}(a)(x-a)^{2}
$$

3.4.11.6. Hint. One way to go about this is to approximate the function $f(x)=$ $5 \cdot x^{1 / 3}$, because then $5^{4 / 3}=5 \cdot 5^{1 / 3}=f(5)$.
3.4.11.7. Hint. For 3.4.11.7.c, look for cancellations.
3.4.11.8. Hint. Compare (c) to (b).

Compare (e) and (f) to (d).
To get an alternating sign, consider powers of $(-1)$.

## Exercises - Stage 3

3.4.11.9. Hint. You can evaluate $f(1)$ exactly. Recall $\frac{\mathrm{d}}{\mathrm{d} x} \arcsin x=\frac{1}{\sqrt{1-x^{2}}}$.
3.4.11.10. Hint. Let $f(x)=e^{x}$, and use the quadratic approximation of $f(x)$ about $x=0$ (given in your text, or you can reproduce it) to approximate $f(1)$.
3.4.11.11. Hint. Be wary of indices: for example $\sum_{n=1}^{3} n=\sum_{n=5}^{7}(n-4)$.

## - Exercises for § 3.4.4

## Exercises - Stage 1

3.4.11.1. Hint. $T_{3}^{\prime \prime}(x)$ and $f^{\prime \prime}(x)$ agree when $x=1$.
3.4.11.2. Hint. The $n$th degree Taylor polynomial for $f(x)$ about $x=5$ is

$$
T_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(5)}{k!}(x-5)^{k}
$$

Match up the terms.

## Exercises - Stage 3

3.4.11.3. Hint. The fourth-degree Maclaurin polynomial for $f(x)$ is

$$
T_{4}(x)=f(0)+f^{\prime}(0) x+\frac{1}{2} f^{\prime \prime}(0) x^{2}+\frac{1}{3!} f^{\prime \prime \prime}(0) x^{3}+\frac{1}{4!} f^{(4)}(0) x^{4}
$$

while the third-degree Maclaurin polynomial for $f(x)$ is

$$
T_{3}(x)=f(0)+f^{\prime}(0) x+\frac{1}{2} f^{\prime \prime}(0) x^{2}+\frac{1}{3!} f^{\prime \prime \prime}(0) x^{3}
$$

3.4.11.4. Hint. The third-degree Taylor polynomial for $f(x)$ about $x=1$ is

$$
T_{3}(x)=f(1)+f^{\prime}(1)(x-1)+\frac{1}{2} f^{\prime \prime}(1)(x-1)^{2}+\frac{1}{3!} f^{\prime \prime \prime}(1)(x-1)^{3}
$$

How can you recover $f(1), f^{\prime}(1), f^{\prime \prime}(1)$, and $f^{\prime \prime \prime}(1)$ from $T_{4}(x)$ ?
3.4.11.5. Hint. Compare the given polynomial to the more standard form of the $n$th degree Taylor polynomial,

$$
\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(5)(x-5)^{k}
$$

and notice that the term you want (containing $\left.f^{(10)}(5)\right)$ corresponds to $k=10$ in the standard form, but is not the term corresponding to $k=10$ in the polynomial given in the question.
3.4.11.6. Hint. $T_{3}^{\prime \prime \prime}(a)=f^{\prime \prime \prime}(a)$

## - Exercises for § 3.4.5

## Exercises - Stage 1

3.4.11.1. Hint. The derivatives of $f(x)$ repeat themselves.
3.4.11.2. Hint. You are approximating a polynomial with a polynomial.
3.4.11.3. Hint. Recall $\frac{\mathrm{d}}{\mathrm{d} x}\left\{2^{x}\right\}=2^{x} \log 2$, where $\log 2$ is the constant $\log _{e} 2$.
3.4.11.4. Hint. Just keep differentiating-it gets easier!
3.4.11.5. Hint. Start by differentiating, and finding the pattern for $f^{(k)}(0)$. Remember the chain rule!

## Exercises - Stage 3

3.4.11.6. Hint. You'll need to differentiate $x^{x}$. This is accomplished using logarithmic differentiation, covered in Section 2.10.
3.4.11.7. Hint. What is $6 \arctan \left(\frac{1}{\sqrt{3}}\right)$ ?
3.4.11.8. Hint. After a few derivatives, this will be very similar to Example 3.4.13.
3.4.11.9. Hint. Treat the even and odd powers separately.
3.4.11.10. Hint. Compare this to the Maclaurin polynomial for $e^{x}$.
3.4.11.11. Hint. Compare this to the Maclaurin polynomial for cosine.

## - Exercises for § 3.4.6

## Exercises - Stage 1

3.4.11.1. Hint. $\Delta x$ and $\Delta y$ represent changes in $x$ and $y$, respectively, while $f(x)$ and $f(x+\Delta x)$ are the $y$-values the function takes.
3.4.11.2. Hint. Let $f(x)$ be the number of problems finished after $x$ minutes of work.

## Exercises - Stage 2

3.4.11.3. Hint. $\Delta y=f(5.1)-f(5)$
3.4.11.4. Hint. Use the approximation $\Delta y \approx s^{\prime}(4) \Delta x$ when $x$ is near 4 .

## - Exercises for § 3.4.7

## Exercises - Stage 1

3.4.11.1. Hint. Is the linear approximation exact, or approximate?
3.4.11.2. Hint. When an exact value $Q_{0}$ is measured as $Q_{0}+\Delta Q$, Definition 3.4.25
gives us the absolute error as $|\Delta Q|$, and the percentage error as $100 \frac{|\Delta Q|}{Q_{0}}$.
3.4.11.3. Hint. Let $\Delta y$ is the change in $f(x)$ associated to a change in $x$ from $a$ to $a+\Delta x$. The linear approximation tells us

$$
\Delta y \approx f^{\prime}(a) \Delta x
$$

while the quadratic approximation tells us

$$
\Delta y \approx f^{\prime}(a) \Delta x+\frac{1}{2} f^{\prime \prime}(a)(\Delta x)^{2}
$$

## Exercises - Stage 2

3.4.11.4. Hint. The exact area desired is $A_{0}$. Let the corresponding exact radius desired be $r_{0}$. The linear approximation tells us $\Delta A \approx A^{\prime}\left(r_{0}\right) \Delta r$. Use this relationship, and what you know about the error allowable in $A$, to find the error allowable in $r$.
3.4.11.5. Hint. For part (b), cut the triangle (with angle $\theta$ and side $d$ ) into two right triangles.
3.4.11.6. Hint. The volume of a cone of height $h$ and radius $r$ is $\frac{1}{3} \pi r^{2} h$.

## Exercises - Stage 3

3.4.11.7. Hint. Remember that the amount of the isotope present at time $t$ is $Q(t)=Q(0) e^{-k t}$ for some constant $k$. The measured quantity after 3 years will allow you to replace $k$ in the equation, then solving $Q(t)=\frac{1}{2} Q(0)$ for $t$ will give you the half-life of the isotope.

## - Exercises for § 3.4.8

## Exercises - Stage 1

3.4.11.1. Hint. $R(10)=f(10)-F(10)=-3-5$
3.4.11.2. Hint. Equation 3.4.33 tells us

$$
\left|f(2)-T_{3}(2)\right|=\left|\frac{f^{(4)}(c)}{4!}(2-0)^{4}\right|
$$

for some $c$ strictly between 0 and 2 .
3.4.11.3. Hint. You are approximating a third-degree polynomial with a fifthdegree Taylor polynomial. You should be able to tell how good your approximation will be without a long calculation.
3.4.11.4. Hint. Draw a picture - it should be clear how the two approximations behave.

## Exercises - Stage 2

3.4.11.5. Hint. In this case, Equation 3.4.33 tells us that

$$
\left|f(11.5)-T_{5}(11.5)\right|=\left|\frac{f^{(6)}(c)}{6!}(11.5-11)^{6}\right|
$$

for some $c$ strictly between 11 and 11.5.
3.4.11.6. Hint. In this case, Equation 3.4.33 tells us that $\left|f(0.1)-T_{2}(0.1)\right|=$ $\left|\frac{f^{(3)}(c)}{3!}(0.1-0)^{3}\right|$ for some $c$ strictly between 0 and 0.1 .
3.4.11.7. Hint. In our case, Equation 3.4 .33 tells us

$$
\left|f\left(-\frac{1}{4}\right)-T_{5}\left(-\frac{1}{4}\right)\right|=\left|\frac{f^{(6)}(c)}{6!}\left(-\frac{1}{4}-0\right)^{6}\right|
$$

for some $c$ between $-\frac{1}{4}$ and 0 .
3.4.11.8. Hint. In this case, Equation 3.4.33 tells us that $\left|f(30)-T_{3}(30)\right|=$ $\left|\frac{f^{(4)}(c)}{4!}(30-32)^{4}\right|$ for some $c$ strictly between 30 and 32 .
3.4.11.9. Hint. In our case, Equation 3.4.33 tells us $\left|f(0.01)-T_{1}(0.01)\right|=$ $\left|\frac{f^{(2)}(c)}{2!}\left(0.01-\frac{1}{\pi}\right)^{2}\right|$ for some $c$ between 0.01 and $\frac{1}{\pi}$.
3.4.11.10. Hint. Using Equation 3.4.33, $\left|f\left(\frac{1}{2}\right)-T_{2}\left(\frac{1}{2}\right)\right|=$ $\left|\frac{f^{(3)}(c)}{3!}\left(\frac{1}{2}-0\right)^{3}\right|$ for some $c$ in $\left(0, \frac{1}{2}\right)$.

## Exercises - Stage 3

3.4.11.11. Hint. It helps to have a formula for $f^{(n)}(x)$. You can figure it out by taking several derivatives and noticing the pattern, but also this has been given previously in the text.
3.4.11.12. Hint. You can approximate the function $f(x)=x^{\frac{1}{7}}$.

It's a good bit of trivia to know $3^{7}=2187$.
A low-degree Taylor approximation will give you a good enough estimation. If you guess a degree, and take that Taylor polynomial, the error will probably be less than 0.001 (but you still need to check).
3.4.11.13. Hint. Use the 6 th-degree Maclaurin approximation for $f(x)=\sin x$.
3.4.11.14. Hint. For part 3.4.11.14.c, after you plug in the appropriate values to Equation 3.4.33, simplify the upper and lower bounds for $e$ separately. In particular, for the upper bound, you'll have to solve for $e$.

## - Further problems for § 3.4

## Exercises - Stage 1

3.4.11.1. *. Hint. Compare the given polynomial to the definition of a Maclaurin polynomial.
3.4.11.3. *. Hint. Compare the given polynomial to the definition of a Taylor polynomial.

## Exercises - Stage 2

3.4.11.5. *. Hint. You can use the error formula to determine whether the approximation is too large or too small.
3.4.11.6. *. Hint. Use the function $f(x)=\sqrt{x}$.
3.4.11.7. *. Hint. Use the function $f(x)=x^{1 / 3}$. What is a good choice of centre?
3.4.11.8. *. Hint. Try using the function $f(x)=x^{5}$.
3.4.11.9. *. Hint. If you use the function $f(x)=\sin (x)$, what is a good centre?
3.4.11.10. *. Hint. Recall $\frac{\mathrm{d}}{\mathrm{d} x}\{\arctan x\}=\frac{1}{1+x^{2}}$.
3.4.11.11. *. Hint. Try using the function $f(x)=(2+x)^{3}$.
3.4.11.12. *. Hint. You can try using $f(x)=(8+x)^{1 / 3}$. What is a suitable centre for your approximation?
3.4.11.13. *. Hint. This is the same as the Maclaurin polynomial.
3.4.11.14. *. Hint. This is a straightforward application of Equation 3.4.33.
3.4.11.17. *. Hint. $5^{2 / 3}=f\left(5^{2}\right)=f(25)$

## Exercises - Stage 3

3.4.11.18. Hint. The fourth-degree Maclaurin polynomial for $f(x)$ is

$$
T_{4}(x)=f(0)+f^{\prime}(0) x+\frac{1}{2} f^{\prime \prime}(0) x^{2}+\frac{1}{3!} f^{\prime \prime \prime}(0) x^{3}+\frac{1}{4!} f^{(4)}(0) x^{4}
$$

while the third-degree Maclaurin polynomial for $f(x)$ is

$$
T_{3}(x)=f(0)+f^{\prime}(0) x+\frac{1}{2} f^{\prime \prime}(0) x^{2}+\frac{1}{3!} f^{\prime \prime \prime}(0) x^{3}
$$

3.4.11.19. *. Hint. For part 3.4.11.19.c, think about what the quadratic approximation looks like - is it pointing up or down?
3.4.11.22. *. Hint. For (c), you can write $f(x)$ as the sum of $Q(x)$ and its error term.
For (d), you can use the linear approximation of $e^{x}$ centred at 0 , with its error term when $x=0.1$.

## 3.5 - Optimisation

### 3.5.4 • Exercises

- Exercises for § 3.5.1


## Exercises - Stage 1

3.5.4.1. Hint. Estimate $f^{\prime}(0)$.
3.5.4.2. Hint. If the graph is discontinuous at a point, it is not differentiable at that point.
3.5.4.3. Hint. Try making a little bump at $x=2$, the letting the function get quite large somewhere else.

## Exercises - Stage 2

3.5.4.4. Hint. Critical points are those values of $x$ for which $f^{\prime}(x)=0$.

Singular points are those values of $x$ for which $f(x)$ is not differentiable.

## Exercises - Stage 3

3.5.4.5. Hint. We're only after local extrema, not global. Let $f(x)$ be our function. If there is some interval around $x=2$ where nothing is bigger than $f(2)$, then $f(2)$ is a local maximum, whether or not it is a maximum overall.
3.5.4.6. Hint. By Theorem 3.5.4, if $x=2$ not a critical point, then it must be a singular point.
3.5.4.7. Hint. You should be able to figure out the global minima of $f(x)$ in your head.
Remember with absolute values, $|X|=\left\{\begin{array}{ll}X & X \geq 0 \\ -X & X<0\end{array}\right.$.
3.5.4.8. Hint. Review the definitions of critical points and extrema: Definition 3.5.6 and Definition 3.5.3.

## - Exercises for § 3.5.2

## Exercises - Stage 1

3.5.4.1. Hint. One way to avoid a global minimum is to have $\lim _{x \rightarrow \infty} f(x)=-\infty$. Since $f(x)$ keeps getting lower and lower, there is no one value that is the lowest.
3.5.4.2. Hint. Try allowing the function to approach the $x$-axis without ever touching it.
3.5.4.3. Hint. Since the global minimum value occurs at $x=5$ and $x=-5$, it must be true that $f(5)=f(-5)$.

## Exercises - Stage 2

3.5.4.4. Hint. Global extrema will either occur at critical points in the interval $(-5,5)$ or at the endpoints $x=5, x=-5$.
3.5.4.5. Hint. You only need to consider critical points that are in the interval $(-4,0)$.

## - Exercises for § 3.5.3

## Exercises - Stage 1

3.5.4.1. *. Hint. Factor the derivative.
3.5.4.2. *. Hint. Remember to test endpoints.
3.5.4.4. *. Hint. One way to decide whether a critical point $x=c$ is a local extremum is to consider the first derivative. For example: if $f^{\prime}(x)$ is negative for all $x$ just to the left of $c$, and positive for all $x$ just to the right of $c$, then $f(x)$ decreases up till $c$, then increases after $c$, so $f(x)$ has a local minimum at $c$.
3.5.4.5. *. Hint. One way to decide whether a critical point $x=c$ is a local extremum is to consider the first derivative. For example: if $f^{\prime}(x)$ is negative for all $x$ just to the left of $c$, and positive for all $x$ just to the right of $c$, then $f(x)$ decreases up till $c$, then increases after $c$, so $f(x)$ has a local minimum at $c$.
3.5.4.6. *. Hint. Start with a formula for travel time from $P$ to $B$. You might want to assign a variable to the distance from $A$ where your buggy first reaches the road.
3.5.4.7. *. Hint. A box has three dimensions; make variables for them, and write the relations given in the problem in terms of these variables.
3.5.4.8. *. Hint. Find a formula for the cost of the base, and another formula for the cost of the other sides. The total cost is the sum of these two formulas.
3.5.4.9. *. Hint. The setup is this:

3.5.4.10. *. Hint. Put the whole system on $x y$-axes, so that you can easily describe the pieces using $(x, y)$-coordinates.
3.5.4.11. *. Hint. The surface area consists of two discs and a strip. Find the areas of these pieces.
The volume of a cylinder with radius $r$ and height $h$ is $\pi r^{2} h$.

3.5.4.12. *. Hint. If the circle has radius $r$, and the entire window has perimeter $P$, what is the height of the rectangle?

Exercises - Stage 3
3.5.4.14. *. Hint. Use logarithmic differentiation to find $f^{\prime}(x)$.
3.5.4.15. *. Hint. When you are finding the global extrema of a function, remember to check endpoints as well as critical points.

## 3.6 • Sketching Graphs

### 3.6.7 • Exercises

## - Exercises for § 3.6.1

## Exercises - Stage 1

3.6.7.1. Hint. What happens if $g(x)=x+3$ ?

## Exercises - Stage 2

3.6.7.2. Hint. Use domains and intercepts to distinguish between the functions.
3.6.7.3. Hint. To find $p$, the equation $f(0)=2$ gives you two possible values of $p$. Consider the domain of $f(x)$ to decide between them.
3.6.7.4. Hint. Check for horizontal asymptotes by evaluating $\lim _{x \rightarrow \pm \infty} f(x)$, and check for vertical asymptotes by finding any value of $x$ near which $f(x)$ blows up.
3.6.7.5. Hint. Check for horizontal asymptotes by evaluating $\lim _{x \rightarrow \pm \infty} f(x)$, and check for vertical asymptotes by finding any value of $x$ near which $f(x)$ blows up.

## - Exercises for § 3.6.2

Exercises - Stage 1
3.6.7.1. Hint. For each of the graphs, consider where the derivative is positive, negative, and zero.

## Exercises - Stage 2

3.6.7.2. *. Hint. Where is $f^{\prime}(x)>0$ ?
3.6.7.3. *. Hint. Consider the signs of the numerator and the denominator of $f^{\prime}(x)$.
3.6.7.4. *. Hint. Remember $\frac{\mathrm{d}}{\mathrm{d} x}\{\arctan x\}=\frac{1}{1+x^{2}}$.

## - Exercises for § 3.6.3

Exercises - Stage 1
3.6.7.1. Hint. There are two intervals where the function is concave up, and two where it is concave down.
3.6.7.2. Hint. Try allowing your graph to have horizontal asymptotes. For example, let the function get closer and closer to the $x$-axis (or another horizontal line) without touching it.
3.6.7.3. Hint. Consider $f(x)=(x-3)^{4}$.

## Exercises - Stage 3

3.6.7.5. *. Hint. You must show it has at least one inflection point (try the Intermediate Value Theorem), and at most one inflection point (consider whether the second derivative is increasing or decreasing).
3.6.7.6. *. Hint. Use 3.6.7.6.a in proving 3.6.7.6.b.
3.6.7.7. Hint. Since $x=3$ is an inflection point, we know the concavity of $f(x)$ changes at $x=3$. That is, there is some interval around 3 , with endpoints $a$ and $b$, such that

- $f^{\prime \prime}(a)<0$ and $f^{\prime \prime}(x)<0$ for every $x$ between $a$ and 3 , and
- $f^{\prime \prime}(b)>0$ and $f^{\prime \prime}(x)>0$ for every $x$ between $b$ and 3 .

Use the IVT to show that $f^{\prime \prime}(x)=3$ for some $x$ between $a$ and $b$; then show that this value of $x$ can't be anything except $x=3$.

## - Exercises for § 3.6.4

## Exercises - Stage 1

3.6.7.1. Hint. This function is symmetric across the $y$-axis.
3.6.7.2. Hint. There are two.
3.6.7.3. Hint. Since the function is even, you only have to reflect the portion shown across the $y$-axis to complete the sketch.
3.6.7.4. Hint. Since the function is odd, to complete the sketch, reflect the portion shown across the $y$-axis, then the $x$-axis.

## Exercises - Stage 2

3.6.7.5. Hint. A function is even if $f(-x)=f(x)$.
3.6.7.6. Hint. Its period is not $2 \pi$.
3.6.7.7. Hint. Simplify $f(-x)$ to see whether it is the same as $f(x),-f(x)$, or neither.
3.6.7.8. Hint. Simplify $f(-x)$ to see whether it is the same as $f(x),-f(x)$, or neither.
3.6.7.9. Hint. Find the smallest value $k$ such that $f(x+k)=f(x)$ for any $x$ in the domain of $f$.
You may use the fact that the period of $g(X)=\tan X$ is $\pi$.

## Exercises - Stage 3

3.6.7.10. Hint. It is true that $f(x)=f(x+2 \pi)$ for every $x$ in the domain of $f(x)$, but the period is not $2 \pi$.

## - Exercises for § 3.6.6

## Exercises - Stage 1

3.6.7.1. *. Hint. You'll find the intervals of increase and decrease. These will give you a basic outline of the behaviour of the function. Use concavity to refine your picture.
3.6.7.2. *. Hint. The local maximum is also a global maximum.
3.6.7.3. *. Hint. The sign of the first derivative is determined entirely by the numerator, but the sign of the second derivative depends on both the numerator and the denominator.
3.6.7.4. *. Hint. The function is odd.
3.6.7.5. *. Hint. The function is continuous at $x=0$, but its derivative is not.
3.6.7.6. *. Hint. Since you aren't asked to find the intervals of concavity exactly, sketch the intervals of increase and decrease, and turn them into a smooth curve. You might not get exactly the intervals of concavity that are given in the solution, but there should be the same number of intervals as the solution, and they should have the same positions relative to the local extrema.
3.6.7.7. *. Hint. Use intervals of increase and decrease, concavity, and asymptotes to sketch the curve.
3.6.7.8. Hint. Although the function exhibits a certain kind of repeating behaviour, it is not periodic.
3.6.7.9. *. Hint. The period of this function is $2 \pi$. So, it's enough to graph the curve $y=f(x)$ over the interval $[-\pi, \pi]$, because that figure will simply repeat.
Use trigonometric identities to write $f^{\prime \prime}(x)=-4\left(4 \sin ^{2} x+\sin x-2\right)$. Then you can find where $f^{\prime \prime}(x)=0$ by setting $y=\sin x$ and solving $0=4 y^{2}+y-2$.
3.6.7.10. Hint. There is one point where the curve is continuous but has a vertical tangent line.

## Exercises - Stage 3

3.6.7.11. *. Hint. Use $\lim _{x \rightarrow-\infty} f^{\prime}(x)$ to determine $\lim _{x \rightarrow-\infty} f(x)$.
3.6.7.12. *. Hint. Once you have the graph of a function, reflect it over the line $y=x$ to graph its inverse. Be careful of the fact that $f(x)$ is only defined in this problem for $x \geq 0$.
3.6.7.14. *. Hint. For (a), don't be intimidated by the new names: we can graph these functions using the methods learned in this section.
For (b), remember that to define an inverse of a function, we need to restrict the domain of that function to an interval where it is one-to-one. Then to graph the inverse, we can simply reflect the original function over the line $y=x$.
For (c), set $y(x)=\cosh ^{-1}(x)$, so $\cosh (y(x))=x$. The differentiate using the chain rule. To get your final answer in terms of $x$ (instead of $y$ ), use the identity $\cosh ^{2}(y)-\sinh ^{2}(y)=1$.

## 3.7 • L'Hôpital's Rule, Indeterminate Forms

### 3.7.4 • Exercises

## Exercises - Stage 1

3.7.4.1. Hint. Try making one function a multiple of the other.
3.7.4.2. Hint. Try making one function a multiple of the other, but not a constant multiple.
3.7.4.3. Hint. Try modifying the function from Example 3.7.20.

## Exercises - Stage 2

3.7.4.4. *. Hint. Plugging in $x=1$ to the numerator and denominator makes both zero. This is exactly one of the indeterminate forms where l'Hôpital's rule can be directly applied.
3.7.4.5. *. Hint. Is this an indeterminate form?
3.7.4.6. *. Hint. First, rearrange the expression to a more natural form (without a negative exponent).
3.7.4.7. *. Hint. If at first you don't succeed, try, try again.
3.7.4.8. *. Hint. Keep at it!
3.7.4.9. Hint. Rather than use l'Hôpital, try factoring out $x^{2}$ from the numerator and denominator.
3.7.4.10. *. Hint. Keep going!
3.7.4.12. Hint. Try plugging in $x=0$. Is this an indeterminate form?
3.7.4.13. Hint. Simplify the trigonometric part first.
3.7.4.14. Hint. Rationalize, then remember your training.
3.7.4.15. *. Hint. If it is too difficult to take a derivative for l'Hôpital's Rule, try splitting up the function into smaller chunks and evaluating their limits independently.
3.7.4.17. *. Hint. Try manipulating the function to get it into a nicer form
3.7.4.19. Hint. $\lim _{x \rightarrow 0} \sqrt[x^{2}]{\sin ^{2} x}=\left(\sin ^{2} x\right)^{\frac{1}{x^{2}}}$; what form is this?
3.7.4.20. Hint. $\lim _{x \rightarrow 0} \sqrt[x^{2}]{\cos x}=\lim _{x \rightarrow 0}(\cos x)^{\frac{1}{x^{2}}}$
3.7.4.21. Hint. logarithms
3.7.4.22. Hint. Introduce yet another logarithm.
3.7.4.23. *. Hint. If the denominator tends to zero, and the limit exists, what must be the limit of the numerator?
3.7.4.24. *. Hint. Start with one application of l'Hôpital's Rule. After that, you need to consider three distinct cases: $k>2, k<2$, and $k=2$.

## Exercises - Stage 3

3.7.4.25. Hint. Percentage error: $100\left|\frac{\text { exact-approx }}{\text { exact }}\right|$. Absolute error: |exact approx|. (We'll see these definitions again in 3.4.25.)

## 4 - Towards Integral Calculus

## 4.1 • Introduction to Antiderivatives

### 4.1.2 • Exercises

## Exercises - Stage 1

4.1.2.1. Hint. The function $f(x)$ is an antiderivative of $f^{\prime}(x)$, but it is not the most general one.
4.1.2.2. Hint. When $f(x)$ is positive, its antiderivative $F(x)$ is increasing. When $f(x)$ is negative, its antiderivative $F(x)$ is decreasing. When $f(x)=0, F(x)$ has a horizontal tangent line.

## Exercises - Stage 2

4.1.2.3. Hint. For any constant $n \neq-1$, an antiderivative of $x^{n}$ is $\frac{1}{n+1} x^{n+1}$.
4.1.2.4. Hint. For any constant $n \neq-1$, an antiderivative of $x^{n}$ is $\frac{1}{n+1} x^{n+1}$.
4.1.2.5. Hint. For any constant $n \neq-1$, an antiderivative of $x^{n}$ is $\frac{1}{n+1} x^{n+1}$. The constant $n$ does not have to be an integer.
4.1.2.6. Hint. What is the derivative of $\sqrt{x}$ ?
4.1.2.7. Hint. The derivative of $e^{5 x+11}$ is close to, but not exactly the same as, $f(x)$. Don't be afraid to just make a guess. But be sure to check by differentiating your guess. If the derivative isn't what you want, you will often still learn enough to be able to then guess the correct antiderivative.
4.1.2.8. Hint. From the table in the text, an antiderivative of $\sin x$ is $-\cos x$, and an antiderivative of $\cos x$ is $\sin x$.
4.1.2.9. Hint. What is the derivative of tangent?
4.1.2.10. Hint. What is an antiderivative of $\frac{1}{x}$ ?
4.1.2.11. Hint. $\frac{7}{\sqrt{3-3 x^{2}}}=\frac{7}{\sqrt{3}}\left(\frac{1}{\sqrt{1-x^{2}}}\right)$
4.1.2.12. Hint. How is this similar to the derivative of the arctangent function?
4.1.2.13. Hint. First, find the antiderivative of $f^{\prime}(x)$. Your answer will have an unknown $+C$ in it. Figure out which value of $C$ gives $f(1)=10$.
4.1.2.14. Hint. Remember that one antiderivative of $\cos x$ is $\sin x($ not $-\sin x)$.
4.1.2.15. Hint. An antiderivative of $\frac{1}{x}$ is $\log (x)+C$, but only for $x>0$.
4.1.2.16. Hint. What is the derivative of the arcsine function?
4.1.2.17. Hint. If $P(t)$ is the population at time $t$, then the information given in the problem is $P^{\prime}(t)=100 e^{2 t}$.
4.1.2.18. Hint. You can't know exactly - there will be a constant involved.
4.1.2.19. Hint. If $P(t)$ is the amount of power in kW -hours the house has used since time $t=0$, where $t$ is measured in hours, then the information given is that $P^{\prime}(t)=0.5 \sin \left(\frac{\pi}{24} t\right)+0.25 \mathrm{~kW}$.

## Exercises - Stage 3

4.1.2.20. *. Hint. The derivatives should match. Remember $\sin ^{-1}$ is another way of writing arcsine.
4.1.2.21. Hint. Think about the product rule.
4.1.2.22. Hint. Think about the quotient rule for derivatives.
4.1.2.23. Hint. Notice that the derivative of $x^{3}$ is $3 x^{2}$.
4.1.2.24. Hint. Think about the chain rule for derivatives. You might need to multiply your first guess by a constant.
4.1.2.25. Hint. Simplify.
4.1.2.26. Hint. This problem is similar to Question 4.1.2.11, but you'll have to do some harder factoring. Try getting $f(x)$ into the form $a\left(\frac{1}{\sqrt{1-(b x)^{2}}}\right)$ for some constants $a$ and $b$.
4.1.2.27. Hint. Following Example 4.1.7 let $V(H)$ be the volume of the solid formed by rotating the segment of the parabola from $x=-H$ to $x=H$. The plan is to evaluate

$$
V^{\prime}(H)=\lim _{h \rightarrow H} \frac{V(H)-V(h)}{H-h}
$$

and then antidifferentiate $V^{\prime}(H)$ to find $V(H)$. Since you don't know $V(H)-V(h)$ (yet), first find upper and lower bounds on it when $h<H$. These bounds can be the volumes of two cylinders, one with radius $H$ (and what height?) and the other with radius $h$.

## Answers to Exercises

## 1 . Limits

## 1.1 • Drawing Tangents and a First Limit

### 1.1.2 • Exercises

Exercises - Stage 1
1.1.2.1. Answer.

1.1.2.2. Answer.
a True
b In general, this is false. For most functions $f(x)$ this will be false, but there are some functions for which it is true.
1.1.2.3. Answer. At least once.

## 1.2 • Another Limit and Computing Velocity

### 1.2.2 • Exercises

## Exercises - Stage 1

1.2.2.1. Answer. Speed is nonnegative; velocity has a sign (positive or negative) that indicates direction.
1.2.2.2. Answer. Yes-an object that is not moving has speed 0 .
1.2.2.3. Answer. 0 kph
1.2.2.4. Answer. The speed at the one second mark is larger than the average speed.
1.2.2.5. Answer. The slope of a curve is given by $\frac{\text { change in vertical component }}{\text { change in horizontal component }}$. The change in the vertical component is exactly $s(b)-s(a)$, and the change in the horizontal component is exactly $b-a$.
1.2.2.6. Answer. $(0,2) \cup(6,7)$

## Exercises - Stage 2

1.2.2.7. Answer.
a 24 units per second.
b 6 units per second

### 1.2.2.8. Answer.

a $\frac{1}{4}$ units per second
b $\frac{1}{2}$ units per second
c $\frac{1}{6}$ units per second
Remark: the average velocity is not the average of the two instantaneous velocities.

## 1.3 - The Limit of a Function

### 1.3.2 • Exercises

## Exercises - Stage 1

1.3.2.1. Answer.
a $\lim _{x \rightarrow-2} f(x)=1$
b $\lim _{x \rightarrow 0} f(x)=0$
c $\lim _{x \rightarrow 2} f(x)=2$
1.3.2.2. Answer. DNE

### 1.3.2.3. Answer.

a $\lim _{x \rightarrow-1^{-}} f(x)=2$
b $\lim _{x \rightarrow-1^{+}} f(x)=-2$
c $\lim _{x \rightarrow-1} f(x)=$ DNE
d $\lim _{x \rightarrow-2^{+}} f(x)=0$
e $\lim _{x \rightarrow 2^{-}} f(x)=0$
1.3.2.4. Answer. Many answers are possible; here is one.

|  |  |  |
| :---: | :---: | :---: |

1.3.2.5. Answer. Many answers are possible; here is one.

1.3.2.6. Answer. In general, this is false.
1.3.2.7. Answer. False
1.3.2.8. Answer. $\lim _{x \rightarrow-2^{-}} f(x)=16$
1.3.2.9. Answer. Not enough information to say.

Exercises - Stage 2
1.3.2.10. Answer. $\lim _{t \rightarrow 0} \sin t=0$
1.3.2.11. Answer. $\lim _{x \rightarrow 0^{+}} \log x=-\infty$
1.3.2.12. Answer. $\lim _{y \rightarrow 3} y^{2}=9$
1.3.2.13. Answer. $\lim _{x \rightarrow 0^{-}} \frac{1}{x}=-\infty$
1.3.2.14. Answer. $\lim _{x \rightarrow 0} \frac{1}{x}=\mathrm{DNE}$
1.3.2.15. Answer. $\lim _{x \rightarrow 0} \frac{1}{x^{2}}=\infty$
1.3.2.16. Answer. $\lim _{x \rightarrow 3} \frac{1}{10}=\frac{1}{10}$
1.3.2.17. Answer. 9

## 1.4 - Calculating Limits with Limit Laws 1.4.2 • Exercises

Exercises - Stage 1
1.4.2.1. Answer. (1.4.2.1.a) and (1.4.2.1.d)
1.4.2.2. Answer. There are many possible answers; one is $f(x)=10(x-3)$, $g(x)=x-3$.
1.4.2.3. Answer. There are many possible answers; one is $f(x)=(x-3)^{2}$ and $g(x)=x-3$. Another is $f(x)=0$ and $g(x)=x-3$.
1.4.2.4. Answer. There are many possible answers; one is $f(x)=x-3, g(x)=$ $(x-3)^{3}$.
1.4.2.5. Answer. Any real number; positive infinity; negative infinity; does not exist.

## Exercises - Stage 2

1.4.2.6. Answer. 0
1.4.2.7. Answer. 6
1.4.2.8. Answer. 16
1.4.2.9. *. Answer. $4 / \cos (3)$
1.4.2.10. *. Answer. 2
1.4.2.11. *. Answer. $-7 / 2$
1.4.2.12. *. Answer. 3
1.4.2.13. *. Answer. $-\frac{3}{2}$
1.4.2.14. *. Answer. $\log (2)-1$
1.4.2.15. *. Answer. $\frac{1}{4}$
1.4.2.16. *. Answer. $\frac{1}{2}$
1.4.2.17. *. Answer. 5
1.4.2.18. *. Answer. -6
1.4.2.19. Answer. -14
1.4.2.20. *. Answer. $-\frac{1}{3}$
1.4.2.21. *. Answer. $\frac{1}{6}$
1.4.2.22. *. Answer. $\frac{1}{\sqrt{3}}$
1.4.2.23. *. Answer. 1
1.4.2.24. *. Answer. 12
1.4.2.25. Answer. 0
1.4.2.26. Answer. $\frac{1}{2}$
1.4.2.27. *. Answer. 0
1.4.2.28. Answer. 5
1.4.2.29. Answer. $-\infty$
1.4.2.30. Answer. $\sqrt{\frac{2}{3}}$
1.4.2.31. Answer. DNE
1.4.2.32. Answer. $\infty$
1.4.2.33. Answer. $x^{5}-32 x+15$
1.4.2.34. Answer. 0
1.4.2.35. *. Answer. 0
1.4.2.36. *. Answer. 2
1.4.2.37. Answer. 0
1.4.2.38. Answer. $-\frac{32}{9}$
1.4.2.39. Answer. DNE
1.4.2.40. Answer. DNE
1.4.2.41. Answer. $-\frac{9}{2}$
1.4.2.42. Answer. -4
1.4.2.43. *. Answer. $a=\frac{7}{2}$

### 1.4.2.44. Answer.

a $\lim _{x \rightarrow 0} f(x)=0$
b $\lim _{x \rightarrow 0} g(x)=$ DNE
c $\lim _{x \rightarrow 0} f(x) g(x)=2$
d $\lim _{x \rightarrow 0} \frac{f(x)}{g(x)}=0$
e $\lim _{x \rightarrow 2} f(x)+g(x)=\frac{9}{2}$
$\mathrm{f} \lim _{x \rightarrow 0} \frac{f(x)+1}{g(x+1)}=1$

## Exercises - Stage 3

1.4.2.45. Answer.


Pictures may vary somewhat; the important points are the values of the function at integer values of $x$, and the vertical asymptotes.

### 1.4.2.46. Answer.


1.4.2.47. Answer. 10
1.4.2.48. Answer. 1.4.2.48.a DNE, DNE
1.4.2.48.b 0
1.4.2.48.c No: it is only true when both $\lim _{x \rightarrow a} f(x)$ and $\lim _{x \rightarrow a} g(x)$ exist.
1.4.2.49. Answer. 1.4.2.49.a $\lim _{x \rightarrow 0^{-}} f(x)=-3$
1.4.2.49.b $\lim _{x \rightarrow 0^{+}} f(x)=3$
1.4.2.49.c $\lim _{x \rightarrow 0} f(x)=$ DNE
1.4.2.50. Answer. 1.4.2.50.a $\lim _{x \rightarrow-4^{-}} f(x)=0$
1.4.2.50.b $\lim _{x \rightarrow-4^{+}} f(x)=0$
1.4.2.50.c $\lim _{x \rightarrow-4} f(x)=0$

## 1.5 - Limits at Infinity

1.5.2 • Exercises

## Exercises - Stage 1

1.5.2.1. Answer. There are many answers: any constant polynomial has this property. One answer is $f(x)=1$.
1.5.2.2. Answer. There are many answers: any odd-degree polynomial has this property. One answer is $f(x)=x$.

## Exercises - Stage 2

1.5.2.3. Answer. 0
1.5.2.4. Answer. $\infty$
1.5.2.5. Answer. 0
1.5.2.6. Answer. DNE
1.5.2.7. Answer. $-\infty$
1.5.2.8. Answer. $\sqrt{3}$
1.5.2.9. *. Answer. 3
1.5.2.10. *. Answer. $-\frac{3}{4}$
1.5.2.11. *. Answer. $-\frac{1}{2}$
1.5.2.12. *. Answer. $\frac{1}{2}$
1.5.2.13. *. Answer. $\frac{5}{3}$
1.5.2.14. *. Answer. 0
1.5.2.15. *. Answer. $\frac{4}{7}$
1.5.2.16. Answer. 1
1.5.2.17. *. Answer. 0
1.5.2.18. Answer. -1
1.5.2.19. Answer. 1
1.5.2.20. *. Answer. -1
1.5.2.21. *. Answer. $-\frac{3}{2}$
1.5.2.22. *. Answer. $-\frac{5}{3}$
1.5.2.23. Answer. $-\infty$
1.5.2.24. *. Answer. $\frac{5}{2}$
1.5.2.25. Answer. $\lim _{a \rightarrow 0^{+}} \frac{a^{2}-\frac{1}{a}}{a-1}=\infty$
1.5.2.26. Answer. $\lim _{x \rightarrow 3} \frac{2 x+8}{\frac{1}{x-3}+\frac{1}{x^{2}-9}}=0$

## Exercises - Stage 3

1.5.2.27. Answer. No such rational function exists.
1.5.2.28. Answer. This is the amount of the substance that will linger longterm. Since it's nonzero, the substance would be something that would stay in your body. Something like "tattoo ink" is a reasonable answer, while "penicillin" is not.

## 1.6 • Continuity

### 1.6.4 • Exercises

## Exercises - Stage 1

1.6.4.1. Answer. Many answers are possible; the tangent function behaves like this.
1.6.4.2. Answer. At some time between my birth and now, I was exactly one meter tall.
1.6.4.3. Answer. One example is $f(x)=\left\{\begin{array}{ll}0 & \text { when } 0 \leq x \leq 1 \\ 2 & \text { when } 1<x \leq 2\end{array}\right.$. The IVT only guarantees $f(c)=1$ for some $c$ in $[0,2]$ when $f$ is continuous over [0,2]. If $f$ is not continuous, the IVT says nothing.

1.6.4.4. Answer. Yes
1.6.4.5. Answer. No
1.6.4.6. Answer. No
1.6.4.7. Answer. True.
1.6.4.8. Answer. True.
1.6.4.9. Answer. In general, false.
1.6.4.10. Answer. $\lim _{x \rightarrow 0^{+}} h(x)=0$

## Exercises - Stage 2

1.6.4.11. Answer. $k=0$
1.6.4.12. Answer. Since $f$ is a polynomial, it is continuous over all real numbers. $f(0)=1<12345$ and $f(12345)=12345^{3}+12345^{2}+12345+1>12345$ (since all terms are positive). So by the IVT, $f(c)=12345$ for some $c$ between 0 and 12345 .
1.6.4.13. *. Answer. $(-\infty,-1) \cup(-1,1) \cup(1,+\infty)$
1.6.4.14. *. Answer. $(-\infty,-1) \cup(1,+\infty)$
1.6.4.15. *. Answer. The function is continuous except at $x=$ $\pm \pi, \pm 3 \pi, \pm 5 \pi, \cdots$.
1.6.4.16. *. Answer. $x \neq n \pi$, where $n$ is any integer
1.6.4.17. *. Answer. $\pm 2$
1.6.4.18. *. Answer. $c=1$
1.6.4.19. *. Answer. $-1,4$
1.6.4.20. *. Answer. $c=1, c=-1$

## Exercises - Stage 3

1.6.4.21. Answer. This isn't the kind of equality that we can just solve; we'll need a trick, and that trick is the IVT. The general idea is to show that $\sin x$ is somewhere bigger, and somewhere smaller, than $x-1$. However, since the IVT can only show us that a function is equal to a constant, we need to slightly adjust our language. Showing $\sin x=x-1$ is equivalent to showing $\sin x-x+1=0$, so let $f(x)=\sin x-x+1$, and let's show that it has a real root.
First, we need to note that $f(x)$ is continuous (otherwise we can't use the IVT). Now, we need to find a value of $x$ for which it is positive, and for which it's negative. By checking a few values, we find $f(0)$ is positive, and $f(100)$ is negative. So, by the IVT, there exists a value of $x$ (between 0 and 100) for which $f(x)=0$. Therefore, there exists a value of $x$ for which $\sin x=x-1$.
1.6.4.22. *. Answer. We let $f(x)=3^{x}-x^{2}$. Then $f(x)$ is a continuous function, since both $3^{x}$ and $x^{2}$ are continuous for all real numbers.
We want a value $a$ such that $f(a)>0$. We see that $a=0$ works since

$$
f(0)=3^{0}-0=1>0 .
$$

We want a value $b$ such that $f(b)<0$. We see that $b=-1$ works since

$$
f(-1)=\frac{1}{3}-1<0 .
$$

So, because $f(x)$ is continuous on $(-\infty, \infty)$ and $f(0)>0$ while $f(-1)<0$, then the Intermediate Value Theorem guarantees the existence of a real number $c \in(-1,0)$ such that $f(c)=0$.
1.6.4.23. *. Answer. We let $f(x)=2 \tan (x)-x-1$. Then $f(x)$ is a continuous function on the interval $(-\pi / 2, \pi / 2)$ since $\tan (x)=\sin (x) / \cos (x)$ is continuous on this interval, while $x+1$ is a polynomial and therefore continuous for all real numbers.
We find a value $a \in(-\pi / 2, \pi / 2)$ such that $f(a)<0$. We observe immediately that $a=0$ works since

$$
f(0)=2 \tan (0)-0-1=0-1=-1<0 .
$$

We find a value $b \in(-\pi / 2, \pi / 2)$ such that $f(b)>0$. We see that $b=\pi / 4$ works since

$$
\begin{aligned}
f(\pi / 4) & =2 \tan (\pi / 4)-\pi / 4-1=2-\pi / 4-1=1-\pi / 4 \\
& =(4-\pi) / 4>0
\end{aligned}
$$

because $3<\pi<4$.
So, because $f(x)$ is continuous on $[0, \pi / 4]$ and $f(0)<0$ while $f(\pi / 4)>0$, then the Intermediate Value Theorem guarantees the existence of a real number $c \in(0, \pi / 4)$ such that $f(c)=0$.
1.6.4.24. *. Answer. Let $f(x)=\sqrt{\cos (\pi x)}-\sin (2 \pi x)-1 / 2$. This function is continuous provided $\cos (\pi x) \geq 0$. This is true for $0 \leq x \leq \frac{1}{2}$.
Now $f$ takes positive values on $[0,1 / 2]$ :

$$
f(0)=\sqrt{\cos (0)}-\sin (0)-1 / 2=\sqrt{1}-1 / 2=1 / 2
$$

And $f$ takes negative values on $[0,1 / 2]$ :

$$
f(1 / 2)=\sqrt{\cos (\pi / 2)}-\sin (\pi)-1 / 2=0-0-1 / 2=-1 / 2
$$

(Notice that $f(1 / 3)=(\sqrt{2}-\sqrt{3}) / 2-1 / 2$ also works)
So, because $f(x)$ is continuous on $[0,1 / 2)$ and $f(0)>0$ while $f(1 / 2)<0$, then the Intermediate Value Theorem guarantees the existence of a real number $c \in(0,1 / 2)$ such that $f(c)=0$.
1.6.4.25. *. Answer. We let $f(x)=\frac{1}{\cos ^{2}(\pi x)}-x-\frac{3}{2}$. Then $f(x)$ is a continuous function on the interval $(-1 / 2,1 / 2)$ since $\cos x$ is continuous everywhere and nonzero on that interval.
The function $f$ takes negative values. For example, when $x=0$ :

$$
f(0)=\frac{1}{\cos ^{2}(0)}-0-\frac{3}{2}=1-\frac{3}{2}=-\frac{1}{2}<0
$$

It also takes positive values, for instance when $x=1 / 4$ :

$$
\begin{aligned}
f(1 / 4) & =\frac{1}{(\cos \pi / 4)^{2}}-\frac{1}{4}-\frac{3}{2} \\
& =\frac{1}{1 / 2}-\frac{1+6}{4} \\
& =2-7 / 4=1 / 4>0
\end{aligned}
$$

By the IVT there is $c, 0<c<1 / 4$ such that $f(c)=0$, in which case

$$
\frac{1}{(\cos \pi c)^{2}}=c+\frac{3}{2} .
$$

1.6.4.26. Answer. $[0,1]$ is the easiest answer to find. Also acceptable are $[-2,-1]$ and $[14,15]$.
1.6.4.27. Answer. 1.91
1.6.4.28. Answer.

- If $f(a)=g(a)$, or $f(b)=g(b)$, then we simply take $c=a$ or $c=b$.
- Suppose $f(a) \neq g(a)$ and $f(b) \neq g(b)$. Then $f(a)<g(a)$ and $g(b)<f(b)$, so if we define $h(x)=f(x)-g(x)$, then $h(a)<0$ and $h(b)>0$. Since $h$ is the difference of two functions that are continuous over $[a, b]$, also $h$ is continuous over $[a, b]$. So, by the Intermediate Value Theorem, there exists some $c \in(a, b)$ with $h(c)=0$; that is, $f(c)=g(c)$.


## 2 - Derivatives

## 2.1 $\cdot$ Revisiting Tangent Lines

### 2.1.2 • Exercises

## Exercises - Stage 1

2.1.2.1. Answer. If $Q$ is to the left of the $y$ axis, the secant line has positive slope; if $Q$ is to the right of the $y$ axis, the secant line has negative slope.

### 2.1.2.2. Answer. 2.1.2.2.a closer

2.1.2.2.b the tangent line has the larger slope
2.1.2.3. Answer. $\{(a),(c),(e)\},\{(b),(f)\},\{(d)\}$

## Exercises - Stage 2

2.1.2.4. Answer. Something like 1.5. A reasonable answer would be between 1 and 2 .
2.1.2.5. Answer. There is only one tangent line to $f(x)$ at $P$ (shown in blue), but there are infinitely many choices of $Q$ and $R$ (one possibility shown in red).


### 2.1.2.6. Answer.



## 2.2 • Definition of the Derivative

### 2.2.4 • Exercises

## Exercises - Stage 1

2.2.4.1. Answer. (a), (d)
2.2.4.2. Answer. (e)
2.2.4.3. Answer. (b)
2.2.4.4. *. Answer. By definition, $f(x)=x^{3}$ is differentiable at $x=0$ if the limit

$$
\lim _{h \rightarrow 0} \frac{f(h)-f(0)}{h}=\lim _{h \rightarrow 0} \frac{h^{3}-0}{h}
$$

exists.
2.2.4.5. Answer. $x=-1$ and $x=3$
2.2.4.6. Answer. True. (Contrast to Question 2.2.4.7.)
2.2.4.7. Answer. In general, false. (Contrast to Question 2.2.4.6.)
2.2.4.8. Answer. metres per second

## Exercises - Stage 2

2.2.4.9. Answer. $y-6=3(x-1)$, or $y=3 x+3$
2.2.4.10. Answer. $\frac{-1}{x^{2}}$
2.2.4.11. *. Answer. By definition

$$
f^{\prime}(0)=\lim _{h \rightarrow 0} \frac{f(h)-f(0)}{h}=\lim _{h \rightarrow 0} \frac{h|h|}{h}=\lim _{h \rightarrow 0}|h|=0
$$

In particular, the limit exists, so the derivative exists (and is equal to zero).
2.2.4.12. *. Answer. $\frac{-2}{(x+1)^{2}}$
2.2.4.13. *. Answer. $\frac{-2 x}{\left[x^{2}+3\right]^{2}}$
2.2.4.14. Answer. 1
2.2.4.15. *. Answer. $f^{\prime}(x)=-\frac{2}{x^{3}}$
2.2.4.16. *. Answer. $a=4, b=-4$
2.2.4.17. *. Answer. $f^{\prime}(x)=\frac{1}{2 \sqrt{1+x}}$ when $x>-1 ; f^{\prime}(x)$ does not exist when $x \leq-1$.

## Exercises - Stage 3

2.2.4.18. Answer. $v(t)=4 t^{3}-2 t$
2.2.4.19. *. Answer. No, it does not.
2.2.4.20. *. Answer. No, it does not.
2.2.4.21. *. Answer. Yes, it is.
2.2.4.22. *. Answer. Yes, it is.
2.2.4.23. . Answer. Many answers are possible; here is one.

2.2.4.24. Answer.

$$
p^{\prime}(x)=\lim _{h \rightarrow 0} \frac{p(x+h)-p(x)}{h}
$$

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} \\
& =\lim _{h \rightarrow 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right] \\
(*) & =\left[\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}\right]+\left[\lim _{h \rightarrow 0} \frac{g(x+h)-g(x)}{h}\right] \\
& =f^{\prime}(x)+g^{\prime}(x)
\end{aligned}
$$

At step $(*)$, we use the limit law that $\lim _{x \rightarrow a}[F(x)+G(x)]=\lim _{x \rightarrow a} F(x)+\lim _{x \rightarrow a} G(x)$, as long as $\lim _{x \rightarrow a} F(x)$ and $\lim _{x \rightarrow a} G(x)$ exist. Because the problem states that $f^{\prime}(x)$ and $g^{\prime}(x)$ exist, we know that $\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ and $\lim _{h \rightarrow 0} \frac{g(x+h)-g(x)}{h}$ exist, so our work is valid.
2.2.4.25. Answer. 2.2.4.25.a $f^{\prime}(x)=2$ and $g^{\prime}(x)=1$
2.2.4.25.b $p^{\prime}(x)=4 x$
2.2.4.25.c no
2.2.4.26. *. Answer. $y=6 x-9$ and $y=-2 x-1$
2.2.4.27. *. Answer. $a>1$

## 2.3 - Interpretations of the Derivative

### 2.3.3 • Exercises

## Exercises - Stage 2

2.3.3.1. Answer. 2.3.3.1.a The average rate of change of the height of the water over the single day starting at $t=0$, measured in $\frac{\mathrm{m}}{\mathrm{hr}}$.
2.3.3.1.b The instantaneous rate of change of the height of the water at the time $t=0$.
2.3.3.2. Answer. Profit per additional widget sold, when $t$ widgets are being sold. This is called the marginal profit per widget, when $t$ widgets are being sold.
2.3.3.3. Answer. $T^{\prime}(d)$ measures how quickly the temperature is changing per unit change of depth, measured in degrees per metre. $\left|T^{\prime}(d)\right|$ will probably be largest when $d$ is near zero, unless there are hot springs or other underwater heat sources.
2.3.3.4. Answer. Calories per additional gram, when there are $w$ grams
2.3.3.5. Answer. The acceleration of the object.
2.3.3.6. Answer. Degrees Celsius temperature change per joule of heat added. (This is closely related to heat capacity and to specific heat - there's a nice explanation of this on Wikipedia.)
2.3.3.7. Answer. Number of bacteria added per degree. That is: the number of extra bacteria (possibly negative) that will exist in the population by raising the temperature by one degree.

## Exercises - Stage 3

2.3.3.8. Answer. $360 R^{\prime}(t)$
2.3.3.9. Answer. If $P^{\prime}(t)$ is positive, your sample is below the ideal temperature, and if $P^{\prime}(t)$ is negative, your sample is above the ideal temperature. If $P^{\prime}(t)=0$, you don't know whether the sample is exactly at the ideal temperature, or way above or below it with no living bacteria.

## 2.4 • Arithmetic of Derivatives - a Differentiation Toolbox

### 2.4.2 $\cdot$ Exercises

## Exercises - Stage 1

2.4.2.1. Answer. True
2.4.2.2. Answer. False, in general.
2.4.2.3. Answer. True
2.4.2.4. Answer. If you're creative, you can find lots of ways to differentiate!

- Constant multiple: $g^{\prime}(x)=3 f^{\prime}(x)$.
- Product rule: $g^{\prime}(x)=\frac{\mathrm{d}}{\mathrm{d} x}\{3\} f(x)+3 f^{\prime}(x)=0 f(x)+3 f^{\prime}(x)=3 f^{\prime}(x)$.
- Sum rule: $g^{\prime}(x)=\frac{\mathrm{d}}{\mathrm{d} x}\{f(x)+f(x)+f(x)\}=f^{\prime}(x)+f^{\prime}(x)+f^{\prime}(x)=3 f^{\prime}(x)$.
- Quotient rule: $g^{\prime}(x)=\frac{\mathrm{d}}{\mathrm{d} x}\left\{\frac{f(x)}{\frac{1}{3}}\right\}=\frac{\frac{1}{3} f^{\prime}(x)-f(x)(0)}{\frac{1}{9}}=\frac{\frac{1}{3} f^{\prime}(x)}{\frac{1}{9}}=9\left(\frac{1}{3}\right) f^{\prime}(x)=$ $3 f^{\prime}(x)$.

All rules give $g^{\prime}(x)=3 f^{\prime}(x)$.

Exercises - Stage 2
2.4.2.5. Answer. $f^{\prime}(x)=6 x+\frac{2}{\sqrt{x}}$
2.4.2.6. Answer. $-36 x+24 \sqrt{x}+\frac{20}{\sqrt{x}}-45$
2.4.2.7. *. Answer. $y-\frac{1}{8}=\frac{3}{4} \cdot\left(x-\frac{1}{2}\right)$, or $y=\frac{3}{4} x-\frac{1}{4}$
2.4.2.8. *. Answer. 2.4.2.8.a 4
2.4.2.8.b left
2.4.2.8.c decreasing
2.4.2.9. *. Answer. $\frac{1}{(x+1 / 2)^{2}}$, or $\frac{4}{(2 x+1)^{2}}$
2.4.2.10. Answer. -72
2.4.2.11. Answer. $y-\frac{1}{2}=-\frac{1}{8}(x-1)$, or $y=-\frac{1}{8} x+\frac{5}{8}$
2.4.2.12. Answer. $b^{\prime}(t)-d^{\prime}(t)$
2.4.2.13. *. Answer. $\{(1,3),(3,27)\}$
2.4.2.14. *. Answer. $\frac{1}{2 \sqrt{100180}}$
2.4.2.15. Answer. $20 t+7$ square metres per second.
2.4.2.16. Answer. 0
2.4.2.17. Answer. First expression, $f(x)=\frac{g(x)}{h(x)}$ :

$$
f^{\prime}(x)=\frac{h(x) g^{\prime}(x)-g(x) h^{\prime}(x)}{h^{2}(x)}
$$

Second expresson, $f(x)=\frac{g(x)}{k(x)} \cdot \frac{k(x)}{h(x)}$ :

$$
\begin{aligned}
& f^{\prime}(x)=\left(\frac{k(x) g^{\prime}(x)-g(x) k^{\prime}(x)}{k^{2}(x)}\right)\left(\frac{k(x)}{h(x)}\right) \\
& +\left(\frac{g(x)}{k(x)}\right)\left(\frac{h(x) k^{\prime}(x)-k(x) h^{\prime}(x)}{h^{2}(x)}\right) \\
& =\frac{k(x) g^{\prime}(x)-g(x) k^{\prime}(x)}{k(x) h(x)}+\frac{g(x) h(x) k^{\prime}(x)-g(x) k(x) h^{\prime}(x)}{k(x) h^{2}(x)} \\
& =\frac{h(x) k(x) g^{\prime}(x)-h(x) g(x) k^{\prime}(x)}{k(x) h^{2}(x)}+\frac{g(x) h(x) k^{\prime}(x)-g(x) k(x) h^{\prime}(x)}{k(x) h^{2}(x)} \\
& =\frac{h(x) k(x) g^{\prime}(x)-h(x) g(x) k^{\prime}(x)+g(x) h(x) k^{\prime}(x)-g(x) k(x) h^{\prime}(x)}{k(x) h^{2}(x)} \\
& =\frac{h(x) k(x) g^{\prime}(x)-g(x) k(x) h^{\prime}(x)}{k(x) h^{2}(x)} \\
& =\frac{h(x) g^{\prime}(x)-g(x) h^{\prime}(x)}{h^{2}(x)}
\end{aligned}
$$

and this is exactly what we got from differentiating the first expression.

## 2.6 • Using the Arithmetic of Derivatives - Examples

### 2.6.2 • Exercises

## Exercises - Stage 1

2.6.2.1. Answer. In the quotient rule, there is a minus, not a plus. Also, $2(x+$ 1) $+2 x$ is not the same as $2(x+1)$.

The correct version is:

$$
\begin{aligned}
f(x) & =\frac{2 x}{x+1} \\
f^{\prime}(x) & =\frac{2(x+1)-2 x}{(x+1)^{2}}
\end{aligned}
$$

$$
=\frac{2}{(x+1)^{2}}
$$

### 2.6.2.2. Answer. False

## Exercises - Stage 2

2.6.2.3. Answer. $4 x\left(x^{2}+2\right)\left(x^{2}+3\right)$
2.6.2.4. Answer. $12 t^{3}+15 t^{2}+\frac{1}{t^{2}}$
2.6.2.5. Answer. $x^{\prime}(y)=8 y^{3}+2 y$
2.6.2.6. Answer. $T^{\prime}(x)=\frac{\left(x^{2}+3\right)\left(\frac{1}{2 \sqrt{x}}\right)-(\sqrt{x}+1)(2 x)}{\left(x^{2}+3\right)^{2}}$
2.6.2.7. *. Answer. $\frac{21-4 x-7 x^{2}}{\left(x^{2}+3\right)^{2}}$

### 2.6.2.8. Answer. 7

2.6.2.9. Answer. $\frac{3 x^{4}+30 x^{3}-2 x-5}{\left(x^{2}+5 x\right)^{2}}$
2.6.2.10. *. Answer. $\frac{-3 x^{2}+12 x+5}{(2-x)^{2}}$
2.6.2.11. *. Answer. $\frac{-22 x}{\left(3 x^{2}+5\right)^{2}}$
2.6.2.12. *. Answer. $\frac{4 x^{3}+12 x^{2}-1}{(x+2)^{2}}$
2.6.2.13. *. Answer. The derivative of the function is

$$
\frac{\left(1-x^{2}\right) \cdot \frac{1}{2 \sqrt{x}}-\sqrt{x} \cdot(-2 x)}{\left(1-x^{2}\right)^{2}}=\frac{\left(1-x^{2}\right)-2 x \cdot(-2 x)}{2 \sqrt{x}\left(1-x^{2}\right)^{2}}
$$

The derivative is undefined if either $x<0$ or $x=0, \pm 1$ (since the square-root is undefined for $x<0$ and the denominator is zero when $x=0,1,-1$. Putting this together - the derivative exists for $x>0, x \neq 1$.
2.6.2.14. Answer. $\left(\frac{3}{5} x^{\frac{-4}{5}}+5 x^{\frac{-2}{3}}\right)\left(3 x^{2}+8 x-5\right)+(3 \sqrt[5]{x}+15 \sqrt[3]{x}+8)(6 x+8)$
2.6.2.15. Answer. $f^{\prime}(x)=(2 x+5)\left(x^{-1 / 2}+x^{-2 / 3}\right)+\left(x^{2}+5 x+\right.$ 1) $\left(\frac{-1}{2} x^{-3 / 2}-\frac{2}{3} x^{-5 / 3}\right)$
2.6.2.16. Answer. $x=-5$ and $x=1$

## Exercises - Stage 3

2.6.2.17. *. Answer. $y=x-\frac{1}{4}$
2.6.2.18. Answer.

$y=4 x-4$ and $y=-2 x-1$
2.6.2.19. *. Answer. $2015 \cdot 2^{2014}$

## 2.7 • Derivatives of Exponential Functions

### 2.7.3 • Exercises

Exercises - Stage 1
2.7.3.1. Answer. A- $(a)$ and $(d), \mathrm{B}-(e), \mathrm{C}-(c), \mathrm{D}-(b)$
2.7.3.2. Answer. (b), (d), (e)
2.7.3.3. Answer. False
2.7.3.4. Answer. increasing

Exercises - Stage 2
2.7.3.5. Answer. $\frac{(x-1) e^{x}}{2 x^{2}}$
2.7.3.6. Answer. $2 e^{2 x}$
2.7.3.7. Answer. $e^{a+x}$
2.7.3.8. Answer. $x>-1$
2.7.3.9. Answer. $-e^{-x}$
2.7.3.10. Answer. $2 e^{2 x}$
2.7.3.11. Answer. When $t$ is in the interval $(-2,0)$.

## Exercises - Stage 3

2.7.3.12. Answer. $g^{\prime}(x)=\left[f(x)+f^{\prime}(x)\right] e^{x}$
2.7.3.13. Answer. (b) and (d)
2.7.3.14. *. Answer. $a=b=\frac{e}{2}$

## 2.8 - Derivatives of Trigonometric Functions

### 2.8.8 • Exercises

## Exercises - Stage 1 <br> 2.8.8.1. Answer.



The graph $f(x)=\sin x$ has horizontal tangent lines precisely at those points where $\cos x=0$.

### 2.8.8.2. Answer.



The graph $f(x)=\sin x$ has maximum slope at those points where $\cos x$ has a maximum. That is, where $\cos x=1$.

## Exercises - Stage 2

2.8.8.3. Answer. $f^{\prime}(x)=\cos x-\sin x+\sec ^{2} x$
2.8.8.4. Answer. $x=\frac{\pi}{4}+\pi n$, for any integer $n$.
2.8.8.5. Answer. 0
2.8.8.6. Answer. $f^{\prime}(x)=2\left(\cos ^{2} x-\sin ^{2} x\right)$
2.8.8.7. Answer. $f^{\prime}(x)=e^{x}\left(\cot x-\csc ^{2} x\right)$
2.8.8.8. Answer. $f^{\prime}(x)=\frac{2+3 \sec x+2 \sin x-2 \tan x \sec x+3 \sin x \tan x}{(\cos x+\tan x)^{2}}$
2.8.8.9. Answer. $f^{\prime}(x)=\frac{5 \sec x \tan x-5 \sec x-1}{e^{x}}$
2.8.8.10. Answer. $f^{\prime}(x)=\left(e^{x}+\cot x\right)\left(30 x^{5}+\csc x \cot x\right)+\left(e^{x}-\csc ^{2} x\right)\left(5 x^{6}-\right.$ $\csc x$ )
2.8.8.11. Answer. $-\sin (\theta)$
2.8.8.12. Answer. $f^{\prime}(x)=-\cos x-\sin x$
2.8.8.13. Answer. $\left(\frac{\cos \theta+\sin \theta}{\cos \theta-\sin \theta}\right)^{2}+1$
2.8.8.14. *. Answer. $a=0, b=1$.
2.8.8.15. *. Answer. $y-\pi=1 \cdot(x-\pi / 2)$

## Exercises - Stage 3

2.8.8.16. *. Answer. $-\sin (2015)$
2.8.8.17. *. Answer. $-\sqrt{3} / 2$
2.8.8.18. *. Answer. -1
2.8.8.19. Answer.

$$
\tan \theta=\frac{\sin \theta}{\cos \theta}
$$

So, using the quotient rule,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} \theta}\{\tan \theta\} & =\frac{\cos \theta \cos \theta-\sin \theta(-\sin \theta)}{\cos ^{2} \theta}=\frac{\cos ^{2} \theta+\sin ^{2} \theta}{\cos ^{2} \theta} \\
& =\left(\frac{1}{\cos \theta}\right)^{2}=\sec ^{2} \theta
\end{aligned}
$$

2.8.8.20. *. Answer. $a=-\frac{2}{3}, b=2$
2.8.8.21. *. Answer. All values of $x$ except $x=\frac{\pi}{2}+n \pi$, for any integer $n$.
2.8.8.22. *. Answer. The function is differentiable whenever $x^{2}+x-6 \neq 0$ since the derivative equals

$$
\frac{10 \cos (x) \cdot\left(x^{2}+x-6\right)-10 \sin (x) \cdot(2 x+1)}{\left(x^{2}+x-6\right)^{2}}
$$

which is well-defined unless $x^{2}+x-6=0$. We solve $x^{2}+x-6=(x-2)(x+3)=0$, and get $x=2$ and $x=-3$. So, the function is differentiable for all real values $x$ except for $x=2$ and for $x=-3$.
2.8.8.23. *. Answer. The function is differentiable whenever $\sin (x) \neq 0$ since the derivative equals

$$
\frac{\sin (x) \cdot(2 x+6)-\cos (x) \cdot\left(x^{2}+6 x+5\right)}{(\sin x)^{2}}
$$

which is well-defined unless $\sin x=0$. This happens when $x$ is an integer multiple
of $\pi$. So, the function is differentiable for all real values $x$ except $x=n \pi$, where $n$ is any integer.
2.8.8.24. *. Answer. $y-1=2 \cdot(x-\pi / 4)$
2.8.8.25. *. Answer. $y=2 x+2$
2.8.8.26. Answer. $x=\frac{3 \pi}{4}+n \pi$ for any integer $n$.
2.8.8.27. Answer. $f^{\prime}(0)=0$
2.8.8.28. *. Answer. $h^{\prime}(x)=\left\{\begin{array}{rl}\cos x & x>0 \\ -\cos x & x<0\end{array}\right.$ It exists for all $x \neq 0$.
2.8.8.29. *. Answer. 2.8.8.29.iii
2.8.8.30. *. Answer. 2

## 2.9 - One More Tool - the Chain Rule

### 2.9.4 • Exercises

## Exercises - Stage 1

2.9.4.1. Answer. 2.9.4.1.a $\frac{\mathrm{d} K}{\mathrm{~d} U}$ is negative
2.9.4.1.b $\frac{\mathrm{d} U}{\mathrm{~d} O}$ is negative
2.9.4.1.c $\frac{\mathrm{d} K}{\mathrm{~d} O}$ is positive
2.9.4.2. Answer. negative

## Exercises - Stage 2

2.9.4.3. Answer. $-5 \sin (5 x+3)$
2.9.4.4. Answer. $10 x\left(x^{2}+2\right)^{4}$
2.9.4.5. Answer. $17\left(4 k^{4}+2 k^{2}+1\right)^{16} \cdot\left(16 k^{3}+4 k\right)$
2.9.4.6. Answer. $\frac{-2 x}{\left(x^{2}-1\right) \sqrt{x^{4}-1}}$
2.9.4.7. Answer. $-e^{\cos \left(x^{2}\right)} \cdot \sin \left(x^{2}\right) \cdot 2 x$
2.9.4.8. *. Answer. -4
2.9.4.9. *. Answer. $[\cos x-x \sin x] e^{x \cos (x)}$
2.9.4.10. *. Answer. $[2 x-\sin x] e^{x^{2}+\cos (x)}$
2.9.4.11. *. Answer. $\frac{3}{2 \sqrt{x-1} \sqrt{x+2}}{ }^{3}$
2.9.4.12. *. Answer. $f^{\prime}(x)=-\frac{2}{x^{3}}+\frac{x}{\sqrt{x^{2}-1}}$ is defined for $x$ in $(-\infty, 1) \cup(1, \infty)$.
2.9.4.13. *. Answer. $f^{\prime}(x)=\frac{\left(1+x^{2}\right)(5 \cos 5 x)-(\sin 5 x)(2 x)}{\left(1+x^{2}\right)^{2}}$
2.9.4.14. Answer. $2 e^{2 x+7} \sec \left(e^{2 x+7}\right) \tan \left(e^{2 x+7}\right)$
2.9.4.15. Answer. $y=1$
2.9.4.16. Answer. $t=\frac{2}{3}$ and $t=4$
2.9.4.17. Answer. $2 e \sec ^{2}(e)$
2.9.4.18. *. Answer. $y^{\prime}=4 e^{4 x} \tan x+e^{4 x} \sec ^{2} x$
2.9.4.19. *. Answer. $\frac{3}{\left(1+e^{3}\right)^{2}}$
2.9.4.20. *. Answer. $2 \sin (x) \cdot \cos (x) \cdot e^{\sin ^{2}(x)}$
2.9.4.21. *. Answer. $\cos \left(e^{5 x}\right) \cdot e^{5 x} \cdot 5$
2.9.4.22. *. Answer. $-e^{\cos \left(x^{2}\right)} \cdot \sin \left(x^{2}\right) \cdot 2 x$
2.9.4.23. *. Answer. $y^{\prime}=-\sin \left(x^{2}+\sqrt{x^{2}+1}\right)\left(2 x+\frac{x}{\sqrt{x^{2}+1}}\right)$
2.9.4.24. *. Answer. $y^{\prime}=2 x \cos ^{2} x-2\left(1+x^{2}\right) \sin x \cos x$
2.9.4.25. *. Answer. $y^{\prime}=\frac{e^{3 x}\left(3 x^{2}-2 x+3\right)}{\left(1+x^{2}\right)^{2}}$
2.9.4.26. *. Answer. -40
2.9.4.27. *. Answer. $(1,1)$ and $(-1,-1)$.
2.9.4.28. Answer. Always
2.9.4.29. Answer. $e^{x} \sec ^{3}(5 x-7)(1+15 \tan (5 x-7))$
2.9.4.30. *. Answer. $e^{2 x} \cos 4 x+2 x e^{2 x} \cos 4 x-4 x e^{2 x} \sin 4 x$

## Exercises - Stage 3

2.9.4.31. Answer. $t=\frac{\pi}{4}$
2.9.4.32. *. Answer. Let $f(x)=e^{x+x^{2}}$ and $g(x)=1+x$. Then $f(0)=g(0)=1$. $f^{\prime}(x)=(1+2 x) e^{x+x^{2}}$ and $g^{\prime}(x)=1$. When $x>0$,

$$
f^{\prime}(x)=(1+2 x) e^{x+x^{2}}>1 \cdot e^{x+x^{2}}=e^{x+x^{2}}>e^{0+0^{2}}=1=g^{\prime}(x) .
$$

Since $f(0)=g(0)$, and $f^{\prime}(x)>g^{\prime}(x)$ for all $x>0$, that means $f$ and $g$ start at the same place, but $f$ always grows faster. Therefore, $f(x)>g(x)$ for all $x>0$.
2.9.4.33. Answer. $\cos (2 x)=\cos ^{2} x-\sin ^{2} x$
2.9.4.34. Answer.

$$
f^{\prime}(x)=\frac{1}{3}\left(\frac{\sqrt{x^{3}-9} \tan x}{e^{\csc x^{2}}}\right)^{\frac{2}{3}}
$$

$$
\frac{e^{\csc x^{2}}\left(-2 x \sqrt{x^{3}-9} \tan x \frac{\cos \left(x^{2}\right)}{\sin ^{2}\left(x^{2}\right)}-\frac{3 x^{2} \tan x}{2 \sqrt{x^{3}-9}}-\sqrt{x^{3}-9} \sec ^{2} x\right)}{\left(\tan ^{2} x\right)\left(x^{3}-9\right)}
$$

2.9.4.35. Answer. 2.9.4.35.a


The particle traces the curve $y=1-x^{2}$ restricted to domain $[-1,1]$. At $t=0$, the particle is at the top of the curve, $(1,0)$. Then it moves to the right, and goes back and forth along the curve, repeating its path every $2 \pi$ units of time.
2.9.4.35.b $\sqrt{3}$

### 2.10 • The Natural Logarithm

### 2.10.3 • Exercises

## Exercises - Stage 1

2.10.3.1. Answer. Ten speakers: 13 dB . One hundred speakers: 23 dB .
2.10.3.2. Answer. $20 \log 2 \approx 14$ years
2.10.3.3. Answer. (b)

## Exercises - Stage 2

2.10.3.4. Answer. $f^{\prime}(x)=\frac{1}{x}$
2.10.3.5. Answer. $f^{\prime}(x)=\frac{2}{x}$
2.10.3.6. Answer. $f^{\prime}(x)=\frac{2 x+1}{x^{2}+x}$
2.10.3.7. Answer. $f^{\prime}(x)=\frac{1}{x \log 10}$
2.10.3.8. *. Answer. $y^{\prime}=\frac{1-3 \log x}{x^{4}}$
2.10.3.9. Answer. $\frac{d}{d \theta} \log (\sec \theta)=\tan \theta$
2.10.3.10. Answer. $f^{\prime}(x)=\frac{-e^{\cos (\log x)} \sin (\log x)}{x}$
2.10.3.11. $*$. Answer. $y^{\prime}=\frac{2 x+\frac{4 x^{3}}{2 \sqrt{x^{4}+1}}}{x^{2}+\sqrt{x^{4}+1}}$
2.10.3.12. *. Answer. $\frac{\tan x}{2 \sqrt{-\log (\cos x)}}$
2.10.3.13. *. Answer. $\frac{\sqrt{x^{2}+4}+x}{x \sqrt{x^{2}+4}+x^{2}+4}=\frac{1}{\sqrt{x^{2}+4}}$
2.10.3.14. *. Answer. $g^{\prime}(x)=\frac{2 x e^{x^{2}} \sqrt{1+x^{4}}+2 x^{3}}{e^{x^{2}} \sqrt{1+x^{4}}+1+x^{4}}$
2.10.3.15. *. Answer. $\frac{4}{3}$
2.10.3.16. Answer. $f^{\prime}(x)=\frac{3 x}{x^{2}+5}-\frac{2 x^{3}}{x^{4}+10}$
2.10.3.17. Answer. $\frac{40}{3}$
2.10.3.18. *. Answer. $g^{\prime}(x)=\pi^{x} \log \pi+\pi x^{\pi-1}$
2.10.3.19. Answer. $f^{\prime}(x)=x^{x}(\log x+1)$
2.10.3.20. *. Answer. $\quad x^{x}(\log x+1)+\frac{1}{x \log 10}$
2.10.3.21. Answer. $f^{\prime}(x)=\frac{1}{4}\left(\sqrt[4]{\frac{\left(x^{4}+12\right)\left(x^{4}-x^{2}+2\right)}{x^{3}}}\right)\left(\frac{4 x^{3}}{x^{4}+12}+\frac{4 x^{3}-2 x}{x^{4}-x^{2}+2}-\frac{3}{x}\right)$
2.10.3.22. Answer.

$$
\begin{aligned}
& f^{\prime}(x)=(x+1)\left(x^{2}+1\right)^{2}\left(x^{3}+1\right)^{3}\left(x^{4}+1\right)^{4}\left(x^{5}+1\right)^{5} \\
& \quad\left[\frac{1}{x+1}+\frac{4 x}{x^{2}+1}+\frac{9 x^{2}}{x^{3}+1}+\frac{16 x^{3}}{x^{4}+1}+\frac{25 x^{4}}{x^{5}+1}\right]
\end{aligned}
$$

2.10.3.23. Answer. $\left(\frac{x^{2}+2 x+3}{3 x^{4}+4 x^{3}+5}\right)\left(\frac{1}{x^{2}+2 x+3}-\frac{6 x^{2}}{3 x^{4}+4 x^{3}+5}-\frac{1}{2(x+1)^{2}}\right)$
2.10.3.24. *. Answer. $f^{\prime}(x)=(\cos x)^{\sin x}[(\cos x) \log (\cos x)-\sin x \tan x]$
2.10.3.25. *. Answer. $\frac{\mathrm{d}}{\mathrm{d} x}\left\{(\tan x)^{x}\right\}=(\tan x)^{x}\left(\log (\tan x)+\frac{x}{\sin x \cos x}\right)$
2.10.3.26. *. Answer. $2 x\left(x^{2}+1\right)^{x^{2}+1}\left(1+\log \left(x^{2}+1\right)\right)$
2.10.3.27. *. Answer. $f^{\prime}(x)=\left(x^{2}+1\right)^{\sin (x)} \cdot\left(\cos x \cdot \log \left(x^{2}+1\right)+\frac{2 x \sin x}{x^{2}+1}\right)$
2.10.3.28. *. Answer. $x^{\cos ^{3}(x)} \cdot\left(-3 \cos ^{2}(x) \sin (x) \log (x)+\frac{\cos ^{3}(x)}{x}\right)$
2.10.3.29. *. Answer. $(3+\sin (x))^{x^{2}-3} \cdot\left[2 x \log (3+\sin (x))+\frac{\left(x^{2}-3\right) \cos (x)}{3+\sin (x)}\right]$

## Exercises - Stage 3

2.10.3.30. Answer. $\frac{\mathrm{d}}{\mathrm{d} x}\left\{[f(x)]^{g(x)}\right\}=[f(x)]^{g(x)}\left[g^{\prime}(x) \log (f(x))+\frac{g(x) f^{\prime}(x)}{f(x)}\right]$
2.10.3.31. Answer. Let $g(x):=\log (f(x))$. Notice $g^{\prime}(x)=\frac{f^{\prime}(x)}{f(x)}$.

In order to show that the two curves have horizontal tangent lines at the same values of $x$, we will show two things: first, that if $f(x)$ has a horizontal tangent line at some value of $x$, then also $g(x)$ has a horizontal tangent line at that value of $x$. Second, we will show that if $g(x)$ has a horizontal tangent line at some value of $x$, then also $f(x)$ has a horizontal tangent line at that value of $x$.
Suppose $f(x)$ has a horizontal tangent line where $x=x_{0}$ for some point $x_{0}$. This means $f^{\prime}\left(x_{0}\right)=0$. Then $g^{\prime}\left(x_{0}\right)=\frac{f^{\prime}\left(x_{0}\right)}{f\left(x_{0}\right)}$. Since $f\left(x_{0}\right) \neq 0, \frac{f^{\prime}\left(x_{0}\right)}{f\left(x_{0}\right)}=\frac{0}{f\left(x_{0}\right)}=0$, so $g(x)$ also has a horizontal tangent line when $x=x_{0}$. This shows that whenever $f$ has a horizontal tangent line, $g$ has one too.
Now suppose $g(x)$ has a horizontal tangent line where $x=x_{0}$ for some point $x_{0}$. This means $g^{\prime}\left(x_{0}\right)=0$. Then $g^{\prime}\left(x_{0}\right)=\frac{f^{\prime}\left(x_{0}\right)}{f\left(x_{0}\right)}=0$, so $f^{\prime}\left(x_{0}\right)$ exists and is equal to zero. Therefore, $f(x)$ also has a horizontal tangent line when $x=x_{0}$. This shows that whenever $g$ has a horizontal tangent line, $f$ has one too.

### 2.11 • Implicit Differentiation

### 2.11.2 • Exercises

## Exercises - Stage 1

### 2.11.2.1. Answer. (a) and (b)

2.11.2.2. Answer. At $(0,4)$ and $(0,-4), \frac{\mathrm{d} y}{\mathrm{~d} x}$ is 0 ; at $(0,0), \frac{\mathrm{d} y}{\mathrm{~d} x}$ does not exist.
2.11.2.3. Answer. (a) no
(b) no $\frac{\mathrm{d} y}{\mathrm{~d} x}=-\frac{x}{y}$. It is not possible to write $\frac{\mathrm{d} y}{\mathrm{~d} x}$ as a function of $x$, because (as stated in (b)) one value of $x$ may give two values of $\frac{\mathrm{d} y}{\mathrm{~d} x}$. For instance, when $x=\pi / 4$, at the point $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ the circle has slope $\frac{\mathrm{d} y}{\mathrm{~d} x}=-1$, while at the point $\left(\frac{\pi}{4}, \frac{-1}{\sqrt{2}}\right)$ the circle has slope $\frac{\mathrm{d} y}{\mathrm{~d} x}=1$.

## Exercises - Stage 2

2.11.2.4. *. Answer. $\frac{\mathrm{d} y}{\mathrm{~d} x}=-\frac{e^{x}+y}{e^{y}+x}$
2.11.2.5. *. Answer. $\frac{\mathrm{d} y}{\mathrm{~d} x}=\frac{y^{2}+1}{e^{y}-2 x y}$
2.11.2.6. *. Answer. At $(x, y)=(4,1), y^{\prime}=-\frac{1}{\pi+1} . \quad$ At $(x, y)=(-4,1)$, $y^{\prime}=\frac{1}{\pi-1}$.
2.11.2.7. *. Answer. $-\frac{2 x \sin \left(x^{2}+y\right)+3 x^{2}}{4 y^{3}+\sin \left(x^{2}+y\right)}$
2.11.2.8. *. Answer. At $(x, y)=(1,0), y^{\prime}=-6$, and at $(x, y)=(-5,0)$, $y^{\prime}=\frac{6}{25}$.
2.11.2.9. *. Answer. $\frac{d y}{d x}=\frac{\cos (x+y)-2 x}{2 y-\cos (x+y)}$
2.11.2.10. *. Answer. At $(x, y)=(2,0)$ we have $y^{\prime}=-\frac{3}{2}$, and at $(x, y)=$ $(-4,0)$ we have $y^{\prime}=-\frac{3}{4}$.
2.11.2.11. Answer. $\left(\frac{\sqrt{3}}{2}, \frac{-1}{2 \sqrt{3}}\right),\left(\frac{-\sqrt{3}}{2}, \frac{1}{2 \sqrt{3}}\right)$
2.11.2.12. *. Answer. $-\frac{28}{3}$
2.11.2.13. *. Answer. $\frac{\mathrm{d} y}{\mathrm{~d} x}=-\frac{2 x y^{2}+\sin y}{2 x^{2} y+x \cos y}$

Exercises - Stage 3
2.11.2.14. *. Answer. At $(x, y)=(2,0), y^{\prime}=-2$. At $(x, y)=(-2,0), y^{\prime}=2$.
2.11.2.15. Answer. $x=0, x=1, x=-1$

### 2.12 • Inverse Trigonometric Functions

### 2.12.2 • Exercises

## Exercises - Stage 1

2.12.2.1. Answer. (a) $(-\infty, \infty)$
(b) all integer multiples of $\pi$
(c) $[-1,1]$
2.12.2.2. Answer. False
2.12.2.3. Answer.


### 2.12.2.4. Answer.

- If $|a|>1$, there is no point where the curve has horizontal tangent line.
- If $|a|=1$, the curve has a horizontal tangent line where $x=2 \pi n+\frac{a \pi}{2}$ for any integer $n$.
- If $|a|<1$, the curve has a horizontal tangent line where $x=2 \pi n+\arcsin (a)$ or $x=(2 n+1) \pi-\arcsin (a)$ for any integer $n$.
2.12.2.5. Answer. Domain: $x= \pm 1$. Not differentiable anywhere.


## Exercises - Stage 2

2.12.2.6. Answer. $f^{\prime}(x)=\frac{1}{\sqrt{9-x^{2}}}$; domain of $f$ is $[-3,3]$.
2.12.2.7. Answer. $f^{\prime}(t)=\frac{-\frac{t^{2}-1}{\sqrt{1-t^{2}}}-2 t \arccos t}{\left(t^{2}-1\right)^{2}}$, and the domain of $f(t)$ is $(-1,1)$.
2.12.2.8. Answer. The domain of $f(x)$ is all real numbers, and $f^{\prime}(x)=$ $\frac{-2 x}{\left(x^{2}+2\right) \sqrt{x^{4}+4 x^{2}+3}}$.
2.12.2.9. Answer. $f^{\prime}(x)=\frac{1}{a^{2}+x^{2}}$ and the domain of $f(x)$ is all real numbers.
2.12.2.10. Answer. $f^{\prime}(x)=\arcsin x$, and the domain of $f(x)$ is $[-1,1]$.
2.12.2.11. Answer. $x=0$
2.12.2.12. Answer. $\frac{\mathrm{d}}{\mathrm{d} x}\{\arcsin x+\arccos x\}=0$
2.12.2.13. *. Answer. $y^{\prime}=\frac{-1}{x^{2} \sqrt{1-\frac{1}{x^{2}}}}$
2.12.2.14. *. Answer. $y^{\prime}=\frac{-1}{1+x^{2}}$
2.12.2.15. *. Answer. $2 x \arctan x+1$
2.12.2.16. Answer. Let $\theta=\arctan x$. Then $\theta$ is the angle of a right triangle that gives $\tan \theta=x$. In particular, the ratio of the opposite side to the adjacent side is $x$. So, we have a triangle that looks like this:

where the length of the hypotenuse came from the Pythagorean Theorem. Now,

$$
\sin (\arctan x)=\sin \theta=\frac{\text { opp }}{\text { hyp }}=\frac{x}{\sqrt{x^{2}+1}}
$$

From here, we differentiate using the quotient rule:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{x}{\sqrt{x^{2}+1}}\right\} & =\frac{\sqrt{x^{2}+1}-x \frac{2 x}{2 \sqrt{x^{2}+1}}}{x^{2}+1} \\
& =\left(\frac{\sqrt{x^{2}+1}-\frac{x^{2}}{\sqrt{x^{2}+1}}}{x^{2}+1}\right) \cdot \frac{\sqrt{x^{2}+1}}{\sqrt{x^{2}+1}} \\
& =\frac{\left(x^{2}+1\right)-x^{2}}{\left(x^{2}+1\right)^{3 / 2}} \\
& =\frac{1}{\left(x^{2}+1\right)^{3 / 2}}=\left(x^{2}+1\right)^{-3 / 2}
\end{aligned}
$$

2.12.2.17. Answer. Let $\theta=\arcsin x$. Then $\theta$ is the angle of a right triangle that gives $\sin \theta=x$. In particular, the ratio of the opposite side to the hypotenuse is $x$. So, we have a triangle that looks like this:

where the length of the adjacent side came from the Pythagorean Theorem. Now,

$$
\cot (\arcsin x)=\cot \theta=\frac{\mathrm{adj}}{\mathrm{opp}}=\frac{\sqrt{1-x^{2}}}{x}
$$

From here, we differentiate using the quotient rule:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{\sqrt{1-x^{2}}}{x}\right\} & =\frac{x \frac{-2 x}{2 \sqrt{1-x^{2}}}-\sqrt{1-x^{2}}}{x^{2}} \\
& =\frac{-x^{2}-\left(1-x^{2}\right)}{x^{2} \sqrt{1-x^{2}}} \\
& =\frac{-1}{x^{2} \sqrt{1-x^{2}}}
\end{aligned}
$$

2.12.2.18. *. Answer. $(x, y)= \pm\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$
2.12.2.19. Answer. $x=\frac{(2 n+1) \pi}{2}$ for any integer $n$

## Exercises - Stage 3

2.12.2.20. *. Answer. $\quad g^{\prime}(y)=\frac{1}{1-\sin g(y)}$
2.12.2.21. *. Answer. $\frac{1}{2}$
2.12.2.22. *. Answer. $\frac{1}{e+1}$
2.12.2.23. Answer. $f^{\prime}(x)=[\sin x+2]^{\operatorname{arcsec} x}\left(\frac{\log [\sin x+2]}{|x| \sqrt{x^{2}-1}}+\frac{\operatorname{arcsec} x \cdot \cos x}{\sin x+2}\right)$. The domain of $f(x)$ is $|x| \geq 1$.
2.12.2.24. Answer. The function $\frac{1}{\sqrt{x^{2}-1}}$ exists only for those values of $x$ with $x^{2}-1>0$ : that is, the domain of $\frac{1}{\sqrt{x^{2}-1}}$ is $|x|>1$. However, the domain of arcsine is $|x| \leq 1$. So, there is not one single value of $x$ where $\arcsin x$ and $\frac{1}{\sqrt{x^{2}-1}}$
are both defined.
If the derivative of $\arcsin (x)$ were given by $\frac{1}{\sqrt{x^{2}-1}}$, then the derivative of $\arcsin (x)$ would not exist anywhere, so we would probably just write "derivative does not exist," instead of making up a function with a mismatched domain. Also, the function $f(x)=\arcsin (x)$ is a smooth curve-its derivative exists at every point strictly inside its domain. (Remember not all curves are like this: for instance, $g(x)=|x|$ does not have a derivative at $x=0$, but $x=0$ is strictly inside its domain.) So, it's a pretty good bet that the derivative of arcsine is not $\frac{1}{\sqrt{x^{2}-1}}$.
2.12.2.25. Answer. $\frac{1}{2}$
2.12.2.26. Answer. $f^{-1}(7)=-\frac{25}{4}$
2.12.2.27. Answer. $f(0)=-7$
2.12.2.28. Answer. $y^{\prime}=\frac{2 x \sqrt{1-(x+2 y)^{2}}-1}{2-2 y \sqrt{1-(x+2 y)^{2}}}$, or equivalently, $y^{\prime}=$ $\frac{2 x \cos \left(x^{2}+y^{2}\right)-1}{2-2 y \cos \left(x^{2}+y^{2}\right)}$

### 2.13 - The Mean Value Theorem <br> 2.13.5 • Exercises

## Exercises - Stage 1

2.13.5.1. Answer. The caribou spent at least about 71 and a half hours travelling during its migration (probably much more) in one year.
2.13.5.2. Answer. At some point during the day, the crane was travelling at exactly 10 kph .
2.13.5.3. Answer. One possible answer:


Another possible answer:

2.13.5.4. Answer. One possible answer: $f(x)= \begin{cases}0 & x \neq 10 \\ 10 & x=10\end{cases}$

Another answer: $f(x)= \begin{cases}10 & x \neq 0 \\ 0 & x=0\end{cases}$
Yet another answer: $f(x)= \begin{cases}5 & x \neq 0,10 \\ 10 & x=10 \\ 0 & x=0\end{cases}$
2.13.5.5. Answer. (a) No such function is possible: Rolle's Theorem guarantees
$f^{\prime}(c)=0$ for at least one point $c \in(1,2)$.
For the other functions, examples are below, but many answers are possible.
(b)

(c)

(d)

2.13.5.6. Answer. The function $f(x)$ is continuous over all real numbers, but it is only differentiable when $x \neq 0$. So, if we want to apply the MVT, our interval must consist of only positive numbers or only negative numbers: the interval $(-4,13)$ is not valid.
It is possible to use the mean value theorem to prove what we want: if $a=1$ and $b=144$, then $f(x)$ is differentiable over the interval $(1,144)$ (since 0 is not contained in that interval), and $f(x)$ is continuous everywhere, so by the mean value theorem there exists some point $c$ where $f^{\prime}(x)=\frac{\sqrt{|144|}-\sqrt{|1|}}{144-1}=\frac{11}{143}=\frac{1}{13}$.
That being said, an easier way to prove that a point exists is to simply find it-
without using the MVT. When $x>0, f(x)=\sqrt{x}$, so $f^{\prime}(x)=\frac{1}{2 \sqrt{x}}$. Then $f^{\prime}\left(\frac{169}{4}\right)=\frac{1}{13}$.

## Exercises - Stage 2

2.13.5.7. *. Answer. We note that $f(0)=f(2 \pi)=0$. Then using the Mean Value Theorem (note that the function is differentiable for all real numbers), we conclude that there exists $c$ in $(0,2 \pi)$ such that

$$
f^{\prime}(c)=\frac{f(2 \pi)-f(0)}{2 \pi-0}=0
$$

2.13.5.8. *. Answer. We note that $f(0)=f(1)=0$. Then using the Mean Value Theorem (note that the function is differentiable for all real numbers), we get that there exists $c \in(0,1)$ such that

$$
f^{\prime}(c)=\frac{f(1)-f(0)}{1-0}=0
$$

2.13.5.9. *. Answer. We note that $f(0)=f(2 \pi)=\sqrt{3}+\pi^{2}$. Then using the Mean Value Theorem (note that the function is differentiable for all real numbers since $3+\sin x>0)$, we get that there exists $c \in(0,2 \pi)$ such that

$$
f^{\prime}(c)=\frac{f(2 \pi)-f(0)}{2 \pi-0}=0
$$

2.13.5.10. *. Answer. We note that $f(0)=0$ and $f(\pi / 4)=0$. Then using the Mean Value Theorem (note that the function is differentiable for all real numbers), we get that there exists $c \in(0, \pi / 4)$ such that

$$
f^{\prime}(c)=\frac{f(\pi / 4)-f(0)}{\pi / 4-0}=0
$$

2.13.5.11. Answer. 1
2.13.5.12. Answer. 2
2.13.5.13. Answer. 1
2.13.5.14. Answer. 1
2.13.5.15. *. Answer. 2.13.5.15.a

$$
f^{\prime}(x)=15 x^{4}-30 x^{2}+15=15\left(x^{4}-2 x^{2}+1\right)=15\left(x^{2}-1\right)^{2} \geq 0
$$

The derivative is nonnegative everywhere. The only values of $x$ for which $f^{\prime}(x)=0$ are 1 and -1 , so $f^{\prime}(x)>0$ for every $x$ in $(-1,1)$.
2.13.5.15.b If $f(x)$ has two roots $a$ and $b$ in $[-1,1]$, then by Rolle's Theorem, $f^{\prime}(c)=$ 0 for some $x$ strictly between $a$ and $b$. But since $a$ and $b$ are in $[-1,1]$, and $c$ is between $a$ and $b$, that means $c$ is in $(-1,1)$; however, we know for every $c$ in $(-1,1)$, $f^{\prime}(c)>0$, so this can't happen. Therefore, $f(x)$ does not have two roots $a$ and $b$ in $[-1,1]$. This means $f(x)$ has at most one root in $[-1,1]$.
2.13.5.16. *. Answer. $\log \left(\frac{e^{T}-1}{T}\right)$
2.13.5.17. Answer. See the solution for the $\operatorname{argument}$ that $\operatorname{arcsec} x=C-$ $\operatorname{arccsc} x$ for some constant $C$.
The constant $C=\frac{\pi}{2}$.

## Exercises - Stage 3

2.13.5.18. *. Answer. Since $e^{-f(x)}$ is always positive (regardless of the value of $f(x)$ ),

$$
f^{\prime}(x)=\frac{1}{1+e^{-f(x)}}<\frac{1}{1+0}=1
$$

for every $x$.
Since $f^{\prime}(x)$ exists for every $x$, we see that $f$ is differentiable, so the Mean Value Theorem applies. If $f(100)$ is greater than or equal to 100 , then by the Mean Value Theorem, there would have to be some $c$ between 0 and 100 such that

$$
f^{\prime}(c)=\frac{f(100)-f(0)}{100} \geq \frac{100}{100}=1
$$

Since $f^{\prime}(x) \leq 1$ for every $x$, there is no value of $c$ as described. Therefore, it is not possible that $f(100) \geq 100$. So, $f(100)<100$.
2.13.5.19. Answer. Domain of $f^{-1}(x):(-\infty, \infty)$ Interval where $f$ is one-to-one, and range of $f^{-1}(x):(-\infty, \infty)$
2.13.5.20. Answer. One-to-one interval, and range of $f^{-1}:\left[-\frac{2 \pi}{3}, \frac{2 \pi}{3}\right]$ Domain of $f^{-1}$ : $\left[-\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right),\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)\right]$
2.13.5.21. Answer. Define $h(x)=f(x)-g(x)$, and notice $h(a)=f(a)-g(a)<0$ and $h(b)=f(b)-g(b)>0$. Since $h$ is the difference of two functions that are continuous over $[a, b]$ and differentiable over $(a, b)$, also $h$ is continuous over $[a, b]$ and differentiable over $(a, b)$. So, by the Mean Value Theorem, there exists some $c \in(a, b)$ with

$$
h^{\prime}(c)=\frac{h(b)-h(a)}{b-a}
$$

Since $(a, b)$ is an interval, $b>a$, so the denominator of the above expression is positive; since $h(b)>0>h(a)$, also the numerator of the above expression is positive. So, $h^{\prime}(c)>0$ for some $c \in(a, b)$. Since $h^{\prime}(c)=f^{\prime}(c)-g^{\prime}(c)$, we conclude $f^{\prime}(c)>g^{\prime}(c)$ for some $c \in(a, b)$.
2.13.5.22. Answer. 3
2.13.5.23. Answer. 2

### 2.14 • Higher Order Derivatives

### 2.14.2 • Exercises

## Exercises - Stage 1

2.14.2.1. Answer. $e^{x}$
2.14.2.2. Answer. 2.14.2.2.ii, 2.14.2.2.iv
2.14.2.3. Answer. $\frac{3}{15!}$
2.14.2.4. Answer. The derivative $\frac{\mathrm{d} y}{\mathrm{~d} x}$ is $\frac{11}{4}$ only at the point $(1,3)$ : it is not constantly $\frac{11}{4}$, so it is wrong to differentiate the constant $\frac{11}{4}$ to find $\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}$. Below is a correct solution.

$$
-28 x+2 y+2 x y^{\prime}+2 y y^{\prime}=0
$$

Plugging in $x=1, y=3$ :

$$
\begin{aligned}
-28+6+2 y^{\prime}+6 y^{\prime} & =0 \\
y^{\prime} & =\frac{11}{4} \quad \text { at the point }(1,3)
\end{aligned}
$$

Differentiating the equation $-28 x+2 y+2 x y^{\prime}+2 y y^{\prime}=0$ :

$$
\begin{aligned}
-28+2 y^{\prime}+2 y^{\prime}+2 x y^{\prime \prime}+2 y^{\prime} y^{\prime}+2 y y^{\prime \prime} & =0 \\
4 y^{\prime}+2\left(y^{\prime}\right)^{2}+2 x y^{\prime \prime}+2 y y^{\prime \prime} & =28
\end{aligned}
$$

At the point $(1,3), y^{\prime}=\frac{11}{4}$. Plugging in:

$$
\begin{aligned}
4\left(\frac{11}{4}\right)+2\left(\frac{11}{4}\right)^{2}+2(1) y^{\prime \prime}+2(3) y^{\prime \prime} & =28 \\
y^{\prime \prime} & =\frac{15}{64}
\end{aligned}
$$

## Exercises - Stage 2

2.14.2.5. Answer. $f^{\prime \prime}(x)=\frac{1}{x}$
2.14.2.6. Answer. $\frac{\mathrm{d}^{2}}{\mathrm{~d} x^{2}}\{\arctan x\}=\frac{-2 x}{\left(1+x^{2}\right)^{2}}$
2.14.2.7. Answer. $\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}=\frac{-1}{y^{3}}$
2.14.2.8. Answer. 0
2.14.2.9. Answer. $\frac{\mathrm{d}^{3}}{\mathrm{~d} x^{3}}\left\{\log \left(5 x^{2}-12\right)\right\}=\frac{100 x\left(5 x^{2}+36\right)}{\left(5 x^{2}-12\right)^{3}}$
2.14.2.10. Answer. speeding up
2.14.2.11. Answer. slower
2.14.2.12. Answer. -4
2.14.2.13. Answer. (a) true
(b) true
(c) false

## Exercises - Stage 3

2.14.2.14. Answer. (ii)
2.14.2.15. Answer. $f^{(n)}=2^{x}(\log 2)^{n}$
2.14.2.16. Answer. $n=4$
2.14.2.17. *. Answer.

- 2.14.2.17.a $f^{\prime}(x)=(1+2 x) e^{x+x^{2}} f^{\prime \prime}(x)=\left(4 x^{2}+4 x+3\right) e^{x+x^{2}} h^{\prime}(x)=1+3 x$ $h^{\prime \prime}(x)=3$
- 2.14.2.17.b $f(0)=h(0)=1 ; f^{\prime}(0)=h^{\prime}(0)=1 ; f^{\prime \prime}(0)=h^{\prime \prime}(0)=3$
- 2.14.2.17.c $f$ and $h$ "start at the same place", since $f(0)=h(0)$. Also $f^{\prime}(0)=$ $h^{\prime}(0)$, and $f^{\prime \prime}(x)=\left(4 x^{2}+4 x+3\right) e^{x+x^{2}}>3 e^{x+x^{2}}>3=h^{\prime \prime}(x)$ when $x>0$. Since $f^{\prime}(0)=h^{\prime}(0)$, and since $f^{\prime}$ grows faster than $h^{\prime}$ for positive $x$, we conclude $f^{\prime}(x)>h^{\prime}(x)$ for all positive $x$. Now we can conclude that (since $f(0)=h(0)$ and $f$ grows faster than $h$ when $x>0)$ also $f(x)>h(x)$ for all positive $x$.
2.14.2.18. *. Answer.
a $y^{\prime}(1)=\frac{4}{13}$
b
(


### 2.14.2.19. Answer.

- 2.14.2.19.a $g^{\prime \prime}(x)=\left[f(x)+2 f^{\prime}(x)+f^{\prime \prime}(x)\right] e^{x}$
- 2.14.2.19.b $g^{\prime \prime \prime}(x)=\left[f(x)+3 f^{\prime}(x)+3 f^{\prime \prime}(x)+f^{\prime \prime \prime}(x)\right] e^{x}$
- 2.14.2.19.c $g^{(4)}(x)=\left[f(x)+4 f^{\prime}(x)+6 f^{\prime \prime}(x)+4 f^{\prime \prime \prime}(x)+f^{(4)}(x)\right] e^{x}$
2.14.2.20. Answer. $m+n$
2.14.2.21. Answer. 2
2.14.2.22. *. Answer. 2.14.2.22.a In order to make $f(x)$ a little more tractable, let's change the format. Since $|x|=\left\{\begin{array}{rl}x & x \geq 0 \\ -x & x<0\end{array}\right.$, then:

$$
f(x)=\left\{\begin{array}{rl}
-x^{2} & x<0 \\
x^{2} & x \geq 0
\end{array}\right.
$$

Now, we turn to the definition of the derivative to figure out whether $f^{\prime}(0)$ exists.

$$
f^{\prime}(0)=\lim _{h \rightarrow 0} \frac{f(0+h)-f(0)}{h}=\lim _{h \rightarrow 0} \frac{f(h)-0}{h}=\lim _{h \rightarrow 0} \frac{f(h)}{h} \quad \text { if it exists. }
$$

Since $f$ looks different to the left and right of 0 , in order to evaluate this limit, we look at the corresponding one-sided limits. Note that when $h$ approaches 0 from the right, $h>0$ so $f(h)=h^{2}$. By contrast, when $h$ approaches 0 from the left, $h<0$ so $f(h)=-h^{2}$.

$$
\begin{aligned}
\lim _{h \rightarrow 0^{+}} \frac{f(h)}{h} & =\lim _{h \rightarrow 0^{+}} \frac{h^{2}}{h}=\lim _{h \rightarrow 0^{+}} h=0 \\
\lim _{h \rightarrow 0^{-}} \frac{f(h)}{h} & =\lim _{h \rightarrow 0^{-}} \frac{-h^{2}}{h}=\lim _{h \rightarrow 0^{-}}-h=0
\end{aligned}
$$

Since both one-sided limits exist and are equal to 0 ,

$$
\lim _{h \rightarrow 0} \frac{f(0+h)-f(0)}{h}=0
$$

and so $f$ is differentiable at $x=0$ and $f^{\prime}(0)=0$.
2.14.2.22.b From 2.14.2.22.a, $f^{\prime}(0)=0$ and

$$
f(x)=\left\{\begin{array}{rl}
-x^{2} & x<0 \\
x^{2} & x \geq 0
\end{array}\right.
$$

So,

$$
f^{\prime}(x)=\left\{\begin{array}{rl}
-2 x & x<0 \\
2 x & x \geq 0
\end{array}\right.
$$

Then, we know the second derivative of $f$ everywhere except at $x=0$ :

$$
f^{\prime \prime}(x)=\left\{\begin{array}{cc}
-2 & x<0 \\
? ? & x=0 \\
2 & x>0
\end{array}\right.
$$

So, whenever $x \neq 0, f^{\prime \prime}(x)$ exists. To investigate the differentiability of $f^{\prime}(x)$ when $x=0$, again we turn to the definition of a derivative. If

$$
\lim _{h \rightarrow 0} \frac{f^{\prime}(0+h)-f^{\prime}(0)}{h}
$$

exists, then $f^{\prime \prime}(0)$ exists.

$$
\lim _{h \rightarrow 0} \frac{f^{\prime}(0+h)-f^{\prime}(0)}{h}=\lim _{h \rightarrow 0} \frac{f^{\prime}(h)-0}{h}=\lim _{h \rightarrow 0} \frac{f^{\prime}(h)}{h}
$$

Since $f(h)$ behaves differently when $h$ is greater than or less than zero, we look at the one-sided limits.

$$
\begin{aligned}
\lim _{h \rightarrow 0^{+}} \frac{f^{\prime}(h)}{h} & =\lim _{h \rightarrow 0^{+}} \frac{2 h}{h}=2 \\
\lim _{h \rightarrow 0^{-}} \frac{f^{\prime}(h)}{h} & =\lim _{h \rightarrow 0^{-}} \frac{-2 h}{h}=-2
\end{aligned}
$$

Since the one-sided limits do not agree,

$$
\lim _{h \rightarrow 0} \frac{f^{\prime}(0+h)-f^{\prime}(0)}{h}=D N E
$$

So, $f^{\prime \prime}(0)$ does not exist. Now we have a complete picture of $f^{\prime \prime}(x)$ :

$$
f^{\prime \prime}(x)= \begin{cases}-2 & x<0 \\ D N E & x=0 \\ 2 & x>0\end{cases}
$$

## 3 - Applications of derivatives

## 3.1 . Velocity and Acceleration

### 3.1.2 • Exercises

## Exercises - Stage 1

3.1.2.1. Answer. False (but its velocity is zero)
3.1.2.2. Answer. It takes 10 seconds to accelerate from $2 \frac{\mathrm{~m}}{\mathrm{~s}}$ to $3 \frac{\mathrm{~m}}{\mathrm{~s}}$, and 100 seconds to accelerate from $3 \frac{\mathrm{~m}}{\mathrm{~s}}$ to $13 \frac{\mathrm{~m}}{\mathrm{~s}}$.
3.1.2.3. Answer. In general, false.
3.1.2.4. Answer. True

## Exercises - Stage 2

3.1.2.5. Answer. The pot is falling at 14 metres per second, just as it hits the ground.
3.1.2.6. Answer. (a) $4.9 x^{2}$ metres
(b) $4.9 x^{2}+x$ metres
3.1.2.7. Answer. $8-4 \sqrt{3} \approx 1 \mathrm{sec}$
3.1.2.8. Answer. 7.2 sec
3.1.2.9. Answer. 72000 kph per hour
3.1.2.10. Answer. $\frac{100}{7} \approx 14 \mathrm{kph}$
3.1.2.11. Answer. about 1240 miles
3.1.2.12. Answer. 49 metres per second
3.1.2.13. Answer. About 416 metres
3.1.2.14. Answer. $v_{0}=\sqrt{1960} \approx 44$ metres per second
3.1.2.15. Answer. $\approx 74.2 \mathrm{kph}$

## Exercises - Stage 3

3.1.2.16. Answer. Time elapsed: $\frac{1}{4.9}+\frac{1}{9.8} \approx 0.3$ seconds
3.1.2.17. Answer. The acceleration is given by $2^{t} v_{0} \log 2$, where $v_{0}$ is the velocity of the object at time $t=0$.

## 3.2 - Related Rates

### 3.2.2 • Exercises

## Exercises - Stage 1

3.2.2.1. Answer. ii and iv

## Exercises - Stage 2

3.2.2.2. *. Answer. $-\frac{3}{2}$
3.2.2.3. *. Answer. $6 \%$
3.2.2.4. *. Answer. 3.2.2.4.a 0
3.2.2.4.b $100 \frac{F^{\prime}}{F}=15 \%$, or $F^{\prime}=0.15 F$
3.2.2.5. *. Answer. $-\frac{17}{5}$ units per second
3.2.2.6. *. Answer. $\frac{4}{5}$ units per second
3.2.2.7. *. Answer. increasing at 7 mph
3.2.2.8. *. Answer. 8 cm per minute
3.2.2.9. *. Answer. $-\frac{13}{6}$ metres per second
3.2.2.10. Answer. The height of the water is decreasing at $\frac{3}{16}=0.1875 \frac{\mathrm{~cm}}{\mathrm{~min}}$.
3.2.2.11. Answer. $\frac{1}{29200}$ metres per second (or about 1 centimetre every five minutes)
3.2.2.12. Answer. $\left(\frac{2}{\left(\frac{1235}{72}\right)^{2}+4}\right)\left(\frac{6175}{3}\right) \approx 13.8 \frac{\mathrm{rad}}{\mathrm{hour}} \approx 0.0038 \frac{\mathrm{rad}}{\mathrm{sec}}$
3.2.2.13. *. Answer. 3.2 .2 .13 .a $\frac{24}{13} \approx 1.85 \mathrm{~km} / \mathrm{min}$
3.2.2.13.b about .592 radians $/ \mathrm{min}$
3.2.2.14. Answer. $\frac{55 \sqrt{21} \pi}{42} \approx 19$ centimetres per hour.
3.2.2.15. *. Answer. $\frac{\mathrm{d} A}{\mathrm{~d} t}=-2 \pi \frac{\mathrm{~cm}^{2}}{\mathrm{~s}}$
3.2.2.16. Answer. $288 \pi$ cubic units per unit time
3.2.2.17. Answer. 0 square centimetres per minute
3.2.2.18. Answer. $-\frac{7 \pi}{12} \approx-1.8 \frac{\mathrm{~cm}^{3}}{\sec ^{2}}$
3.2.2.19. Answer. The flow is decreasing at a rate of $\frac{\sqrt{7}}{1000} \frac{\mathrm{~m}^{3}}{\mathrm{sec}^{2}}$.
3.2.2.20. Answer. $\frac{-15}{49 \pi} \approx-0.097 \mathrm{~cm}$ per minute

## Exercises - Stage 3

3.2.2.21. Answer.
a $\frac{\mathrm{d} D}{\mathrm{~d} t}=\frac{1}{2 \sqrt{2}}$ metres per hour
b The river is higher than 2 metres.
c The river's flow has reversed direction. (This can happen near an ocean at high tide.)
3.2.2.22. Answer. (a) 2 units per second
(b) Its $y$-coordinate is decreasing at $\frac{1}{2}$ unit per second. The point is moving at $\frac{\sqrt{5}}{2}$ units per second.
3.2.2.23. Answer. (a) $10 \pi=\pi[3(a+b)-\sqrt{(a+3 b)(3 a+b)}]$ or equivalently, $10=3(a+b)-\sqrt{(a+3 b)(3 a+b)}$
(b) $20 \pi a b$
(c) The water is spilling out at about 375.4 cubic centimetres per second. The exact amount is $-\frac{200 \pi}{9-\sqrt{35}}\left(1-2\left(\frac{3 \sqrt{35}-11}{3 \sqrt{35}-13}\right)\right) \frac{\mathrm{cm}^{3}}{\mathrm{sec}}$.
3.2.2.24. Answer. $B(10)=0$

## 3.3 - Exponential Growth and Decay - a First Look at Differential Equations

### 3.3.4 • Exercises <br> - Exercises for § 3.3.1

## Exercises - Stage 1

3.3.4.1. Answer. (a), (b)
3.3.4.2. Answer. (a), (d)
3.3.4.3. Answer. If $C=0$, then there was none to start out with, and $Q(t)=0$ for all values of $t$.
If $C \neq 0$, then $Q(t)$ will never be 0 (but as $t$ gets bigger and bigger, $Q(t)$ gets closer and closer to 0 ).

## Exercises - Stage 2

3.3.4.4. *. Answer. $A=5, k=\frac{1}{7} \cdot \log (\pi / 5)$
3.3.4.5. *. Answer. $y(t)=2 e^{-3(t-1)}$, or equivalently, $y(t)=2 e^{3} e^{-3 t}$
3.3.4.6. Answer. $5 \cdot 2^{-\frac{10000}{5730}} \approx 1.5 \mu g$
3.3.4.7. Answer. Radium- 226 has a half life of about 1600 years.
3.3.4.8. *. Answer. $\frac{\log 2}{\log 6}=\log _{6}(2)$ years, which is about 139 days
3.3.4.9. Answer. $120 \cdot \frac{\log 10}{\log 2}$ seconds, or about six and a half minutes.

## Exercises - Stage 3

3.3.4.10. Answer. About $0.5 \%$ of the sample decays in a day. The exact amount is $\left[100\left(1-2^{-\frac{1}{138}}\right)\right] \%$.
3.3.4.11. Answer. After ten years, the sample contains between 6.2 and $6.8 \mu \mathrm{~g}$ of Uranium-232.

## - Exercises for § 3.3.2

## Exercises - Stage 1

3.3.4.1. Answer. (a), (c), (d)
3.3.4.2. Answer. The temperature of the room is -10 degrees, and the room is colder than the object.
3.3.4.3. Answer. $K$ is a negative number. It cannot be positive or zero.
3.3.4.4. Answer. If the object has a different initial temperature than its surroundings, then $T(t)$ is never equal to $A$. (But as time goes on, it gets closer and closer.)
If the object starts out with the same temperature as its surrounding, then $T(t)=A$ for all values of $t$.

## Exercises - Stage 2

3.3.4.5. Answer. $\frac{-10 \log (750)}{\log \left(\frac{2}{15}\right)} \approx 32.9$ seconds
3.3.4.6. Answer. $10 \frac{\log (10)}{\log (5)} \approx 14.3$ minutes

## Exercises - Stage 3

3.3.4.7. *. Answer. If Newton adds his cream just before drinking, the coffee ends up $\left\{\right.$ cooler by $\left.0.85^{\circ} \mathrm{C}\right\}$.
3.3.4.8. *. Answer.
a $\frac{\mathrm{d} T}{\mathrm{~d} t}=\frac{1}{5} \log \left(\frac{4}{5}\right)(T-30)$
b $\frac{5 \log (2 / 5)}{\log (4 / 5)} \approx 20.53 \mathrm{~min}$
3.3.4.9. Answer. positive

## - Exercises for § 3.3.3

## Exercises - Stage 1

3.3.4.1. Answer. If $P(0)=0$, yes. If $P(0) \neq 0$, no: it does not take into account external constraints on population growth.

## Exercises - Stage 2

3.3.4.2. Answer. The Malthusian model predicts the herd will number 217 individuals in 2020.
3.3.4.3. Answer. $\frac{\log (3)}{\log (2)} \approx 1.6$ hours
3.3.4.4. Answer. 1912 or 1913
3.3.4.5. Answer. $\frac{10^{6}}{5^{4}-1} \approx 1603$

## Exercises - Stage 3

3.3.4.6. Answer. (a) At $t=0$, there are 100 units of the radioactive isotope in the sample. $k$ is negative.
(b) At $t=0$, there are 100 individuals in the population. $k$ is positive.
(c) The ambient temperature is 0 degrees. $k$ is negative.

## - Further problems for § 3.3

3.3.4.1. *. Answer. $f(2)=2 e^{2 \pi}$
3.3.4.2. Answer. Solutions to the differential equation have the form

$$
T(t)=\left[T(0)+\frac{9}{7}\right] e^{7 t}-\frac{9}{7}
$$

for some constant $T(0)$.
3.3.4.3. *. Answer. $\frac{8 \log (0.4)}{\log (0.8)} \approx 32.85$ days
3.3.4.4. Answer. $25^{\circ} \mathrm{C}$

### 3.3.4.5. *. Answer.

a $A(t)=90,000 \cdot e^{0.05 t}-40,000$. When the graduate is 65 , they will have $\$ 625,015.05$ in the account.
b $\$ 49,437.96$
3.3.4.6. *. Answer. 3.3.4.6.a $A(t)=150,000-30,000 e^{0.06 t}$
3.3.4.6.b after $\{26.8 \mathrm{yrs}\}$
3.3.4.7. *. Answer. $\frac{9 \log 2}{\log 3} \approx 5.68 \mathrm{hr}$
3.3.4.8. *. Answer. (a) $v(t)=\left[v_{0}+\frac{g}{k}\right] e^{-k t}-\frac{g}{k}$
(b) $\lim _{t \rightarrow \infty} v(t)=-\frac{g}{k}$

## 3.4 . Approximating Functions Near a Specified Point - Taylor Polynomials

### 3.4.11 • Exercises

- Exercises for § 3.4.1


## Exercises - Stage 1

3.4.11.1. Answer. Since $f(0)$ is closer to $g(0)$ than it is to $h(0)$, you would probably want to estimate $f(0) \approx g(0)=1+2 \sin (1)$ if you had the means to efficiently figure out what $\sin (1)$ is, and if you were concerned with accuracy. If you had a calculator, you could use this estimation. Also, later in this chapter we will learn methods of approximating $\sin (1)$ that do not require a calculator, but they do require time.
Without a calculator, or without a lot of time, using $f(0) \approx h(0)=0.7$ probably
makes the most sense. It isn't as accurate as $f(0) \approx g(0)$, but you get an estimate very quickly, without worrying about figuring out what $\sin (1)$ is.

## Exercises - Stage 2

3.4.11.2. Answer. $\log (0.93) \approx \log (1)=0$

3.4.11.3. Answer. $\arcsin (0.1) \approx 0$
3.4.11.4. Answer. $\sqrt{3} \tan (1) \approx 3$

## Exercises - Stage 3

3.4.11.5. Answer. $10.1^{3} \approx 10^{3}=1000$

## - Exercises for § 3.4.2

## Exercises - Stage 1

3.4.11.1. Answer. (a) $f(5)=6$
(b) $f^{\prime}(5)=3$
(c) not enough information to know

### 3.4.11.2. Answer.



The linear approximation is shown in red.
3.4.11.3. Answer. $f(x)=2 x+5$
3.4.11.4. Answer. $\log (0.93) \approx-0.07$

3.4.11.5. Answer. $\sqrt{5} \approx \frac{9}{4}$
3.4.11.6. Answer. $\sqrt[5]{30} \approx \frac{79}{40}$

Exercises - Stage 3
3.4.11.7. Answer. $10.1^{3} \approx 1030,10.1^{3}=1030.301$
3.4.11.8. Answer. There are many possible answers. One is $f(x)=\sin x, a=0$, and $b=\pi$.
3.4.11.9. Answer. $a=\sqrt{3}$

## - Exercises for § 3.4.3

## Exercises - Stage 1

3.4.11.1. Answer. $f(3)=9, f^{\prime}(3)=0, f^{\prime \prime}(3)=-2$; there is not enough information to know $f^{\prime \prime \prime}(3)$.
3.4.11.2. Answer. $f(x) \approx 2 x+5$

## Exercises - Stage 2

3.4.11.3. Answer. $\log (0.93) \approx-0.07245$
3.4.11.4. Answer. $\cos \left(\frac{1}{15}\right) \approx \frac{449}{450}$
3.4.11.5. Answer. $e^{2 x} \approx 1+2 x+2 x^{2}$
3.4.11.6. Answer. One approximation: $e^{\frac{4}{3}} \approx \frac{275}{32}$
3.4.11.7. Answer. 3.4.11.7.a 26
3.4.11.7.b 16
3.4.11.7.c $\frac{10}{\frac{11}{75}}$
3.4.11.7.d $\frac{75}{64}$
3.4.11.8. Answer. For each of these, there are many solutions. We provide some below.
a $1+2+3+4+5=\sum_{n=1}^{5} n$
b $2+4+6+8=\sum_{n=1}^{4} 2 n$
c $3+5+7+9+11=\sum_{n=1}^{5}(2 n+1)$
d $9+16+25+36+49=\sum_{n=3}^{7} n^{2}$
e $9+4+16+5+25+6+36+7+49+8=\sum_{n=3}^{7}\left(n^{2}+n+1\right)$
$\mathrm{f} 8+15+24+35+48=\sum_{n=3}^{7}\left(n^{2}-1\right)$
g $3-6+9-12+15-18=\sum_{n=1}^{6}(-1)^{n+1} 3 n$

## Exercises - Stage 3

3.4.11.9. Answer. $f(1) \approx 2, f(1)=\pi$
3.4.11.10. Answer. $e \approx 2.5$
3.4.11.11. Answer.

- [3.4.11.11. $a=3 \cdot 4 \cdot 11.11 . d=3 \cdot 4 \cdot 11.11 . e]$, and
- [3.4.11.11. $\mathrm{b}=3.4 .11 .11 . \mathrm{g}]$, and
- [3.4.11.11. $\mathrm{c}=3 \cdot 4.11 .11 . \mathrm{f}]$


## - Exercises for § 3.4.4

## Exercises - Stage 1

3.4.11.1. Answer. $f^{\prime \prime}(1)=-4$
3.4.11.2. Answer. $f^{(10)}(5)=10$ !

## Exercises - Stage 3

3.4.11.3. Answer. $T_{3}(x)=-x^{3}+x^{2}-x+1$
3.4.11.4. Answer. $T_{3}(x)=-7+7(x-1)+9(x-1)^{2}+5(x-1)^{3}$, or equivalently, $T_{3}(x)=5 x^{3}-6 x^{2}+4 x-10$
3.4.11.5. Answer. $f^{(10)}(5)=\frac{11 \cdot 10!}{6}$
3.4.11.6. Answer. $a=\sqrt{e}$

## - Exercises for § 3.4.5

## Exercises - Stage 1

3.4.11.1. Answer.

$$
\begin{aligned}
T_{16}(x)=1 & +x-\frac{1}{2} x^{2}-\frac{1}{3!} x^{3}+\frac{1}{4!} x^{4}+\frac{1}{5!} x^{5}-\frac{1}{6!} x^{6}-\frac{1}{7!} x^{7}+\frac{1}{8!} x^{8}+\frac{1}{9!} x^{9} \\
& -\frac{1}{10!} x^{10}-\frac{1}{11!} x^{11}+\frac{1}{12!} x^{12}+\frac{1}{13!} x^{13}-\frac{1}{14!} x^{14}-\frac{1}{15!} x^{15} \\
& +\frac{1}{16!} x^{16}
\end{aligned}
$$

3.4.11.2. Answer. $T_{100}(t)=127.5+48(t-5)+4.9(t-5)^{2}=4.9 t^{2}-t+10$
3.4.11.3. Answer. $T_{n}(x)=\sum_{k=0}^{n} \frac{2(\log 2)^{k}}{k!}(x-1)^{k}$
3.4.11.4. Answer.

$$
\begin{aligned}
T_{6}(x)=7 & +5(x-1)+\frac{7}{2}(x-1)^{2}+\frac{1}{3}(x-1)^{3}-\frac{1}{12}(x-1)^{4} \\
& +\frac{1}{30}(x-1)^{5}-\frac{1}{60}(x-1)^{6}
\end{aligned}
$$

3.4.11.5. Answer. $T_{n}(x)=\sum_{k=0}^{n} x^{k}$

## Exercises - Stage 3

3.4.11.6. Answer. $T_{3}(x)=1+(x-1)+(x-1)^{2}+\frac{1}{2}(x-1)^{3}$
3.4.11.7. Answer. $\pi=6 \arctan \left(\frac{1}{\sqrt{3}}\right) \approx \frac{82}{45} \sqrt{3} \approx 3.156$
3.4.11.8. Answer. $T_{100}(x)=-1+\sum_{k=2}^{100} \frac{(-1)^{k}}{k(k-1)}(x-1)^{k}$
3.4.11.9. Answer.

$$
T_{2 n}(x)=\sum_{\ell=0}^{n} \frac{(-1)^{\ell}}{(2 \ell)!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{2 \ell}+\sum_{\ell=0}^{n-1} \frac{(-1)^{\ell}}{(2 \ell+1)!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{2 \ell+1}
$$

### 3.4.11.10. Answer.

$$
1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{157!} \approx e-1
$$

3.4.11.11. Answer. We estimate that the sum is close to $-\frac{1}{\sqrt{2}}$.

## - Exercises for § 3.4.6

## Exercises - Stage 1

3.4.11.1. Answer.

3.4.11.2. Answer. Let $f(x)$ be the number of problems finished after $x$ minutes of work. The question tells us that $5 \Delta y \approx \Delta x$. So, if $\Delta x=15, \Delta y \approx 3$. That is, in 15 minutes more, you will finish about 3 more problems.

## Exercises - Stage 2

3.4.11.3. Answer. (a) $\Delta y \approx \frac{1}{260} \approx 0.003846$
(b) $\Delta y \approx \frac{51}{13520} \approx 0.003772$
3.4.11.4. Answer. (a) $\Delta y \approx 1.1$ metres per second
(b) The increase from the first to the second jump is bigger.

## - Exercises for § 3.4.7

## Exercises - Stage 1

3.4.11.1. Answer. False.
3.4.11.2. Answer. Absolute error: 0.17; percentage error: $2.92 \%$
3.4.11.3. Answer. The linear approximates estimates the error in $f(x)$ to be about 60 , while the quadratic approximates estimates the error in $f(x)$ to be about 63.

## Exercises - Stage 2

3.4.11.4. Answer. $1 \%$
3.4.11.5. Answer. (a) $\frac{9}{2} \theta$
(b) $\theta=2 \arcsin \left(\frac{d}{6}\right)$
(c) $\frac{9}{\sqrt{36-0.68^{2}}} \cdot 0.02 \approx 0.03$
3.4.11.6. Answer. We estimate that the volume decreased by about 0.00245 cubic metres, or about 2450 cubic centimetres.

## Exercises - Stage 3

3.4.11.7. Answer. Correct to within about 10.4 years (or about $53 \%$ )

## - Exercises for § 3.4.8

## Exercises - Stage 1

3.4.11.1. Answer. (a) False
(b) True
(c) True
(d) True
3.4.11.2. Answer. Equation 3.4 .33 gives us the bound $\left|f(2)-T_{3}(2)\right|<6$. A calculator tells us actually $\left|f(2)-T_{3}(2)\right| \approx 1.056$.
3.4.11.3. Answer. $|f(37)-T(37)|=0$
3.4.11.4. Answer. You do, you clever goose!

## Exercises - Stage 2

3.4.11.5. Answer. $\left|f(11.5)-T_{5}(11.5)\right|<\frac{9}{7 \cdot 2^{6}}<0.02$
3.4.11.6. Answer. $\left|f(0.1)-T_{2}(0.1)\right|<\frac{1}{1125}$
3.4.11.7. Answer. $\left|f\left(-\frac{1}{4}\right)-T_{5}\left(-\frac{1}{4}\right)\right|<\frac{1}{6 \cdot 4^{6}}<0.00004$
3.4.11.8. Answer. Your answer may vary. One reasonable answer is
$\left|f(30)-T_{3}(30)\right|<\frac{14}{5^{7} \cdot 9 \cdot 15}<0.000002$.
Another reasonable answer is $\left|f(30)-T_{3}(30)\right|<\frac{14}{5^{7} \cdot 9}<0.00002$.
3.4.11.9. Answer. Equation 3.4 .33 gives the bound $\left|f(0.01)-T_{n}(0.01)\right| \leq$ $100^{2}\left(\frac{100}{\pi}-1\right)^{2}$.
A more reasonable bound on the error is that it is less than 5 .
3.4.11.10. Answer. Using Equation 3.4.33,

$$
\left|f\left(\frac{1}{2}\right)-T_{2}\left(\frac{1}{2}\right)\right|<\frac{1}{10} .
$$

The actual error is

$$
\left|f\left(\frac{1}{2}\right)-T_{2}\left(\frac{1}{2}\right)\right|=\frac{\pi}{6}-\frac{1}{2}
$$

which is about 0.02 .

## Exercises - Stage 3

3.4.11.11. Answer. Any $n$ greater than or equal to 3 .
3.4.11.12. Answer. $\sqrt[7]{2200} \approx 3+\frac{13}{7 \cdot 3^{6}} \approx 3.00255$
3.4.11.13. Answer. If we're going to use Equation 3.4.33, then we'll probably be taking a Taylor polynomial. Using Example 3.4.16, the 6th-degree Maclaurin polynomial for $\sin x$ is

$$
T_{6}(x)=T_{5}(x)=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}
$$

so let's play with this a bit. Equation 3.4.33 tells us that the error will depend on the seventh derivative of $f(x)$, which is $-\cos x$ :

$$
\begin{aligned}
f(1)-T_{6}(1) & =f^{(7)}(c) \frac{1^{7}}{7!} \\
\sin (1)-\left(1-\frac{1}{3!}+\frac{1}{5!}\right) & =\frac{-\cos c}{7!} \\
\sin (1)-\frac{101}{5!} & =\frac{-\cos c}{7!} \\
\sin (1) & =\frac{4242-\cos c}{7!}
\end{aligned}
$$

for some $c$ between 0 and 1 . Since $-1 \leq \cos c \leq 1$,

$$
\begin{aligned}
& \frac{4242-1}{7!} \leq \sin (1) \leq \frac{4242+1}{7!} \\
& \frac{4241}{7!} \leq \sin (1) \leq \frac{4243}{7!} \\
& \frac{4241}{5040} \leq \sin (1) \leq \frac{4243}{5040}
\end{aligned}
$$

Remark: there are lots of ways to play with this idea to get better estimates. One way is to take a higher-degree Maclaurin polynomial. Another is to note that, since $0<c<1<\frac{\pi}{3}$, then $\frac{1}{2}<\cos c<1$, so

$$
\begin{aligned}
& \frac{4242-1}{7!}<\sin (1)<\frac{4242-\frac{1}{2}}{7!} \\
& \frac{4241}{5040}<\sin (1)<\frac{8483}{10080}<\frac{4243}{5040}
\end{aligned}
$$

If you got tighter bounds than asked for in the problem, congratulations!
3.4.11.14. Answer. (a) $T_{4}(x)=\sum_{n=0}^{4} \frac{x^{n}}{n!}$
(b) $T_{4}(1)=\frac{65}{24}$
(c) See the solution.

## - Further problems for $\S 3.4$

## Exercises - Stage 1

3.4.11.1. *. Answer. $f^{\prime}(0)=0$ and $f^{\prime \prime}(0)=6$.
3.4.11.2. *. Answer. 4
3.4.11.3. *. Answer. $\quad h^{\prime}(2)=\frac{1}{2}, h^{\prime \prime}(2)=0$

## Exercises - Stage 2

3.4.11.4. *. Answer. (a) 1.92
(b) 1.918
3.4.11.5. *. Answer. $10^{1 / 3} \approx \frac{13}{6}$; this approximation is too big.
3.4.11.6. *. Answer. $\sqrt{2} \approx \frac{3}{2}$
3.4.11.7. *. Answer. $\sqrt[3]{26} \approx \frac{80}{27}$
3.4.11.8. *. Answer. $(10.1)^{5} \approx 105,000$
3.4.11.9. *. Answer. $\sin \left(\frac{101 \pi}{100}\right) \approx-\frac{\pi}{100}$
3.4.11.10. *. Answer. $\arctan (1.1) \approx\left(\frac{\pi}{4}+\frac{1}{20}\right)$
3.4.11.11. *. Answer. $\frac{8012}{1000}$
3.4.11.12. *. Answer. $(8.06)^{2 / 3} \approx \frac{402}{100}=\frac{201}{50}$
3.4.11.13. *. Answer. $1+x+2 x^{2}+\frac{14}{3} x^{3}$

### 3.4.11.14. *. Answer.

- By Equation 3.4.33, the absolute value of the error is

$$
\left|\frac{f^{\prime \prime \prime}(c)}{3!} \cdot(2-1)^{3}\right|=\left|\frac{c}{6\left(22-c^{2}\right)}\right|
$$

for some $c \in(1,2)$.

- When $1 \leq c \leq 2$, we know that $18 \leq 22-c^{2} \leq 21$, and that numerator and denominator are non-negative, so

$$
\begin{aligned}
\left|\frac{c}{6\left(22-c^{2}\right)}\right| & =\frac{c}{6\left(22-c^{2}\right)} \leq \frac{2}{6\left(22-c^{2}\right)} \leq \frac{2}{6 \cdot 18} \\
& =\frac{1}{54} \leq \frac{1}{50}
\end{aligned}
$$

as required.

- Alternatively, notice that $c$ is an increasing function of $c$, while $22-c^{2}$ is a decreasing function of $c$. Hence the fraction is an increasing function of $c$ and takes its largest value at $c=2$. Hence

$$
\left|\frac{c}{6\left(22-c^{2}\right)}\right| \leq \frac{2}{6 \times 18}=\frac{1}{54} \leq \frac{1}{50}
$$

### 3.4.11.15. *. Answer.

- By Equation 3.4.33, there is $c \in(0,0.5)$ such that the error is

$$
\begin{aligned}
R_{4} & =\frac{f^{(4)}(c)}{4!}(0.5-0)^{4} \\
& =\frac{1}{24 \cdot 16} \cdot \frac{\cos \left(c^{2}\right)}{3-c}
\end{aligned}
$$

- For any $c$ we have $\left|\cos \left(c^{2}\right)\right| \leq 1$, and for $c<0.5$ we have $3-c>2.5$, so that

$$
\left|\frac{\cos \left(c^{2}\right)}{3-c}\right| \leq \frac{1}{2.5}
$$

- We conclude that

$$
\left|R_{4}\right| \leq \frac{1}{2.5 \cdot 24 \cdot 16}=\frac{1}{60 \cdot 16}<\frac{1}{60 \cdot 10}=\frac{1}{600}<\frac{1}{500}
$$

3.4.11.16. *. Answer.

- By Equation 3.4.33, there is $c \in(0,1)$ such that the error is

$$
\left|\frac{f^{\prime \prime \prime}(c)}{3!} \cdot(1-0)^{3}\right|=\left|\frac{e^{-c}}{6\left(8+c^{2}\right)}\right| .
$$

- When $0<c<1$, we know that $1>e^{-c}>e^{-1}$ and $8 \leq 8+c^{2}<9$, so

$$
\begin{aligned}
\left|\frac{e^{-c}}{6\left(8+c^{2}\right)}\right| & =\frac{e^{-c}}{6\left(8+c^{2}\right)} \\
& <\frac{1}{6\left|8+c^{2}\right|} \\
& <\frac{1}{6 \times 8}=\frac{1}{48}<\frac{1}{40}
\end{aligned}
$$

as required.
3.4.11.17. *. Answer. 3.4.11.17.a 2.9259
3.4.11.17.b 2.9241
3.4.11.17.c $\frac{4}{9} 25^{-5 / 3}$

## Exercises - Stage 3

3.4.11.18. Answer. $T_{3}(x)=5 x^{2}-9$
3.4.11.19. *. Answer. (a) 1.05
(b) 1.0483
3.4.11.19.c

3.4.11.20. *. Answer. 3.4.11.20.a 0.9
3.4.11.20.b $\{0.8867\}$
3.4.11.20.c

$$
y=y(x) \approx-1-(x+1)-\frac{4}{3}(x+1)^{2}
$$

3.4.11.21. *. Answer. $\log 10.3 \approx 2.33259$ The error is between -0.00045 and -0.00042 .
3.4.11.22. *. Answer. (a) $L(x)=e+e x$
(b) $Q(x)=e+e x+e x^{2}$
(c) Since $e x^{2}>0$ for all $x>0, L(x)<Q(x)$ for all $x>0$.

From the error formula, we know that

$$
\begin{aligned}
f(x) & =f(0)+f^{\prime}(0) x+\frac{1}{2} f^{\prime \prime}(0) x^{2}+\frac{1}{3!} f^{\prime \prime \prime}(c) x^{3} \\
& =Q(x)+\frac{1}{6}\left(e^{c}+3 e^{2 c}+e^{3 c}\right) e^{e^{c}} x^{3}
\end{aligned}
$$

for some $c$ between 0 and $x$. Since $\frac{1}{6}\left(e^{c}+3 e^{2 c}+e^{3 c}\right) e^{e^{c}}$ is positive for any $c$, for all $x>0, \frac{1}{6}\left(e^{c}+3 e^{2 c}+e^{3 c}\right) e^{e^{c}} x^{3}>0$, so $Q(x)<f(x)$.
(d) $1.105<e^{0.1}<1.115$

## $3.5 \cdot$ Optimisation

### 3.5.4 • Exercises

- Exercises for § 3.5.1


## Exercises - Stage 1 <br> 3.5.4.1. Answer.



There is a critical point at $x=0$. The $x$-value of the red dot is a singular point, and a local maximum occurs there.

### 3.5.4.2. Answer.



The $x$-coordinate corresponding to the blue dot (let's call it $a$ ) is a critical point, and $f(x)$ has a local and global minimum at $x=a$. The $x$-coordinate corresponding to the discontinuity (let's call it $b$ ) is a singular point, but there is not a global or local extremum at $x=b$.
3.5.4.3. Answer. One possible answer is shown below.


## Exercises - Stage 2

3.5.4.4. Answer. The critical points are $x=3$ and $x=-1$. These two points are the only places where local extrema might exist. There are no singular points.

## Exercises - Stage 3

3.5.4.5. Answer.

3.5.4.6. Answer. There are many possible answers. Every answer must have $x=2$ as a singular point strictly inside the domain of $f(x)$. Two possibilities are shown below.

3.5.4.7. Answer. $\quad x=-7, x=-1$, and $x=5$
3.5.4.8. Answer. Every real number $c$ is a critical point of $f(x)$, and $f(x)$ has a local and global maximum and minimum at $x=c$. There are no singular points.

## - Exercises for § 3.5.2

Exercises - Stage 1
3.5.4.1. Answer. Two examples are given below, but many are possible.

3.5.4.2. Answer. Two examples are given below, but many are possible.


3.5.4.3. Answer. One possible answer:


## Exercises - Stage 2

3.5.4.4. Answer. The global maximum is 45 at $x=5$ and the global minimum is -19 at $x=-3$.
3.5.4.5. Answer. The global maximum over the interval is 61 at $x=-3$, and the global minimum is 7 at $x=0$.

## - Exercises for § 3.5.3

## Exercises - Stage 1

3.5.4.1. *. Answer. The global maximum is $f(-1)=6$, the global minimum is $f(-2)=-20$.
3.5.4.2. *. Answer. Global maximum is $f(2)=12$, global minimum is $f(1)=$ -14 .
3.5.4.3. *. Answer. Global maximum is $f(4)=30$, global minimum is $f(2)=$ -10 .
3.5.4.4. *. Answer. Local max at $(-2,20)$, local min at $(2,-12)$.
3.5.4.5. *. Answer. $(-2,33) \max$, and $(2,-31)$ min
3.5.4.6. *. Answer. $Q$ should be $4 \sqrt{3}$ kilometres from $A$
3.5.4.7. *. Answer. $10 \times 30 \times 15$
3.5.4.8. *. Answer. $2 \times 2 \times 6$
3.5.4.9. *. Answer. $X=Y=\sqrt{2}$
3.5.4.10. *. Answer. The largest possible perimeter is $2 \sqrt{5} R$ and the smallest possible perimeter is $2 R$.
3.5.4.11. *. Answer. $\frac{A^{3 / 2}}{3 \sqrt{6 \pi}}$
3.5.4.12. *. Answer. $\frac{P^{2}}{2(\pi+4)}$
3.5.4.13. *. Answer.

$$
\text { (a) } x=\sqrt{\frac{A}{3 p}}, y=\sqrt{\frac{A p}{3}} \text {, and } z=\frac{\sqrt{A p}}{\sqrt{3}(1+p)}
$$

(b) $p=1$
(The dimensions of the resulting baking pan are $x=y=\sqrt{\frac{A}{3}}$ and $z=\frac{1}{2} \sqrt{\frac{A}{3}}$.)

## Exercises - Stage 3

3.5.4.14. *. Answer. 3.5.4.14.a $x^{x}(1+\log x)$
3.5.4.14.b $x=\frac{1}{e} 3 \cdot 5.4 .14$.c local minimum
3.5.4.15. *. Answer. Maximum area: do not cut, make a circle and no square. Minimum area: make a square out of a piece that is $\frac{4}{4+\pi}$ of the total length of the wire.

## 3.6 • Sketching Graphs

### 3.6.7 • Exercises

- Exercises for § 3.6.1


## Exercises - Stage 1

3.6.7.1. Answer. In general, false.

## Exercises - Stage 2

3.6.7.2. Answer.
$f(x)=A(x)$
$g(x)=C(x)$
$h(x)=B(x)$
$k(x)=D(x)$
3.6.7.3. Answer. (a) $p=e^{2}$
(b) $b=-e^{2}, 1-e^{2}$
3.6.7.4. Answer. vertical asymptote at $x=3$; horizontal asymptotes $\lim _{x \rightarrow \pm \infty} f(x)=\frac{2}{3}$
3.6.7.5. Answer. horizontal asymptote $y=0$ as $x \rightarrow-\infty$; no other asymptotes

## - Exercises for § 3.6.2

## Exercises - Stage 1

### 3.6.7.1. Answer.

$$
\begin{array}{lll}
A^{\prime}(x)=l(x) & B^{\prime}(x)=p(x) & C^{\prime}(x)=n(x) \\
D^{\prime}(x)=o(x) & E^{\prime}(x)=m(x) &
\end{array}
$$

Exercises - Stage 2
3.6.7.2. *. Answer. $(-2, \infty)$
3.6.7.3. *. Answer. $(1,4)$
3.6.7.4. *. Answer. $(-\infty, 1)$

## - Exercises for § 3.6.3

## Exercises - Stage 1 <br> 3.6.7.1. Answer.



### 3.6.7.2. Answer.


3.6.7.3. Answer. In general, false.

## Exercises - Stage 2

3.6.7.4. *. Answer. $x=1, y=11$

## Exercises - Stage 3

3.6.7.5. *. Answer. Let

$$
g(x)=f^{\prime \prime}(x)=x^{3}+5 x-20
$$

Then $g^{\prime}(x)=3 x^{2}+5$, which is always positive. That means $g(x)$ is strictly increasing for all $x$. So, $g(x)$ can change signs once, from negative to positive, but it can never change back to negative. An inflection point of $f(x)$ occurs when $g(x)$ changes signs. So, $f(x)$ has at most one inflection point.
Since $g(x)$ is continuous, we can apply the Intermediate Value Theorem to it. Notice $g(3)>0$ while $g(0)<0$. By the IVT, $g(x)=0$ for at least one $x \in(0,3)$. Since $g(x)$ is strictly increasing, at the point where $g(x)=0, g(x)$ changes from negative to positive. So, the concavity of $f(x)$ changes. Therefore, $f(x)$ has at least one inflection point.
Now that we've shown that $f(x)$ has at most one inflection point, and at least one inflection point, we conclude it has exactly one inflection point.
3.6.7.6. *. Answer. 3.6.7.6.a Let

$$
g(x)=f^{\prime}(x)
$$

Then $f^{\prime \prime}(x)$ is the derivative of $g(x)$. Since $f^{\prime \prime}(x)>0$ for all $x, g(x)=f^{\prime}(x)$ is strictly increasing for all $x$. In other words, if $y>x$ then $g(y)>g(x)$.
Suppose $g(x)=0$. Then for every $y$ that is larger than $x, g(y)>g(x)$, so $g(y) \neq 0$. Similarly, for every $y$ that is smaller than $x, g(y)<g(x)$, so $g(y) \neq 0$. Therefore, $g(x)$ can only be zero for at most one value of $x$. Since $g(x)=f^{\prime}(x)$, that means $f(x)$ can have at most one critical point.
Suppose $f^{\prime}(c)=0$. Since $f^{\prime}(x)$ is a strictly increasing function, $f^{\prime}(x)<0$ for all
$x<c$ and $f^{\prime}(x)>0$ for all $x>c$.


Then $f(x)$ is decreasing for $x<c$ and increasing for $x>c$. So $f(x)>f(c)$ for all $x \neq c$.


Since $f(x)>f(c)$ for all $x \neq c$, so $c$ is an absolute minimum for $f(x)$.
3.6.7.6.b We know that the maximum over an interval occurs at an endpoint, a critical point, or a singular point.

- Since $f^{\prime}(x)$ exists everywhere, there are no singular points.
- If the maximum were achieved at a critical point, that critical point would have to provide both the absolute maximum and the absolute minimum (by part (a)). So, the function would have to be a constant and consequently could not have a nonzero second derivative. So the maximum is not at a critical point.

That leaves only the endpoints of the interval.
3.6.7.7. Answer. If $x=3$ is an inflection point, then the concavity of $f(x)$ changes at $x=3$. That is, there is some interval strictly containing 3 , with endpoints $a$ and $b$, such that

- $f^{\prime \prime}(a)<0$ and $f^{\prime \prime}(x)<0$ for every $x$ between $a$ and 3 , and
- $f^{\prime \prime}(b)>0$ and $f^{\prime \prime}(x)>0$ for every $x$ between $b$ and 3 .

Since $f^{\prime \prime}(a)<0$ and $f^{\prime \prime}(b)>0$, and since $f^{\prime \prime}(x)$ is continuous, the Intermediate Value Theorem tells us that there exists some $x$ strictly between $a$ and $b$ with $f^{\prime \prime}(x)=0$. So, we know $f^{\prime \prime}(x)=0$ somewhere between $a$ and $b$. The question is,
where exactly could that be?

- $f^{\prime \prime}(x)<0$ (and hence $f^{\prime \prime}(x) \neq 0$ ) for all $x$ between $a$ and 3
- $f^{\prime \prime}(x)>0$ (and hence $f^{\prime \prime}(x) \neq 0$ ) for all $x$ between $b$ and 3
- So, any number between $a$ and $b$ that is not 3 has $f^{\prime \prime}(x) \neq 0$.

So, $x=3$ is the only possible place between $a$ and $b$ where $f^{\prime \prime}(x)$ could be zero. Therefore, $f^{\prime \prime}(3)=0$.

## - Exercises for § 3.6.4

## Exercises - Stage 1

3.6.7.1. Answer. even
3.6.7.2. Answer. odd, periodic

### 3.6.7.3. Answer.


3.6.7.4. Answer.


Exercises - Stage 2
3.6.7.5. Answer. A function is even if $f(-x)=f(x)$.

$$
\begin{aligned}
f(-x) & =\frac{(-x)^{4}-(-x)^{6}}{e^{(-x)^{2}}} \\
& =\frac{x^{4}-x^{6}}{e^{x^{2}}} \\
& =f(x)
\end{aligned}
$$

So, $f(x)$ is even.
3.6.7.6. Answer. For any real number $x$, we will show that $f(x)=f(x+4 \pi)$.

$$
\begin{aligned}
f(x+4 \pi) & =\sin (x+4 \pi)+\cos \left(\frac{x+4 \pi}{2}\right) \\
& =\sin (x+4 \pi)+\cos \left(\frac{x}{2}+2 \pi\right) \\
& =\sin (x)+\cos \left(\frac{x}{2}\right) \\
& =f(x)
\end{aligned}
$$

So, $f(x)$ is periodic.
3.6.7.7. Answer. even
3.6.7.8. Answer. none
3.6.7.9. Answer. 1

## Exercises - Stage 3

3.6.7.10. Answer. $\pi$

## - Exercises for § 3.6.6

## Exercises - Stage 1

3.6.7.1. *. Answer. 3.6.7.1.a $(-\infty, 3]$
3.6.7.1.b $f(x)$ in increasing on $(-\infty, 2)$ and decreasing on $(2,3)$. There is a local maximum at $x=2$ and a local minimum at the endpoint $x=3$.
3.6.7.1.c $f(x)$ is always concave down and has no inflection points.
3.6.7.1.d (3, 0)

3.6.7.2. *. Answer. The open dot is the inflection point, and the closed dot is the local and global maximum.

3.6.7.3. *. Answer. The open dot marks the inflection point.

3.6.7.4. *. Answer.

3.6.7.5. *. Answer. 3.6.7.5.a One branch of the function, the exponential function $e^{x}$, is continuous everywhere. So $f(x)$ is continuous for $x<0$. When $x \geq 0$, $f(x)=\frac{x^{2}+3}{3(x+1)}$, which is continuous whenever $x \neq-1$ (so it's continuous for all
$x>0)$. So, $f(x)$ is continuous for $x>0$. To see that $f(x)$ is continuous at $x=0$, we see:

$$
\begin{aligned}
\lim _{x \rightarrow 0-} f(x)=\lim _{x \rightarrow 0-} e^{x} & =1 \\
\lim _{x \rightarrow 0+} f(x)=\lim _{x \rightarrow 0+} \frac{x^{2}+3}{3(x+1)} & =1 \\
\text { So, } \lim _{x \rightarrow 0} f(x) & =1=f(0)
\end{aligned}
$$

Hence $f(x)$ is continuous at $x=0$, so $f(x)$ is continuous everywhere.
3.6.7.5.b

- i. $f(x)$ is increasing for $x<0$ and $x>1$, decreasing for $0<x<1$, has a local max at $(0,1)$, and has a local min at $\left(1, \frac{2}{3}\right)$.
- ii. $f(x)$ is concave upwards for all $x \neq 0$.
- iii. The $x$-axis is a horizontal asymptote as $x \rightarrow-\infty$.
3.6.7.5.c



### 3.6.7.6. *. Answer.


3.6.7.7. *. Answer. 3.6.7.7.a

- Increasing: $(-1,1)$, decreasing: $(-\infty,-1) \cup(1, \infty)$
- concave up: $(-\sqrt{3}, 0) \cup(\sqrt{3}, \infty)$, concave down: $(-\infty,-\sqrt{3}) \cup(0, \sqrt{3})$
- inflection points: $x= \pm \sqrt{3}, 0$
3.6.7.7.b The local and global minimum of $f(x)$ is at $\left(-1, \frac{-1}{\sqrt{e}}\right)$, and the local and global maximum of $f(x)$ is at $\left(1, \frac{1}{\sqrt{e}}\right)$.
3.6.7.7.c In the graph below, open dots are inflection points, and solid dots are extrema.

3.6.7.8. Answer. Local maxima occur at $x=\frac{2 \pi}{3}+2 \pi n$ for all integers $n$, and local minima occur at $x=-\frac{2 \pi}{3}+2 \pi n$ for all integers $n$. Inflection points occur at every integer multiple of $\pi$.

3.6.7.9. *. Answer. Below is the graph $y=f(x)$ over the interval $[-\pi, \pi]$. The sketch of the curve over a larger domain is simply a repetition of this figure.


On the interval $[0, \pi]$, the maximum value of $f(x)$ is 6 and the minimum value is -2 .
Let $a=\arcsin \left(\frac{-1+\sqrt{33}}{8}\right) \approx 0.635 \approx 0.2 \pi$ and $b=\arcsin \left(\frac{-1-\sqrt{33}}{8}\right) \approx$ $-1.003 \approx-0.3 \pi$. The points $-\pi-b, b, a$, and $\pi-a$ are inflection points.
3.6.7.10. Answer. The closed dot is the local minimum, and the open dots are inflection points at $x=-1$ and $x=-2 \pm \sqrt{1.5}$. The graph has horizontal asymptotes $y=0$ as $x$ goes to $\pm \infty$.


## Exercises - Stage 3

3.6.7.11. *. Answer.

- 3.6.7.11.a decreasing for $x<0$ and $x>2$, increasing for $0<x<2$, minimum at $(0,0)$, maximum at $(2,2)$.
- 3.6.7.11.b concave up for $x<2-\sqrt{2}$ and $x>2+\sqrt{2}$, concave down for $2-\sqrt{2}<x<2+\sqrt{2}$, inflection points at $x=2 \pm \sqrt{2}$.
- 3.6.7.11.c $\infty$
3.6.7.11.d

Open dots indicate inflection points, and closed dots indicate local extrema.

3.6.7.12. *. Answer. 3.6.7.12.a


There are no inflection points or extrema, except the endpoint $(0,1)$. 3.6.7.12.b


There are no inflection points or extrema, except the endpoint $(1,0)$.
3.6.7.12.c The domain of $g$ is $(0,1]$. The range of $g$ is $[0, \infty)$.
3.6.7.12.d $g^{\prime}\left(\frac{1}{2}\right)=-2$
3.6.7.13. *. Answer. (a)


Local maximum at $x=-\frac{1}{\sqrt[4]{5}}$; local minimum at $x=\frac{1}{\sqrt[4]{5}}$; inflection point at the origin; concave down for $x<0$; concave up for $x>0$.
(b) The number of distinct real roots of $x^{5}-x+k$ is:

- 1 when $|k|>\frac{4}{5 \sqrt[4]{5}}$
- 2 when $|k|=\frac{4}{5 \sqrt[4]{5}}$
- 3 when $|k|<\frac{4}{5 \sqrt[4]{5}}$


### 3.6.7.14. *. Answer. (a)


(b) For any real $x$, define $\sinh ^{-1}(x)$ to be the unique solution of $\sinh (y)=x$. For every $x \in[1, \infty)$, define $\cosh ^{-1}(x)$ to be the unique $y \in[0, \infty)$ that obeys $\cosh (y)=x$.

(c) $\frac{\mathrm{d}}{\mathrm{d} x}\left\{\cosh ^{-1}(x)\right\}=\frac{1}{\sqrt{x^{2}-1}}$

## 3.7 • L'Hôpital's Rule, Indeterminate Forms <br> 3.7.4 • Exercises

## Exercises - Stage 1

3.7.4.1. Answer. There are many possible answers. Here is one: $f(x)=5 x$, $g(x)=2 x$.
3.7.4.2. Answer. There are many possible answers. Here is one: $f(x)=x$, $g(x)=x^{2}$.
3.7.4.3. Answer. There are many possible answers. Here is one: $f(x)=1+\frac{1}{x}$, $g(x)=x \log 5$ (recall we use $\log$ to mean logarithm base $e$ ).

## Exercises - Stage 2

3.7.4.4. *. Answer. $-\frac{2}{\pi}$
3.7.4.5. *. Answer. $-\infty$
3.7.4.6. *. Answer. 0
3.7.4.7. *. Answer. 0
3.7.4.8. *. Answer. 3
3.7.4.9. Answer. 2
3.7.4.10. *. Answer. 0
3.7.4.11. *. Answer. $\frac{1}{2}$
3.7.4.12. Answer. 0
3.7.4.13. Answer. 5
3.7.4.14. Answer. $\infty$
3.7.4.15. *. Answer. 3
3.7.4.16. *. Answer. $\frac{3}{2}$
3.7.4.17. *. Answer. 0
3.7.4.18. *. Answer. $\frac{1}{3}$
3.7.4.19. Answer. 0
3.7.4.20. Answer. $\frac{1}{\sqrt{e}}$
3.7.4.21. Answer. 1
3.7.4.22. Answer. 1
3.7.4.23. *. Answer. $c=0$
3.7.4.24. *. Answer.

$$
\lim _{x \rightarrow 0} \frac{e^{k \sin \left(x^{2}\right)}-\left(1+2 x^{2}\right)}{x^{4}}=\left\{\begin{array}{rr}
-\infty & k<2 \\
2 & k=2 \\
\infty & k>2
\end{array}\right.
$$

Exercises - Stage 3
3.7.4.25. Answer.

- We want to find the limit as $n$ goes to infinity of the percentage error, $\lim _{n \rightarrow \infty} 100 \frac{|S(n)-A(n)|}{|S(n)|}$. Since $A(n)$ is a nicer function than $S(n)$, let's simplify:
$\lim _{n \rightarrow \infty} 100 \frac{|S(n)-A(n)|}{|S(n)|}=100\left|1-\lim _{n \rightarrow \infty} \frac{A(n)}{S(n)}\right|$.
We figure out this limit the natural way:

$$
\begin{aligned}
100\left|1-\lim _{n \rightarrow \infty} \frac{A(n)}{S(n)}\right| & =100|1-\lim _{n \rightarrow \infty} \underbrace{\frac{5 n^{4}}{5 n^{4}-13 n^{3}-4 n+\log (n)}}_{\substack{\text { num } \rightarrow \infty \\
\text { den } \rightarrow \infty}}| \\
& =100\left|1-\lim _{n \rightarrow \infty} \frac{20 n^{3}}{20 n^{3}-39 n^{2}-4+\frac{1}{n}}\right| \\
& =100\left|1-\lim _{n \rightarrow \infty} \frac{n^{3}}{n^{3}} \cdot \frac{20}{20-\frac{39}{n}-\frac{4}{n^{3}}+\frac{1}{n^{4}}}\right| \\
& =100|1-1|=0
\end{aligned}
$$

So, as $n$ gets larger and larger, the relative error in the approximation gets closer and closer to 0 .

- Now, let's look at the absolute error.

$$
\lim _{n \rightarrow \infty}|S(n)-A(n)|=\lim _{n \rightarrow \infty}\left|-13 n^{3}-4 n+\log n\right|=\infty
$$

So although the error gets small relative to the giant numbers we're talking about, the absolute error grows without bound.

## 4 - Towards Integral Calculus

## 4.1 • Introduction to Antiderivatives

### 4.1.2 • Exercises

## Exercises - Stage 1

4.1.2.1. Answer. $F(x)=f(x)+C$
4.1.2.2. Answer.
$C(x)$

Exercises - Stage 2
4.1.2.3. Answer. $F(x)=x^{3}+x^{5}+5 x^{2}-9 x+C$
4.1.2.4. Answer. $\quad F(x)=\frac{3}{40} x^{8}-\frac{18}{5} x^{5}+\frac{1}{2} x^{2}+C$
4.1.2.5. Answer. $F(x)=3 x^{\frac{4}{3}}+\frac{45}{17 x^{1.7}}+C$
4.1.2.6. Answer. $F(x)=\frac{2}{7} \sqrt{x}+C$
4.1.2.7. Answer. $F(x)=\frac{1}{5} e^{5 x+11}+C$
4.1.2.8. Answer. $F(x)=-\frac{3}{5} \cos (5 x)+\frac{7}{13} \sin (13 x)+C$
4.1.2.9. Answer. $F(x)=\tan (x+1)+C$
4.1.2.10. Answer. $F(x)=\log |x+2|+C$
4.1.2.11. Answer. $F(x)=\frac{7}{\sqrt{3}} \arcsin (x)+C$
4.1.2.12. Answer. $F(x)=\frac{1}{5} \arctan (5 x)+C$
4.1.2.13. Answer. $f(x)=x^{3}-\frac{9}{2} x^{2}+4 x+\frac{19}{2}$
4.1.2.14. Answer. $f(x)=\frac{1}{2} \sin (2 x)+\pi$
4.1.2.15. Answer. $f(x)=\log |x|$
4.1.2.16. Answer. $f(x)=\arcsin x+x-\pi-1$
4.1.2.17. Answer. It takes $\frac{1}{2} \log 7$ hours (about 58 minutes) for the initial colony to increase by 300 individuals.
4.1.2.18. Answer. At time $t$, the amount of money in your bank account is $75000 e^{\frac{t}{50}}+C$ dollars, for some constant $C$.
4.1.2.19. Answer. $\frac{24}{\pi}+6 \approx 13.6 \mathrm{kWh}$

## Exercises - Stage 3

4.1.2.20. *. Answer. $f^{\prime}(x)=g^{\prime}(x)=\frac{1}{\sqrt{x-x^{2}}} ; f$ and $g$ differ only by a constant.
4.1.2.21. Answer. $F(x)=\sin (2 x) \cos (3 x)+C$
4.1.2.22. Answer. $F(x)=\frac{e^{x}}{x^{2}+1}+C$
4.1.2.23. Answer. $F(x)=e^{x^{3}}+C$
4.1.2.24. Answer. $\quad F(x)=-\frac{5}{2} \cos \left(x^{2}\right)+C$
4.1.2.25. Answer. $F(x)=\frac{1}{2} x^{2}+C$.
4.1.2.26. Answer. $F(x)=7 \arcsin \left(\frac{x}{\sqrt{3}}\right)+C$
4.1.2.27. Answer. $V(H)=2 \pi\left(\frac{1}{5} H^{5}+\frac{2}{3} H^{3}+H\right)$

## SOLUTIONS TO EXERCISES

## 1 . Limits

## 1.1 • Drawing Tangents and a First Limit

### 1.1.2 • Exercises

## Exercises - Stage 1

1.1.2.1. Solution.


The tangent line to $y=f(x)$ at a point should go through the point, and be "in the same direction" as $f$ at that point. The secant line through $P$ and $Q$ is simply the straight line passing through $P$ and $Q$.

### 1.1.2.2. Solution.

a True: since $y=2 x+3$ is the tangent line to $y=f(x)$ at the point $x=2$, this means the function and the tangent line have the same value at $x=2$. So $f(2)=2(2)+3=7$.
b In general, this is false. We are only guaranteed that the curve $y=f(x)$ and its tangent line $y=2 x+3$ agree at $x=2$. The functions $f(x)$ and $2 x+3$ may or may not take the same values when $x \neq 2$. For example, if $f(x)=2 x+3$, then of course $f(x)$ and $2 x+3$ agree for all values of $x$. But if $f(x)=2 x+3+(x-2)^{2}$, then $f(x)$ and $2 x+3$ agree only for $x=2$.
1.1.2.3. Solution. Since the tangent line to the curve at point $P$ passes through point $P$, the curve and the tangent line touch at point $P$. So, they must intersect at least once. By drawing various examples, we can see that different curves may touch their tangent lines exactly once, exactly twice, exactly three times, etc.


## 1.2 • Another Limit and Computing Velocity 1.2.2 • Exercises

## Exercises - Stage 1

1.2.2.1. Solution. Speed is nonnegative; velocity has a sign (positive or negative) that indicates direction.
1.2.2.2. Solution. Yes-an object that is not moving has speed 0 .
1.2.2.3. Solution. Since you started and ended in the same place, your difference in position was 0 , and your difference in time was 24 hours. So, your average velocity was $\frac{0}{24}=0 \mathrm{kph}$.
1.2.2.4. Solution. Objects accelerate as they fall - their speed gets bigger and bigger. So in the entire first second of falling, the object is at its fastest at the one-second mark. We have defined the average speed over a time interval to be $\frac{\text { distance moved }}{\text { time taken }}$. We do not yet know how to compute the distanced travelled by time taken distance travelled is no more than the maximum speed times the time taken, so that the average speed is no larger that the maximum speed. In the next chapter we will verify that intuition mathematically. See Example 2.13.7. So the average speed will be a smaller number then the speed at the one-second mark, which is the maximum speed.
1.2.2.5. Solution. The slope of a curve is given by $\frac{\text { change in vertical component }}{\text { change in horizontal component }}$. The change in the vertical component is exactly $s(b)-s(a)$, and the change in the horizontal component is exactly $b-a$.
1.2.2.6. Solution. The velocity is positive when the object is going in the increasing direction; it is going "up" on the graph when $t$ is between 0 and 2 , and when $t$ is between 6 and 7 . So, the velocity is positive when $t$ is in $(0,2) \cup(6,7)$.

## Exercises - Stage 2

### 1.2.2.7. Solution.

a Average velocity:

$$
\begin{aligned}
\frac{\text { change in position }}{\text { change in time }} & =\frac{s(5)-s(3)}{5-3}=\frac{\left(3 \cdot 5^{2}+5\right)-\left(3 \cdot 3^{2}+5\right)}{5-3} \\
& =24 \text { units per second. }
\end{aligned}
$$

b From the notes, we know the velocity of an object at time $a$ is

$$
v(a)=\lim _{h \rightarrow 0} \frac{s(a+h)-s(a)}{h}
$$

So, in our case:

$$
\begin{aligned}
v(1) & =\lim _{h \rightarrow 0} \frac{s(1+h)-s(1)}{h}=\lim _{h \rightarrow 0} \frac{\left[3(1+h)^{2}+5\right]-\left[3(1)^{2}+5\right]}{h} \\
& =\lim _{h \rightarrow 0} \frac{6 h+3 h^{2}}{h}=\lim _{h \rightarrow 0} 6+3 h=6
\end{aligned}
$$

So the velocity when $t=1$ is 6 units per second.

### 1.2.2.8. Solution.

a Average velocity:

$$
\begin{aligned}
\frac{\text { change in position }}{\text { change in time }} & =\frac{s(9)-s(1)}{9-1}=\frac{3-1}{9-1} \\
& =\frac{1}{4} \text { units per second. }
\end{aligned}
$$

b From the notes, we know the velocity of an object at time $a$ is

$$
v(a)=\lim _{h \rightarrow 0} \frac{s(a+h)-s(a)}{h}
$$

So, in our case:

$$
\begin{aligned}
v(1) & =\lim _{h \rightarrow 0} \frac{s(1+h)-s(1)}{h}=\lim _{h \rightarrow 0} \frac{\sqrt{1+h}-1}{h} \\
& =\lim _{h \rightarrow 0} \frac{\sqrt{1+h}-1}{h} \cdot\left(\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}\right) \\
& =\lim _{h \rightarrow 0} \frac{(1+h)-1}{h(\sqrt{1+h}+1)} \\
& =\lim _{h \rightarrow 0} \frac{1}{\sqrt{1+h}+1}=\frac{1}{2} \text { units per second }
\end{aligned}
$$

c

$$
\begin{aligned}
v(9) & =\lim _{h \rightarrow 0} \frac{s(9+h)-s(9)}{h}=\lim _{h \rightarrow 0} \frac{\sqrt{9+h}-3}{h} \\
& =\lim _{h \rightarrow 0} \frac{\sqrt{9+h}-3}{h} \cdot\left(\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}\right) \\
& =\lim _{h \rightarrow 0} \frac{(9+h)-9}{h(\sqrt{9+h}+3)} \\
& =\lim _{h \rightarrow 0} \frac{1}{\sqrt{9+h}+3}=\frac{1}{6} \text { units per second }
\end{aligned}
$$

Remark: the average velocity is not the average of the two instantaneous velocities.

## 1.3 - The Limit of a Function

### 1.3.2 • Exercises

## Exercises - Stage 1

### 1.3.2.1. Solution.

a $\lim _{x \rightarrow-2} f(x)=1$ : as $x$ gets very close to $-2, y$ gets very close to 1 .
b $\lim _{x \rightarrow 0} f(x)=0$ : as $x$ gets very close to $0, y$ also gets very close to 0 .
c $\lim _{x \rightarrow 2} f(x)=2$ : as $x$ gets very close to $2, y$ gets very close to 2 . We ignore the value of the function where $x$ is exactly 2 .
1.3.2.2. Solution. The limit does not exist. As $x$ approaches 0 from the left, $y$ approaches -1 ; as $x$ approaches 0 from the right, $y$ approaches 1 . This tells us $\lim _{x \rightarrow 0^{-}} f(x)=-1$ and $\lim _{x \rightarrow 0^{+}} f(x)=1$, but neither of these are what the question asked. Since the limits from left and right do not agree, the limit does not exist. Put another way, there is no single number $y$ approaches as $x$ approaches 0 , so the limit $\lim _{x \rightarrow 0} f(x)$ does not exist.

### 1.3.2.3. Solution.

a $\lim _{x \rightarrow-1^{-}} f(x)=2$ : as $x$ approaches -1 from the left, $y$ approaches 2 . It doesn't matter that the function isn't defined at $x=-1$, and it doesn't matter what happens to the right of $x=-1$.
b $\lim _{x \rightarrow-1^{+}} f(x)=-2$ : as $x$ approaches -1 from the right, $y$ approaches -2 . It doesn't matter that the function isn't defined at -1 , and it doesn't matter what happens to the left of -1 .
c $\lim _{x \rightarrow-1} f(x)=$ DNE: since the limits from the left and right don't agree, the limit does not exist.
$\mathrm{d} \lim _{x \rightarrow-2^{+}} f(x)=0$ : as $x$ approaches -2 from the right, $y$ approaches 0 . It doesn't
matter that the function isn't defined at 2 , or to the left of 2 .
e $\lim _{x \rightarrow 2^{-}} f(x)=0$ : as $x$ approaches 2 from the left, $y$ approaches 0 . It doesn't matter that the function isn't defined at 2 , or to the right of 2 .
1.3.2.4. Solution. Many answers are possible; here is one.


As $x$ gets closer and closer to $3, y$ gets closer and closer to 10: this shows $\lim _{x \rightarrow 3} f(x)=$ 10. Also, at 3 itself, the function takes the value 10 ; this shows $f(3)=10$.
1.3.2.5. Solution. Many answers are possible; here is one.


Note that, as $x$ gets closer and closer to 3 except at 3 itself, $y$ gets closer and closer to 10 : this shows $\lim _{x \rightarrow 3} f(x)=10$. Then, when $x=3$, the function has value 0 : this shows $f(3)=0$.
1.3.2.6. Solution. In general, this is false. The limit as $x$ goes to 3 does not take into account the value of the function at $3: f(3)$ can be anything.
1.3.2.7. Solution. False. The limit as $x$ goes to 3 does not take into account the value of the function at $3: f(3)$ tells us nothing about $\lim _{x \rightarrow 3} f(x)$.
1.3.2.8. Solution. $\lim _{x \rightarrow-2^{-}} f(x)=16$ : in order for the limit $\lim _{x \rightarrow 2} f(x)$ to exist and be equal to 16 , both one sided limits must exist and be equal to 16 .
1.3.2.9. Solution. Not enough information to say. If $\lim _{x \rightarrow-2^{+}} f(x)=16$, then $\lim _{x \rightarrow-2} f(x)=16$. If $\lim _{x \rightarrow-2^{+}} f(x) \neq 16$, then $\lim _{x \rightarrow-2} f(x)$ does not exist.

## Exercises - Stage 2

1.3.2.10. Solution. $\lim _{t \rightarrow 0} \sin t=0$ : as $t$ approaches $0, \sin t$ approaches 0 as well.
1.3.2.11. Solution. $\quad \lim _{x \rightarrow 0^{+}} \log x=-\infty$ : as $x$ approaches 0 from the right, $\log x$ is negative and increasingly large, growing without bound.
1.3.2.12. Solution. $\lim _{y \rightarrow 3} y^{2}=9$ : as $y$ gets closer and closer to $3, y^{2}$ gets closer and closer to $3^{2}$.
1.3.2.13. Solution. $\lim _{x \rightarrow 0^{-}} \frac{1}{x}=-\infty$ : as $x$ gets closer and closer to 0 from the left, $\frac{1}{x}$ becomes a larger and larger negative number.
1.3.2.14. Solution. $\lim _{x \rightarrow 0} \frac{1}{x}=$ DNE: as $x$ gets closer and closer to 0 from the left, $\frac{1}{x}$ becomes a larger and larger negative number; but as $x$ gets closer and closer to 0 from the right, $\frac{1}{x}$ becomes a larger and larger positive number. So the limit from the left is not the same as the limit from the right, and so $\lim _{x \rightarrow 0} \frac{1}{x}=$ DNE. Contrast this with Question 1.3.2.15.
1.3.2.15. Solution. $\lim _{x \rightarrow 0} \frac{1}{x^{2}}=\infty$ : as $x$ gets closer and closer to 0 from the either side, $\frac{1}{x^{2}}$ becomes a larger and larger positive number, growing without bound. Contrast this with Question 1.3.2.14.
1.3.2.16. Solution. $\lim _{x \rightarrow 3} \frac{1}{10}=\frac{1}{10}$ : no matter what $x$ is, $\frac{1}{10}$ is always $\frac{1}{10}$. In particular, as $x$ approaches $3, \frac{1}{10}$ stays put at $\frac{1}{10}$.
1.3.2.17. Solution. When $x$ is very close to $3, f(x)$ looks like the function $x^{2}$. So: $\lim _{x \rightarrow 3} f(x)=\lim _{x \rightarrow 3} x^{2}=9$

## 1.4 - Calculating Limits with Limit Laws

### 1.4.2 • Exercises

## Exercises - Stage 1

1.4.2.1. Solution. Zeroes cause a problem when they show up in the denominator, so we can only compute (1.4.2.1.a) and (1.4.2.1.d). (Both these limits are zero.) Be careful: there is no such rule as "zero divided by zero is one," or "zero divided by zero is zero."
1.4.2.2. Solution. The statement $\lim _{x \rightarrow 3} \frac{f(x)}{g(x)}=10$ tells us that, as $x$ gets very close to $3, f(x)$ is 10 times as large as $g(x)$. We notice that if $f(x)=10 g(x)$, then $\frac{f(x)}{g(x)}=10$, so $\lim _{x \rightarrow} \frac{f(x)}{g(x)}=10$ wherever $f$ and $g$ exist. So it's enough to find a function $g(x)$ that has limit 0 at 3 . Such a function is (for example) $g(x)=x-3$. So, we take $f(x)=10(x-3)$ and $g(x)=x-3$. It is easy now to check that $\lim _{x \rightarrow 3} f(x)=\lim _{x \rightarrow 3} g(x)=0$ and $\lim _{x \rightarrow 3} \frac{f(x)}{g(x)}=\lim _{x \rightarrow 3} \frac{10(x-3)}{x-3}=\lim _{x \rightarrow 3} 10=10$.

### 1.4.2.3. Solution.

- As we saw in Question 1.4.2.2, $x-3$ is a function with limit 0 at $x=3$. So one way of thinking about this question is to try choosing $f(x)$ so that $\frac{f(x)}{g(x)}=g(x)=x-3$ too, which leads us to the solution $f(x)=(x-3)^{2}$ and $g(x)=x-3$. This is one of many, many possible answers.
- Another way of thinking about this problem is that $f(x)$ should go to 0 "more strongly" than $g(x)$ when $x$ approaches 3 . One way of a function going to 0 really strongly is to make that function identically zero. So we can set $f(x)=0$ and $g(x)=x-3$. Now $\frac{f(x)}{g(x)}$ is equal to 0 whenever $x \neq 3$, and is undefined at $x=3$. Since the limit as $x$ goes to three does not take into account the value of the function at 3 , we have $\lim _{x \rightarrow 3} \frac{f(x)}{g(x)}=0$.

There are many more possible answers.
1.4.2.4. Solution. One way to start this problem is to remember $\lim _{x \rightarrow 0} \frac{1}{x^{2}}=\infty$. (Using $\frac{1}{x^{2}}$ as opposed to $\frac{1}{x}$ is important, since $\lim _{x \rightarrow 0} \frac{1}{x}$ does not exist.) Then by "shifting" by three, we find $\lim _{x \rightarrow 3} \frac{1}{(x-3)^{2}}=\infty$. So it is enough to arrange that $\frac{f(x)}{g(x)}=\frac{1}{(x-3)^{2}}$. We can achieve this with $f(x)=x-3$ and $g(x)=(x-3)^{3}$, and maintain $\lim _{x \rightarrow 3} f(x)=\lim _{x \rightarrow 3} g(x)=0$. Again, this is one of many possible solutions.
1.4.2.5. Solution. Any real number; positive infinity; negative infinity; does not exist.
This is an important thing to remember: often, people see limits that look like $\frac{0}{0}$ and think that the limit must be 1 , or 0 , or infinite. In fact, this limit could be anything-it depends on the relationship between $f$ and $g$.
Questions 1.4.2.2 and 1.4.2.3 show us examples where the limit is 10 and 0 ; they can easily be modified to make the limit any real number.
Question 1.4.2.4 show us an example where the limit is $\infty$; it can easily be modified to make the limit $-\infty$ or DNE.

## Exercises - Stage 2

1.4.2.6. Solution. Since we're not trying to divide by 0 , or multiply by infinity: $\lim _{t \rightarrow 10} \frac{2(t-10)^{2}}{t}=\frac{2 \cdot 0}{10}=0$
1.4.2.7. Solution. Since we're not doing anything dodgy like putting 0 in the denominator, $\lim _{y \rightarrow 0} \frac{(y+1)(y+2)(y+3)}{\cos y}=\frac{(0+1)(0+2)(0+3)}{\cos 0}=\frac{6}{1}=6$.
1.4.2.8. Solution. Since the limits of the numerator and denominator exist, and since the limit of the denominator is nonzero: $\lim _{x \rightarrow 3}\left(\frac{4 x-2}{x+2}\right)^{4}=\left(\frac{4(3)-2}{3+2}\right)^{4}=16$

### 1.4.2.9. *. Solution.

$$
\lim _{t \rightarrow-3}\left(\frac{1-t}{\cos (t)}\right)=\frac{\lim _{t \rightarrow-3}(1-t)}{\lim _{t \rightarrow-3} \cos (t)}=4 / \cos (-3)=4 / \cos (3)
$$

1.4.2.10. *. Solution. If try naively then we get $0 / 0$, so we expand and then simplify:

$$
\frac{(2+h)^{2}-4}{2 h}=\frac{h^{2}+4 h+4-4}{2 h}=\frac{h}{2}+2
$$

Hence the limit is $\lim _{h \rightarrow 0}\left(\frac{h}{2}+2\right)=2$.

### 1.4.2.11. *. Solution.

$$
\lim _{t \rightarrow-2}\left(\frac{t-5}{t+4}\right)=\frac{\lim _{t \rightarrow-2}(t-5)}{\lim _{t \rightarrow-2}(t+4)}=-7 / 2
$$

### 1.4.2.12. *. Solution.

$$
\lim _{t \rightarrow 1} \sqrt{5 x^{3}+4}=\sqrt{\lim _{t \rightarrow 1}\left(5 x^{3}+4\right)}=\sqrt{5 \lim _{t \rightarrow 1}\left(x^{3}\right)+4}=\sqrt{9}=3 .
$$

1.4.2.13. *. Solution.

$$
\lim _{t \rightarrow-1}\left(\frac{t-2}{t+3}\right)=\frac{\lim _{t \rightarrow-1}(t-2)}{\lim _{t \rightarrow-1}(t+3)}=-3 / 2
$$

1.4.2.14. *. Solution. We simply plug in $x=1: \lim _{x \rightarrow 1}\left[\frac{\log (1+x)-x}{x^{2}}\right]=$ $\log (2)-1$.
1.4.2.15. *. Solution. If we try naively then we get $0 / 0$, so we simplify first:

$$
\frac{x-2}{x^{2}-4}=\frac{x-2}{(x-2)(x+2)}=\frac{1}{x+2}
$$

Hence the limit is $\lim _{x \rightarrow 2} \frac{1}{x+2}=1 / 4$.
1.4.2.16. *. Solution. If we try to plug in $x=4$, we find the denominator is zero. So to get a better idea of what's happening, we factor the numerator and denominator:

$$
\begin{aligned}
\lim _{x \rightarrow 4} \frac{x^{2}-4 x}{x^{2}-16} & =\lim _{x \rightarrow 4} \frac{x(x-4)}{(x+4)(x-4)} \\
& =\lim _{x \rightarrow 4} \frac{x}{x+4} \\
& =\frac{4}{8}=\frac{1}{2}
\end{aligned}
$$

1.4.2.17. *. Solution. If we try to plug in $x=2$, we find the denominator is zero. So to get a better idea of what's happening, we factor the numerator:

$$
\begin{aligned}
\lim _{x \rightarrow 2} \frac{x^{2}+x-6}{x-2} & =\lim _{x \rightarrow 2} \frac{(x+3)(x-2)}{x-2} \\
& =\lim _{x \rightarrow 2}(x+3)=5
\end{aligned}
$$

1.4.2.18. *. Solution. If we try naively then we get $0 / 0$, so we simplify first:

$$
\frac{x^{2}-9}{x+3}=\frac{(x-3)(x+3)}{(x+3)}=x-3
$$

Hence the limit is $\lim _{x \rightarrow-3}(x-3)=-6$.
1.4.2.19. Solution. To calculate the limit of a polynomial, we simply evaluate the polynomial: $\lim _{t \rightarrow 2} \frac{1}{2} t^{4}-3 t^{3}+t=\frac{1}{2} \cdot 2^{4}-3 \cdot 2^{3}+2=-14$

### 1.4.2.20. *. Solution.

$$
\frac{\sqrt{x^{2}+8}-3}{x+1}=\frac{\sqrt{x^{2}+8}-3}{x+1} \cdot \frac{\sqrt{x^{2}+8}+3}{\sqrt{x^{2}+8}+3}
$$

$$
\begin{aligned}
& =\frac{\left(x^{2}+8\right)-3^{2}}{(x+1)\left(\sqrt{x^{2}+8}+3\right)} \\
& =\frac{x^{2}-1}{(x+1)\left(\sqrt{x^{2}+8}+3\right)} \\
& =\frac{(x-1)(x+1)}{(x+1)\left(\sqrt{x^{2}+8}+3\right)} \\
& =\frac{(x-1)}{\sqrt{x^{2}+8}+3} \\
\lim _{x \rightarrow-1} \frac{\sqrt{x^{2}+8}-3}{x+1} & =\lim _{x \rightarrow-1} \frac{(x-1)}{\sqrt{x^{2}+8}+3} \\
& =\frac{-2}{\sqrt{9}+3} \\
& =-\frac{2}{6}=-\frac{1}{3} .
\end{aligned}
$$

1.4.2.21. *. Solution. If we try to do the limit naively we get $0 / 0$. Hence we must simplify.

$$
\begin{aligned}
\frac{\sqrt{x+7}-\sqrt{11-x}}{2 x-4} & =\frac{\sqrt{x+7}-\sqrt{11-x}}{2 x-4} \cdot\left(\frac{\sqrt{x+7}+\sqrt{11-x}}{\sqrt{x+7}+\sqrt{11-x}}\right) \\
& =\frac{(x+7)-(11-x)}{(2 x-4)(\sqrt{x+7}+\sqrt{11-x})} \\
& =\frac{2 x-4}{(2 x-4)(\sqrt{x+7}+\sqrt{11-x})} \\
& =\frac{1}{\sqrt{x+7}+\sqrt{11-x}} \\
\text { So, } \lim _{x \rightarrow 2} \frac{\sqrt{x+7}-\sqrt{11-x}}{2 x-4} & =\lim _{x \rightarrow 2} \frac{1}{\sqrt{x+7}+\sqrt{11-x}} \\
& =\frac{1}{\sqrt{9}+\sqrt{9}} \\
& =\frac{1}{6}
\end{aligned}
$$

1.4.2.22. *. Solution. If we try to do the limit naively we get $0 / 0$. Hence we must simplify.

$$
\begin{aligned}
\frac{\sqrt{x+2}-\sqrt{4-x}}{x-1} & =\frac{\sqrt{x+2}-\sqrt{4-x}}{x-1} \cdot \frac{\sqrt{x+2}+\sqrt{4-x}}{\sqrt{x+2}+\sqrt{4-x}} \\
& =\frac{(x+2)-(4-x)}{(x-1)(\sqrt{x+2}+\sqrt{4-x})} \\
& =\frac{2 x-2}{(x-1)(\sqrt{x+2}+\sqrt{4-x})} \\
& =\frac{2}{\sqrt{x+2}+\sqrt{4-x}}
\end{aligned}
$$

So the limit is

$$
\begin{aligned}
\lim _{x \rightarrow 1} \frac{\sqrt{x+2}-\sqrt{4-x}}{x-1} & =\lim _{x \rightarrow 1} \frac{2}{\sqrt{x+2}+\sqrt{4-x}} \\
& =\frac{2}{\sqrt{3}+\sqrt{3}} \\
& =\frac{1}{\sqrt{3}}
\end{aligned}
$$

1.4.2.23. *. Solution. If we try to do the limit naively we get $0 / 0$. Hence we must simplify.

$$
\begin{aligned}
\frac{\sqrt{x-2}-\sqrt{4-x}}{x-3} & =\frac{\sqrt{x-2}-\sqrt{4-x}}{x-3} \cdot \frac{\sqrt{x-2}+\sqrt{4-x}}{\sqrt{x-2}+\sqrt{4-x}} \\
& =\frac{(x-2)-(4-x)}{(x-3)(\sqrt{x-2}+\sqrt{4-x})} \\
& =\frac{2 x-6}{(x-3)(\sqrt{x-2}+\sqrt{4-x})} \\
& =\frac{2}{\sqrt{x-2}+\sqrt{4-x}}
\end{aligned}
$$

$$
\text { So, } \lim _{x \rightarrow 3} \frac{\sqrt{x-2}-\sqrt{4-x}}{x-3}=\lim _{x \rightarrow 3} \frac{2}{\sqrt{x-2}+\sqrt{4-x}}
$$

$$
=\frac{2}{1+1}
$$

$$
=1
$$

1.4.2.24. *. Solution. Here we get $0 / 0$ if we try the naive approach. Hence we must simplify.

$$
\begin{aligned}
\frac{3 t-3}{2-\sqrt{5-t}} & =\frac{3 t-3}{2-\sqrt{5-t}} \times \frac{2+\sqrt{5-t}}{2+\sqrt{5-t}} \\
& =(2+\sqrt{5-t}) \frac{3 t-3}{2^{2}-(5-t)} \\
& =(2+\sqrt{5-t}) \frac{3 t-3}{t-1} \\
& =(2+\sqrt{5-t}) \frac{3(t-1)}{t-1}
\end{aligned}
$$

So there is a cancelation. Hence the limit is

$$
\begin{aligned}
\lim _{t \rightarrow 1} \frac{3 t-3}{2-\sqrt{5-t}} & =\lim _{t \rightarrow 1}(2+\sqrt{5-t}) \cdot 3 \\
& =12
\end{aligned}
$$

1.4.2.25. Solution. This is a classic example of the Squeeze Theorem. It is tempting to try to use arithmetic of limits: $\lim _{x \rightarrow 0}-x^{2}=0$, and $\lim _{x \rightarrow 0} \cos \left(\frac{3}{x}\right)=$
something, and zero times something is 0 . However, this is invalid reasoning, because we can only use arithmetic of limits when those limits exist, and $\lim _{x \rightarrow 0} \cos \left(\frac{3}{x}\right)$ does not exist. So, we need the Squeeze Theorem.
Since $-1 \leq \cos \left(\frac{3}{x}\right) \leq 1$, we can bound our function of interest from above and below (being careful of the sign!):

$$
-x^{2}(1) \leq-x^{2} \cos \left(\frac{3}{x}\right) \leq-x^{2}(-1)
$$

So our function of interest is between $-x^{2}$ and $x^{2}$. Since $\lim _{x \rightarrow 0}-x^{2}=\lim _{x \rightarrow 0} x^{2}=0$, by the Squeeze Theorem, also $\lim _{x \rightarrow 0}-x^{2} \cos \left(\frac{3}{x}\right)=0$.
Advice about writing these up: whenever we use the Squeeze Theorem, we need to explicitly write that two things are true: that the function we're interested is bounded above and below by two other functions, and that both of those functions have the same limit. Then we can conclude (and we need to write this down as well!) that our original function also shares that limit.
1.4.2.26. Solution. Recall that sine and cosine, no matter what (real-number) input we feed them, spit out numbers between -1 and 1 . So we can bound our horrible numerator, rather than trying to deal with it directly.

$$
x^{4}(-1)+5 x^{2}(-1)+2 \leq x^{4} \sin \left(\frac{1}{x}\right)+5 x^{2} \cos \left(\frac{1}{x}\right)+2 \leq x^{4}(1)+5 x^{2}(1)+2
$$

Further, notice that our bounded functions tend to the same value as $x$ goes to 0 :

$$
\lim _{x \rightarrow 0} x^{4}(-1)+5 x^{2}(-1)+2=\lim _{x \rightarrow 0} x^{4}+5 x^{2}+2=2
$$

So, by the Squeeze Theorem, also

$$
\lim _{x \rightarrow 0} x^{4} \sin \left(\frac{1}{x}\right)+5 x^{2} \cos \left(\frac{1}{x}\right)+2=2
$$

Now, we evaluate our original limit:

$$
\lim _{x \rightarrow 0} \frac{x^{4} \sin \left(\frac{1}{x}\right)+5 x^{2} \cos \left(\frac{1}{x}\right)+2}{(x-2)^{2}}=\frac{2}{(-2)^{2}}=\frac{1}{2}
$$

1.4.2.27. *. Solution. $\lim _{x \rightarrow 0} \sin ^{2}\left(\frac{1}{x}\right)=D N E$, so we think about using the Squeeze Theorem. We'll need to bound the expression $x \sin ^{2}\left(\frac{1}{x}\right)$, but the bounding is a little delicate. For any non-zero value we plug in for $x, \sin ^{2}\left(\frac{1}{x}\right)$ is a number in the interval $[0,1]$. If $a$ is a number in the interval $[0,1]$, then:

$$
0<x a<x \quad \text { when } x \text { is positive }
$$

$$
x<x a<0 \quad \text { when } x \text { is negative }
$$

We'll show you two ways to use this information to create bound that will allow you to apply the Squeeze Theorem.

- Solution 1: We will evaluate separately the limit from the right and from the left.
When $x>0$,

$$
0 \leq x \sin ^{2}\left(\frac{1}{x}\right) \leq x
$$

because $0 \leq \sin ^{2}\left(\frac{1}{x}\right) \leq 1$. Since

$$
\lim _{x \rightarrow 0^{+}} 0=0 \quad \text { and } \quad \lim _{x \rightarrow 0^{+}} x=0
$$

then by the Squeeze Theorem, also

$$
\lim _{x \rightarrow 0^{+}} x \sin ^{2}\left(\frac{1}{x}\right)=0
$$

Similarly, When $x<0$,

$$
x \leq x \sin ^{2}\left(\frac{1}{x}\right) \leq 0
$$

because $0 \leq \sin ^{2}\left(\frac{1}{x}\right) \leq 1$. Since

$$
\lim _{x \rightarrow 0^{-}} x=0 \quad \text { and } \quad \lim _{x \rightarrow 0^{-}} 0=0
$$

then by the Squeeze Theorem, also

$$
\lim _{x \rightarrow 0^{-}} x \sin ^{2}\left(\frac{1}{x}\right)=0
$$

Since the one-sided limits are both equal to zero,

$$
\lim _{x \rightarrow 0} x \sin ^{2}\left(\frac{1}{x}\right)=0
$$

Remark: this is a perfectly fine proof, but it seems to repeat itself. Since the cases $x<0$ and $x>0$ are so similar, we would like to take care of them together. This can be done as shown below.

- Solution 2: If $x \neq 0$, then $0 \leq \sin ^{2}\left(\frac{1}{x}\right) \leq 1$, so

$$
-|x| \leq x \sin ^{2}\left(\frac{1}{x}\right) \leq|x|
$$

Since

$$
\lim _{x \rightarrow 0}-|x|=0 \quad \text { and } \quad \lim _{x \rightarrow 0}|x|=0
$$

then by the Squeeze Theorem, also

$$
\lim _{x \rightarrow 0} x \sin ^{2}\left(\frac{1}{x}\right)=0
$$

1.4.2.28. Solution. When we plug $w=5$ in to the numerator and denominator, we find that each becomes zero. Since we can't divide by zero, we have to dig a little deeper. When a polynomial has a root, that also means it has a factor: we can factor $(w-5)$ out of the top. That lets us cancel:

$$
\lim _{w \rightarrow 5} \frac{2 w^{2}-50}{(w-5)(w-1)}=\lim _{w \rightarrow 5} \frac{2(w-5)(w+5)}{(w-5)(w-1)}=\lim _{w \rightarrow 5} \frac{2(w+5)}{(w-1)}
$$

Note that the function $\frac{2 w^{2}-50}{(w-5)(w-1)}$ is NOT defined at $w=5$, while the function $\frac{2(w+5)}{(w-1)}$ IS defined at $w=5$; so strictly speaking, these two functions are not equal. However, for every value of $w$ that is not 5 , the functions are the same, so their limits are equal. Furthermore, the limit of the second function is quite easy to calculate, since we've eliminated the zero in the denominator: $\lim _{w \rightarrow 5} \frac{2(w+5)}{(w-1)}=\frac{2(5+5)}{5-1}=5$. So $\lim _{w \rightarrow 5} \frac{2 w^{2}-50}{(w-5)(w-1)}=\lim _{w \rightarrow 5} \frac{2(w+5)}{(w-1)}=5$.
1.4.2.29. Solution. When we plug in $r=-5$ to the denominator, we find that it becomes 0 , so we need to dig deeper. The numerator is not zero, so cancelling is out. Notice that the denominator is factorable: $r^{2}+10 r+25=(r+5)^{2}$. As $r$ approaches -5 from either side, the denominator gets very close to zero, but stays positive. The numerator gets very close to -5 . So, as $r$ gets closer to -5 , we have something close to -5 divided by a very small, positive number. Since the denominator is small, the fraction will have a large magnitude; since the numerator is negative and the denominator is positive, the fraction will be negative. So, $\lim _{r \rightarrow-5} \frac{r}{r^{2}+10 r+25}=-\infty$
1.4.2.30. Solution. First, we find $\lim _{x \rightarrow-1} \frac{x^{3}+x^{2}+x+1}{3 x+3}$. When we plug in $x=$ -1 to the top and the bottom, both become zero. In a polynomial, where there is a root, there is a factor, so this tells us we can factor out $(x+1)$ from both the top and the bottom. It's pretty easy to see how to do this in the bottom. For the top, if you're having a hard time, one factoring method (of many) to try is long division of polynomials; another is to factor out $(x+1)$ from the first two terms and the last two terms. (Detailed examples of long division are given in Appendix A. 16 and Examples 1.10.2 and 1.10.3 of the CLP-2 Integral Calculus text.)

$$
\lim _{x \rightarrow-1} \frac{x^{3}+x^{2}+x+1}{3 x+3}=\lim _{x \rightarrow-1} \frac{x^{2}(x+1)+(x+1)}{3 x+3}=\lim _{x \rightarrow-1} \frac{(x+1)\left(x^{2}+1\right)}{3(x+1)}
$$

$$
=\lim _{x \rightarrow-1} \frac{x^{2}+1}{3}=\frac{(-1)^{2}+1}{3}=\frac{2}{3} .
$$

One thing to note here is that the function $\frac{x^{3}+x^{2}+x+1}{3 x+3}$ is not defined at $x=-1$ (because we can't divide by zero). So we replaced it with the function $\frac{x^{2}+1}{3}$, which IS defined at $x=-1$. These functions only differ at $x=-1$; they are the same at every other point. That is why we can use the second function to find the limit of the first function.
Now we're ready to find the actual limit asked in the problem:

$$
\lim _{x \rightarrow-1} \sqrt{\frac{x^{3}+x^{2}+x+1}{3 x+3}}=\sqrt{\frac{2}{3}}
$$

1.4.2.31. Solution. When we plug $x=0$ into the denominator, we get 0 , which means we need to look harder. The numerator is not zero, so we won't be able to cancel our problems away. Let's factor to make things clearer.

$$
\frac{x^{2}+2 x+1}{3 x^{5}-5 x^{3}}=\frac{(x+1)^{2}}{x^{3}\left(3 x^{2}-5\right)}
$$

As $x$ gets close to 0 , the numerator is close to 1 ; the term $\left(3 x^{2}-5\right)$ is negative; and the sign of $x^{3}$ depends on the direction we're approaching 0 from. Since we're dividing a numerator that is very close to 1 by something that's getting very close to 0 , the magnitude of the fraction is getting bigger and bigger without bound. Since the sign of the fraction flips depending on whether we are using numbers slightly bigger than 0 , or slightly smaller than 0 , that means the one-sided limits are $\infty$ and $-\infty$, respectively. (In particular, $\lim _{x \rightarrow 0^{-}} \frac{x^{2}+2 x+1}{3 x^{5}-5 x^{3}}=\infty$ and $\lim _{x \rightarrow 0^{+}} \frac{x^{2}+2 x+1}{3 x^{5}-5 x^{3}}=-\infty$.) Since the one-sided limits don't agree, the limit does not exist.
1.4.2.32. Solution. As usual, we first try plugging in $t=7$, but the denominator is 0 , so we need to think harder. The top and bottom are both squares, so let's go ahead and factor: $\frac{t^{2} x^{2}+2 t x+1}{t^{2}-14 t+49}=\frac{(t x+1)^{2}}{(t-7)^{2}}$. Since $x$ is positive, the numerator is nonzero. Also, the numerator is positive near $t=7$. So, we have something positive and nonzero on the top, and we divide it by the bottom, which is positive and getting closer and closer to zero. The quotient is always positive near $t=7$, and it is growing in magnitude without bound, so $\lim _{t \rightarrow 7} \frac{t^{2} x^{2}+2 t x+1}{t^{2}-14 t+49}=\infty$.
Remark: there is an important reason we specified that $x$ must be a positive constant. Suppose $x$ were $-\frac{1}{7}$ (which is negative and so was not allowed in the question posed). In this case, we would have

$$
\begin{aligned}
\lim _{t \rightarrow 7} \frac{t^{2} x^{2}+2 t x+1}{t^{2}-14 t+49} & =\lim _{t \rightarrow 7} \frac{(t x+1)^{2}}{(t-7)^{2}} \\
& =\lim _{t \rightarrow 7} \frac{(-t / 7+1)^{2}}{(t-7)^{2}}
\end{aligned}
$$

$$
\begin{aligned}
& =\lim _{t \rightarrow 7} \frac{(-1 / 7)^{2}(t-7)^{2}}{(t-7)^{2}} \\
& =\lim _{t \rightarrow 7}(-1 / 7)^{2} \\
& =\frac{1}{49} \\
& \neq \infty
\end{aligned}
$$

1.4.2.33. Solution. The function whose limit we are taking does not depend on d. Since $x$ is a constant, $x^{5}-32 x+15$ is also a constant-it's just some number, that doesn't change, regardless of what $d$ does. So $\lim _{d \rightarrow 0} x^{5}-32 x+15=x^{5}-32 x+15$.
1.4.2.34. Solution. There's a lot going on inside that sine function... and we don't have to care about any of it. No matter what horrible thing we put inside a sine function, the sine function will spit out a number between -1 and 1. So that means we can bound our horrible function like this:

$$
(x-1)^{2} \cdot(-1) \leq(x-1)^{2} \sin \left[\left(\frac{x^{2}-3 x+2}{x^{2}-2 x+1}\right)^{2}+15\right] \leq(x-1)^{2} \cdot(1)
$$

Since $\lim _{x \rightarrow 1}(x-1)^{2} \cdot(-1)=\lim _{x \rightarrow 1}(x-1)^{2} \cdot(1)=0$, the Squeeze Theorem tells us that

$$
\lim _{x \rightarrow 1}(x-1)^{2} \sin \left[\left(\frac{x^{2}-3 x+2}{x^{2}-2 x+1}\right)^{2}+15\right]=0
$$

as well.
1.4.2.35. *. Solution. Since $-1 \leq \sin x \leq 1$ for all values of $x$,

$$
\begin{aligned}
-1 & \leq \sin \left(x^{-100}\right) \leq 1 \\
(-1) x^{1 / 101} & \leq x^{1 / 101} \sin \left(x^{-100}\right) \leq(1) x^{1 / 101} \quad \text { when } x>0, \text { and } \\
(1) x^{1 / 101} & \leq x^{1 / 101} \sin \left(x^{-100}\right) \leq(-1) x^{1 / 101} \quad \text { when } x<0 . \text { Also, } \\
\lim _{x \rightarrow 0} x^{1 / 101} & =\lim _{x \rightarrow 0}-x^{1 / 101}=0 \quad \text { So, by the Squeeze Theorem, } \\
\lim _{x \rightarrow 0^{-}} x^{1 / 101} \sin \left(x^{-100}\right) & =0=\lim _{x \rightarrow 0^{+}} x^{1 / 101} \sin \left(x^{-100}\right) \quad \text { and so } \\
0 & =\lim _{x \rightarrow 0} x^{1 / 101} \sin \left(x^{-100}\right) .
\end{aligned}
$$

Remark: there is a technical point here. When $x$ is a positive number, $-x$ is negative, so $(-1) x<x$. But, when $x$ is negative, $-x$ is positive, so $(-1) x>x$. This is why we take the one-sided limits of our function, and apply the Squeeze Theorem to them separately. It is not true to say that, for instance, $(-1) x^{1 / 101} \leq$ $x^{1 / 101} \sin \left(x^{-100}\right)$ when $x$ is near zero, because this does not hold when $x$ is less than zero.

### 1.4.2.36. *. Solution.

$$
\lim _{x \rightarrow 2} \frac{x^{2}-4}{x^{2}-2 x}=\lim _{x \rightarrow 2} \frac{(x-2)(x+2)}{x(x-2)}=\lim _{x \rightarrow 2} \frac{x+2}{x}=2
$$

1.4.2.37. Solution. When we plug in $x=5$ to the top and the bottom, both limits exist, and the bottom is nonzero. So $\lim _{x \rightarrow 5} \frac{(x-5)^{2}}{x+5}=\frac{0}{10}=0$.
1.4.2.38. Solution. Since we can't plug in $t=\frac{1}{2}$, we'll simplify. One way to start is to add the fractions in the numerator. We'll need a common demoninator, such as $3 t^{2}\left(t^{2}-1\right)$.

$$
\begin{aligned}
\lim _{t \rightarrow \frac{1}{2}} \frac{\frac{1}{3 t^{2}}+\frac{1}{t^{2}-1}}{2 t-1} & =\lim _{t \rightarrow \frac{1}{2}} \frac{\frac{t^{2}-1}{3 t^{2}\left(t^{2}-1\right)}+\frac{3 t^{2}}{3 t^{2}\left(t^{2}-1\right)}}{2 t-1} \\
& =\lim _{t \rightarrow \frac{1}{2}} \frac{\frac{4 t^{2}-1}{3 t^{2}\left(t^{2}-1\right)}}{2 t-1} \\
& =\lim _{t \rightarrow \frac{1}{2}} \frac{4 t^{2}-1}{3 t^{2}\left(t^{2}-1\right)(2 t-1)} \\
& =\lim _{t \rightarrow \frac{1}{2}} \frac{(2 t+1)(2 t-1)}{3 t^{2}\left(t^{2}-1\right)(2 t-1)} \\
& =\lim _{t \rightarrow \frac{1}{2}} \frac{2 t+1}{3 t^{2}\left(t^{2}-1\right)}
\end{aligned}
$$

Since we cancelled out the term that was causing the numerator and denominator to be zero when $t=\frac{1}{2}$, now $t=\frac{1}{2}$ is in the domain of our function, so we simply plug it in:

$$
\begin{aligned}
& =\frac{1+1}{\frac{3}{4}\left(\frac{1}{4}-1\right)} \\
& =\frac{2}{\frac{3}{4}\left(-\frac{3}{4}\right)} \\
& =-\frac{32}{9}
\end{aligned}
$$

1.4.2.39. Solution. We recall that

$$
|x|=\left\{\begin{aligned}
x & , \quad x \geq 0 \\
-x & , x<0
\end{aligned}\right.
$$

So,

$$
\begin{aligned}
\frac{|x|}{x} & =\left\{\begin{aligned}
\frac{x}{x}, & x>0 \\
\frac{-x}{x}, & x<0
\end{aligned}\right. \\
& =\left\{\begin{aligned}
1, & x>0 \\
-1, & x<0
\end{aligned}\right.
\end{aligned}
$$

Therefore,

$$
3+\frac{|x|}{x}= \begin{cases}4, & x>0 \\ 2 & ,\end{cases}
$$

Since our function gives a value of 4 when $x$ is to the right of zero, and a value of 2 when $x$ is to the left of zero, $\lim _{x \rightarrow 0}\left(3+\frac{|x|}{x}\right)$ does not exist.
To further clarify the situation, the graph of $y=f(x)$ is sketched below:

1.4.2.40. Solution. If we factor out 3 from the numerator, our function becomes $3 \frac{|d+4|}{d+4}$. We recall that

$$
|X|=\left\{\begin{array}{rl}
X & ,
\end{array} \quad X \geq 0\right.
$$

So, with $X=d+4$,

$$
\begin{aligned}
3 \frac{|d+4|}{d+4} & = \begin{cases}3 \frac{d+4}{d+4} & , \\
3 \frac{-(d+4)}{d+4} & , \\
d+4<0\end{cases} \\
& =\left\{\begin{array}{rr}
3, & d>-4 \\
-3, & d<-4
\end{array}\right.
\end{aligned}
$$

Since our function gives a value of 3 when $d>-4$, and a value of -3 when $d<-4$, $\lim _{d \rightarrow-4} \frac{|3 d+12|}{d+4}$ does not exist.
To further clarify the situation, the graph of $y=f(x)$ is sketched below:

1.4.2.41. Solution. Note that $x=0$ is in the domain of our function, and nothing "weird" is happening there: we aren't dividing by zero, or taking the square root of a negative number, or joining two pieces of a piecewise-defined function. So, as $x$ gets extremely close to zero, $\frac{5 x-9}{|x|+2}$ is getting extremely close to $\frac{0-9}{0+2}=\frac{-9}{2}$.
That is, $\lim _{x \rightarrow 0} \frac{5 x-9}{|x|+2}=-\frac{9}{2}$.
1.4.2.42. Solution. Since we aren't dividing by zero, and all these limits exist:

$$
\lim _{x \rightarrow-1} \frac{x f(x)+3}{2 f(x)+1}=\frac{(-1)(-1)+3}{2(-1)+1}=-4
$$

1.4.2.43. *. Solution. As $x \rightarrow-2$, the denominator goes to 0 , and the numerator goes to $-2 a+7$. For the ratio to have a limit, the numerator must also converge to 0 , so we need $a=\frac{7}{2}$. Then,

$$
\begin{aligned}
\lim _{x \rightarrow-2} \frac{x^{2}+a x+3}{x^{2}+x-2} & =\lim _{x \rightarrow-2} \frac{x^{2}+\frac{7}{2} x+3}{(x+2)(x-1)} \\
& =\lim _{x \rightarrow-2} \frac{(x+2)\left(x+\frac{3}{2}\right)}{(x+2)(x-1)} \\
& =\lim _{x \rightarrow-2} \frac{x+\frac{3}{2}}{x-1} \\
& =\frac{1}{6}
\end{aligned}
$$

so the limit exists when $a=\frac{7}{2}$.

### 1.4.2.44. Solution.

a $\lim _{x \rightarrow 0} f(x)=0$ : as $x$ approaches 0 , so does $2 x$.
b $\lim _{x \rightarrow 0} g(x)=$ DNE: the left and right limits do not agree, so the limit does not exist. In particular: $\lim _{x \rightarrow 0^{-}} g(x)=-\infty$ and $\lim _{x \rightarrow 0^{+}} g(x)=\infty$.
c $\lim _{x \rightarrow 0} f(x) g(x)=\lim _{x \rightarrow 0} 2 x \cdot \frac{1}{x}=\lim _{x \rightarrow 0} 2=2$. Remark: although the limit of $g(x)$ does not exist here, the limit of $f(x) g(x)$ does.
d $\lim _{x \rightarrow 0} \frac{f(x)}{g(x)}=\lim _{x \rightarrow 0} \frac{2 x}{\frac{1}{x}}=\lim _{x \rightarrow 0} 2 x^{2}=0$
e $\lim _{x \rightarrow 2} f(x)+g(x)=\lim _{x \rightarrow 2} 2 x+\frac{1}{x}=4+\frac{1}{2}=\frac{9}{2}$
f $\lim _{x \rightarrow 0} \frac{f(x)+1}{g(x+1)}=\lim _{x \rightarrow 0} \frac{2 x+1}{\frac{1}{x+1}}=\frac{1}{1}=1$

## Exercises - Stage 3

1.4.2.45. Solution. We can begin by plotting the points that are easy to read off the diagram.

| $x$ | $f(x)$ | $\frac{1}{f(x)}$ |
| :---: | :---: | :---: |
| -3 | -3 | $\frac{-1}{3}$ |
| -2 | 0 | $U N D$ |
| -1 | 3 | $\frac{1}{3}$ |
| 0 | 3 | $\frac{1}{3}$ |
| 1 | $\frac{3}{2}$ | $\frac{2}{3}$ |
| 2 | 0 | $U N D$ |
| 3 | 1 | 1 |

Note that $\frac{1}{f(x)}$ is undefined when $f(x)=0$. So $\frac{1}{f(x)}$ is undefined at $x=-2$ and $x=2$. We shall look more closely at the behaviour of $\frac{1}{f(x)}$ for $x$ near $\pm 2$ shortly. Plotting the above points, we get the following picture:


Since $f(x)$ is constant when $x$ is between -1 and 0 , then also $\frac{1}{f(x)}$ is constant between -1 and 0 , so we update our picture:


The big question that remains is the behaviour of $\frac{1}{f(x)}$ when $x$ is near -2 and 2 . We can answer this question with limits. As $x$ approaches -2 from the left, $f(x)$ gets closer to zero, and is negative. So $\frac{1}{f(x)}$ will be negative, and will increase in magnitude without bound; that is, $\lim _{x \rightarrow-2^{-}} \frac{1}{f(x)}=-\infty$. Similarly, as $x$ approaches -2 from the right, $f(x)$ gets closer to zero, and is positive. So $\frac{1}{f(x)}$ will be positive,
and will increase in magnitude without bound; that is, $\lim _{x \rightarrow-2^{+}} \frac{1}{f(x)}=\infty$. We add this behaviour to our graph:


Now, we consider the behaviour at $x=2$. Since $f(x)$ gets closer and closer to 0 AND is positive as $x$ approaches 2 , we conclude $\lim _{x \rightarrow 2} \frac{1}{f(x)}=\infty$. Adding to our picture:


Now the only remaining blank space is between $x=0$ and $x=1$. Since $f(x)$ is a smooth curve that stays away from 0 , we can draw some kind of smooth curve here,
and call it good enough. (Later on we'll go into more details about drawing graphs. The purpose of this exercise was to utilize what we've learned about limits.)

1.4.2.46. Solution. We can start by examining points.

| $x$ | $f(x)$ | $g(x)$ | $\frac{f(x)}{g(x)}$ |
| :--- | :--- | :--- | :--- |
| -3 | -3 | -1.5 | 2 |
| -2 | 0 | 0 | UND |
| -1 | 3 | 1.5 | 2 |
| -0 | 3 | 1.5 | 2 |
| 1 | 1.5 | .75 | 2 |
| 2 | 0 | 0 | UND |
| 3 | 1 | .5 | 2 |

We cannot divide by zero, so $\frac{f(x)}{g(x)}$ is not defined when $x= \pm 2$. But for every other value of $x$ that we plotted, $f(x)$ is twice as large as $g(x), \frac{f(x)}{g(x)}=2$. With this in mind, we see that the graph of $f(x)$ is exactly the graph of $2 g(x)$.
This gives us the graph below.


Remark: $f(2)=g(2)=0$, so $\frac{f(2)}{g(2)}$ does not exist, but $\lim _{x \rightarrow 2} \frac{f(x)}{g(x)}=2$. Although we are trying to "divide by zero" at $x= \pm 2$, it would be a mistake here to interpret this as a vertical asymptote.
1.4.2.47. Solution. Velocity of white ball when $t=1$ is $\lim _{h \rightarrow 0} \frac{s(1+h)-s(1)}{h}$, so the given information tells us $\lim _{h \rightarrow 0} \frac{s(1+h)-s(1)}{h}=5$. Then the velocity of the red ball when $t=1$ is $\lim _{h \rightarrow 0} \frac{2 s(1+h)-2 s(1)}{h}=\lim _{h \rightarrow 0} 2 \cdot \frac{s(1+h)-s(1)}{h}=2 \cdot 5=10$.
1.4.2.48. Solution. 1.4.2.48.a Neither limit exists. When $x$ gets close to 0 , these limits go to positive infinity from one side, and negative infinity from the other.
1.4.2.48.b $\lim _{x \rightarrow 0}[f(x)+g(x)]=\lim _{x \rightarrow 0}\left[\frac{1}{x}-\frac{1}{x}\right]=\lim _{x \rightarrow 0} 0=0$.
1.4.2.48.c No: this is an example of a time when the two individual functions have limits that don't exist, but the limit of their sum does exist. This "sum rule" is only true when both $\lim _{x \rightarrow a} f(x)$ and $\lim _{x \rightarrow a} g(x)$ exist.
1.4.2.49. Solution. 1.4.2.49.a When we evaluate the limit from the left, we only consider values of $x$ that are less than zero. For these values of $x$, our function is $x^{2}-3$. So, $\lim _{x \rightarrow 0^{-}} f(x)=\lim _{x \rightarrow 0^{-}}\left(x^{2}-3\right)=-3$.
1.4.2.49.b When we evaluate the limit from the right, we only consider values of $x$ that are greater than zero. For these values of $x$, our function is $x^{2}+3$. So, $\lim _{x \rightarrow 0^{+}} f(x)=\lim _{x \rightarrow 0^{+}}\left(x^{2}+3\right)=3$.
1.4.2.49.c Since the limits from the left and right do not agree, $\lim _{x \rightarrow 0} f(x)=$ DNE.

1.4.2.50. Solution. 1.4.2.50.a When we evaluate $\lim _{x \rightarrow-4^{-}} f(x)$, we only consider values of $x$ that are less than -4 . For these values, $f(x)=x^{3}+8 x^{2}+16 x$. So,

$$
\lim _{x \rightarrow 4^{-}} f(x)=\lim _{x \rightarrow-4^{-}}\left(x^{3}+8 x^{2}+16 x\right)=(-4)^{3}+8(-4)^{2}+16(-4)=0
$$

Note that, because $x^{3}+8 x^{2}+16 x$ is a polynomial, we can evaluate the limit by directly substituting in $x=-4$.
1.4.2.50.b When we evaluate $\lim _{x \rightarrow-4^{+}} f(x)$, we only consider values of $x$ that are greater than -4 . For these values,

$$
f(x)=\frac{x^{2}+8 x+16}{x^{2}+30 x-4}
$$

So

$$
\lim _{x \rightarrow-4^{+}} f(x)=\lim _{x \rightarrow-4^{+}} \frac{x^{2}+8 x+16}{x^{2}+30 x-4}
$$

This is a rational function, and $x=-4$ is in its domain (we aren't doing anything suspect, like dividing by 0 ), so again we can directly substitute $x=-4$ to evaluate
the limit:

$$
\lim _{x \rightarrow-4^{+}} f(x)=\frac{(-4)^{2}+8(-4)+16}{(-4)^{2}+30(-4)-4}=\frac{0}{-108}=0
$$

1.4.2.50.c Since $\lim _{x \rightarrow-4^{-}} f(x)=\lim _{x \rightarrow-4^{+}} f(x)=0$, we conclude that $\lim _{x \rightarrow-4} f(x)=0$

## $1.5 \cdot$ Limits at Infinity

### 1.5.2 • Exercises

## Exercises - Stage 1

1.5.2.1. Solution. Any polynomial of degree one or higher will go to $\infty$ or $-\infty$ as $x$ goes to $\infty$. So, we need a polynomial of degree 0 -that is, $f(x)$ is a constant. One possible answer is $f(x)=1$.
1.5.2.2. Solution. This will be the case for any polynomial of odd degree. For instance, $f(x)=x$.
Many answers are possible: also $f(x)=x^{15}-32 x^{2}+9$ satisfies $\lim _{x \rightarrow \infty} f(x)=\infty$ and $\lim _{x \rightarrow-\infty} f(x)=-\infty$.

## Exercises - Stage 2

1.5.2.3. Solution. $\lim _{x \rightarrow \infty} 2^{-x}=\lim _{x \rightarrow \infty} \frac{1}{2^{x}}=0$
1.5.2.4. Solution. As $x$ gets larger and larger, $2^{x}$ grows without bound. (For integer values of $x$, you can imagine multiplying 2 by itself more and more times.) So, $\lim _{x \rightarrow \infty} 2^{x}=\infty$.
1.5.2.5. Solution. Write $X=-x$. As $x$ becomes more and more negative, $X$ becomes more and more positive. From Question 1.5.2.4, we know that $2^{X}$ grows without bound as $X$ gets larger and larger. Since $2^{x}=2^{-(-x)}=2^{-X}=\frac{1}{2^{X}}$, as we let $x$ become a huge negative number, we are in effect dividing by a huge positive number; hence $\lim _{x \rightarrow-\infty} 2^{x}=0$.
A more formulaic way to describe the above is this: $\lim _{x \rightarrow-\infty} 2^{x}=\lim _{X \rightarrow \infty} 2^{-X}=$ $\lim _{X \rightarrow \infty} \frac{1}{2^{X}}=0$.
1.5.2.6. Solution. There is no single number that $\cos x$ approaches as $x$ becomes more and more strongly negative: as $x$ grows in the negative direction, the function oscillates between -1 and +1 , never settling close to one particular number. So, this limit does not exist.
1.5.2.7. Solution. The highest-order term in this polynomial is $-3 x^{5}$, so this dominates the function's behaviour as $x$ goes to infinity. More formally:

$$
\lim _{x \rightarrow \infty}\left(x-3 x^{5}+100 x^{2}\right)=\lim _{x \rightarrow \infty}-3 x^{5}\left(1-\frac{1}{3 x^{4}}-\frac{100}{3 x^{3}}\right)
$$

$$
=\lim _{x \rightarrow \infty}-3 x^{5}=-\infty
$$

because

$$
\lim _{x \rightarrow \infty}\left(1-\frac{1}{3 x^{4}}-\frac{100}{3 x^{3}}\right)=1-0-0=1
$$

1.5.2.8. Solution. Our standard trick is to factor out the highest power of $x$ in the denominator: $x^{4}$. We just have to be a little careful with the square root. Since we are taking the limit as $x$ goes to positive infinity, we have positive $x$-values, so $\sqrt{x^{2}}=x$ and $\sqrt{x^{8}}=x^{4}$.

$$
\begin{aligned}
\lim _{x \rightarrow \infty} \frac{\sqrt{3 x^{8}+7 x^{4}}+10}{x^{4}-2 x^{2}+1} & =\lim _{x \rightarrow \infty} \frac{\sqrt{x^{8}\left(3+\frac{7}{x^{4}}\right)}+10}{x^{4}\left(1-\frac{2}{x^{2}}+\frac{1}{x^{4}}\right)} \\
& =\lim _{x \rightarrow \infty} \frac{\sqrt{x^{8}} \sqrt{3+\frac{7}{x^{4}}}+10}{x^{4}\left(1-\frac{2}{x^{2}}+\frac{1}{x^{4}}\right)} \\
& =\lim _{x \rightarrow \infty} \frac{x^{4} \sqrt{3+\frac{7}{x^{4}}}+10}{x^{4}\left(1-\frac{2}{x^{2}}+\frac{1}{x^{4}}\right.} \\
& =\lim _{x \rightarrow \infty} \frac{x^{4}\left(\sqrt{3+\frac{7}{x^{4}}}+\frac{10}{x^{4}}\right)}{x^{4}\left(1-\frac{2}{x^{2}}+\frac{1}{x^{4}}\right)} \\
& =\lim _{x \rightarrow \infty} \frac{\sqrt{3+\frac{7}{x^{4}}}+\frac{10}{x^{4}}}{1-\frac{2}{x^{2}}+\frac{1}{x^{4}}} \\
& =\frac{\sqrt{3+0}+0}{1-0+0}=\sqrt{3}
\end{aligned}
$$

1.5.2.9. *. Solution. We have two terms, each getting extremely large. It's unclear at first what happens when we subtract them. To get this equation into another form, we multiply and divide by the conjugate, $\sqrt{x^{2}+5 x}+\sqrt{x^{2}-x}$.

$$
\begin{aligned}
\lim _{x \rightarrow \infty} & {\left[\sqrt{x^{2}+5 x}-\sqrt{x^{2}-x}\right] } \\
& =\lim _{x \rightarrow \infty}\left[\frac{\left(\sqrt{x^{2}+5 x}-\sqrt{x^{2}-x}\right)\left(\sqrt{x^{2}+5 x}+\sqrt{x^{2}-x}\right)}{\sqrt{x^{2}+5 x}+\sqrt{x^{2}-x}}\right] \\
& =\lim _{x \rightarrow \infty} \frac{\left(x^{2}+5 x\right)-\left(x^{2}-x\right)}{\sqrt{x^{2}+5 x}+\sqrt{x^{2}-x}} \\
& =\lim _{x \rightarrow \infty} \frac{6 x}{\sqrt{x^{2}+5 x}+\sqrt{x^{2}-x}}
\end{aligned}
$$

Now we divide the numerator and denominator by $x$. In the case of the denominator, since $x>0, x=\sqrt{x^{2}}$.

$$
=\lim _{x \rightarrow \infty} \frac{6(x)}{\sqrt{x^{2}} \sqrt{1+\frac{5}{x}}+\sqrt{x^{2}} \sqrt{1-\frac{1}{x}}}
$$

$$
\begin{aligned}
& =\lim _{x \rightarrow \infty} \frac{6(x)}{(x) \sqrt{1+\frac{5}{x}}+(x) \sqrt{1-\frac{1}{x}}} \\
& =\lim _{x \rightarrow \infty} \frac{6}{\sqrt{1+\frac{5}{x}}+\sqrt{1-\frac{1}{x}}} \\
& =\frac{6}{\sqrt{1+0}+\sqrt{1-0}}=3
\end{aligned}
$$

1.5.2.10. *. Solution. Note that for large negative $x$, the first term in the denominator $\sqrt{4 x^{2}+x} \approx \sqrt{4 x^{2}}=|2 x|=-2 x$ not $+2 x$. A good way to avoid incorrectly computing $\sqrt{x^{2}}$ when $x$ is negative is to define $y=-x$ and express everything in terms of $y$. That's what we'll do.

$$
\begin{aligned}
\lim _{x \rightarrow-\infty} \frac{3 x}{\sqrt{4 x^{2}+x}-2 x} & =\lim _{y \rightarrow+\infty} \frac{-3 y}{\sqrt{4 y^{2}-y}+2 y} \\
& =\lim _{y \rightarrow+\infty} \frac{-3 y}{y \sqrt{4-\frac{1}{y}}+2 y} \\
& =\lim _{y \rightarrow+\infty} \frac{-3}{\sqrt{4-\frac{1}{y}}+2} \\
& =\frac{-3}{\sqrt{4-0}+2} \quad \text { since } 1 / y \rightarrow 0 \text { as } y \rightarrow+\infty \\
& =-\frac{3}{4}
\end{aligned}
$$

1.5.2.11. *. Solution. The highest power of $x$ in the denominator is $x^{2}$, so we divide the numerator and denominator by $x^{2}$ :

$$
\begin{aligned}
\lim _{x \rightarrow-\infty} \frac{1-x-x^{2}}{2 x^{2}-7} & =\lim _{x \rightarrow-\infty} \frac{1 / x^{2}-1 / x-1}{2-7 / x^{2}} \\
& =\frac{0-0-1}{2-0}=-\frac{1}{2}
\end{aligned}
$$

### 1.5.2.12. *. Solution.

$$
\begin{aligned}
\lim _{x \rightarrow \infty}\left(\sqrt{x^{2}+x}-x\right) & =\lim _{x \rightarrow \infty} \frac{\left(\sqrt{x^{2}+x}-x\right)\left(\sqrt{x^{2}+x}+x\right)}{\sqrt{x^{2}+x}+x} \\
& =\lim _{x \rightarrow \infty} \frac{\left(x^{2}+x\right)-x^{2}}{\sqrt{x^{2}+x}+x}=\lim _{x \rightarrow \infty} \frac{x}{\sqrt{x^{2}+x}+x} \\
& =\lim _{x \rightarrow \infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1}=\frac{1}{2}
\end{aligned}
$$

1.5.2.13. *. Solution. We have, after dividing both numerator and denominator
by $x^{2}$ (which is the highest power of the denominator) that

$$
\frac{5 x^{2}-3 x+1}{3 x^{2}+x+7}=\frac{5-\frac{3}{x}+\frac{1}{x^{2}}}{3+\frac{1}{x}+\frac{7}{x^{2}}} .
$$

Since $1 / x \rightarrow 0$ and also $1 / x^{2} \rightarrow 0$ as $x \rightarrow+\infty$, we conclude that

$$
\lim _{x \rightarrow+\infty} \frac{5 x^{2}-3 x+1}{3 x^{2}+x+7}=\frac{5}{3} .
$$

1.5.2.14. *. Solution. We have, after dividing both numerator and denominator by $x$ (which is the highest power of the denominator) that

$$
\frac{\sqrt{4 x+2}}{3 x+4}=\frac{\sqrt{\frac{4}{x}+\frac{2}{x^{2}}}}{3+\frac{4}{x}} .
$$

Since $1 / x \rightarrow 0$ and also $1 / x^{2} \rightarrow 0$ as $x \rightarrow+\infty$, we conclude that

$$
\lim _{x \rightarrow+\infty} \frac{\sqrt{4 x+2}}{3 x+4}=\frac{0}{3}=0
$$

1.5.2.15. *. Solution. The dominant terms in the numerator and denominator have order $x^{3}$. Taking out that common factor we get

$$
\frac{4 x^{3}+x}{7 x^{3}+x^{2}-2}=\frac{4+\frac{1}{x^{2}}}{7+\frac{1}{x}-\frac{2}{x^{3}}} .
$$

Since $1 / x^{a} \rightarrow 0$ as $x \rightarrow+\infty$ (for $a>0$ ), we conclude that

$$
\lim _{x \rightarrow+\infty} \frac{4 x^{3}+x}{7 x^{3}+x^{2}-2}=\frac{4}{7}
$$

### 1.5.2.16. Solution.

- Solution 1

We want to factor out $x$, the highest power in the denominator. Since our limit only sees negative values of $x$, we must remember that $\sqrt[4]{x^{4}}=|x|=-x$, although $\sqrt[3]{x^{3}}=x$.

$$
\begin{aligned}
\lim _{x \rightarrow-\infty} \frac{\sqrt[3]{x^{2}+x}-\sqrt[4]{x^{4}+5}}{x+1} & =\lim _{x \rightarrow-\infty} \frac{\left.\sqrt[3]{x^{3}\left(\frac{1}{x}+\frac{1}{x^{2}}\right.}\right)-\sqrt[4]{x^{4}\left(1+\frac{5}{x^{4}}\right)}}{x\left(1+\frac{1}{x}\right)} \\
& =\lim _{x \rightarrow-\infty} \frac{\sqrt[3]{x^{3}} \sqrt[3]{\frac{1}{x}+\frac{1}{x^{2}}}-\sqrt[4]{x^{4}} \sqrt[4]{1+\frac{5}{x^{4}}}}{x\left(1+\frac{1}{x}\right)} \\
& =\lim _{x \rightarrow-\infty} \frac{x \sqrt[3]{\frac{1}{x}+\frac{1}{x^{2}}}-(-x) \sqrt[4]{1+\frac{5}{x^{4}}}}{x\left(1+\frac{1}{x}\right)}
\end{aligned}
$$

$$
\begin{aligned}
& =\lim _{x \rightarrow-\infty} \frac{\sqrt[3]{\frac{1}{x}+\frac{1}{x^{2}}}+\sqrt[4]{1+\frac{5}{x^{4}}}}{1+\frac{1}{x}} \\
& =\frac{\sqrt[3]{0+0}+\sqrt[4]{1+0}}{1+0}=1
\end{aligned}
$$

## - Solution 2

Alternately, we can use the transformation $\lim _{x \rightarrow-\infty} f(x)=\lim _{x \rightarrow \infty} f(-x)$. Then we only look at positive values of $x$, so roots behave nicely: $\sqrt[4]{x^{4}}=|x|=x$.

$$
\begin{aligned}
\lim _{x \rightarrow-\infty} \frac{\sqrt[3]{x^{2}+x}-\sqrt[4]{x^{4}+5}}{x+1} & =\lim _{x \rightarrow \infty} \frac{\sqrt[3]{(-x)^{2}-x}-\sqrt[4]{(-x)^{4}+5}}{-x+1} \\
& =\lim _{x \rightarrow \infty} \frac{\sqrt[3]{x^{2}-x}-\sqrt[4]{x^{4}+5}}{-x+1} \\
& =\lim _{x \rightarrow \infty} \frac{\sqrt[3]{x^{3}} \sqrt[3]{\frac{1}{x}-\frac{1}{x^{2}}}-\sqrt[4]{x^{4}} \sqrt[4]{1+\frac{5}{x^{4}}}}{x\left(-1+\frac{1}{x}\right)} \\
& =\lim _{x \rightarrow \infty} \frac{x \sqrt[3]{\frac{1}{x}-\frac{1}{x^{2}}}-x \sqrt[4]{1+\frac{5}{x^{4}}}}{x\left(-1+\frac{1}{x}\right)} \\
& =\lim _{x \rightarrow \infty} \frac{\sqrt[3]{\frac{1}{x}-\frac{1}{x^{2}}}-\sqrt[4]{1+\frac{5}{x^{4}}}}{-1+\frac{1}{x}} \\
& =\frac{\sqrt[3]{0-0}-\sqrt[4]{1+0}}{-1+0}=\frac{-1}{-1}=1
\end{aligned}
$$

1.5.2.17. *. Solution. We have, after dividing both numerator and denominator by $x^{3}$ (which is the highest power of the denominator) that:

$$
\lim _{x \rightarrow \infty} \frac{5 x^{2}+10}{3 x^{3}+2 x^{2}+x}=\lim _{x \rightarrow \infty} \frac{\frac{5}{x}+\frac{10}{x^{3}}}{3+\frac{2}{x}+\frac{1}{x^{2}}}=\frac{0}{3}=0 .
$$

1.5.2.18. Solution. Since we only consider negative values of $x, \sqrt{x^{2}}=|x|=-x$.

$$
\begin{aligned}
\lim _{x \rightarrow-\infty} \frac{x+1}{\sqrt{x^{2}}} & =\lim _{x \rightarrow-\infty} \frac{x+1}{-x} \\
& =\lim _{x \rightarrow-\infty} \frac{x}{-x}+\frac{1}{-x} \\
& =\lim _{x \rightarrow-\infty}-1-\frac{1}{x} \\
& =-1
\end{aligned}
$$

1.5.2.19. Solution. Since we only consider positive values of $x, \sqrt{x^{2}}=|x|=x$.

$$
\lim _{x \rightarrow \infty} \frac{x+1}{\sqrt{x^{2}}}=\lim _{x \rightarrow \infty} \frac{x+1}{x}
$$

$$
=\lim _{x \rightarrow \infty} 1+\frac{1}{x}=1+0=1
$$

1.5.2.20. *. Solution. When $x<0,|x|=-x$ and so $\lim _{x \rightarrow \infty} \sin \left(\frac{\pi}{2} \cdot \frac{|x|}{x}\right)+\frac{1}{x}=$ $\sin (-\pi / 2)=-1$.
1.5.2.21. *. Solution. We divide both the numerator and the denominator by the highest power of $x$ in the denominator, which is $x$. Since $x<0$, we have $\sqrt{x^{2}}=|x|=-x$, so that

$$
\frac{\sqrt{x^{2}+5}}{x}=-\sqrt{\frac{x^{2}+5}{x^{2}}}=-\sqrt{1+\frac{5}{x^{2}}} .
$$

Since $1 / x \rightarrow 0$ and also $1 / x^{2} \rightarrow 0$ as $x \rightarrow-\infty$, we conclude that

$$
\lim _{x \rightarrow-\infty} \frac{3 x+5}{\sqrt{x^{2}+5}-x}=\lim _{x \rightarrow-\infty} \frac{3+\frac{5}{x}}{-\sqrt{1+\frac{5}{x^{2}}}-1}=\frac{3}{-1-1}=-\frac{3}{2}
$$

1.5.2.22. *. Solution. We divide both the numerator and the denominator by the highest power of $x$ in the denominator, which is $x$. Since $x<0$, we have $\sqrt{x^{2}}=|x|=-x$, so that

$$
\frac{\sqrt{4 x^{2}+15}}{x}=\frac{\sqrt{4 x^{2}+15}}{-\sqrt{x^{2}}}=-\sqrt{\frac{4 x^{2}+15}{x^{2}}}=-\sqrt{4+\frac{15}{x^{2}}} .
$$

Since $1 / x \rightarrow 0$ and also $1 / x^{2} \rightarrow 0$ as $x \rightarrow-\infty$, we conclude that

$$
\lim _{x \rightarrow-\infty} \frac{5 x+7}{\sqrt{4 x^{2}+15}-x}=\lim _{x \rightarrow-\infty} \frac{5+\frac{7}{x}}{-\sqrt{4+\frac{15}{x^{2}}}-1}=\frac{5}{-2-1}=-\frac{5}{3}
$$

### 1.5.2.23. Solution.

$$
\begin{aligned}
\lim _{x \rightarrow-\infty} \frac{3 x^{7}+x^{5}-15}{4 x^{2}+32 x} & =\lim _{x \rightarrow-\infty} \frac{x^{2}\left(3 x^{5}+x^{3}-\frac{15}{x^{2}}\right)}{x^{2}\left(4+\frac{32}{x}\right)} \\
& =\lim _{x \rightarrow-\infty} \frac{3 x^{5}+x^{3}-\frac{15}{x^{2}}}{4+\frac{32}{x}} \\
& =\lim _{x \rightarrow+\infty} \frac{3(-x)^{5}+(-x)^{3}-\frac{15}{(-x)^{2}}}{4+\frac{32}{-x}} \\
& =\lim _{x \rightarrow+\infty} \frac{-3 x^{5}-x^{3}-\frac{15}{x^{2}}}{4-\frac{32}{x}} \\
& =-\infty
\end{aligned}
$$

1.5.2.24. *. Solution. We multiply and divide the expression by its conjugate,
$\left(\sqrt{n^{2}+5 n}+n\right)$.

$$
\begin{aligned}
\lim _{n \rightarrow \infty}\left(\sqrt{n^{2}+5 n}-n\right) & =\lim _{n \rightarrow \infty}\left(\sqrt{n^{2}+5 n}-n\right)\left(\frac{\sqrt{n^{2}+5 n}+n}{\sqrt{n^{2}+5 n}+n}\right) \\
& =\lim _{n \rightarrow \infty} \frac{\left(n^{2}+5 n\right)-n^{2}}{\sqrt{n^{2}+5 n}+n} \\
& =\lim _{n \rightarrow \infty} \frac{5 n}{\sqrt{n^{2}+5 n}+n} \\
& =\lim _{n \rightarrow \infty} \frac{5 \cdot n}{\sqrt{n^{2}} \sqrt{1+\frac{5}{n}}+n}
\end{aligned}
$$

Since $n>0$, we can simplify $\sqrt{n^{2}}=n$.

$$
\begin{aligned}
& =\lim _{n \rightarrow \infty} \frac{5 \cdot n}{n \sqrt{1+\frac{5}{n}}+n} \\
& =\lim _{n \rightarrow \infty} \frac{5}{\sqrt{1+\frac{5}{n}}+1} \\
& =\frac{5}{\sqrt{1+0}+1}=\frac{5}{2}
\end{aligned}
$$

### 1.5.2.25. Solution.

## - Solution 1:

When $a$ approaches 0 from the right, the numerator approaches negative infinity, and the denominator approaches -1 . So, $\lim _{a \rightarrow 0^{+}} \frac{a^{2}-\frac{1}{a}}{a-1}=\infty$.
More precisely, using Theorem 1.5.9:

$$
\begin{aligned}
\lim _{a \rightarrow 0^{+}} \frac{1}{a} & =+\infty \\
\text { Also, } \lim _{a \rightarrow 0^{+}} a^{2} & =0
\end{aligned}
$$

So, using Theorem 1.5.9,

$$
\begin{array}{r}
\qquad \lim _{a \rightarrow 0^{+}} a^{2}-\frac{1}{a}=-\infty \\
\text { Furthermore, } \lim _{a \rightarrow 0^{+}} a-1=-1 \\
\text { So, using our theorem, } \lim _{a \rightarrow 0^{+}} \frac{a^{2}-\frac{1}{a}}{a-1}=\infty
\end{array}
$$

## - Solution 2:

Since $a=0$ is not in the domain of our function, a reasonable impulse is to simplify.

$$
\frac{a^{2}-\frac{1}{a}}{a-1}\left(\frac{a}{a}\right)=\frac{a^{3}-1}{a(a-1)}=\frac{(a-1)\left(a^{2}+a+1\right)}{a(a-1)}
$$

```
So,
```

$$
\begin{aligned}
\lim _{a \rightarrow 0^{+}} \frac{a^{2}-\frac{1}{a}}{a-1} & =\lim _{a \rightarrow 0^{+}} \frac{(a-1)\left(a^{2}+a+1\right)}{a(a-1)} \\
& =\lim _{a \rightarrow 0^{+}} \frac{a^{2}+a+1}{a} \\
& =\lim _{a \rightarrow 0^{+}} a+1+\frac{1}{a}=\infty
\end{aligned}
$$

1.5.2.26. Solution. Since $x=3$ is not in the domain of the function, we simplify, hoping we can cancel a problematic term.

$$
\begin{aligned}
\lim _{x \rightarrow 3} \frac{2 x+8}{\frac{1}{x-3}+\frac{1}{x^{2}-9}} & =\lim _{x \rightarrow 3} \frac{2 x+8}{\frac{x+3}{x^{2}-9}+\frac{1}{x^{2}-9}} \\
& =\lim _{x \rightarrow 3} \frac{2 x+8}{\frac{x+4}{x^{2}-9}} \\
& =\lim _{x \rightarrow 3} \frac{(2 x+8)\left(x^{2}-9\right)}{x+4}=0
\end{aligned}
$$

## Exercises - Stage 3

1.5.2.27. Solution. First, we need a rational function whose limit at infinity is a real number. This means that the degree of the bottom is greater than or equal to the degree of the top. There are two cases: the denominator has higher degree than the numerator, or the denominator has the same degree as the numerator.
If the denominator has higher degree than the numerator, then $\lim _{x \rightarrow \infty} f(x)=$ $\lim _{x \rightarrow-\infty} f(x)=0$, so the limits are equal-not what we're looking for.
If the denominator has the same degree as the numerator, then the limit as $x$ goes to $\pm \infty$ is the ratio of the leading terms: again, the limits are equal. So no rational function exists as described.
1.5.2.28. Solution. The amount of the substance that will linger long-term is some positive number-the substance will stick around. One example of a substance that does this is the ink in a tattoo. (If the injection was of medicine, probably it will be metabolized, and $\lim _{t \rightarrow \infty} c(t)=0$.)
Remark: it actually doesn't make much sense to let $t$ go to infinity: after a few million hours, you won't even have a body, so what is $c(t)$ measuring? Often when we use formulas in the real world, there is an understanding that they are only good for some fixed range. We often use the limit as $t$ goes to infinity as a stand-in for the function's long-term behaviour.

## 1.6 • Continuity

### 1.6.4 • Exercises

## Exercises - Stage 1

1.6.4.1. Solution. Many answers are possible; the tangent function behaves like this.
1.6.4.2. Solution. If we let $f$ be my height, the time of my birth is $a$, and now is $b$, then we know that $f(a) \leq 1 \leq f(b)$. It is reasonable to assume that my height is a continuous function. So by the IVT, there is some value $c$ between $a$ and $b$ where $f(c)=1$. That is, there is some time (we called it $c$ ) between my birth and today when I was exactly one meter tall.
Notice the IVT does not say precisely what day I was one meter tall; it only guarantees that such a day occurred between my birth and today.
1.6.4.3. Solution. One example is $f(x)=\left\{\begin{array}{ll}0 & \text { when } 0 \leq x \leq 1 \\ 2 & \text { when } 1<x \leq 2\end{array}\right.$. The IVT only guarantees $f(c)=1$ for some $c$ in $[0,2]$ when $f$ is continuous over $[0,2]$. If $f$ is not continuous, the IVT says nothing.

1.6.4.4. Solution. Yes. This is a straightforward application of IVT.
1.6.4.5. Solution. No. IVT says that $f(x)=0$ for some $x$ between 10 and 20, but it doesn't have to be exactly half way between.
1.6.4.6. Solution. No. IVT says nothing about functions that are not guaranteed to be continuous at the outset. It's quite easy to construct a function that is as described, but not continuous. For example, the function pictured below, whose equation format is somewhat less enlightening than its graph: $f(x)=\left\{\begin{array}{ll}-\frac{26}{5} x+65, & 10 \leq x<15 \\ -\frac{26}{5} x+91, & 15 \leq x \leq 20\end{array}\right.$.

1.6.4.7. Solution. True. Since $f(t)$ is continuous at $t=5$, that means $\lim _{t \rightarrow 5} f(t)=$ $f(5)$. For that to be true, $f(5)$ must exist - that is, 5 is in the domain of $f(x)$.
1.6.4.8. Solution. True. Using the definition of continuity, $\lim _{t \rightarrow 5} f(t)=f(5)=17$.
1.6.4.9. Solution. In general, false. If $f(t)$ is continuous at $t=5$, then $f(5)=17$; if $f(t)$ is discontinuous at $t=5$, then $f(5)$ either does not exist, or is a number other than 17.
An example of a function with $\lim _{t \rightarrow 5} f(t)=17 \neq f(5)$ is $f(t)=\left\{\begin{array}{ll}17 & , t \neq 5 \\ 0, & t=5\end{array}\right.$, shown below.

1.6.4.10. Solution. Since $f(x)$ and $g(x)$ are continuous at zero, and since $g^{2}(x)+$ 1 must be nonzero, then $h(x)$ is continuous at 0 as well. According to the definition of continuity, then $\lim _{x \rightarrow 0} h(x)$ exists and is equal to $h(0)=\frac{0 f(0)}{g^{2}(0)+1}=0$.
Since the limit $\lim _{x \rightarrow 0} h(x)$ exists and is equal to zero, also the one-sided limit $\lim _{x \rightarrow 0^{+}} h(x)$ exists and is equal to zero.

## Exercises - Stage 2

1.6.4.11. Solution. Using the definition of continuity, we need $k=\lim _{x \rightarrow 0} f(x)$. Since the limit is blind to what actually happens to $f(x)$ at $x=0$, this is equivalent to $k=\lim _{x \rightarrow 0} x \sin \left(\frac{1}{x}\right)$. So if we find the limit, we solve the problem.
As $x$ gets small, $\sin \left(\frac{1}{x}\right)$ goes a little crazy (see example 1.3.5), so let's get rid of it by using the Squeeze Theorem. We can bound the function above and below, but we should be a little careful about whether we're going from the left or the right. The reason we need to worry about direction is illustrated with the following observation:
If $a \leq b$ and $x>0$, then $x a \leq x b$. (For example, plug in $x=1, a=2, b=3$.) But if $a \leq b$ and $x<0$, then $x a \geq x b$. (For example, plug in $x=-1, a=2, b=3$.) So first, let's find $\lim _{x \rightarrow 0^{-}} \sin \left(\frac{1}{x}\right)$. When $x<0$,

$$
1 \geq \sin \left(\frac{1}{x}\right) \geq-1
$$

$$
\text { so: } \quad x(1) \leq x \sin \left(\frac{1}{x}\right) \leq x(-1)
$$

and $\lim _{x \rightarrow 0^{-}} x=\lim _{x \rightarrow 0^{-}}-x=0$, so by the Squeeze Theorem, also $\lim _{x \rightarrow 0^{-}} x \sin \left(\frac{1}{x}\right)=0$. Now, let's find $\lim _{x \rightarrow 0^{+}} x \sin \left(\frac{1}{x}\right)$. When $x>0$,

$$
\left.\begin{array}{rl}
-1 & \leq \sin \left(\frac{1}{x}\right)
\end{array}\right)=1
$$

and $\lim _{x \rightarrow 0^{+}} x=\lim _{x \rightarrow 0^{+}}-x=0$, so by the Squeeze Theorem, also $\lim _{x \rightarrow 0^{+}} x \sin \left(\frac{1}{x}\right)=0$.
Since the limits from the left and right agree, we conclude $\lim _{x \rightarrow 0} x \sin \left(\frac{1}{x}\right)=0$, so when $k=0$, the function is continuous at $x=0$.
1.6.4.12. Solution. Since $f$ is a polynomial, it is continuous over all real numbers. $f(0)=1<12345$ and $f(12345)=12345^{3}+12345^{2}+12345+1>12345$ (since all terms are positive). So by the IVT, $f(c)=12345$ for some $c$ between 0 and 12345 .
1.6.4.13. *. Solution. $f(x)$ is a rational function and so is continuous except when its denominator is zero. That is, except when $x=1$ and $x=-1$.
1.6.4.14. *. Solution. The function is continuous when $x^{2}-1>0$, i.e. $(x-$ $1)(x+1)>0$, which yields the intervals $(-\infty,-1) \cup(1,+\infty)$.
1.6.4.15. *. Solution. The function $1 / \sqrt{x}$ is continuous on $(0,+\infty)$ and the function $\cos (x)+1$ is continuous everywhere.
So $1 / \sqrt{\cos (x)+1}$ is continuous except when $\cos x=-1$. This happens when $x$ is an odd multiple of $\pi$. Hence the function is continuous except at $x= \pm \pi, \pm 3 \pi, \pm 5 \pi, \cdots$.
1.6.4.16. *. Solution. The function is continuous when $\sin (x) \neq 0$. That is, when $x$ is not an integer multiple of $\pi$.
1.6.4.17. *. Solution. The function is continuous for $x \neq c$ since each of those two branches are polynomials. So, the only question is whether the function is continuous at $x=c$; for this we need

$$
\lim _{x \rightarrow c^{-}} f(x)=f(c)=\lim _{x \rightarrow c+} f(x)
$$

We compute

$$
\begin{gathered}
\lim _{x \rightarrow c^{-}} f(x)=\lim _{x \rightarrow c^{-}} 8-c x=8-c^{2} \\
f(c)=8-c \cdot c=8-c^{2} \text { and } \\
\lim _{x \rightarrow c^{+}} f(x)=\lim _{x \rightarrow c^{+}} x^{2}=c^{2}
\end{gathered}
$$

So, we need $8-c^{2}=c^{2}$, which yields $c^{2}=4$, i.e. $c=-2$ or $c=2$.
1.6.4.18. *. Solution. The function is continuous for $x \neq 0$ since $x^{2}+c$ and $\cos c x$ are continuous everywhere. It remains to check continuity at $x=0$. To do this we must check that the following three are equal.

$$
\begin{aligned}
\lim _{x \rightarrow 0^{+}} f(x) & =\lim _{x \rightarrow 0^{+}} x^{2}+c=c \\
f(0) & =c \\
\lim _{x \rightarrow 0^{-}} f(x) & =\lim _{x \rightarrow 0^{-}} \cos c x=\cos 0=1
\end{aligned}
$$

Hence when $c=1$ we have the limits agree.
1.6.4.19. *. Solution. The function is continuous for $x \neq c$ since each of those two branches are defined by polynomials. Thus, the only question is whether the function is continuous at $x=c$. Furthermore,

$$
\lim _{x \rightarrow c^{-}} f(x)=c^{2}-4
$$

and

$$
\lim _{x \rightarrow c^{+}} f(x)=f(c)=3 c
$$

For continunity we need both limits and the value to agree, so $f$ is continuous if and only if $c^{2}-4=3 c$, that is if and only if

$$
c^{2}-3 c-4=0
$$

Factoring this as $(c-4)(c+1)=0$ yields $c=-1$ or $c=+4$.
1.6.4.20. *. Solution. The function is continuous for $x \neq c$ since each of those two branches are polynomials. So, the only question is whether the function is continuous at $x=c$; for this we need

$$
\lim _{x \rightarrow 2 c^{-}} f(x)=f(2 c)=\lim _{x \rightarrow 2 c+} f(x)
$$

We compute

$$
\begin{gathered}
\lim _{x \rightarrow 2 c^{-}} f(x)=\lim _{x \rightarrow 2 c^{-}} 6-c x=6-2 c^{2} \\
f(2 c)=6-c \cdot 2 c=6-2 c^{2} \text { and } \\
\lim _{x \rightarrow 2 c^{+}} f(x)=\lim _{x \rightarrow 2 c^{+}} x^{2}=4 c^{2}
\end{gathered}
$$

So, we need $6-2 c^{2}=4 c^{2}$, which yields $c^{2}=1$, i.e. $c=-1$ or $c=1$.

## Exercises - Stage 3

1.6.4.21. Solution. This isn't the kind of equality that we can just solve; we'll need a trick, and that trick is the IVT. The general idea is to show that $\sin x$ is somewhere bigger, and somewhere smaller, than $x-1$. However, since the IVT can only show us that a function is equal to a constant, we need to slightly adjust our language. Showing $\sin x=x-1$ is equivalent to showing $\sin x-x+1=0$, so let
$f(x)=\sin x-x+1$, and let's show that it has a real root.
First, we need to note that $f(x)$ is continuous (otherwise we can't use the IVT).
Now, we need to find a value of $x$ for which it is positive, and for which it's negative. By checking a few values, we find $f(0)$ is positive, and $f(100)$ is negative. So, by the IVT, there exists a value of $x$ (between 0 and 100) for which $f(x)=0$. Therefore, there exists a value of $x$ for which $\sin x=x-1$.
1.6.4.22. *. Solution. We let $f(x)=3^{x}-x^{2}$. Then $f(x)$ is a continuous function, since both $3^{x}$ and $x^{2}$ are continuous for all real numbers.
We find a value $a$ such that $f(a)>0$. We observe immediately that $a=0$ works since

$$
f(0)=3^{0}-0=1>0 .
$$

We find a value $b$ such that $f(b)<0$. We see that $b=-1$ works since

$$
f(-1)=\frac{1}{3}-1<0 .
$$

So, because $f(x)$ is continuous on $(-\infty, \infty)$ and $f(0)>0$ while $f(-1)<0$, then the Intermediate Value Theorem guarantees the existence of a real number $c$ in the interval $(-1,0)$ such that $f(c)=0$.
1.6.4.23. *. Solution. We let $f(x)=2 \tan (x)-x-1$. Then $f(x)$ is a continuous function on the interval $(-\pi / 2, \pi / 2)$ since $\tan (x)=\sin (x) / \cos (x)$ is continuous on this interval, while $x+1$ is a polynomial and therefore continuous for all real numbers.
We find a value $a \in(-\pi / 2, \pi / 2)$ such that $f(a)<0$. We observe immediately that $a=0$ works since

$$
f(0)=2 \tan (0)-0-1=0-1=-1<0 .
$$

We find a value $b \in(-\pi / 2, \pi / 2)$ such that $f(b)>0$. We see that $b=\pi / 4$ works since

$$
\begin{aligned}
f(\pi / 4) & =2 \tan (\pi / 4)-\pi / 4-1=2-\pi / 4-1=1-\pi / 4 \\
& =(4-\pi) / 4>0
\end{aligned}
$$

because $3<\pi<4$.
So, because $f(x)$ is continuous on $[0, \pi / 4]$ and $f(0)<0$ while $f(\pi / 4)>0$, then the Intermediate Value Theorem guarantees the existence of a real number $c \in(0, \pi / 4)$ such that $f(c)=0$.
1.6.4.24. *. Solution. Let $f(x)=\sqrt{\cos (\pi x)}-\sin (2 \pi x)-1 / 2$. This function is continuous provided $\cos (\pi x) \geq 0$. This is true for $0 \leq x \leq \frac{1}{2}$.
Now $f$ takes positive values on $[0,1 / 2]$ :

$$
f(0)=\sqrt{\cos (0)}-\sin (0)-1 / 2=\sqrt{1}-1 / 2=1 / 2 .
$$

And $f$ takes negative values on $[0,1 / 2]$ :

$$
f(1 / 2)=\sqrt{\cos (\pi / 2)}-\sin (\pi)-1 / 2=0-0-1 / 2=-1 / 2
$$

(Notice that $f(1 / 3)=(\sqrt{2}-\sqrt{3}) / 2-1 / 2$ also works)
So, because $f(x)$ is continuous on $[0,1 / 2)$ and $f(0)>0$ while $f(1 / 2)<0$, then the Intermediate Value Theorem guarantees the existence of a real number $c \in(0,1 / 2)$ such that $f(c)=0$.
1.6.4.25. *. Solution. We let $f(x)=\frac{1}{\cos ^{2}(\pi x)}-x-\frac{3}{2}$. Then $f(x)$ is a continuous function on the interval $(-1 / 2,1 / 2)$ since $\cos x$ is continuous everywhere and nonzero on that interval.
The function $f$ takes negative values. For example, when $x=0$ :

$$
f(0)=\frac{1}{\cos ^{2}(0)}-0-\frac{3}{2}=1-\frac{3}{2}=-\frac{1}{2}<0 .
$$

It also takes positive values, for instance when $x=1 / 4$ :

$$
\begin{aligned}
f(1 / 4) & =\frac{1}{(\cos \pi / 4)^{2}}-\frac{1}{4}-\frac{3}{2} \\
& =\frac{1}{1 / 2}-\frac{1+6}{4} \\
& =2-7 / 4=1 / 4>0 .
\end{aligned}
$$

By the IVT there is $c, 0<c<1 / 4$ such that $f(c)=0$, in which case

$$
\frac{1}{(\cos \pi c)^{2}}=c+\frac{3}{2}
$$

1.6.4.26. Solution. $f(x)$ is a polynomial, so it's continuous everywhere. If we can find values $a$ and $b$ so that $f(a)>0$ and $f(b)<0$, then by the IVT, there will exist some $c$ in $(a, b)$ where $f(c)=0$; that is, there is a root in the interval $[a, b]$. Let's start plugging in easy values of $x$.
$f(0)=15$, and $f(1)=1-15+9-18+15=-8$. Since 0 is between $f(0)$ and $f(1)$, and since $f$ is continuous, by IVT there must be some $x$ in $[0,1]$ for which $f(x)=0$ : that is, there is some root in $[0,1]$.
That was the easiest interval to find. If you keep playing around, you find two more. $f(-1)=26$ (positive) and $f(-2)=-1001$ (negative), so there is a root in $[-2,-1]$. The arithmetic is nasty, but there is also a root in $[14,15]$.
This is an important trick. For high-degree polynomials, it is often impossible to get the exact values of the roots. Using the IVT, we can at least give a range where a root must be.
1.6.4.27. Solution. Let $f(x)=x^{3}$. Since $f$ is a polynomial, it is continuous everywhere. If $f(a)<7<f(b)$, then $\sqrt[3]{7}$ is somewhere between $a$ and $b$. If we can find $a$ and $b$ that satisfy these inequalities, and are very close together, that will give us a good approximation for $\sqrt[3]{7}$.

- Let's start with integers. $1^{3}<7<2^{3}$, so $\sqrt[3]{7}$ is in the interval $(1,2)$.
- Let's narrow this down, say by testing $f(1.5) .(1.5)^{3}=3.375<7$, so $\sqrt[3]{7}$ is in the interval $(1.5,2)$.
- Let's narrow further, say by testing $f(1.75) .(1.75)^{3} \approx 5.34<7$, so $\sqrt[3]{7}$ is in the interval $(1.75,2)$.
- Testing various points, we find $f(1.9)<7<f(2)$, so $\sqrt[3]{7}$ is between 1.9 and 2.
- By testing more, we find $f(1.91)<7<f(1.92)$, so $\sqrt[3]{7}$ is in $(1.91,1.92)$.
- In order to get an approximation for $\sqrt[3]{7}$ that is rounded to two decimal places, we have to know whether $\sqrt[3]{7}$ is greater or less than 1.915 ; indeed $f(1.915) \approx 7.02>7$, so $\sqrt[3]{7}<1.915$; then rounded to two decimal places, $\sqrt[3]{7} \approx 1.91$.

If this seems like the obvious way to approximate $\sqrt[3]{7}$, that's good. The IVT is a formal statement of a very intuitive principle.

### 1.6.4.28. Solution.

- If $f(a)=g(a)$, or $f(b)=g(b)$, then we simply take $c=a$ or $c=b$.
- Suppose $f(a) \neq g(a)$ and $f(b) \neq g(b)$. Then $f(a)<g(a)$ and $g(b)<f(b)$, so if we define $h(x)=f(x)-g(x)$, then $h(a)<0$ and $h(b)>0$. Since $h$ is the difference of two functions that are continuous over $[a, b]$, also $h$ is continuous over $[a, b]$. So, by the Intermediate Value Theorem, there exists some $c \in(a, b)$ with $h(c)=0$; that is, $f(c)=g(c)$.


## 2 - Derivatives

## 2.1 $\cdot$ Revisiting Tangent Lines

### 2.1.2 • Exercises

## Exercises - Stage 1

2.1.2.1. Solution. If $Q$ is to the left of the $y$ axis, the line through $Q$ and $P$ is increasing, so the secant line has positive slope. If $Q$ is to the right of the $y$ axis, the line through $Q$ and $P$ is decreasing, so the secant line has negative slope.
2.1.2.2. Solution. 2.1.2.2.a By drawing a few pictures, it's easy to see that sliding $Q$ closer to $P$, the slope of the secant line increases.
2.1.2.2.b Since the slope of the secant line increases the closer $Q$ gets to $P$, that means the tangent line (which is the limit as $Q$ approaches $P$ ) has a larger slope than the secant line between $Q$ and $P$ (using the location where $Q$ is right now). Alternately, by simply sketching the tangent line at $P$, we can see that has a steeper slope than the secant line between $P$ and $Q$.
2.1.2.3. Solution. The slope of the secant line will be $\frac{f(2)-f(-2)}{2-(-2)}=$
$\frac{f(2)-f(-2)}{4}$, in every part. So, if two lines have the same slope, that means their differences $f(2)-f(-2)$ will be the same.
The graphs in (a),(c), and (e) all have $f(2)-f(-2)=1$, so they all have the same secant line slope. The graphs in (b) and (f) both have $f(2)-f(-2)=-1$, so they both have the same secant line slope. The graph in (d) has $f(2)-f(-2)=0$, and it is the only graph with this property, so it does not share its secant line slope with any of the other graphs.

## Exercises - Stage 2

2.1.2.4. Solution. A good approximation from the graph is $f(5)=0.5$. We want to find a secant line whose endpoints are both very close to $x=5$, but that also give us clear $y$-values. It looks like $f(5.25) \approx 1$, and $f(4.75) \approx \frac{1}{8}$. The secant line from $x=5$ to $x=5.25$ has approximate slope $\frac{f(5.25)-f(5)}{5.25-5} \approx \frac{1-.5}{.25}=2$. The secant line from $x=5$ to $x=4.75$ has approximate slope $\frac{0.5-\frac{1}{8}}{5-4.75}=\frac{3}{2}$.
The graph increases more and more quickly (gets steeper and steeper), so it makes sense that the secant line to the left of $x=5$ has a smaller slope than the secant line to the right of $x=5$. Also, if you're taking secant lines that have endpoints farther out from $x=5$, you'll notice that the slopes of the secant lines change quite dramatically. You have to be very, very close to $x=5$ to get any kind of accuracy. If we split the difference, we might approximate the slope of the secant line to be the average of $\frac{3}{2}$ and 2 , which is $\frac{7}{4}$.
Another way to try to figure out the tangent line is by carefully drawing it in with a ruler. This is shown here in blue:


It's much easier to take the slope of a line than a curve, and this one looks like it has slope about 1.5. However, we drew this with a computer: by hand it's much
harder to draw an accurate tangent line. (That's why we need calculus!)
The actual slope of the tangent line to the function at $x=5$ is about 1.484. This is extremely hard to figure out just from the graph-by hand, a guess between 1.25 and 1.75 would be very accurate.
2.1.2.5. Solution. There is only one tangent line to $f(x)$ at $P$ (shown in blue), but there are infinitely many choices of $Q$ and $R$ (one possibility shown in red). One easy way to sketch the secant line on paper is to draw any line parallel to the tangent line, and choose two intercepts with $y=f(x)$.

2.1.2.6. Solution. Any place the graph looks flat (if you imagine zooming in) is where the tangent line has slope 0 . This occurs three times.


Notice that two of the indicated points are at a low point and a high point, respectively. Later, we'll use these places where the tangent line has slope zero to find where a graph achieves its biggest and smallest values.

## 2.2 • Definition of the Derivative

### 2.2.4 • Exercises

## Exercises - Stage 1

2.2.4.1. Solution. The function shown is a line, so it has a constant slope-(a) . Since the function is always increasing, $f^{\prime}$ is always positive, so also (d) holds. Remark: it does not matter that the function itself is sometimes negative; the slope is always positive because the function is always increasing. Also, since the slope is constant, $f^{\prime}$ is neither increasing nor decreasing: it is the function that is increasing, not its derivative.
2.2.4.2. Solution. The function is always decreasing, so $f^{\prime}$ is always negative, option (e). However, the function alternates between being more and less steep, so $f^{\prime}$ alternates between increasing and decreasing several times, and no other options hold.
Remark: $f$ is always positive, but (d) does not hold!
2.2.4.3. Solution. At the left end of the graph, $f$ is decreasing rapidly, so $f^{\prime}$ is a strongly negative number. Then as we move towards $x=0, f$ decreases less rapidly, so $f^{\prime}$ is a less strongly negative number. As we pass $0, f$ increases, so $f^{\prime}$ is a positive number. As we move to the right, $f$ increases more and more rapidly, so $f^{\prime}$ is an increasing positive number. This description tells us that $f^{\prime}$ increases for the entire range shown. So (b) holds, but not (a) or (c). Since $f^{\prime}$ is negative to the left of the $y$ axis, and positive to the right of it, also (d) and (e) do not hold.
2.2.4.4. *. Solution. By definition, $f(x)=x^{3}$ is differentiable at $x=0$ if the limit

$$
\lim _{h \rightarrow 0} \frac{f(h)-f(0)}{h}=\lim _{h \rightarrow 0} \frac{h^{3}-0}{h}
$$

exists.
2.2.4.5. Solution. $f^{\prime}(-1)$ does not exist, because to the left of $x=-1$ the slope is a pretty big positive number (looks like around +1 ) and to the right the slope is $-1 / 4$. Since the derivative involves a limit, that limit needs to match the limit from the left and the limit from the right. The sharp angle made by the graph at $x=-1$ indicates that the left and right limits do not match, so the derivative does not exist.
$f^{\prime}(3)$ also does not exist. One way to see this is to notice that the function is discontinuous here. More viscerally, note that $f(3)=1$, so as we take secant lines with one endpoint $(3,1)$, and the other endpoint just to the right of $x=3$, we get slopes that are more and more strongly negative, as shown in the picture below. If we take the limit of the slopes of these secant lines as $x$ goes to 3 from the right, we get $-\infty$. (This certainly doesn't match the slope from the left, which is $-\frac{1}{4}$.)


At $x=-3$, there is some kind of "change" in the graph; however, it is a smooth curve, so the derivative exists here.
2.2.4.6. Solution. True. The definition of the derivative tells us that

$$
f^{\prime}(a)=\lim _{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}
$$

if it exists. We know from our work with limits that if both one-sided limits $\lim _{h \rightarrow 0^{-}} \frac{f(a+h)-f(a)}{h}$ and $\lim _{h \rightarrow 0^{+}} \frac{f(a+h)-f(a)}{h}$ exist and are equal to each other, then $\lim _{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}$ exists and has the same value as the one-sided limits. So, since the one-sided limits exist and are equal to one, we conclude $f^{\prime}(a)$ exists and is equal to one.
2.2.4.7. Solution. In general, this is false. The key problem that can arise is that $f(x)$ might not be continuous at $x=1$. One example is the function

$$
f(x)= \begin{cases}x & x<0 \\ x-1 & x \geq 0\end{cases}
$$

where $f^{\prime}(x)=1$ whenever $x \neq 0$ (so in particular, $\lim _{x \rightarrow 0^{-}} f^{\prime}(x)=\lim _{x \rightarrow 0^{+}} f^{\prime}(x)=1$ ) but $f^{\prime}(0)$ does not exist.
There are two ways to see that $f^{\prime}(0)$ does not exist. One is to notice that $f$ is not continuous at $x=0$.


Another way to see that $f^{\prime}(0)$ does not exist is to use the definition of the derivative. Remember, in order for a limit to exist, both one-sided limits must exist. Let's consider the limit from the left. If $h \rightarrow 0^{-}$, then $h<0$, so $f(h)$ is equal to $h$ (not $h-1)$.

$$
\begin{aligned}
\lim _{h \rightarrow 0^{-}} \frac{f(0+h)-f(0)}{h} & =\lim _{h \rightarrow 0^{-}} \frac{(h)-(-1)}{h} \\
& =\lim _{h \rightarrow 0^{-}} \frac{h+1}{h} \\
& =\lim _{h \rightarrow 0^{-}} 1+\frac{1}{h} \\
& =-\infty
\end{aligned}
$$

In particular, this limit does not exist. Since the one-sided limit does not exist,

$$
\lim _{h \rightarrow 0} \frac{f(0+h)-f(0)}{h}=D N E
$$

and so $f^{\prime}(0)$ does not exist.
2.2.4.8. Solution. Using the definition of the derivative,

$$
s^{\prime}(t)=\lim _{h \rightarrow 0} \frac{s(t+h)-s(t)}{h}
$$

The units of the numerator are meters, and the units of the denominator are seconds (since the denominator comes from the change in the input of the function). So, the units of $s^{\prime}(t)$ are metres per second.
Remark: we learned that the derivative of a position function gives velocity. In this example, the position is given in metres, and the velocity is measured in metres per second.

## Exercises - Stage 2

2.2.4.9. Solution. We can use point-slope form to get the equation of the line, if we have a point and its slope. The point is given: $(1,6)$. The slope is the derivative:

$$
\begin{aligned}
y^{\prime}(1) & =\lim _{h \rightarrow 0} \frac{y(1+h)-y(1)}{h} \\
& =\lim _{h \rightarrow 0} \frac{\left[(1+h)^{3}+5\right]-\left[1^{3}+5\right]}{h} \\
& =\lim _{h \rightarrow 0} \frac{\left[1+3 h+3 h^{2}+h^{3}+5\right]-[1+5]}{h} \\
& =\lim _{h \rightarrow 0} \frac{3 h+3 h^{2}+h^{3}}{h} \\
& =\lim _{h \rightarrow 0} 3+3 h+h^{2} \\
& =3
\end{aligned}
$$

So our slope is 3 , which gives a line of equation $y-6=3(x-1)$.
2.2.4.10. Solution. We set up the definition of the derivative.

$$
\begin{aligned}
f^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} \\
& =\lim _{h \rightarrow 0} \frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{h} \\
& =\lim _{h \rightarrow 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} \\
& =\lim _{h \rightarrow 0} \frac{\frac{-h}{x(x+h)}}{h} \\
& =\lim _{h \rightarrow 0} \frac{-1}{x(x+h)} \\
& =\frac{-1}{x^{2}}
\end{aligned}
$$

2.2.4.11. *. Solution. By definition

$$
f^{\prime}(0)=\lim _{h \rightarrow 0} \frac{f(h)-f(0)}{h}=\lim _{h \rightarrow 0} \frac{h|h|}{h}=\lim _{h \rightarrow 0}|h|=0
$$

In particular, the limit exists, so the derivative exists (and is equal to zero).
2.2.4.12. *. Solution. We set up the definition of the derivative.

$$
\begin{aligned}
f^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}=\lim _{h \rightarrow 0} \frac{1}{h}\left(\frac{2}{x+h+1}-\frac{2}{x+1}\right) \\
& =\lim _{h \rightarrow 0} \frac{2}{h} \frac{(x+1)-(x+h+1)}{(x+h+1)(x+1)}
\end{aligned}
$$

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{2}{h} \frac{-h}{(x+h+1)(x+1)} \\
& =\lim _{h \rightarrow 0} \frac{-2}{(x+h+1)(x+1)} \\
& =\frac{-2}{(x+1)^{2}}
\end{aligned}
$$

### 2.2.4.13. *. Solution.

$$
\begin{aligned}
f^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}=\lim _{h \rightarrow 0} \frac{1}{h}\left(\frac{1}{(x+h)^{2}+3}-\frac{1}{x^{2}+3}\right) \\
& =\lim _{h \rightarrow 0} \frac{1}{h} \frac{x^{2}-(x+h)^{2}}{\left[(x+h)^{2}+3\right]\left[x^{2}+3\right]} \\
& =\lim _{h \rightarrow 0} \frac{1}{h} \frac{-2 x h-h^{2}}{\left[(x+h)^{2}+3\right]\left[x^{2}+3\right]} \\
& =\lim _{h \rightarrow 0} \frac{-2 x-h}{\left[(x+h)^{2}+3\right]\left[x^{2}+3\right]} \\
& =\frac{-2 x}{\left[x^{2}+3\right]^{2}}
\end{aligned}
$$

2.2.4.14. Solution. The slope of the tangent line is the derivative. We set this up using the same definition of the derivative that we always do. This limit is hard to take for general $x$, but easy when $x=0$.

$$
\begin{aligned}
f^{\prime}(0) & =\lim _{h \rightarrow 0} \frac{f(0+h)-f(0)}{h} \\
& =\lim _{h \rightarrow 0} \frac{h \log _{10}(2 h+10)-0}{h} \\
& =\lim _{h \rightarrow 0} \log _{10}(2 h+10)=\log _{10}(10)=1
\end{aligned}
$$

So, the slope of the tangent line is 1 .

### 2.2.4.15. *. Solution.

$$
\begin{aligned}
f^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}=\lim _{h \rightarrow 0} \frac{\frac{1}{(x+h)^{2}}-\frac{1}{x^{2}}}{h}=\lim _{h \rightarrow 0} \frac{x^{2}-(x+h)^{2}}{(x+h)^{2} x^{2} h} \\
& =\lim _{h \rightarrow 0} \frac{-2 x h-h^{2}}{(x+h)^{2} x^{2} h}=\lim _{h \rightarrow 0} \frac{-2 x-h}{(x+h)^{2} x^{2}}=\frac{-2 x}{x^{4}} \\
& =-\frac{2}{x^{3}}
\end{aligned}
$$

2.2.4.16. *. Solution. When $x$ is not equal to 2 , then the function is differentiable the only place we have to worry about is when $x$ is exactly 2 .
In order for $f$ to be differentiable at $x=2$, it must also be continuous at $x=2$. This forces $\left.x^{2}\right|_{x=2}=[a x+b]_{x=2}$ or

$$
2 a+b=4 .
$$

In order for a limit to exist, the left- and right-hand limits must exist and be equal
to each other. Since a derivative is a limit, in order for $f$ to be differentiable at $x=2$, the left hand derivative of $a x+b$ at $x=2$ must be the same as the right hand derivative of $x^{2}$ at $x=2$. Since $a x+b$ is a line, its derivative is $a$ everywhere. We've already seen the derivative of $x^{2}$ is $2 x$, so we need

$$
a=\left.2 x\right|_{x=2}=4
$$

So, the values of $a$ and $b$ that makes $f$ differentiable everywhere are $a=4$ and $b=-4$.
2.2.4.17. *. Solution. We plug in $f(x)$ to the definition of a derivative. To evaluate the limit, we multiply and divide by the conjugate of the numerator, then simplify.

$$
\begin{aligned}
f^{\prime}(x)= & \lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}=\lim _{h \rightarrow 0} \frac{\sqrt{1+x+h}-\sqrt{1+x}}{h} \\
& =\lim _{h \rightarrow 0} \frac{\sqrt{1+x+h}-\sqrt{1+x}}{h}\left(\frac{\sqrt{1+x+h}+\sqrt{1+x}}{\sqrt{1+x+h}+\sqrt{1+x}}\right) \\
& =\lim _{h \rightarrow 0} \frac{(1+x+h)-(1+x)}{h(\sqrt{1+x+h}+\sqrt{1+x})} \\
& =\lim _{h \rightarrow 0} \frac{h}{h(\sqrt{1+x+h}+\sqrt{1+x})} \\
& =\lim _{h \rightarrow 0} \frac{1}{\sqrt{1+x+h}+\sqrt{1+x}} \\
& =\frac{1}{\sqrt{1+x+0}+\sqrt{1+x}}=\frac{1}{2 \sqrt{1+x}}
\end{aligned}
$$

The domain of the function is $[-1, \infty)$. In particular, $f(x)$ is defined when $x=-1$. However, $f^{\prime}(x)$ is not defined when $x=-1$, so $f^{\prime}(x)$ only exists over $(-1, \infty)$.
Remark: $\lim _{x \rightarrow-1^{+}} f^{\prime}(x)=\infty$, so the tangent line to $f(x)$ at the point $x=-1$ has a vertical slope.

## Exercises - Stage 3

2.2.4.18. Solution. From Section 1.2 , we see that the velocity is exactly the derivative.

$$
\begin{aligned}
v(t) & =\lim _{h \rightarrow 0} \frac{s(t+h)-s(t)}{h} \\
& =\lim _{h \rightarrow 0} \frac{(t+h)^{4}-(t+h)^{2}-t^{4}+t^{2}}{h} \\
& =\lim _{h \rightarrow 0} \frac{\left(t^{4}+4 t^{3} h+6 t^{2} h^{2}+4 t h^{3}+h^{4}\right)-\left(t^{2}+2 t h+h^{2}\right)-t^{4}+t^{2}}{h} \\
& =\lim _{h \rightarrow 0} \frac{4 t^{3} h+6 t^{2} h^{2}+4 t h^{3}+h^{4}-2 t h-h^{2}}{h} \\
& =\lim _{h \rightarrow 0} 4 t^{3}+6 t^{2} h+4 t h^{2}+h^{3}-2 t-h
\end{aligned}
$$

$$
=4 t^{3}-2 t
$$

So, the velocity is given by $v(t)=4 t^{3}-2 t$.
2.2.4.19. *. Solution. The function is differentiable at $x=0$ if the following limit:

$$
\lim _{x \rightarrow 0} \frac{f(x)-f(0)}{x-0}=\lim _{x \rightarrow 0} \frac{f(x)-0}{x}=\lim _{x \rightarrow 0} \frac{f(x)}{x}
$$

exists (note that we used the fact that $f(0)=0$ as per the definition of the first branch which includes the point $x=0$ ). We start by computing the left limit. For this computation, recall that if $x<0$ then $\sqrt{x^{2}}=|x|=-x$.

$$
\begin{aligned}
\lim _{x \rightarrow 0^{-}} \frac{f(x)}{x} & =\lim _{x \rightarrow 0^{-}} \frac{\sqrt{x^{2}+x^{4}}}{x}=\lim _{x \rightarrow 0^{-}} \frac{\sqrt{x^{2}} \sqrt{1+x^{2}}}{x} \\
& =\lim _{x \rightarrow 0} \frac{-x \sqrt{1+x^{2}}}{x}=-1
\end{aligned}
$$

Now, from the right:

$$
\lim _{x \rightarrow 0^{+}} \frac{x \cos x}{x}=\lim _{x \rightarrow 0^{+}} \cos x=1
$$

Since the limit from the left does not equal the limit from the right, the derivative does not exist at $x=0$.
2.2.4.20. *. Solution. The function is differentiable at $x=0$ if the following limit:

$$
\lim _{x \rightarrow 0} \frac{f(x)-f(0)}{x-0}=\lim _{x \rightarrow 0} \frac{f(x)-0}{x}=\lim _{x \rightarrow 0} \frac{f(x)}{x}
$$

exists (note that we used the fact that $f(0)=0$ as per the definition of the first branch which includes the point $x=0$ ).
We start by computing the left limit.

$$
\lim _{x \rightarrow 0^{-}} \frac{f(x)}{x}=\lim _{x \rightarrow 0^{-}} \frac{x \cos x}{x}=\lim _{x \rightarrow 0^{-}} \cos x=1
$$

Now, from the right:

$$
\begin{aligned}
\lim _{x \rightarrow 0^{+}} \frac{\sqrt{1+x}-1}{x} & =\lim _{x \rightarrow 0^{+}} \frac{\sqrt{1+x}-1}{x} \cdot \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} \\
& =\lim _{x \rightarrow 0^{+}} \frac{1+x-1}{x(\sqrt{1+x}+1)}=\lim _{x \rightarrow 0^{+}} \frac{1}{\sqrt{1+x}+1}=\frac{1}{2}
\end{aligned}
$$

Since the limit from the left does not equal the limit from the right, the derivative does not exist at $x=0$.
2.2.4.21. *. Solution. The function is differentiable at $x=0$ if the following limit:

$$
\lim _{x \rightarrow 0} \frac{f(x)-f(0)}{x-0}=\lim _{x \rightarrow 0} \frac{f(x)-0}{x}=\lim _{x \rightarrow 0} \frac{f(x)}{x}
$$

exists (note that we used the fact that $f(0)=0$ as per the definition of the first
branch which includes the point $x=0$ ). We compute left and right limits; so

$$
\lim _{x \rightarrow 0^{-}} \frac{f(x)}{x}=\lim _{x \rightarrow 0^{-}} \frac{x^{3}-7 x^{2}}{x}=\lim _{x \rightarrow 0^{-}} x^{2}-7 x=0
$$

and

$$
\lim _{x \rightarrow 0^{+}} \frac{x^{3} \cos \left(\frac{1}{x}\right)}{x}=\lim _{x \rightarrow 0^{+}} x^{2} \cdot \cos \left(\frac{1}{x}\right) .
$$

This last limit equals 0 by the Squeeze Theorem since

$$
-1 \leq \cos \left(\frac{1}{x}\right) \leq 1
$$

and so,

$$
-x^{2} \leq x^{2} \cdot \cos \left(\frac{1}{x}\right) \leq x^{2}
$$

where in these inequalities we used the fact that $x^{2} \geq 0$. Finally, since $\lim _{x \rightarrow 0^{+}}-x^{2}=\lim _{x \rightarrow 0^{+}} x^{2}=0$, the Squeeze Theorem yields that also $\lim _{x \rightarrow 0^{+}} x^{2} \cos \left(\frac{1}{x}\right)=0$, as claimed.
Since the left and right limits match (they're both equal to 0 ), we conclude that indeed $f(x)$ is differentiable at $x=0$ (and its derivative at $x=0$ is actually equal to 0 ).
2.2.4.22. *. Solution. The function is differentiable at $x=1$ if the following limit:

$$
\lim _{x \rightarrow 1} \frac{f(x)-f(1)}{x-1}=\lim _{x \rightarrow 1} \frac{f(x)-0}{x-1}=\lim _{x \rightarrow 1} \frac{f(x)}{x-1}
$$

exists (note that we used the fact that $f(1)=0$ as per the definition of the first branch which includes the point $x=0$ ). We compute left and right limits; so

$$
\begin{aligned}
\lim _{x \rightarrow 1^{-}} \frac{f(x)}{x-1} & =\lim _{x \rightarrow 1^{-}} \frac{4 x^{2}-8 x+4}{x-1}=\lim _{x \rightarrow 1^{-}} \frac{4(x-1)^{2}}{x-1} \\
& =\lim _{x \rightarrow 1^{-}} 4(x-1)=0
\end{aligned}
$$

and

$$
\lim _{x \rightarrow 1^{+}} \frac{(x-1)^{2} \sin \left(\frac{1}{x-1}\right)}{x-1}=\lim _{x \rightarrow 1^{+}}(x-1) \cdot \sin \left(\frac{1}{x-1}\right) .
$$

This last limit equals 0 by the Squeeze Theorem since

$$
-1 \leq \sin \left(\frac{1}{x-1}\right) \leq 1
$$

and so,

$$
-(x-1) \leq(x-1) \cdot \sin \left(\frac{1}{x-1}\right) \leq x-1
$$

where in these inequalities we used the fact that $x \rightarrow 1^{+}$yields positive values for $x-1$. Finally, since $\lim _{x \rightarrow 1^{+}}-x+1=\lim _{x \rightarrow 1^{+}} x-1=0$, the Squeeze Theorem yields that also $\lim _{x \rightarrow 1^{+}}(x-1) \sin \left(\frac{1}{x-1}\right)=0$, as claimed.
Since the left and right limits match (they're both equal to 0 ), we conclude that indeed $f(x)$ is differentiable at $x=1$ (and its derivative at $x=1$ is actually equal to 0 ).
2.2.4.23. . Solution. Many answers are possible; here is one.


The key is to realize that the few points you're given suggest a pattern, but don't guarantee it. You only know nine points; anything can happen in between.

### 2.2.4.24. Solution.

$$
\begin{aligned}
p^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{p(x+h)-p(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} \\
& =\lim _{h \rightarrow 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right] \\
(*) & =\left[\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}\right]+\left[\lim _{h \rightarrow 0} \frac{g(x+h)-g(x)}{h}\right] \\
& =f^{\prime}(x)+g^{\prime}(x)
\end{aligned}
$$

At step $(*)$, we use the limit law that $\lim _{x \rightarrow a}[F(x)+G(x)]=\lim _{x \rightarrow a} F(x)+\lim _{x \rightarrow a} G(x)$, as long as $\lim _{x \rightarrow a} F(x)$ and $\lim _{x \rightarrow a} G(x)$ exist. Because the problem states that $f^{\prime}(x)$ and
$g^{\prime}(x)$ exist, we know that $\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ and $\lim _{h \rightarrow 0} \frac{g(x+h)-g(x)}{h}$ exist, so our work is valid.
2.2.4.25. Solution. 2.2.4.25. a Since $y=f(x)=2 x$ and $y=g(x)=x$ are straight lines, we don't need the definition of the derivative (although you can use it if you like). $f^{\prime}(x)=2$ and $g^{\prime}(x)=1$.
2.2.4.25.b $p(x)=2 x^{2}$, so $p(x)$ is not a line: we use the definition of a derivative to find $p^{\prime}(x)$.

$$
\begin{aligned}
p^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{p(x+h)-p(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{2(x+h)^{2}-2 x^{2}}{h} \\
& =\lim _{h \rightarrow 0} \frac{2 x^{2}+4 x h+2 h^{2}-2 x^{2}}{h} \\
& =\lim _{h \rightarrow 0} \frac{4 x h+2 h^{2}}{h} \\
& =\lim _{h \rightarrow 0} 4 x+2 h \\
& =4 x
\end{aligned}
$$

2.2.4.25.c No, $p^{\prime}(x)=4 x \neq 2 \cdot 1=f^{\prime}(x) \cdot g^{\prime}(x)$. In general, the derivative of a product is not the same as the derivative of the functions being multiplied.
2.2.4.26. *. Solution. We know that $y^{\prime}=2 x$. So, if we choose a point $\left(\alpha, \alpha^{2}\right)$ on the curve $y=x^{2}$, then the tangent line to the curve at that point has slope $2 \alpha$. That is, the tangent line has equation

$$
\begin{aligned}
\left(y-\alpha^{2}\right) & =2 \alpha(x-\alpha) \\
\text { simplified, } \quad y & =(2 \alpha) x-\alpha^{2}
\end{aligned}
$$

So, if $(1,-3)$ is on the tangent line, then

$$
\begin{aligned}
& & -3 & =(2 \alpha)(1)-\alpha^{2} \\
& \Longleftrightarrow & 0 & =\alpha^{2}-2 \alpha-3 \\
& \Longleftrightarrow & 0 & =(\alpha-3)(\alpha+1) \\
& \Longleftrightarrow & \alpha & =3, \quad \text { or } \quad \alpha=-1 .
\end{aligned}
$$

So, the tangent lines $y=(2 \alpha) x-\alpha^{2}$ are

$$
y=6 x-9 \quad \text { and } \quad y=-2 x-1
$$

2.2.4.27. *. Solution. Using the definition of the derivative, $f$ is differentiable at 0 if and only if

$$
\lim _{h \rightarrow 0} \frac{f(h)-f(0)}{h}
$$

exists. In particular, this means $f$ is differentiable at 0 if and only if both one-sided limits exist and are equal to each other.

When $h<0, f(h)=0$, so

$$
\lim _{h \rightarrow 0^{-}} \frac{f(h)-f(0)}{h}=\lim _{h \rightarrow 0^{-}} \frac{0-0}{h}=0
$$

So, $f$ is differentiable at $x=0$ if and only if

$$
\lim _{h \rightarrow 0^{+}} \frac{f(h)-f(0)}{h}=0
$$

To evaluate the limit above, we note $f(0)=0$ and, when $h>0, f(h)=h^{a} \sin \left(\frac{1}{h}\right)$, so

$$
\begin{aligned}
\lim _{h \rightarrow 0^{+}} \frac{f(h)-f(0)}{h} & =\lim _{h \rightarrow 0^{+}} \frac{h^{a} \sin \left(\frac{1}{h}\right)}{h} \\
& =\lim _{h \rightarrow 0^{+}} h^{a-1} \sin \left(\frac{1}{h}\right)
\end{aligned}
$$

We will spend the rest of this solution evaluating the limit above for different values of $a$, to find when it is equal to zero and when it is not. Let's consider the different values that could be taken by $h^{a-1}$.

- If $a=1$, then $a-1=0$, so $h^{a-1}=h^{0}=1$ for all values of $h$. Then

$$
\lim _{h \rightarrow 0^{+}} h^{a-1} \sin \left(\frac{1}{h}\right)=\lim _{h \rightarrow 0^{+}} \sin \left(\frac{1}{h}\right)=D N E
$$

(Recall that the function $\sin \left(\frac{1}{x}\right)$ oscillates faster and faster as $x$ goes to 0 . We first saw this behaviour in Example 1.3.5.)

- If $a<1$, then $a-1<0$, so $\lim _{h \rightarrow 0^{+}} h^{a-1}=\infty$. (Since we have a negative exponent, we are in effect dividing by a smaller and smaller positive number. For example, if $a=\frac{1}{2}$, then $\lim _{h \rightarrow 0^{+}} h^{a-1}=\lim _{h \rightarrow 0^{+}} h^{-\frac{1}{2}}=\lim _{h \rightarrow 0^{+}} \frac{1}{\sqrt{h}}=\infty$.) Since $\sin \left(\frac{1}{x}\right)$ goes back and forth between one and negative one,

$$
\lim _{h \rightarrow 0^{+}} h^{a-1} \sin \left(\frac{1}{x}\right)=D N E
$$

since as $h$ goes to 0 , the function oscillates between positive and negative numbers of ever-increasing magnitude.

- If $a>1$, then $a-1>0$, so $\lim _{h \rightarrow 0^{+}} h^{a-1}=0$. Although $\sin \left(\frac{1}{x}\right)$ oscillates wildly near $x=0$, it is bounded by -1 and 1 . So,

$$
(-1) h^{a-1} \leq h^{a-1} \sin \left(\frac{1}{h}\right) \leq h^{a-1}
$$

Since both $\lim _{h \rightarrow 0^{+}}(-1) h^{a-1}=0$ and $\lim _{h \rightarrow 0^{+}} h^{a-1}=0$, by the Squeeze Theorem,

$$
\lim _{h \rightarrow 0^{+}} h^{a-1} \sin \left(\frac{1}{x}\right)=0
$$

as well.
In the above cases, we learned $\lim _{h \rightarrow 0^{+}} \frac{f(h)-f(0)}{h}=\lim _{h \rightarrow 0^{+}} h^{a-1} \sin \left(\frac{1}{x}\right)=0$ when $a>1$, and
$\lim _{h \rightarrow 0^{+}} \frac{f(h)-f(0)}{h}=\lim _{h \rightarrow 0^{+}} h^{a-1} \sin \left(\frac{1}{x}\right) \neq 0$ when $a \leq 1$.
So, $f$ is differentiable at $x=0$ if and only if $a>1$.

## 2.3 • Interpretations of the Derivative

### 2.3.3 • Exercises

## Exercises - Stage 2

2.3.3.1. Solution. 2.3.3.1.a The slope of the secant line is $\frac{h(24)-h(0)}{24-0} \quad \frac{\mathrm{~m}}{\mathrm{hr}}$; this is the change in height over the first day divided by the number of hours in the first day. So, it is the average rate of change of the height over the first day, measured in meters per hour.
2.3.3.1.b Consider 2.3.3.1.a. The secant line gives the average rate of change of the height of the dam; as we let the second point of the secant line get closer and closer to $(0, h(0))$, its slope approximates the instantaneous rate of change of the height of the water. So the slope of the tangent line is the instantaneous rate of change of the height of the water at the time $t=0$, measured in $\frac{\mathrm{m}}{\mathrm{hr}}$.
2.3.3.2. Solution. $\quad p^{\prime}(t)=\lim _{h \rightarrow 0} \frac{p(t+h)-p(t)}{h} \approx \frac{p(t+1)-p(t)}{1}=p(t+1)-p(t)$, or the difference in profit caused by the sale of the $(t+1)^{\text {st }}$ widget. So, $p^{\prime}(t)$ is the profit from the $(t+1)^{\text {st }}$ widget. That is, $p^{\prime}(t)$ is the profit per additional widget sold, when $t$ widgets are being sold. This is called the marginal profit per widget, when $t$ widgets are being sold.
2.3.3.3. Solution. How quickly the temperature is changing per unit change of depth, measured in degrees per metre. In an ordinary body of water, the temperature near the surface $(d=0)$ is pretty variable, depending on the sun, but deep down it is more stable (unless there are heat sources). So, one might reasonably expect that $\left|T^{\prime}(d)\right|$ is larger when $d$ is near 0 .

### 2.3.3.4. Solution.

$$
\begin{aligned}
C^{\prime}(w) & =\lim _{h \rightarrow 0} \frac{C(w+h)-C(w)}{h} \approx \frac{C(w+1)-C(w)}{1} \\
& =C(w+1)-C(w)
\end{aligned}
$$

which is the number of calories in $C(w+1)$ grams minus the number of calories in $C(w)$ grams. This is the number of calories per additional gram, when there are $w$ grams.
2.3.3.5. Solution. The rate of change of velocity is acceleration. (If your velocity is increasing, you're accelerating; if your velocity is decreasing, you have negative acceleration.)
2.3.3.6. Solution. The rate of change in this case will be the relationship between the heat added and the temperature change. $\lim _{h \rightarrow 0} \frac{T(j+h)-T(j)}{h} \approx$ $\frac{T(j+1)-T(j)}{1}=T(j+1)-T(j)$, or the change in temperature after the application of one joule. (This is closely related to heat capacity and to specific heat - there's a nice explanation of this on Wikipedia.)
2.3.3.7. Solution. As usual, it is instructive to think about the definition of the derivative:

$$
\begin{aligned}
P^{\prime}(T) & =\lim _{h \rightarrow 0} \frac{P(T+h)=P(T)}{h} \approx \frac{P(T+1)-P(t)}{1} \\
& =P(T+1)-P(T)
\end{aligned}
$$

This is the difference in population between two hypothetical populations, raised one degree in temperature apart. So, it is the number of extra individuals that exist in the hotter experiment (with the understanding that this number could be negative, as one would expect in conditions that are hotter than the bacteria prefer). So $P^{\prime}(T)$ is the number of bacteria added to the colony per degree.

## Exercises - Stage 3

2.3.3.8. Solution. $R^{\prime}(t)$ is the rate at which the wheel is rotating measured in rotations per second. To convert to degrees, we multiply by $360: 360 R^{\prime}(t)$.
2.3.3.9. Solution. If $P^{\prime}(t)$ is positive, your sample is below the ideal temperature, because adding heat increases the population. If $P^{\prime}(t)$ is negative, your sample is above the ideal temperature, because adding heat decreases the population. If $P^{\prime}(t)=0$, then adding a little bit of heat doesn't change the population, but it's unclear why this is. Perhaps your sample is deeply frozen, and adding heat doesn't change the fact that your population is 0 . Perhaps your sample is boiling, and again, changing the heat a little will keep the population constant at "none." But also, at the ideal temperature, you would expect $P^{\prime}(t)=0$. This is best seen by noting in the curve below, the tangent line is horizontal at the peak.


## 2.4 • Arithmetic of Derivatives - a Differentiation Toolbox

### 2.4.2 • Exercises

## Exercises - Stage 1

2.4.2.1. Solution. True: this is exactly what the Sum Rule states.
2.4.2.2. Solution. False, in general. The product rule tells us $\frac{\mathrm{d}}{\mathrm{d} x}\{f(x) g(x)\}=$ $f^{\prime}(x) g(x)+f(x) g^{\prime}(x)$. An easy example of why we can't do it the other way is to take $f(x)=g(x)=x$. Then the equation becomes $\frac{\mathrm{d}}{\mathrm{d} x}\left\{x^{2}\right\}=(1)(1)$, which is false.
2.4.2.3. Solution. True: the quotient rule tells us

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{f(x)}{g(x)}\right\} & =\frac{g(x) f^{\prime}(x)-f(x) g^{\prime}(x)}{g^{2}(x)}=\frac{g(x) f^{\prime}(x)}{g^{2}(x)}-\frac{f(x) g^{\prime}(x)}{g^{2}(x)} \\
& =\frac{f^{\prime}(x)}{g(x)}-\frac{f(x) g^{\prime}(x)}{g^{2}(x)}
\end{aligned}
$$

2.4.2.4. Solution. If you're creative, you can find lots of ways to differentiate!

- Constant multiple: $g^{\prime}(x)=3 f^{\prime}(x)$.
- Product rule:

$$
g^{\prime}(x)=\frac{\mathrm{d}}{\mathrm{~d} x}\{3\} f(x)+3 f^{\prime}(x)=0 f(x)+3 f^{\prime}(x)=3 f^{\prime}(x)
$$

- Sum rule:

$$
\begin{aligned}
g^{\prime}(x) & =\frac{\mathrm{d}}{\mathrm{~d} x}\{f(x)+f(x)+f(x)\}=f^{\prime}(x)+f^{\prime}(x)+f^{\prime}(x) \\
& =3 f^{\prime}(x)
\end{aligned}
$$

- Quotient rule:

$$
g^{\prime}(x)=\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{f(x)}{\frac{1}{3}}\right\}=\frac{\frac{1}{3} f^{\prime}(x)-f(x)(0)}{\frac{1}{9}}=\frac{\frac{1}{3} f^{\prime}(x)}{\frac{1}{9}}
$$

$$
=9\left(\frac{1}{3}\right) f^{\prime}(x)=3 f^{\prime}(x)
$$

All rules give $g^{\prime}(x)=3 f^{\prime}(x)$.

## Exercises - Stage 2

2.4.2.5. Solution. We know, from Examples 2.2.5 and 2.2.9 in the CLP-1 text, that $\frac{\mathrm{d}}{\mathrm{d} x} x^{2}=2 x$ and $\frac{\mathrm{d}}{\mathrm{d} x} x^{1 / 2}=\frac{1}{2 \sqrt{x}}$. So, by linearity,

$$
f^{\prime}(x)=3 \cdot 2 x+4 \cdot \frac{1}{2 \sqrt{x}}=6 x+\frac{2}{\sqrt{x}}
$$

2.4.2.6. Solution. We have already seen $\frac{\mathrm{d}}{\mathrm{d} x}\{\sqrt{x}\}=\frac{1}{2 \sqrt{x}}$ in Example 2.2.9 of the CLP-1 text. Now:

$$
\begin{aligned}
f^{\prime}(x) & =(2)(8 \sqrt{x}-9 x)+(2 x+5)\left(\frac{8}{2 \sqrt{x}}-9\right) \\
& =16 \sqrt{x}-18 x+(2 x+5)\left(\frac{4}{\sqrt{x}}-9\right) \\
& =-36 x+24 \sqrt{x}+\frac{20}{\sqrt{x}}-45
\end{aligned}
$$

2.4.2.7. *. Solution. We already know that $\frac{\mathrm{d}}{\mathrm{d} x} x=1$ and $\frac{\mathrm{d}}{\mathrm{d} x} x^{2}=2 x$, so we can compute the derivative of $x^{3}$ by writing $x^{3}=(x)\left(x^{2}\right)$,

$$
\frac{\mathrm{d}}{\mathrm{~d} x} x^{3}=\frac{\mathrm{d}}{\mathrm{~d} x}(x)\left(x^{2}\right)=(1)\left(x^{2}\right)+(x)(2 x)=3 x^{2}
$$

When this is evaluated at $x=\frac{1}{2}$ we get $\frac{3}{4}$. Since we also compute $\left(\frac{1}{2}\right)^{3}=\frac{1}{8}$, the equation of the tangent line is

$$
y-\frac{1}{8}=\frac{3}{4} \cdot\left(x-\frac{1}{2}\right) .
$$

2.4.2.8. *. Solution. Let $f(t)=t^{3}-4 t^{2}+1$. We saw in Question 2.4.2.7 that $\frac{\mathrm{d}}{\mathrm{d} t} t^{3}=3 t^{2}$. So

$$
\begin{aligned}
f^{\prime}(t) & =3 t^{2}-8 t & f^{\prime}(2) & =3 \times 4-8 \times 2=-4 \\
f^{\prime \prime}(t) & =6 t-8 & f^{\prime \prime}(2) & =6 \times 2-8=4
\end{aligned}
$$

Hence at $t=2$, 2.4.2.8.a the particle has speed of magnitude $\{4\}$, and 2.4.2.8.b is moving \{towards the left\}. At $t=2, f^{\prime \prime}(2)>0$, so $f^{\prime}$ is increasing, i.e. becoming less negative. Since $f^{\prime}$ is getting closer to zero, 2.4.2.8.c the magnitude of the speed is $\{$ decreasing $\}$.
2.4.2.9. *. Solution. We can use the quotient rule here.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{2 x-1}{2 x+1}\right\} & =\frac{(2 x+1)(2)-(2 x-1)(2)}{(2 x+1)^{2}}=\frac{4}{(2 x+1)^{2}} \\
& =\frac{1}{(x+1 / 2)^{2}}
\end{aligned}
$$

2.4.2.10. Solution. First, we find the $y^{\prime}$ for general $x$. Using the corollary to Theorem 2.4.3 and the quotient rule:

$$
\begin{aligned}
y^{\prime} & =2\left(\frac{3 x+1}{3 x-2}\right) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{3 x+1}{3 x-2}\right\} \\
& =2\left(\frac{3 x+1}{3 x-2}\right)\left(\frac{(3 x-2)(3)-(3 x+1)(3)}{(3 x-2)^{2}}\right) \\
& =2\left(\frac{3 x+1}{3 x-2}\right)\left(\frac{-9}{(3 x-2)^{2}}\right) \\
& =\frac{-18(3 x+1)}{(3 x-2)^{3}}
\end{aligned}
$$

So, plugging in $x=1$ :

$$
y^{\prime}(1)=\frac{-18(3+1)}{(3-2)^{3}}=-72
$$

2.4.2.11. Solution. We need $f^{\prime}(1)$, so first we must find $f^{\prime}(x)$. Since $f(x)$ is the reciprocal of $\sqrt{x}+1$, we can use the Corollary 2.4.6:

$$
f^{\prime}(x)=\frac{-\frac{\mathrm{d}}{\mathrm{~d} x}\{\sqrt{x}+1\}}{(\sqrt{x}+1)^{2}}=\frac{-\frac{1}{2 \sqrt{x}}}{(\sqrt{x}+1)^{2}}=\frac{-1}{2 \sqrt{x}(\sqrt{x}+1)^{2}},
$$

so $f^{\prime}(1)=\frac{-1}{2 \sqrt{1}(\sqrt{1}+1)^{2}}=\frac{-1}{8}$.
Now, using the point $\left(1, \frac{1}{2}\right)$ and the slope $\frac{-1}{8}$, our tangent line has equation $y-\frac{1}{2}=$ $-\frac{1}{8}(x-1)$.

## Exercises - Stage 3

2.4.2.12. Solution. Population growth is rate of change of population. Population in year $2000+t$ is given by $P(t)=P_{0}+b(t)-d(t)$, where $P_{0}$ is the initial population of the town. Then $P^{\prime}(t)$ is the expression we're looking for, and $P^{\prime}(t)=b^{\prime}(t)-d^{\prime}(t)$.
It is interesting to note that the initial population does not obviously show up in this calculation. It would probably affect $b(t)$ and $d(t)$, but if we know these we do not need to know $P_{0}$ to answer our question.
2.4.2.13. *. Solution. We already know that $\frac{\mathrm{d}}{\mathrm{d} x} x^{2}=2 x$. So the slope of $y=3 x^{2}$ at $x=a$ is $6 a$. The tangent line to $y=3 x^{2}$ at $x=a, y=3 a^{2}$ is $y-3 a^{2}=6 a(x-a)$.

This tangent line passes through $(2,9)$ if

$$
\begin{aligned}
9-3 a^{2} & =6 a(2-a) \\
3 a^{2}-12 a+9 & =0 \\
a^{2}-4 a+3 & =0 \\
(a-3)(a-1) & =0 \\
\Longrightarrow \quad a & =1,3
\end{aligned}
$$

The points are $\{(1,3)$, $(3,27)\}$.
2.4.2.14. *. Solution. This limit represents the derivative computed at $x=$ 100180 of the function $f(x)=\sqrt{x}$. Since the derivative of $f(x)$ is $\frac{1}{2 \sqrt{x}}$, then its value at $x=100180$ is exactly $\frac{1}{2 \sqrt{100180}}$.
2.4.2.15. Solution. Let $w(t)$ and $l(t)$ be the width and length of the rectangle. Given in the problem is that $w^{\prime}(t)=2$ and $l^{\prime}(t)=5$. Since both functions have constant slopes, both must be lines. Their slopes are given, and their intercepts are $w(0)=l(0)=1$. So, $w(t)=2 t+1$ and $l(t)=5 t+1$.
The area of the rectangle is $A(t)=w(t) \cdot l(t)$, so using the product rule, the rate at which the area is increasing is $A^{\prime}(t)=w^{\prime}(t) l(t)+w(t) l^{\prime}(t)=2(5 t+1)+5(2 t+1)=$ $20 t+7$ square metres per second.
2.4.2.16. Solution. Using the product rule, $f^{\prime}(x)=(2 x) g(x)+x^{2} g^{\prime}(x)$, so $f^{\prime}(0)=$ $0 \cdot g(x)+0 \cdot g^{\prime}(x)=0$. (Since $g$ is differentiable, $g^{\prime}$ exists.)
2.4.2.17. Solution. First expression, $f(x)=\frac{g(x)}{h(x)}$ :

$$
f^{\prime}(x)=\frac{h(x) g^{\prime}(x)-g(x) h^{\prime}(x)}{h^{2}(x)}
$$

Second expresson, $f(x)=\frac{g(x)}{k(x)} \cdot \frac{k(x)}{h(x)}$ :

$$
\begin{aligned}
& f^{\prime}(x)=\left(\frac{k(x) g^{\prime}(x)-g(x) k^{\prime}(x)}{k^{2}(x)}\right)\left(\frac{k(x)}{h(x)}\right) \\
& +\left(\frac{g(x)}{k(x)}\right)\left(\frac{h(x) k^{\prime}(x)-k(x) h^{\prime}(x)}{h^{2}(x)}\right) \\
& =\frac{k(x) g^{\prime}(x)-g(x) k^{\prime}(x)}{k(x) h(x)}+\frac{g(x) h(x) k^{\prime}(x)-g(x) k(x) h^{\prime}(x)}{k(x) h^{2}(x)} \\
& =\frac{h(x) k(x) g^{\prime}(x)-h(x) g(x) k^{\prime}(x)}{k(x) h^{2}(x)}+\frac{g(x) h(x) k^{\prime}(x)-g(x) k(x) h^{\prime}(x)}{k(x) h^{2}(x)} \\
& =\frac{h(x) k(x) g^{\prime}(x)-h(x) g(x) k^{\prime}(x)+g(x) h(x) k^{\prime}(x)-g(x) k(x) h^{\prime}(x)}{k(x) h^{2}(x)} \\
& =\frac{h(x) k(x) g^{\prime}(x)-g(x) k(x) h^{\prime}(x)}{k(x) h^{2}(x)}
\end{aligned}
$$

$$
=\frac{h(x) g^{\prime}(x)-g(x) h^{\prime}(x)}{h^{2}(x)}
$$

and this is exactly what we got from differentiating the first expression.

## 2.6 • Using the Arithmetic of Derivatives - Examples

### 2.6.2 • Exercises

## Exercises - Stage 1

2.6.2.1. Solution. In the quotient rule, there is a minus, not a plus. Also, $2(x+1)+2 x$ is not the same as $2(x+1)$.
The correct version is:

$$
\begin{aligned}
f(x) & =\frac{2 x}{x+1} \\
f^{\prime}(x) & =\frac{2(x+1)-2 x}{(x+1)^{2}} \\
& =\frac{2}{(x+1)^{2}}
\end{aligned}
$$

2.6.2.2. Solution. False: Lemma 2.6 .9 tells us that, for a constant $n, \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{x^{n}\right\}=$ $n x^{n-1}$. Note that the base $x$ is the variable and the exponent $n$ is a constant. In the equation given in the question, the base 2 is a constant, and the exponent $x$ is the variable: this is the opposite of the situation where Lemma 2.6.9 applies.
We do not yet know how to differentiate $2^{x}$. We'll learn about it in Section 2.7

## Exercises - Stage 2

2.6.2.3. Solution. $f(x)=\frac{2}{3} x^{6}+5 x^{4}+12 x^{2}+9$ is a polynomial:

$$
\begin{aligned}
f^{\prime}(x) & =4 x^{5}+20 x^{3}+24 x \\
& =4 x\left(x^{4}+5 x^{2}+6\right) \\
& =4 x\left(\left(x^{2}\right)^{2}+5\left(x^{2}\right)+6\right) \\
& =4 x\left(x^{2}+2\right)\left(x^{2}+3\right)
\end{aligned}
$$

2.6.2.4. Solution. We can rewrite slightly to make every term into a power of $t$ :

$$
\begin{aligned}
s(t) & =3 t^{4}+5 t^{3}-t^{-1} \\
s^{\prime}(t) & =4 \cdot 3 t^{3}+3 \cdot 5 t^{2}-(-1) \cdot t^{-2} \\
& =12 t^{3}+15 t^{2}+\frac{1}{t^{2}}
\end{aligned}
$$

2.6.2.5. Solution. We could use the product rule here, but it's easier to simplify first. Don't be confused by the role reversal of $x$ and $y: x$ is the name of the
function, and $y$ is the variable.

$$
\begin{aligned}
x(y) & =\left(2 y+\frac{1}{y}\right) \cdot y^{3} \\
& =2 y^{4}+y^{2} \\
x^{\prime}(y) & =8 y^{3}+2 y
\end{aligned}
$$

2.6.2.6. Solution. We've already seen that $\frac{\mathrm{d}}{\mathrm{d} x}\{\sqrt{x}\}=\frac{1}{2 \sqrt{x}}$, but if you forget this formula it is easy to figure out: $\sqrt{x}=x^{1 / 2}$, so $\frac{\mathrm{d}}{\mathrm{d} x}\{\sqrt{x}\}=\frac{1}{2} x^{-1 / 2}=\frac{1}{2 \sqrt{x}}$. Using the quotient rule:

$$
\begin{aligned}
T(x) & =\frac{\sqrt{x}+1}{x^{2}+3} \\
T^{\prime}(x) & =\frac{\left(x^{2}+3\right)\left(\frac{1}{2 \sqrt{x}}\right)-(\sqrt{x}+1)(2 x)}{\left(x^{2}+3\right)^{2}}
\end{aligned}
$$

2.6.2.7. *. Solution. We use quotient rule:

$$
\frac{\left(x^{2}+3\right) \cdot 7-2 x \cdot(7 x+2)}{\left(x^{2}+3\right)^{2}}=\frac{21-4 x-7 x^{2}}{\left(x^{2}+3\right)^{2}}
$$

2.6.2.8. Solution. Instead of multiplying to get our usual form of this polynomial, we can use the product rule. If $f_{1}(x)=3 x^{3}+4 x^{2}+x+1$ and $f_{2}(x)=2 x+5$, then $f_{1}^{\prime}(x)=9 x^{2}+8 x+1$ and $f_{2}^{\prime}(x)=2$. Then

$$
\begin{aligned}
f^{\prime}(0) & =f_{1}^{\prime}(0) f_{2}(0)+f_{1}(0) f_{2}^{\prime}(0) \\
& =(1)(5)+(1)(2)=7
\end{aligned}
$$

2.6.2.9. Solution. Using the quotient rule,

$$
f^{\prime}(x)=\frac{\left(x^{2}+5 x\right)\left(9 x^{2}\right)-\left(3 x^{3}+1\right)(2 x+5)}{\left(x^{2}+5 x\right)^{2}}=\frac{3 x^{4}+30 x^{3}-2 x-5}{\left(x^{2}+5 x\right)^{2}}
$$

2.6.2.10. *. Solution. We use quotient rule:

$$
\frac{(2-x)(6 x)-\left(3 x^{2}+5\right)(-1)}{(2-x)^{2}}=\frac{-3 x^{2}+12 x+5}{(x-2)^{2}}
$$

2.6.2.11. *. Solution. We use quotient rule:

$$
\frac{\left(3 x^{2}+5\right)(-2 x)-\left(2-x^{2}\right)(6 x)}{\left(3 x^{2}+5\right)^{2}}=\frac{-22 x}{\left(3 x^{2}+5\right)^{2}}
$$

2.6.2.12. *. Solution. We use quotient rule:

$$
\frac{6 x^{2} \cdot(x+2)-\left(2 x^{3}+1\right) \cdot 1}{(x+2)^{2}}=\frac{4 x^{3}+12 x^{2}-1}{(x+2)^{2}}
$$

2.6.2.13. *. Solution. The derivative of the function is

$$
\frac{\left(1-x^{2}\right) \cdot \frac{1}{2 \sqrt{x}}-\sqrt{x} \cdot(-2 x)}{\left(1-x^{2}\right)^{2}}=\frac{\left(1-x^{2}\right)-2 x \cdot(-2 x)}{2 \sqrt{x}\left(1-x^{2}\right)^{2}}
$$

The derivative is undefined if either $x<0$ or $x=0, \pm 1$ (since the square-root is undefined for $x<0$ and the denominator is zero when $x=0,1,-1$. Putting this together - the derivative exists for $x>0, x \neq 1$.
2.6.2.14. Solution. Using the product rule seems faster than expanding.

$$
\begin{aligned}
& f^{\prime}(x)= \frac{\mathrm{d}}{\mathrm{~d} x}\{3 \sqrt[5]{x}+15 \sqrt[3]{x}+8\}\left(3 x^{2}+8 x-5\right)+(3 \sqrt[5]{x}+15 \sqrt[3]{x}+8) \\
& \frac{\mathrm{d}}{\mathrm{~d} x}\left\{3 x^{2}+8 x-5\right\} \\
&= \frac{\mathrm{d}}{\mathrm{~d} x}\left\{3 x^{\frac{1}{5}}+15 x^{\frac{1}{3}}+8\right\}\left(3 x^{2}+8 x-5\right)+(3 \sqrt[5]{x}+15 \sqrt[3]{x}+8) \\
& \frac{\mathrm{d}}{\mathrm{~d} x}\left\{3 x^{2}+8 x-5\right\} \\
&=\left(\frac{3}{5} x^{\frac{-4}{5}}+5 x^{\frac{-2}{3}}\right)\left(3 x^{2}+8 x-5\right)+(3 \sqrt[5]{x}+15 \sqrt[3]{x}+8)(6 x+8)
\end{aligned}
$$

2.6.2.15. Solution. To avoid the quotient rule, we can divide through the denominator:

$$
\begin{aligned}
f(x) & =\frac{\left(x^{2}+5 x+1\right)(\sqrt{x}+\sqrt[3]{x})}{x}=\left(x^{2}+5 x+1\right) \frac{(\sqrt{x}+\sqrt[3]{x})}{x} \\
& =\left(x^{2}+5 x+1\right)\left(x^{-1 / 2}+x^{-2 / 3}\right)
\end{aligned}
$$

Now, product rule:

$$
f^{\prime}(x)=(2 x+5)\left(x^{-1 / 2}+x^{-2 / 3}\right)+\left(x^{2}+5 x+1\right)\left(\frac{-1}{2} x^{-3 / 2}-\frac{2}{3} x^{-5 / 3}\right)
$$

(If you simplified differently, or used the quotient rule, you probably came up with a different-looking answer. There is only one derivative, though, so all correct answers will look the same after sufficient algebraic manipulation.)
2.6.2.16. Solution. The question asks us to find where $f^{\prime}(x)=0$ and $f(x)$ exists. We can use the formula for the derivative of a reciprocal, Corollary 2.4.6:

$$
\begin{aligned}
f^{\prime}(x) & =\frac{-\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{1}{5} x^{5}+x^{4}-\frac{5}{3} x^{3}\right\}}{\left(\frac{1}{5} x^{5}+x^{4}-\frac{5}{3} x^{3}\right)^{2}} \\
& =\frac{-\left(x^{4}+4 x^{3}-5 x^{2}\right)}{\left(\frac{1}{5} x^{5}+x^{4}-\frac{5}{3} x^{3}\right)^{2}} \\
& =\frac{-x^{2}\left(x^{2}+4 x-5\right)}{\left(\frac{1}{5} x^{5}+x^{4}-\frac{5}{3} x^{3}\right)^{2}}
\end{aligned}
$$

$$
=\frac{-x^{2}(x+5)(x-1)}{\left(\frac{1}{5} x^{5}+x^{4}-\frac{5}{3} x^{3}\right)^{2}}
$$

So our candidates for $x$-values where $f^{\prime}(x)=0$ are $x=0, x=-5$, and $x=1$. However, we need to check that $f$ exists at these places: $f(0)$ is undefined (and $f^{\prime}(0)$ doesn't exist). So $f^{\prime}(x)=0$ only when $x=-5$ and $x=1$.

## Exercises - Stage 3

2.6.2.17. *. Solution. Denote by $m$ the slope of the common tangent, by $\left(x_{1}, y_{1}\right)$ the point of tangency with $y=x^{2}$, and by $\left(x_{2}, y_{2}\right)$ the point of tangency with $y=x^{2}-2 x+2$. Then we must have

$$
y_{1}=x_{1}^{2} \quad y_{2}=x_{2}^{2}-2 x_{2}+2 \quad m=2 x_{1}=2 x_{2}-2=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}
$$

From the " $m$ " equations we get $x_{1}=\frac{m}{2}, x_{2}=\frac{m}{2}+1$ and

$$
\begin{aligned}
m & =\frac{y_{2}-y_{1}}{x_{2}-x_{1}} \\
& =y_{2}-y_{1} \\
& =x_{2}^{2}-2 x_{2}+2-x_{1}^{2} \\
& =\left(x_{2}-x_{1}\right)\left(x_{2}+x_{1}\right)-2\left(x_{2}-1\right) \\
& =\left(\frac{m}{2}+1-\frac{m}{2}\right)\left(\frac{m}{2}+1+\frac{m}{2}\right)-2\left(\frac{m}{2}+1-1\right) \\
& =(1)(m+1)-2 \frac{m}{2} \\
& =1
\end{aligned}
$$

So $m=1$ and

$$
x_{1}=\frac{1}{2}, \quad y_{1}=\frac{1}{4}, \quad x_{2}=\frac{3}{2}, \quad y_{2}=\frac{9}{4}-3+2=\frac{5}{4}
$$

An equation of the common tangent is $y=x-\frac{1}{4}$.
2.6.2.18. Solution. The line $y=m x+b$ is tangent to $y=x^{2}$ at $x=\alpha$ if

$$
2 \alpha=m \text { and } \alpha^{2}=m \alpha+b \Longleftrightarrow m=2 \alpha \text { and } b=-\alpha^{2}
$$

The same line $y=m x+b$ is tangent to $y=-x^{2}+2 x-5$ at $x=\beta$ if

$$
\begin{aligned}
& -2 \beta+2=m \text { and }-\beta^{2}+2 \beta-5=m \beta+b \\
\Longleftrightarrow & m=2-2 \beta \text { and } b=-\beta^{2}+2 \beta-5-(2-2 \beta) \beta=\beta^{2}-5
\end{aligned}
$$

For the line to be simultaneously tangent to the two parabolas we need

$$
m=2 \alpha=2-2 \beta \text { and } b=-\alpha^{2}=\beta^{2}-5
$$

Substituting $\alpha=1-\beta$ into $-\alpha^{2}=\beta^{2}-5$ gives $-(1-\beta)^{2}=\beta^{2}-5$ or $2 \beta^{2}-2 \beta-4=0$ or $\beta=-1,2$. The corresponding values of the other parameters are $\alpha=2,-1$, $m=4,-2$ and $b=-4,-1$. The two lines are $\{y=4 x-4$ and $y=-2 x-1\}$.

2.6.2.19. *. Solution. This limit represents the derivative computed at $x=2$ of the function $f(x)=x^{2015}$. To see this, simply use the definition of the derivative at $a=2$ with $f(x)=x^{2015}$ :

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{~d} x}\{f(x)\}\right|_{a} & =\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a} \\
\left.\frac{\mathrm{~d}}{\mathrm{~d} x}\left\{x^{2015}\right\}\right|_{2} & =\lim _{x \rightarrow 2} \frac{x^{2015}-2^{2015}}{x-2}
\end{aligned}
$$

Since the derivative of $f(x)$ is $2015 \cdot x^{2014}$, then its value at $x=2$ is exactly $2015 \cdot 2^{2014}$.

## 2.7 • Derivatives of Exponential Functions

### 2.7.3 • Exercises

## Exercises - Stage 1

2.7.3.1. Solution. Since $1^{x}=1$ for any $x$, we see that $(b)$ is just the constant function $y=1$, so D matches to (b).
Since $2^{-x}=\frac{1}{2^{x}}=\left(\frac{1}{2}\right)^{x}$, functions $(a)$ and $(d)$ are the same. This is the only function out of the lot that grows as $x \rightarrow-\infty$ and shrinks as $x \rightarrow \infty$, so A matches to (a) and (d).
This leaves B and C to match to $(c)$ and (e). Since $3>2$, when $x>0,3^{x}>2^{x}$. So, (e) matches to the function that grows more quickly to the right of the $x$-axis: B matches to $(e)$, and C matches to (c).
2.7.3.2. Solution. First, let's consider the behaviour of exponential functions $a^{x}$ based on whether $a$ is greater or less than 1. As we know, $\lim _{x \rightarrow \infty} a^{x}= \begin{cases}\infty & a>1 \\ 0 & a<1\end{cases}$ and $\lim _{x \rightarrow-\infty} a^{x}=\left\{\begin{array}{cc}0 & a>1 \\ \infty & a<1\end{array}\right.$. Our function has $\lim _{x \rightarrow \infty} f(x)=\infty$ and $\lim _{x \rightarrow-\infty} f(x)=0$, so we conclude $a>1$ : thus ( $d$ ) and also (b) hold. (We could have also seen that (b)
holds because $a^{x}$ is defined for all real numbers.)
It remains to decide whether $a$ is greater or less than $e$. (If $a$ were equal to $e$, then $f^{\prime}(x)$ would be the same as $f(x)$.) We saw in the text that $\frac{\mathrm{d}}{\mathrm{d} x}\left\{a^{x}\right\}=C(a) a^{x}$ for the function $C(a)=\lim _{h \rightarrow 0} \frac{a^{h}-1}{h}$. We know that $C(e)=1$. (Actually, we chose $e$ to be the number that has this property.) From our graph, we see that $f^{\prime}(x)<f(x)$, so $C(a)<1=C(e)$. In other words, $\lim _{h \rightarrow 0} \frac{a^{h}-1}{h}<\lim _{h \rightarrow 0} \frac{e^{h}-1}{h} ;$ so, $a<e$. Thus (e) holds.
2.7.3.3. Solution. The power rule tells us that $\frac{\mathrm{d}}{\mathrm{d} x}\left\{x^{n}\right\}=n x^{n-1}$. In this equation, the variable is the base, and the exponent is a constant. In the function $e^{x}$, it's reversed: the variable is the exponent, and the base it a constant. So, the power rule does not apply.
2.7.3.4. Solution. $P(t)$ is an increasing function over its domain, so the population is increasing.
There are a few ways to see that $P(t)$ is increasing.
What we really care about is whether $e^{0.2 t}$ is increasing or decreasing, since an increasing function multiplied by 100 is still an increasing function, and a decreasing function multiplied by 100 is still a decreasing function. Since $f(t)=e^{t}$ is an increasing function, we can use what we know about graphing functions to see that $f(0.2 t)=e^{0.2 t}$ is also increasing.

## Exercises - Stage 2

2.7.3.5. Solution. Using the quotient rule,

$$
f^{\prime}(x)=\frac{2 x e^{x}-2 e^{x}}{4 x^{2}}=\frac{e^{x}(2 x-2)}{4 x^{2}}=\frac{(x-1) e^{x}}{2 x^{2}}
$$

### 2.7.3.6. Solution.

$$
f^{\prime}(x)=\frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{2 x}\right\}=\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\left(e^{x}\right)^{2}\right\}=2 \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{e^{x}\right\} e^{x}=2 e^{x} e^{x}=2\left(e^{x}\right)^{2}=2 e^{2 x}
$$

### 2.7.3.7. Solution.

$$
e^{a+x}=e^{a} e^{x}
$$

Since $e^{a}$ is just a constant,

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{a} e^{x}\right\}=e^{a} \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{e^{x}\right\}=e^{a} e^{x}=e^{a+x}
$$

So, $f^{\prime}(x)=f(x)=e^{a+x}$.
2.7.3.8. Solution. If the derivative is positive, the function is increasing, so let's start by finding the derivative. We use the product rule (although Question 2.7.3.12 gives a shortcut).

$$
f^{\prime}(x)=1 \cdot e^{x}+x e^{x}=(1+x) e^{x}
$$

Since $e^{x}$ is always positive, $f^{\prime}(x)>0$ when $1+x>0$. So, $f(x)$ is increasing when $x>-1$.
2.7.3.9. Solution.

$$
e^{-x}=\frac{1}{e^{x}}
$$

Using the rule for differentiating the reciprocal:

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{-x}\right\}=\frac{-e^{x}}{\left(e^{x}\right)^{2}}=\frac{-1}{e^{x}}=-e^{-x}
$$

2.7.3.10. Solution. Using the product rule,

$$
\begin{aligned}
f^{\prime}(x) & =\left(e^{x}\right)\left(e^{x}-1\right)+\left(e^{x}+1\right)\left(e^{x}\right)=e^{x}\left(e^{x}-1+e^{x}+1\right)=2\left(e^{x}\right)^{2} \\
& =2 e^{2 x}
\end{aligned}
$$

Alternate solution: using Question 2.7.3.6:

$$
f(x)=e^{2 x}-1 \Longrightarrow f^{\prime}(x)=2 e^{2 x}
$$

2.7.3.11. Solution. The question asks when $s^{\prime}(t)$ is negative. So, we start by differentiating. Using the product rule:

$$
\begin{aligned}
s^{\prime}(t) & =e^{t}\left(t^{2}+2 t\right) \\
& =e^{t} \cdot t(t+2)
\end{aligned}
$$

$e^{t}$ is always positive, so $s^{\prime}(t)$ is negative when $t$ and $2+t$ have opposite signs. This occurs when $-2<t<0$.

## Exercises - Stage 3

2.7.3.12. Solution. Using the product rule, $g^{\prime}(x)=f^{\prime}(x) e^{x}+f(x) e^{x}=[f(x)+$ $\left.f^{\prime}(x)\right] e^{x}$
2.7.3.13. Solution. We simplify the functions to get a better idea of what's going on.
(a): $y=e^{3 \log x}+1=\left(e^{\log x}\right)^{3}+1=x^{3}+1$. This is not a line.
(b): $2 y+5=e^{3+\log x}=e^{3} e^{\log x}=e^{3} x$. Since $e^{3}$ is a constant, $2 y+5=e^{3} x$ is a line.
(c): There isn't a fancy simplification here-this isn't a line. If that isn't a satisfactory answer, we can check: a line is a function with a constant slope. For our function, $y^{\prime}=\frac{\mathrm{d}}{\mathrm{d} x}\left\{e^{2 x}+4\right\}=\frac{\mathrm{d}}{\mathrm{d} x}\left\{e^{2 x}\right\}=\frac{\mathrm{d}}{\mathrm{d} x}\left\{\left(e^{x}\right)^{2}\right\}=2 e^{x} e^{x}=2 e^{2 x}$. Since the derivative isn't constant, the function isn't a line.
$(d): y=e^{\log x} 3^{e}+\log 2=3^{e} x+\log 2$. Since $e^{3}$ and $\log 2$ are constants, this is a line.
2.7.3.14. *. Solution. When we say a function is differentiable without specifying a range, we mean that it is differentiable over its domain. The function $f(x)$ is differentiable when $x \neq 1$ for any values of $a$ and $b$; it is up to us to figure out which constants make it differentiable when $x=1$.
In order to be differentiable, a function must be continuous. The definition of
continuity tells us that, for $f$ to be continuous at $x=1$, we need $\lim _{x \rightarrow 1} f(x)=f(1)$. From the definition of $f$, we see $f(1)=a+b=\lim _{x \rightarrow 1^{-}} f(x)$, so we need $\lim _{x \rightarrow 1^{+}} f(x)=$ $a+b$. Since $\lim _{x \rightarrow 1^{+}} f(x)=e^{1}=e$, we specifically need

$$
e=a+b
$$

Now, let's consider differentiability of $f$ at $x=1$. We need the following limit to exist:

$$
\lim _{h \rightarrow 0} \frac{f(1+h)-f(1)}{h}
$$

In particular, we need the one-sided limits to exist and be equal:

$$
\lim _{h \rightarrow 0^{-}} \frac{f(1+h)-f(1)}{h}=\lim _{h \rightarrow 0^{+}} \frac{f(1+h)-f(1)}{h}
$$

If $h<0$, then $1+h<1$, so $f(1+h)=a(1+h)^{2}+b$. If $h>0$, then $1+h>1$, so $f(1+h)=e^{1+h}$. With this in mind, we begin to evaluate the one-sided limits:

$$
\begin{aligned}
\lim _{h \rightarrow 0^{-}} \frac{f(1+h)-f(1)}{h} & =\lim _{h \rightarrow 0^{-}} \frac{\left[a(1+h)^{2}+b\right]-[a+b]}{h} \\
& =\lim _{h \rightarrow 0^{-}} \frac{a h^{2}+2 a h}{h}=2 a \\
\lim _{h \rightarrow 0^{+}} \frac{f(1+h)-f(1)}{h} & =\lim _{h \rightarrow 0^{+}} \frac{e^{1+h}-(a+b)}{h}
\end{aligned}
$$

Since we take $a+b$ to be equal to $e$ (to ensure continuity):

$$
\begin{aligned}
& =\lim _{h \rightarrow 0^{+}} \frac{e^{1+h}-e^{1}}{h} \\
& =\left.\frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{x}\right\}\right|_{x=1}=e^{1}=e
\end{aligned}
$$

So, we also need

$$
2 a=e
$$

Therefore, the values of $a$ and $b$ that make $f$ differentiable are $a=b=\frac{e}{2}$.

## 2.8 • Derivatives of Trigonometric Functions

### 2.8.8 • Exercises

## Exercises - Stage 1

### 2.8.8.1. Solution.



The graph $f(x)=\sin x$ has horizontal tangent lines precisely at those points where $\cos x=0$. This must be true, since $\frac{\mathrm{d}}{\mathrm{d} x}\{\sin x\}=\cos x$ : where the derivative of sine is zero, cosine itself is zero.

### 2.8.8.2. Solution.



The graph $f(x)=\sin x$ has maximum slope at those points where $\cos x$ has a maximum. This makes sense, because $f^{\prime}(x)=\cos x$ : the maximum values of the slope of sine correspond to the maximum values of cosine.

## Exercises - Stage 2

2.8.8.3. Solution. You should memorize the derivatives of sine, cosine, and tangent. $f^{\prime}(x)=\cos x-\sin x+\sec ^{2} x$
2.8.8.4. Solution. $f^{\prime}(x)=\cos x-\sin x$, so $f^{\prime}(x)=0$ precisely when $\sin x=\cos x$. This happens at $\pi / 4$, but it also happens at $5 \pi / 4$. By looking at the unit circle, it is clear that $\sin x=\cos x$ whenever $x=\frac{\pi}{4}+\pi n$ for some integer $n$.


### 2.8.8.5. Solution.

- Solution 1: $f(x)=\sin ^{2} x+\cos ^{2} x=1$, so $f^{\prime}(x)=\frac{\mathrm{d}}{\mathrm{d} x}\{1\}=0$.
- Solution 2: Using the formula for the derivative of a squared function,

$$
\begin{aligned}
f^{\prime}(x) & =2 \sin x \cos x+2 \cos x(-\sin x) \\
& =2 \sin x \cos x-2 \sin x \cos x=0
\end{aligned}
$$

2.8.8.6. Solution. It is true that $2 \sin x \cos x=\sin (2 x)$, but we don't know the derivative of $\sin (2 x)$. So, we use the product rule:

$$
f^{\prime}(x)=2 \cos x \cos x+2 \sin x(-\sin x)=2\left(\cos ^{2} x-\sin ^{2} x\right)
$$

### 2.8.8.7. Solution.

- Solution 1: using the product rule,

$$
f^{\prime}(x)=e^{x} \cot x+e^{x}\left(-\csc ^{2} x\right)=e^{x}\left(\cot x-\csc ^{2} x\right)
$$

- Solution 2: using the formula from Question 2.7.3.12, Section 2.7,

$$
f^{\prime}(x)=e^{x}\left(\cot x-\csc ^{2} x\right)
$$

2.8.8.8. Solution. We use the quotient rule.

$$
\begin{aligned}
& f^{\prime}(x) \\
& =\frac{(\cos x+\tan x)\left(2 \cos x+3 \sec ^{2} x\right)-(2 \sin x+3 \tan x)\left(-\sin x+\sec ^{2} x\right)}{(\cos x+\tan x)^{2}} \\
& =\frac{2 \cos ^{2} x+3 \cos x \sec ^{2} x+2 \cos x \tan x+3 \tan x \sec ^{2} x}{(\cos x+\tan x)^{2}} \\
& \quad+\frac{2 \sin ^{2} x-2 \sin x \sec ^{2} x+3 \sin x \tan x-3 \tan x \sec ^{2} x}{(\cos x+\tan x)^{2}} \\
& =\frac{2+3 \sec x+2 \sin x-2 \tan x \sec x+3 \sin x \tan x}{(\cos x+\tan x)^{2}}
\end{aligned}
$$

2.8.8.9. Solution. We use the quotient rule.

$$
\begin{aligned}
f^{\prime}(x) & =\frac{e^{x}(5 \sec x \tan x)-(5 \sec x+1) e^{x}}{\left(e^{x}\right)^{2}} \\
& =\frac{5 \sec x \tan x-5 \sec x-1}{e^{x}}
\end{aligned}
$$

2.8.8.10. Solution. We use the product rule:

$$
f^{\prime}(x)=\left(e^{x}+\cot x\right)\left(30 x^{5}+\csc x \cot x\right)+\left(e^{x}-\csc ^{2} x\right)\left(5 x^{6}-\csc x\right)
$$

2.8.8.11. Solution. We don't know how to differentiate this function as it is written, but an identity helps us. Since $\sin \left(\frac{\pi}{2}-\theta\right)=\cos \theta$, we see $f^{\prime}(\theta)=\frac{\mathrm{d}}{\mathrm{d} \theta}\{\cos \theta\}=$ $-\sin (\theta)$.
2.8.8.12. Solution. We know the derivative of $\sin x$, but not of $\sin (-x)$. So we
re-write $f(x)$ using identities:

$$
\begin{aligned}
f(x) & =\sin (-x)+\cos (-x) \\
& =-\sin x+\cos x \\
f^{\prime}(x) & =-\cos x-\sin x
\end{aligned}
$$

2.8.8.13. Solution. We apply the quotient rule.

$$
\begin{aligned}
s^{\prime}(\theta) & =\frac{(\cos \theta-\sin \theta)(-\sin \theta+\cos \theta)-(\cos \theta+\sin \theta)(-\sin \theta-\cos \theta)}{(\cos \theta-\sin \theta)^{2}} \\
& =\frac{(\cos \theta-\sin \theta)^{2}+(\cos \theta+\sin \theta)^{2}}{(\cos \theta-\sin \theta)^{2}} \\
& =1+\left(\frac{\cos \theta+\sin \theta}{\cos \theta-\sin \theta}\right)^{2}
\end{aligned}
$$

2.8.8.14. *. Solution. In order for $f$ to be differentiable at $x=0$, it must also be continuous at $x=0$. This forces

$$
\begin{array}{ll} 
& \lim _{x \rightarrow 0^{-}} f(x)=\lim _{x \rightarrow 0^{+}} f(x)=f(0) \\
\text { or } \quad & \lim _{x \rightarrow 0^{-}} \cos (x)=\lim _{x \rightarrow 0^{+}}(a x+b)=1
\end{array}
$$

or $b=1$.
In order for $f$ to be differentiable at $x=0$, we need the limit

$$
\lim _{h \rightarrow 0} \frac{f(0+h)-f(0)}{h}
$$

to exist. This is the case if and only if the two one-sided limits

$$
\lim _{h \rightarrow 0^{-}} \frac{f(0+h)-f(0)}{h}=\lim _{h \rightarrow 0^{-}} \frac{\cos (h)-\cos (0)}{h}
$$

and

$$
\lim _{h \rightarrow 0^{+}} \frac{f(0+h)-f(0)}{h}=\lim _{h \rightarrow 0^{+}} \frac{(a h+b)-\cos (0)}{h}=a \quad \text { since } b=1
$$

exist and are equal. Because $\cos (x)$ is differentiable at $x=0$ we have

$$
\lim _{h \rightarrow 0^{-}} \frac{\cos (h)-\cos (0)}{h}=\left.\frac{\mathrm{d}}{\mathrm{~d} x} \cos (x)\right|_{x=0}=-\left.\sin (x)\right|_{x=0}=0
$$

So, we need $a=0$ and $b=1$.
2.8.8.15. *. Solution. We compute the derivative of $\cos (x)+2 x$ as being $-\sin (x)+2$, which evaluated at $x=\frac{\pi}{2}$ yields $-1+2=1$. Since we also compute $\cos (\pi / 2)+2(\pi / 2)=0+\pi$, then the equation of the tangent line is

$$
y-\pi=1 \cdot(x-\pi / 2)
$$

## Exercises - Stage 3

2.8.8.16. *. Solution. This limit represents the derivative computed at $x=2015$ of the function $f(x)=\cos (x)$. To see this, simply use the definition of the derivative at $a=2015$ with $f(x)=\cos x$ :

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{~d} x}\{f(x)\}\right|_{a} & =\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a} \\
\left.\frac{\mathrm{~d}}{\mathrm{~d} x}\{\cos x\}\right|_{2015} & =\lim _{x \rightarrow 2015} \frac{\cos (x)-\cos (2015)}{x-2015}
\end{aligned}
$$

Since the derivative of $f(x)$ is $-\sin (x)$, its value at $x=2015$ is exactly $-\sin (2015)$.
2.8.8.17. *. Solution. This limit represents the derivative computed at $x=\pi / 3$ of the function $f(x)=\cos x$. To see this, simply use the definition of the derivative at $a=\pi / 3$ with $f(x)=\cos x$ :

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{~d} x}\{f(x)\}\right|_{a} & =\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a} \\
\left.\frac{\mathrm{~d}}{\mathrm{~d} x}\{\cos x\}\right|_{\pi / 3} & =\lim _{x \rightarrow \pi / 3} \frac{\cos (x)-\cos (\pi / 3)}{x-\pi / 3} \\
& =\lim _{x \rightarrow \pi / 3} \frac{\cos (x)-1 / 2}{x-\pi / 3}
\end{aligned}
$$

Since the derivative of $f(x)$ is $-\sin x$, then its value at $x=\pi / 3$ is exactly $-\sin (\pi / 3)=-\sqrt{3} / 2$.
2.8.8.18. *. Solution. This limit represents the derivative computed at $x=\pi$ of the function $f(x)=\sin (x)$. To see this, simply use the definition of the derivative at $a=\pi$ with $f(x)=\sin x$ :

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{~d} x}\{f(x)\}\right|_{a} & =\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a} \\
\left.\frac{\mathrm{~d}}{\mathrm{~d} x}\{\sin x\}\right|_{\pi} & =\lim _{x \rightarrow \pi} \frac{\sin (x)-\sin (\pi)}{x-\pi} \\
& =\lim _{x \rightarrow \pi} \frac{\sin (x)}{x-\pi}
\end{aligned}
$$

Since the derivative of $f(x)$ is $\cos (x)$, then its value at $x=\pi$ is exactly $\cos (\pi)=-1$.

### 2.8.8.19. Solution.

$$
\tan \theta=\frac{\sin \theta}{\cos \theta}
$$

So, using the quotient rule,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} \theta}\{\tan \theta\} & =\frac{\cos \theta \cos \theta-\sin \theta(-\sin \theta)}{\cos ^{2} \theta}=\frac{\cos ^{2} \theta+\sin ^{2} \theta}{\cos ^{2} \theta} \\
& =\left(\frac{1}{\cos \theta}\right)^{2}=\sec ^{2} \theta
\end{aligned}
$$

2.8.8.20. *. Solution. In order for the function $f(x)$ to be continuous at $x=0$, the left half formula $a x+b$ and the right half formula $\frac{6 \cos x}{2+\sin x+\cos x}$ must match up at $x=0$. This forces

$$
a \times 0+b=\frac{6 \cos 0}{2+\sin 0+\cos 0}=\frac{6}{3} \Longrightarrow b=2
$$

In order for the derivative $f^{\prime}(x)$ to exist at $x=0$, the limit $\lim _{h \rightarrow 0} \frac{f(h)-f(0)}{h}$ must exist. In particular, the limits $\lim _{h \rightarrow 0^{-}} \frac{f(h)-f(0)}{h}$ and $\lim _{h \rightarrow 0^{+}} \frac{f(h)-f(0)}{h}$ must exist and be equal to each other.
When $h \rightarrow 0^{-}$, this means $h<0$, so $f(h)=a h+b=a h+2$. So:

$$
\lim _{h \rightarrow 0^{-}} \frac{f(h)-f(0)}{h}=\lim _{h \rightarrow 0^{-}} \frac{(a h+2)-2}{h}=\left.\frac{\mathrm{d}}{\mathrm{~d} x}\{a x+2\}\right|_{x=0}=a
$$

Similarly, when $h \rightarrow 0^{+}$, then $h>0$, so $f(h)=\frac{6 \cos h}{1+\sin h+\cos h} \quad$ and

$$
\begin{aligned}
& \lim _{h \rightarrow 0^{+}} \frac{f(h)-f(0)}{h}=\left.\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{6 \cos x}{2+\sin x+\cos x}\right\}\right|_{x=0} \\
& \quad=\left.\frac{-6 \sin x(2+\sin x+\cos x)-6 \cos x(\cos x-\sin x)}{(2+\sin x+\cos x)^{2}}\right|_{x=0}
\end{aligned}
$$

Since the limits from the left and right must be equal, this forces

$$
\begin{aligned}
a & =\frac{-6 \sin 0(2+\sin 0+\cos 0)-6 \cos 0(\cos 0-\sin 0)}{(2+\sin 0+\cos 0)^{2}}=\frac{-6}{(2+1)^{2}} \\
& \Longrightarrow a=-\frac{2}{3}
\end{aligned}
$$

2.8.8.21. *. Solution. In order for $f^{\prime}(x)$ to exist, $f(x)$ has to exist. We already know that $\tan x$ does not exist whenever $x=\frac{\pi}{2}+n \pi$ for any integer $n$. If we look a little deeper, since $\tan x=\frac{\sin x}{\cos x}$, the points where tangent does not exist correspond exactly to the points where cosine is zero.
From its graph, tangent looks like a smooth curve over its domain, so we might guess that everywhere tangent is defined, its derivative is defined. We can check this: $f^{\prime}(x)=\sec ^{2} x=\left(\frac{1}{\cos x}\right)^{2}$. Indeed, wherever $\cos x$ is nonzero, $f^{\prime}$ exists. So, $f^{\prime}(x)$ exists for all values of $x$ except when $x=\frac{\pi}{2}+n \pi$ for some integer $n$.
2.8.8.22. *. Solution. The function is differentiable whenever $x^{2}+x-6 \neq 0$ since the derivative equals

$$
\frac{10 \cos (x) \cdot\left(x^{2}+x-6\right)-10 \sin (x) \cdot(2 x+1)}{\left(x^{2}+x-6\right)^{2}}
$$

which is well-defined unless $x^{2}+x-6=0$. We solve $x^{2}+x-6=(x-2)(x+3)=0$,
and get $x=2$ and $x=-3$. So, the function is differentiable for all real values $x$ except for $x=2$ and for $x=-3$.
2.8.8.23. *. Solution. The function is differentiable whenever $\sin (x) \neq 0$ since the derivative equals

$$
\frac{\sin (x) \cdot(2 x+6)-\cos (x) \cdot\left(x^{2}+6 x+5\right)}{(\sin x)^{2}}
$$

which is well-defined unless $\sin x=0$. This happens when $x$ is an integer multiple of $\pi$. So, the function is differentiable for all real values $x$ except $x=n \pi$, where $n$ is any integer.
2.8.8.24. *. Solution. We compute the derivative of $\tan (x)$ as being $\sec ^{2}(x)$, which evaluated at $x=\frac{\pi}{4}$ yields 2 . Since we also compute $\tan (\pi / 4)=1$, then the equation of the tangent line is

$$
y-1=2 \cdot(x-\pi / 4)
$$

2.8.8.25. *. Solution. We compute the derivative $y^{\prime}=\cos (x)-\sin (x)+e^{x}$, which evaluated at $x=0$ yields $1-0+1=2$. Since we also compute $y(0)=0+1+1=2$, the equation of the tangent line is

$$
y-2=2(x-0)
$$

ie $y=2 x+2$.
2.8.8.26. Solution. We are asked to solve $f^{\prime}(x)=0$. That is, $e^{x}[\sin x+\cos x]=0$. Since $e^{x}$ is always positive, that means we need to find all points where $\sin x+\cos x=$ 0 . That is, we need to find all values of $x$ where $\sin x=-\cos x$. Looking at the unit circle, we see this happens whenever $x=\frac{3 \pi}{4}+n \pi$ for any integer $n$.

2.8.8.27. Solution. First, we note that our function is continuous, because

$$
\lim _{x \rightarrow 0} f(x)=\lim _{x \rightarrow 0} \frac{\sin x}{x}=1=f(0)
$$

This is a handy thing to check: if the function were discontinuous at $x=0$, then
we would automatically know that it was not differentiable there.
Now, on to the derivative. We can use the limit definition:

$$
\begin{aligned}
f^{\prime}(0) & =\lim _{h \rightarrow 0} \frac{f(0+h)-f(0)}{h} \quad \text { if it exists } \\
& =\lim _{h \rightarrow 0} \frac{f(h)-1}{h} \\
& =\lim _{h \rightarrow 0} \frac{\frac{\sin h}{h}-1}{h}
\end{aligned}
$$

As $h$ approaches 0 , both the numerator and the denominator approach 0 . So, to evaluate the limit, we need to do more work. The key insight we can use is a result that was shown in the text while evaluating the derivative of sine. When $h$ is close to $0, \cos h \leq \frac{\sin h}{h} \leq 1$. We use this to bound our limit, and then apply the squeeze theorem.

$$
\begin{aligned}
& \cos h \quad \leq \frac{\sin h}{h} \quad \leq 1 \\
& \text { So, } \quad \cos h-1 \leq \frac{\sin h}{h}-1 \leq 1-1=0
\end{aligned}
$$

and $\frac{1}{h}\left(\frac{\sin h}{h}-1\right)$ is between 0 and $\frac{\cos h-1}{h}$. We can evaluate the limit of $\frac{\cos h-1}{h}$ by noticing its similarity to the definition of the derivative.

$$
\begin{aligned}
\lim _{h \rightarrow 0} \frac{\cos h-1}{h} & =\lim _{h \rightarrow 0} \frac{\cos (0+h)-\cos (0)}{h} \\
& =\left.\frac{\mathrm{d}}{\mathrm{~d} x}\{\cos x\}\right|_{x=0}=0
\end{aligned}
$$

So, by the Squeeze Theorem, $f^{\prime}(0)=\lim _{h \rightarrow 0} \frac{\frac{\sin h}{h}-1}{h}=0$.
2.8.8.28. *. Solution. As usual, when dealing with the absolute value function, we can make things a little clearer by splitting it up into two pieces.

$$
|x|=\left\{\begin{array}{rr}
x & x \geq 0 \\
-x & x<0
\end{array}\right.
$$

So,

$$
\sin |x|=\left\{\begin{array}{rr}
\sin x & x \geq 0 \\
\sin (-x) & x<0
\end{array}=\left\{\begin{array}{rr}
\sin x & x \geq 0 \\
-\sin x & x<0
\end{array}\right.\right.
$$

where we used the identity $\sin (-x)=-\sin x$. From here, it's easy to see $h^{\prime}(x)$ when $x$ is anything other than zero.

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\{\sin |x|\}=\left\{\begin{array}{rl}
\cos x & x>0 \\
? ? & x=0 \\
-\cos x & x<0
\end{array}\right.
$$

To decide whether $h(x)$ is differentiable at $x=0$, we use the definition of the derivative. One word of explanation: usually in the definition of the derivative, $h$ is the tiny "change in $x$ " that is going to zero. Since $h$ is the name of our function, we need another letter to stand for the tiny change in $x$, the size of which is tending to zero. We chose $t$.

$$
\lim _{t \rightarrow 0} \frac{h(t+0)-h(0)}{t}=\lim _{t \rightarrow 0} \frac{\sin |t|}{t}
$$

We consider the behaviour of this function to the left and right of $t=0$ :

$$
\frac{\sin |t|}{t}=\left\{\begin{array}{ll}
\frac{\sin t}{t} & t \geq 0 \\
\frac{\sin (-t)}{t} & t<0
\end{array}= \begin{cases}\frac{\sin t}{t} & t \geq 0 \\
-\frac{\sin t}{t} & t<0\end{cases}\right.
$$

Since we're evaluating the limit as $t$ goes to zero, we need the fact that $\lim _{t \rightarrow 0} \frac{\sin t}{t}=1$. We saw this in Section 2.7, but also we know enough now to evaluate it another way. Using the definition of the derivative:

$$
\lim _{t \rightarrow 0} \frac{\sin t}{t}=\lim _{t \rightarrow 0} \frac{\sin (t+0)-\sin (0)}{t}=\left.\frac{\mathrm{d}}{\mathrm{~d} x}\{\sin x\}\right|_{t=0}=\cos 0=1
$$

At any rate, since we know $\lim _{t \rightarrow 0} \frac{\sin t}{t}=1$, then:

$$
\begin{aligned}
\lim _{t \rightarrow 0^{+}} \frac{h(t+0)-h(0)}{t} & =\lim _{t \rightarrow 0^{+}} \frac{\sin t}{t}=1 \\
\lim _{t \rightarrow 0^{-}} \frac{h(t+0)-h(0)}{t} & =\lim _{t \rightarrow 0^{-}} \frac{-\sin t}{t}=-1
\end{aligned}
$$

So, since the one-sided limits disagree,

$$
\lim _{t \rightarrow 0} \frac{h(t+0)-h(0)}{t}=D N E
$$

so $h(x)$ is not differentiable at $x=0$. Therefore,

$$
h^{\prime}(x)=\left\{\begin{array}{rr}
\cos x & x>0 \\
-\cos x & x<0
\end{array}\right.
$$

2.8.8.29. *. Solution. Statement 2.8.8.29.i is false, since $f(0)=0$. Statement 2.8.8.29.iv cannot hold, since a function that is differentiable is also continuous.

Since $\lim _{x \rightarrow 0+} \frac{\sin x}{x}=1$ (we saw this in Section 2.8 ),

$$
\begin{aligned}
\lim _{x \rightarrow 0+} f(x) & =\lim _{x \rightarrow 0+} \frac{\sin x}{\sqrt{x}} \\
& =\lim _{x \rightarrow 0+} \sqrt{x} \frac{\sin x}{x} \\
& =0 \cdot 1=0
\end{aligned}
$$

So $f$ is continuous at $x=0$, and so Statement 2.8.8.29.ii does not hold. Now let's consider $f^{\prime}(x)$

$$
\begin{aligned}
\lim _{x \rightarrow 0+} \frac{f(x)-f(0)}{x} & =\lim _{x \rightarrow 0+} \frac{\frac{\sin x}{\sqrt{x}}-0}{x} \\
& =\lim _{x \rightarrow 0+} \frac{1}{\sqrt{x}} \frac{\sin x}{x}=+\infty
\end{aligned}
$$

Therefore, using the definition of the derivative,

$$
\begin{aligned}
f^{\prime}(0) & =\lim _{x \rightarrow 0} \frac{f(x)-f(0)}{x} \quad \text { if it exists, but } \\
\lim _{x \rightarrow 0} \frac{f(x)-f(0)}{x} & =D N E
\end{aligned}
$$

since one of the one-sided limits does not exist. So $f$ is continuous but not differentiable at $x=0$. The correct statement is 2.8.8.29.iii.
2.8.8.30. *. Solution. Recall that $\lim _{x \rightarrow 0} \frac{\sin x}{x}=1$. In order to take advantage of this knowledge, we divide the numerator and denominator by $x^{5}$ (because 5 is the power of sine in the denominator, and a denominator that goes to zero generally makes a limit harder).

$$
\lim _{x \rightarrow 0} \frac{\sin x^{27}+2 x^{5} e^{x^{99}}}{\sin ^{5} x}=\lim _{x \rightarrow 0} \frac{\frac{\sin x^{27}}{x^{5}}+2 e^{x^{99}}}{\left(\frac{\sin x}{x}\right)^{5}}
$$

Now the denominator goes to 1 , which is nice, but we need to take care of the fraction $\frac{\sin x^{27}}{x^{5}}$ in the numerator. This fraction isn't very familiar, but we know that, as $x$ goes to zero, $x^{27}$ also goes to zero, so that $\frac{\sin x^{27}}{x^{27}}$ goes to 1 . Consequently,

$$
\lim _{x \rightarrow 0} \frac{\sin x^{27}+2 x^{5} e^{x^{99}}}{\sin ^{5} x}=\lim _{x \rightarrow 0} \frac{x^{22} \frac{\sin x^{27}}{x^{27}}+2 e^{x^{99}}}{\left(\frac{\sin x}{x}\right)^{5}}=\frac{0 \times 1+2 \times e^{0}}{1^{5}}=2
$$

## 2.9 . One More Tool - the Chain Rule 2.9.4 • Exercises

## Exercises - Stage 1

2.9.4.1. Solution. 2.9.4.1.a More urchins means less kelp, and fewer urchins means more kelp. This means kelp and urchins are negatively correlated, so $\frac{\mathrm{d} K}{\mathrm{~d} U}<0$. If you aren't sure why that is, we give a more detailed explanation here, using the definition of the derivative. When $h$ is a positive number, $U+h$ is greater than $U$,
so $K(U+h)$ is less than $U$, hence $K(U+h)-K(U)<0$. Therefore:

$$
\lim _{h \rightarrow 0^{+}} \frac{K(U+h)-K(U)}{h}=\frac{\text { negative }}{\text { positive }}<0
$$

Similarly, when $h$ is negative, $U+h$ is less than $U$, so $K(U+h)-K(U)>0$, and

$$
\lim _{h \rightarrow 0^{-}} \frac{K(U+h)-K(U)}{h}=\frac{\text { positive }}{\text { negative }}<0
$$

Therefore:

$$
\frac{\mathrm{d} K}{\mathrm{~d} U}=\lim _{h \rightarrow 0} \frac{K(U+h)-K(U)}{h}<0
$$

2.9.4.1.b More otters means fewer urchins, and fewer otters means more urchins. So, otters and urchins are negatively correlated: $\frac{\mathrm{d} U}{\mathrm{~d} O}<0$.
2.9.4.1.c Using the chain rule, $\frac{\mathrm{d} K}{\mathrm{~d} O}=\frac{\mathrm{d} K}{\mathrm{~d} U} \cdot \frac{\mathrm{~d} U}{\mathrm{~d} O}$. Parts 2.9.4.1.a and 2.9.4.1.b tell us both these derivatives are negative, so their product is positive: $\frac{\mathrm{d} K}{\mathrm{~d} O}>0$.
We can also see that $\frac{\mathrm{d} K}{\mathrm{~d} O}>0$ by thinking about the relationships as described. When the otter population increases, the urchin population decreases, so the kelp population increases. That means when the otter population increases, the kelp population also increases, so kelp and otters are positively correlated. The chain rule is a formal version of this kind of reasoning.

### 2.9.4.2. Solution.

$$
\frac{\mathrm{d} A}{\mathrm{~d} E}=\frac{\mathrm{d} A}{\mathrm{~d} B} \cdot \frac{\mathrm{~d} B}{\mathrm{~d} C} \cdot \frac{\mathrm{~d} C}{\mathrm{~d} D} \cdot \frac{\mathrm{~d} D}{\mathrm{~d} E}<0
$$

since we multiply three positive quantities and one negative.

## Exercises - Stage 2

2.9.4.3. Solution. Applying the chain rule:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{\cos (5 x+3)\} & =-\sin (5 x+3) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\{5 x+3\} \\
& =-\sin (5 x+3) \cdot 5
\end{aligned}
$$

2.9.4.4. Solution. Using the chain rule,

$$
\begin{aligned}
f^{\prime}(x) & =\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\left(x^{2}+2\right)^{5}\right\} \\
& =5\left(x^{2}+2\right)^{4} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{x^{2}+2\right\} \\
& =5\left(x^{2}+2\right)^{4} \cdot 2 x \\
& =10 x\left(x^{2}+2\right)^{4}
\end{aligned}
$$

2.9.4.5. Solution. Using the chain rule,

$$
T^{\prime}(k)=\frac{\mathrm{d}}{\mathrm{~d} k}\left\{\left(4 k^{4}+2 k^{2}+1\right)^{17}\right\}
$$

$$
\begin{aligned}
& =17\left(4 k^{4}+2 k^{2}+1\right)^{16} \cdot \frac{\mathrm{~d}}{\mathrm{~d} k}\left\{4 k^{4}+2 k^{2}+1\right\} \\
& =17\left(4 k^{4}+2 k^{2}+1\right)^{16} \cdot\left(16 k^{3}+4 k\right)
\end{aligned}
$$

2.9.4.6. Solution. Using the chain rule:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\sqrt{\frac{x^{2}+1}{x^{2}-1}}\right\} & =\frac{1}{2 \sqrt{\frac{x^{2}+1}{x^{2}-1}}} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{\frac{x^{2}+1}{x^{2}-1}\right\} \\
& =\frac{1}{2} \sqrt{\frac{x^{2}-1}{x^{2}+1}} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{\frac{x^{2}+1}{x^{2}-1}\right\}
\end{aligned}
$$

And now, the quotient rule:

$$
\begin{aligned}
& =\frac{1}{2} \sqrt{\frac{x^{2}-1}{x^{2}+1}} \cdot\left(\frac{\left(x^{2}-1\right)(2 x)-\left(x^{2}+1\right) 2 x}{\left(x^{2}-1\right)^{2}}\right) \\
& =\frac{1}{2} \sqrt{\frac{x^{2}-1}{x^{2}+1}} \cdot\left(\frac{-4 x}{\left(x^{2}-1\right)^{2}}\right) \\
& =\sqrt{\frac{x^{2}-1}{x^{2}+1}} \cdot\left(\frac{-2 x}{\left(x^{2}-1\right)^{2}}\right) \\
& =\frac{-2 x}{\left(x^{2}-1\right) \sqrt{x^{4}-1}}
\end{aligned}
$$

2.9.4.7. Solution. If we let $g(x)=e^{x}$ and $h(x)=\cos \left(x^{2}\right)$, then $f(x)=g(h(x))$, so $f^{\prime}(x)=g^{\prime}(h(x)) \cdot h^{\prime}(x)$.

$$
f^{\prime}(x)=e^{\cos \left(x^{2}\right)} \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\left\{\cos \left(x^{2}\right)\right\}
$$

In order to evaluate $\frac{\mathrm{d}}{\mathrm{d} x}\left\{\cos \left(x^{2}\right)\right\}$, we'll need the chain rule again.

$$
\begin{aligned}
& =e^{\cos \left(x^{2}\right)} \cdot\left[-\sin \left(x^{2}\right)\right] \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\left\{x^{2}\right\} \\
& =-e^{\cos \left(x^{2}\right)} \cdot \sin \left(x^{2}\right) \cdot 2 x
\end{aligned}
$$

2.9.4.8. *. Solution. We use the chain rule, followed by the quotient rule:

$$
\begin{aligned}
f^{\prime}(x) & =g^{\prime}\left(\frac{x}{h(x)}\right) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{x}{h(x)}\right\} \\
& =g^{\prime}\left(\frac{x}{h(x)}\right) \cdot \frac{h(x)-x h^{\prime}(x)}{h(x)^{2}}
\end{aligned}
$$

When $x=2$ :

$$
\begin{aligned}
f^{\prime}(2) & =g^{\prime}\left(\frac{2}{h(2)}\right) \frac{h(2)-2 h^{\prime}(2)}{h(2)^{2}} \\
& =4 \frac{2-2 \times 3}{2^{2}}=-4
\end{aligned}
$$

2.9.4.9. *. Solution. Using the chain rule, followed by the product rule:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{x \cos (x)}\right\} & =e^{x \cos x} \frac{\mathrm{~d}}{\mathrm{~d} x}\{x \cos x\} \\
& =[\cos x-x \sin x] e^{x \cos (x)}
\end{aligned}
$$

2.9.4.10. *. Solution. Using the chain rule:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{x^{2}+\cos (x)}\right\} & =e^{x^{2}+\cos x} \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{x^{2}+\cos x\right\} \\
& =[2 x-\sin x] e^{x^{2}+\cos (x)}
\end{aligned}
$$

2.9.4.11. *. Solution. Using the chain rule, followed by the quotient rule:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\sqrt{\frac{x-1}{x+2}}\right\} & =\frac{1}{2 \sqrt{\frac{x-1}{x+2}}} \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{\frac{x-1}{x+2}\right\} \\
& =\frac{\sqrt{x+2}}{2 \sqrt{x-1}} \cdot \frac{(x+2)-(x-1)}{(x+2)^{2}} \\
& =\frac{3}{2 \sqrt{x-1} \sqrt{x+2}^{3}}
\end{aligned}
$$

2.9.4.12. *. Solution. First, we manipulate our function to make it easier to differentiate:

$$
f(x)=x^{-2}+\left(x^{2}-1\right)^{1 / 2}
$$

Now, we can use the power rule to differentiate $\frac{1}{x^{2}}$. This will be easier than differentiating $\frac{1}{x^{2}}$ using quotient rule, but if you prefer, quotient rule will also work.

$$
\begin{aligned}
f^{\prime}(x) & =-2 x^{-3}+\frac{1}{2}\left(x^{2}-1\right)^{-1 / 2} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{x^{2}-1\right\} \\
& =-2 x^{-3}+\frac{1}{2}\left(x^{2}-1\right)^{-1 / 2}(2 x) \\
& =\frac{-2}{x^{3}}+\frac{x}{\sqrt{x^{2}-1}}
\end{aligned}
$$

The function $f(x)$ is only defined when $x \neq 0$ and when $x^{2}-1 \geq 0$. That is, when $x$ is in $(-\infty,-1] \cup[1, \infty)$. We have an added restriction on the domain of $f^{\prime}(x)$ : $x^{2}-1$ must not be zero. So, the domain of $f^{\prime}(x)$ is $(-\infty,-1) \cup(1, \infty)$.
2.9.4.13. *. Solution. We use the quotient rule, noting that $\frac{\mathrm{d}}{\mathrm{d} x}\{\sin 5 x\}=$ $5 \cos 5 x$ :

$$
f^{\prime}(x)=\frac{\left(1+x^{2}\right)(5 \cos 5 x)-(\sin 5 x)(2 x)}{\left(1+x^{2}\right)^{2}}
$$

2.9.4.14. Solution. If we let $g(x)=\sec x$ and $h(x)=e^{2 x+7}$, then $f(x)=g(h(x))$, so by the chain rule, $f^{\prime}(x)=g^{\prime}(h(x)) \cdot h^{\prime}(x)$. Since $g^{\prime}(x)=\sec x \tan x$ :

$$
\begin{aligned}
f^{\prime}(x) & =g^{\prime}(h(x)) \cdot h^{\prime}(x) \\
& =\sec (h(x)) \tan (h(x)) \cdot h^{\prime}(x) \\
& =\sec \left(e^{2 x+7}\right) \tan \left(e^{2 x+7}\right) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{2 x+7}\right\}
\end{aligned}
$$

Here, we need the chain rule again:

$$
\begin{aligned}
& =\sec \left(e^{2 x+7}\right) \tan \left(e^{2 x+7}\right) \cdot\left[e^{2 x+7} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\{2 x+7\}\right] \\
& =\sec \left(e^{2 x+7}\right) \tan \left(e^{2 x+7}\right) \cdot\left[e^{2 x+7} \cdot 2\right] \\
& =2 e^{2 x+7} \sec \left(e^{2 x+7}\right) \tan \left(e^{2 x+7}\right)
\end{aligned}
$$

2.9.4.15. Solution. It is possible to start in on this problem with the product rule and then the chain rule, but it's easier if we simplify first. Since $\tan ^{2} x+1=$ $\sec ^{2} x=\frac{1}{\cos ^{2} x}$, we see

$$
f(x)=\frac{\cos ^{2} x}{\cos ^{2} x}=1
$$

for all values of $x$ for which $\cos x$ is nonzero. That is, $f(x)=1$ for every $x$ that is not an integer multiple of $\pi / 2$ (and $f(x)$ is not defined when $x$ is an integer multiple of $\pi / 2$ ). Therefore, $f^{\prime}(x)=0$ for every $x$ on which $f$ exists, and in particular $f^{\prime}(\pi / 4)=0$. Also, $f(\pi / 4)=1$, so the tangent line to $f$ at $x=\pi / 4$ is the line with slope 0 , passing through the point $(\pi / 4,1)$ :

$$
y=1
$$

2.9.4.16. Solution. Velocity is the derivative of position with respect to time. So, the velocity of the particle is given by $s^{\prime}(t)$. We need to find $s^{\prime}(t)$, and determine when it is zero.
To differentiate, we us the chain rule.

$$
\begin{aligned}
s^{\prime}(t) & =e^{t^{3}-7 t^{2}+8 t} \cdot \frac{\mathrm{~d}}{\mathrm{~d} t}\left\{t^{3}-7 t^{2}+8 t\right\} \\
& =e^{t^{3}-7 t^{2}+8 t} \cdot\left(3 t^{2}-14 t+8\right)
\end{aligned}
$$

To determine where this function is zero, we factor:

$$
=e^{t^{3}-7 t^{2}+8 t} \cdot(3 t-2)(t-4)
$$

So, the velocity is zero when $e^{t^{3}-7 t^{2}+8 t}=0$, when $3 t-2=0$, and when $t-4=0$. Since $e^{t^{3}-7 t^{2}+8 t}$ is never zero, this tells us that the velocity is zero precisely when $t=\frac{2}{3}$ or $t=4$.
2.9.4.17. Solution. The slope of the tangent line is the derivative. If we let

$$
\begin{aligned}
& f(x)=\tan x \text { and } g(x)=e^{x^{2}}, \text { then } f(g(x))=\tan \left(e^{x^{2}}\right), \text { so } y^{\prime}=f^{\prime}(g(x)) \cdot g^{\prime}(x): \\
& \qquad y^{\prime}=\sec ^{2}\left(e^{x^{2}}\right) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{x^{2}}\right\}
\end{aligned}
$$

We find ourselves once more in need of the chain rule:

$$
\begin{aligned}
& =\sec ^{2}\left(e^{x^{2}}\right) \cdot e^{x^{2}} \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{x^{2}\right\} \\
& =\sec ^{2}\left(e^{x^{2}}\right) \cdot e^{x^{2}} \cdot 2 x
\end{aligned}
$$

Finally, we evaluate this derivative at the point $x=1$ :

$$
\begin{aligned}
y^{\prime}(1) & =\sec ^{2}(e) \cdot e \cdot 2 \\
& =2 e \sec ^{2} e
\end{aligned}
$$

2.9.4.18. *. Solution. Using the Product rule,

$$
y^{\prime}=\frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{4 x}\right\} \tan x+e^{4 x} \sec ^{2} x
$$

and the chain rule:

$$
\begin{aligned}
& =e^{4 x} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\{4 x\} \cdot \tan x+e^{4 x} \sec ^{2} x \\
& =4 e^{4 x} \tan x+e^{4 x} \sec ^{2} x
\end{aligned}
$$

2.9.4.19. *. Solution. Using the quotient rule,

$$
f^{\prime}(x)=\frac{\left(3 x^{2}\right)\left(1+e^{3 x}\right)-\left(x^{3}\right) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\left\{1+e^{3 x}\right\}}{\left(1+e^{3 x}\right)^{2}}
$$

Now, the chain rule:

$$
=\frac{\left(3 x^{2}\right)\left(1+e^{3 x}\right)-\left(x^{3}\right)\left(3 e^{3 x}\right)}{\left(1+e^{3 x}\right)^{2}}
$$

So, when $x=1$ :

$$
f^{\prime}(1)=\frac{3\left(1+e^{3}\right)-3 e^{3}}{\left(1+e^{3}\right)^{2}}=\frac{3}{\left(1+e^{3}\right)^{2}}
$$

2.9.4.20. *. Solution. This requires us to apply the chain rule twice.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{\sin ^{2}(x)}\right\} & =e^{\sin ^{2}(x)} \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\left\{\sin ^{2}(x)\right\} \\
& =e^{\sin ^{2}(x)}(2 \sin (x)) \cdot \frac{\mathrm{d}}{\mathrm{~d} x} \sin (x) \\
& =e^{\sin ^{2}(x)}(2 \sin (x)) \cdot \cos (x)
\end{aligned}
$$

2.9.4.21. *. Solution. This requires us to apply the chain rule twice.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\sin \left(e^{5 x}\right)\right\} & =\cos \left(e^{5 x}\right) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{5 x}\right\} \\
& =\cos \left(e^{5 x}\right)\left(e^{5 x}\right) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\{5 x\} \\
& =\cos \left(e^{5 x}\right)\left(e^{5 x}\right) \cdot 5
\end{aligned}
$$

2.9.4.22. *. Solution. We'll use the chain rule twice.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{\cos \left(x^{2}\right)}\right\} & =e^{\cos \left(x^{2}\right)} \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\left\{\cos \left(x^{2}\right)\right\} \\
& =e^{\cos \left(x^{2}\right)} \cdot\left(-\sin \left(x^{2}\right)\right) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\left\{x^{2}\right\} \\
& =-e^{\cos \left(x^{2}\right)} \cdot \sin \left(x^{2}\right) \cdot 2 x
\end{aligned}
$$

2.9.4.23. *. Solution. We start with the chain rule:

$$
\begin{aligned}
y^{\prime} & =-\sin \left(x^{2}+\sqrt{x^{2}+1}\right) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\left\{x^{2}+\sqrt{x^{2}+1}\right\} \\
& =-\sin \left(x^{2}+\sqrt{x^{2}+1}\right) \cdot\left(2 x+\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\sqrt{x^{2}+1}\right\}\right)
\end{aligned}
$$

and find ourselves in need of chain rule a second time:

$$
\begin{aligned}
& =-\sin \left(x^{2}+\sqrt{x^{2}+1}\right) \cdot\left(2 x+\frac{1}{2 \sqrt{x^{2}+1}} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{x^{2}+1\right\}\right) \\
& =-\sin \left(x^{2}+\sqrt{x^{2}+1}\right) \cdot\left(2 x+\frac{2 x}{2 \sqrt{x^{2}+1}}\right)
\end{aligned}
$$

2.9.4.24. *. Solution.

$$
y=\left(1+x^{2}\right) \cos ^{2} x
$$

Using the product rule,

$$
y^{\prime}=(2 x) \cos ^{2} x+\left(1+x^{2}\right) \frac{\mathrm{d}}{\mathrm{~d} x}\left\{\cos ^{2} x\right\}
$$

Here, we'll need to use the chain rule. Remember $\cos ^{2} x=[\cos x]^{2}$.

$$
\begin{aligned}
& =2 x \cos ^{2} x+\left(1+x^{2}\right) 2 \cos x \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\{\cos x\} \\
& =2 x \cos ^{2} x+\left(1+x^{2}\right) 2 \cos x \cdot(-\sin x) \\
& =2 x \cos ^{2} x-2\left(1+x^{2}\right) \sin x \cos x
\end{aligned}
$$

2.9.4.25. *. Solution. We use the quotient rule, noting by the chain rule that $\frac{\mathrm{d}}{\mathrm{d} x}\left\{e^{3 x}\right\}=3 e^{3 x}$ :

$$
y^{\prime}=\frac{\left(1+x^{2}\right) \cdot 3 e^{3 x}-e^{3 x}(2 x)}{\left(1+x^{2}\right)^{2}}
$$

$$
=\frac{e^{3 x}\left(3 x^{2}-2 x+3\right)}{\left(1+x^{2}\right)^{2}}
$$

2.9.4.26. *. Solution. By the chain rule,

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{h\left(x^{2}\right)\right\}=h^{\prime}\left(x^{2}\right) \cdot 2 x
$$

Using the product rules and the result above,

$$
g^{\prime}(x)=3 x^{2} h\left(x^{2}\right)+x^{3} h^{\prime}\left(x^{2}\right) 2 x
$$

Plugging in $x=2$ :

$$
\begin{aligned}
g^{\prime}(2) & =3\left(2^{2}\right) h\left(2^{2}\right)+2^{3} h^{\prime}\left(2^{2}\right) 2 \times 2 \\
& =12 h(4)+32 h^{\prime}(4)=12 \times 2-32 \times 2 \\
& =-40
\end{aligned}
$$

2.9.4.27. *. Solution. Let $f(x)=x e^{-\left(x^{2}-1\right) / 2}=x e^{\left(1-x^{2}\right) / 2}$. Then, using the product rule,

$$
f^{\prime}(x)=e^{\left(1-x^{2}\right) / 2}+x \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{e^{\left(1-x^{2}\right) / 2}\right\}
$$

Here, we need the chain rule:

$$
\begin{aligned}
& =e^{\left(1-x^{2}\right) / 2}+x \cdot e^{\left(1-x^{2}\right) / 2} \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{\frac{1}{2}\left(1-x^{2}\right)\right\} \\
& =e^{\left(1-x^{2}\right) / 2}+x \cdot e^{\left(1-x^{2}\right) / 2} \cdot(-x) \\
& =\left(1-x^{2}\right) e^{\left(1-x^{2}\right) / 2}
\end{aligned}
$$

There is no power of $e$ that is equal to zero; so if the product above is zero, it must be that $1-x^{2}=0$. This happens for $x= \pm 1$. On the curve, when $x=1, y=1$, and when $x=-1, y=-1$. So the points are $(1,1)$ and $(-1,-1)$.
2.9.4.28. Solution. The question asks when $s^{\prime}(t)$ is negative. So, we start by differentiating. Using the chain rule:

$$
\begin{aligned}
s^{\prime}(t) & =\cos \left(\frac{1}{t}\right) \cdot \frac{\mathrm{d}}{\mathrm{~d} t}\left\{\frac{1}{t}\right\} \\
& =\cos \left(\frac{1}{t}\right) \cdot \frac{-1}{t^{2}}
\end{aligned}
$$

When $t \geq 1, \frac{1}{t}$ is between 0 and 1 . Since $\cos \theta$ is positive for $0 \leq \theta<\pi / 2$, and $\pi / 2>1$, we see that $\cos \left(\frac{1}{t}\right)$ is positive for the entire domain of $s(t)$. Also, $\frac{-1}{t^{2}}$ is negative for the entire domain of the function. We conclude that $s^{\prime}(t)$ is negative for the entire domain of $s(t)$, so the particle is always moving in the negative direction.
2.9.4.29. Solution. We present two solutions: one where we dive right in and use the quotient rule, and another where we simplify first and use the product rule.

- Solution 1: We begin with the quotient rule:

$$
\begin{aligned}
f^{\prime}(x) & =\frac{\cos ^{3}(5 x-7) \frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{x}\right\}-e^{x} \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{\cos ^{3}(5 x-7)\right\}}{\cos ^{6}(5 x-7)} \\
& =\frac{\cos ^{3}(5 x-7) e^{x}-e^{x} \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{\cos ^{3}(5 x-7)\right\}}{\cos ^{6}(5 x-7)}
\end{aligned}
$$

Now, we use the chain rule. Since $\cos ^{3}(5 x-7)=[\cos (5 x-7)]^{3}$, our "outside" function is $g(x)=x^{3}$, and our "inside" function is $h(x)=\cos (5 x-1)$.

$$
=\frac{\cos ^{3}(5 x-7) e^{x}-e^{x} \cdot 3 \cos ^{2}(5 x-7) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\{\cos (5 x-7)\}}{\cos ^{6}(5 x-7)}
$$

We need the chain rule again!

$$
\begin{aligned}
& =\frac{\cos ^{3}(5 x-7) e^{x}-e^{x} \cdot 3 \cos ^{2}(5 x-7) \cdot\left[-\sin (5 x-7) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\{5 x-7\}\right]}{\cos ^{6}(5 x-7)} \\
& =\frac{\cos ^{3}(5 x-7) e^{x}-e^{x} \cdot 3 \cos ^{2}(5 x-7) \cdot[-\sin (5 x-7) \cdot 5]}{\cos ^{6}(5 x-7)}
\end{aligned}
$$

We finish by simplifying:

$$
\begin{aligned}
& =\frac{e^{x} \cos ^{2}(5 x-7)(\cos (5 x-7)+15 \sin (5 x-7))}{\cos ^{6}(5 x-7)} \\
& =e^{x} \frac{\cos (5 x-7)+15 \sin (5 x-7)}{\cos ^{4}(5 x-7)} \\
& =e^{x}\left(\sec ^{3}(5 x-7)+15 \tan (5 x-7) \sec ^{3}(5 x-7)\right) \\
& =e^{x} \sec ^{3}(5 x-7)(1+15 \tan (5 x-7))
\end{aligned}
$$

- Solution 2: We simplify to avoid the quotient rule:

$$
\begin{aligned}
f(x) & =\frac{e^{x}}{\cos ^{3}(5 x-7)} \\
& =e^{x} \sec ^{3}(5 x-7)
\end{aligned}
$$

Now we use the product rule to differentiate:

$$
f^{\prime}(x)=e^{x} \sec ^{3}(5 x-7)+e^{x} \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{\sec ^{3}(5 x-7)\right\}
$$

Here, we'll need the chain rule. Since $\sec ^{3}(5 x-7)=[\sec (5 x-7)]^{3}$, our "outside" function is $g(x)=x^{3}$ and our "inside" function is $h(x)=\sec (5 x-7)$, so that $g(h(x))=[\sec (5 x-7)]^{3}=\sec ^{3}(5 x-7)$.

$$
=e^{x} \sec ^{3}(5 x-7)+e^{x} \cdot 3 \sec ^{2}(5 x-7) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\{\sec (5 x-7)\}
$$

We need the chain rule again! Recall $\frac{\mathrm{d}}{\mathrm{d} x}\{\sec x\}=\sec x \tan x$.

$$
\begin{aligned}
= & e^{x} \sec ^{3}(5 x-7) \\
& +e^{x} \cdot 3 \sec ^{2}(5 x-7) \cdot \sec (5 x-7) \tan (5 x-7) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\{5 x-7\} \\
= & e^{x} \sec ^{3}(5 x-7) \\
& +e^{x} \cdot 3 \sec ^{2}(5 x-7) \cdot \sec (5 x-7) \tan (5 x-7) \cdot 5
\end{aligned}
$$

We finish by simplifying:

$$
=e^{x} \sec ^{3}(5 x-7)(1+15 \tan (5 x-7))
$$

### 2.9.4.30. *. Solution.

- Solution 1: In Example 2.6.6, we generalized the product rule to three factors:

$$
\begin{array}{r}
\frac{\mathrm{d}}{\mathrm{~d} x}\{f(x) g(x) h(x)\}=f^{\prime}(x) g(x) h(x)+f(x) g^{\prime}(x) h(x) \\
+f(x) g(x) h^{\prime}(x)
\end{array}
$$

Using this rule:

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} x}\left\{(x)\left(e^{2 x}\right)(\cos 4 x)\right\} \\
& =\frac{\mathrm{d}}{\mathrm{~d} x}\{x\} \cdot e^{2 x} \cos 4 x+x \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{e^{2 x}\right\} \cdot \cos 4 x+x e^{2 x} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\{\cos 4 x\} \\
& =e^{2 x} \cos 4 x+x\left(2 e^{2 x}\right) \cos 4 x+x e^{2 x}(-4 \sin 4 x) \\
& =e^{2 x} \cos 4 x+2 x e^{2 x} \cos 4 x-4 x e^{2 x} \sin 4 x
\end{aligned}
$$

- Solution 2: We can use the product rule twice. In the first step, we split the function $x e^{2 x} \cos 4 x$ into the product of two functions.

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} x}\left\{\left(x e^{2 x}\right)(\cos 4 x)\right\} \\
& =\frac{\mathrm{d}}{\mathrm{~d} x}\left\{x e^{2 x}\right\} \cdot \cos 4 x+x e^{2 x} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\{\cos 4 x\} \\
& =\left(\frac{\mathrm{d}}{\mathrm{~d} x}\{x\} \cdot e^{2 x}+x \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{e^{2 x}\right\}\right) \cdot \cos 4 x+x e^{2 x} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\{\cos 4 x\} \\
& =\left(e^{2 x}+x\left(2 e^{2 x}\right)\right) \cdot \cos 4 x+x e^{2 x}(-4 \sin 4 x) \\
& =e^{2 x} \cos 4 x+2 x e^{2 x} \cos 4 x-4 x e^{2 x} \sin 4 x
\end{aligned}
$$

## Exercises - Stage 3

2.9.4.31. Solution. At time $t$, the particle is at the point $(x(t), y(t))$, with $x(t)=\cos t$ and $y(t)=\sin t$. Over time, the particle traces out a curve; let's call that curve $y=f(x)$. Then $y(t)=f(x(t))$, so the slope of the curve at the point
$(x(t), y(t))$ is $f^{\prime}(x(t))$. You are to determine the values of $t$ for which $f^{\prime}(x(t))=-1$. By the chain rule

$$
y^{\prime}(t)=f^{\prime}(x(t)) \cdot x^{\prime}(t)
$$

Substituting in $x(t)=\cos t$ and $y(t)=\sin t$ gives

$$
\cos t=f^{\prime}(x(t)) \cdot(-\sin t)
$$

so that

$$
f^{\prime}(x(t))=-\frac{\cos t}{\sin t}
$$

is -1 precisely when $\sin t=\cos t$. This happens whenever $t=\frac{\pi}{4}$.
Remark: the path traced by the particle is a semicircle. You can think about the point on the unit circle with angle t , or you can notice that $x^{2}+y^{2}=\sin ^{2} t+\cos ^{2} t=$ 1.
2.9.4.32. *. Solution. Let $f(x)=e^{x+x^{2}}$ and $g(x)=1+x$. Then $f(0)=g(0)=1$. $f^{\prime}(x)=(1+2 x) e^{x+x^{2}}$ and $g^{\prime}(x)=1$. When $x>0$,

$$
f^{\prime}(x)=(1+2 x) e^{x+x^{2}}>1 \cdot e^{x+x^{2}}=e^{x+x^{2}}>e^{0+0^{2}}=1=g^{\prime}(x)
$$

Since $f(0)=g(0)$, and $f^{\prime}(x)>g^{\prime}(x)$ for all $x>0$, that means $f$ and $g$ start at the same place, but $f$ always grows faster. Therefore, $f(x)>g(x)$ for all $x>0$.
2.9.4.33. Solution. Since $\sin 2 x$ and $2 \sin x \cos x$ are the same function, they have the same derivative.

$$
\begin{aligned}
\sin 2 x & =2 \sin x \cos x \\
\Rightarrow \frac{\mathrm{~d}}{\mathrm{~d} x}\{\sin 2 x\} & =\frac{\mathrm{d}}{\mathrm{~d} x}\{2 \sin x \cos x\} \\
2 \cos 2 x & =2\left[\cos ^{2} x-\sin ^{2} x\right] \\
\cos 2 x & =\cos ^{2} x-\sin ^{2} x
\end{aligned}
$$

We conclude $\cos 2 x=\cos ^{2} x-\sin ^{2} x$, which is another common trig identity.
Remark: if we differentiate both sides of this equation, we get the original identity back.

### 2.9.4.34. Solution.

$$
\begin{aligned}
f(x) & =\sqrt[3]{\frac{e^{\csc x^{2}}}{\sqrt{x^{3}-9} \tan x}} \\
& =\left(\frac{e^{\csc x^{2}}}{\sqrt{x^{3}-9} \tan x}\right)^{\frac{1}{3}}
\end{aligned}
$$

To begin the differentiation, we can choose our "outside" function to be $g(x)=x^{\frac{1}{3}}$,
and our "inside" function to be $h(x)=\frac{e^{\csc x^{2}}}{\sqrt{x^{3}-9} \tan x}$. Then $f(x)=g(h(x))$, so $f^{\prime}(x)=g^{\prime}(h(x)) \cdot h^{\prime}(x)=\frac{1}{3}(h(x))^{-\frac{2}{3}} h^{\prime}(x):$

$$
\begin{aligned}
f^{\prime}(x) & =\frac{1}{3}\left(\frac{e^{\csc x^{2}}}{\sqrt{x^{3}-9} \tan x}\right)^{\frac{-2}{3}} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{\frac{e^{\csc x^{2}}}{\sqrt{x^{3}-9} \tan x}\right\} \\
& =\frac{1}{3}\left(\frac{\sqrt{x^{3}-9} \tan x}{e^{\csc x^{2}}}\right)^{\frac{2}{3}} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{\frac{e^{\csc x^{2}}}{\sqrt{x^{3}-9} \tan x}\right\}
\end{aligned}
$$

This leads us to use the quotient rule:

$$
\begin{aligned}
= & \frac{1}{3}\left(\frac{\sqrt{x^{3}-9} \tan x}{e^{\csc x^{2}}}\right)^{\frac{2}{3}} \\
& \left(\frac{\sqrt{x^{3}-9} \tan x \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{e^{\csc x^{2}}\right\}-e^{\csc x^{2}} \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{\sqrt{x^{3}-9} \tan x\right\}}{\left(\tan ^{2} x\right)\left(x^{3}-9\right)}\right)
\end{aligned}
$$

Let's figure out those two derivatives on their own, then plug them in. Using the chain rule twice:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{\csc x^{2}}\right\} & =e^{\csc x^{2}} \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{\csc x^{2}\right\} \\
& =e^{\csc x^{2}} \cdot\left(-\csc \left(x^{2}\right) \cot \left(x^{2}\right)\right) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\left\{x^{2}\right\} \\
& =-2 x e^{\csc x^{2}} \frac{\cos \left(x^{2}\right)}{\sin ^{2}\left(x^{2}\right)}
\end{aligned}
$$

For the other derivative, we start with the product rule, then chain:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\sqrt{x^{3}-9} \tan x\right\} & =\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\sqrt{x^{3}-9}\right\} \cdot \tan x+\sqrt{x^{3}-9} \sec ^{2} x \\
& =\frac{1}{2 \sqrt{x^{3}-9}} \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{x^{3}-9\right\} \cdot \tan x+\sqrt{x^{3}-9} \sec ^{2} x \\
& =\frac{3 x^{2} \tan x}{2 \sqrt{x^{3}-9}}+\sqrt{x^{3}-9} \sec ^{2} x
\end{aligned}
$$

Now, we plug these into our equation for $f^{\prime}(x)$ :

$$
\begin{aligned}
f^{\prime}(x)= & \frac{1}{3}\left(\frac{\sqrt{x^{3}-9} \tan x}{e^{\csc x^{2}}}\right)^{\frac{2}{3}} \\
& \left(\frac{\sqrt{x^{3}-9} \tan x \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{e^{\csc x^{2}}\right\}-e^{\csc x^{2} \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{\sqrt{x^{3}-9} \tan x\right\}}}{\left(\tan ^{2} x\right)\left(x^{3}-9\right)}\right) \\
= & \frac{1}{3}\left(\frac{\sqrt{x^{3}-9} \tan x}{f_{\mathcal{C N}}^{\csc x^{2}}}\right)^{\frac{2}{3}}
\end{aligned}
$$

2.9.4.35. Solution. 2.9.4.35.a The table below gives us a number of points on our graph, and thesc $\frac{\left.x^{2} \text { in (ees they } \sqrt{x^{3} \operatorname{cer}} \tan x \frac{\cos \left(x^{2}\right)}{\sin ^{2}\left(x^{2}\right)}-\frac{3 x^{2} \tan x}{2 \sqrt{x^{3}-9}}-\sqrt{x^{3}-9} \sec ^{2} x\right)}{\text { ) }}$

$$
\left(\tan ^{2} x\right)\left(x^{3}-9\right)
$$

| $t$ | $\left(\sin t, \cos ^{2} t\right)$ |
| :--- | :--- |
| 0 | $(0,1)$ |
| $\pi / 4$ | $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ |
| $\pi / 2$ | $(1,0)$ |
| $3 \pi / 4$ | $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ |
| $\pi$ | $(0,1)$ |
| $5 \pi / 4$ | $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ |
| $3 \pi / 2$ | $(-1,0)$ |
| $7 \pi / 4$ | $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ |
| $2 \pi$ | $(0,1)$ |

These points will repeat with a period of $2 \pi$. With this information, we have a pretty good idea of the particle's motion:


The particle traces out an arc, pointing down. It starts at $t=0$ at the top part of the graph at $(1,0)$, then is moves to the right until it hits $(1,0)$ at time $t=\pi / 2$. From there it reverses direction and moves along the curve to the left, hitting the top at time $t=\pi$ and reaching $(-1,0)$ at time $t=3 \pi / 2$. Then it returns to the top at $t=2 \pi$ and starts again.
So, it starts at the top of the curve, then moves back for forth along the length of the curve. If goes right first, and repeats its cycle every $2 \pi$ units of time.
2.9.4.35.b Let $y=f(x)$ be the curve the particle traces in the $x y$-plane. Since $x$ is a function of $t, y(t)=f(x(t))$. What we want to find is $\frac{\mathrm{d} f}{\mathrm{~d} x}$ when $t=\left(\frac{10 \pi}{3}\right)$. Since $\frac{\mathrm{d} f}{\mathrm{~d} x}$ is a function of $x$, we note that when $t=\left(\frac{10 \pi}{3}\right), x=\sin \left(\frac{10 \pi}{3}\right)=$ $\sin \left(\frac{4 \pi}{3}\right)=-\frac{\sqrt{3}}{2}$. So, the quantity we want to find (the slope of the tangent line to the curve $y=f(x)$ traced by the particle at the time $t=\left(\frac{10 \pi}{3}\right)$ is given by
$\frac{\mathrm{d} f}{\mathrm{~d} x}\left(-\frac{\sqrt{3}}{2}\right)$.
Using the chain rule:

$$
\begin{aligned}
y(t) & =f(x(t)) \\
\frac{\mathrm{d} y}{\mathrm{~d} t}=\frac{\mathrm{d}}{\mathrm{~d} t}\{f(x(t))\} & =\frac{\mathrm{d} f}{\mathrm{~d} x} \cdot \frac{\mathrm{~d} x}{\mathrm{~d} t} \\
\text { so, } \quad \frac{\mathrm{d} f}{\mathrm{~d} x} & =\frac{\mathrm{d} y}{\mathrm{~d} t} \div \frac{\mathrm{d} x}{\mathrm{~d} t}
\end{aligned}
$$

Using $y(t)=\cos ^{2} t$ and $x(t)=\sin t$ :

$$
\frac{\mathrm{d} f}{\mathrm{~d} x}=(-2 \cos t \sin t) \div(\cos t)=-2 \sin t=-2 x
$$

So, when $t=\frac{10 \pi}{3}$ and $x=-\frac{\sqrt{3}}{2}$,

$$
\frac{\mathrm{d} f}{\mathrm{~d} x}\left(\frac{-\sqrt{3}}{2}\right)=-2 \cdot \frac{-\sqrt{3}}{2}=\sqrt{3}
$$

Remark: The standard way to write this problem is to omit the notation $f(x)$, and let the variable $y$ stand for two functions. When $t$ is the variable, $y(t)=\cos ^{2} t$ gives the $y$-coordinate of the particle at time $t$. When $x$ is the variable, $y(x)$ gives the $y$-coordinate of the particle given its position along the $x$-axis. This is an abuse of notation, because if we write $y(1)$, it is not clear whether we are referring to the $y$-coordinate of the particle when $t=1$ (in this case, $y=\cos ^{2}(1) \approx 0.3$ ), or the $y$-coordinate of the particle when $x=1$ (in this case, looking at our table of values, $y=0$ ). Although this notation is not strictly "correct," it is very commonly used. So, you might see a solution that looks like this:

The slope of the curve is $\frac{\mathrm{d} y}{\mathrm{~d} x}$. To find $\frac{\mathrm{d} y}{\mathrm{~d} x}$, we use the chain rule:

$$
\begin{aligned}
\frac{\mathrm{d} y}{\mathrm{~d} t} & =\frac{\mathrm{d} y}{\mathrm{~d} x} \cdot \frac{\mathrm{~d} x}{\mathrm{~d} t} \\
\frac{\mathrm{~d}}{\mathrm{~d} t}\left\{\cos ^{2} t\right\} & =\frac{\mathrm{d} y}{\mathrm{~d} x} \cdot \frac{\mathrm{~d}}{\mathrm{~d} t}\{\sin t\} \\
-2 \cos t \sin t & =\frac{\mathrm{d} y}{\mathrm{~d} x} \cdot \cos t \\
\frac{\mathrm{~d} y}{\mathrm{~d} x} & =-2 \sin t
\end{aligned}
$$

So, when $t=\frac{10 \pi}{3}$,

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=-2 \sin \left(\frac{10 \pi}{3}\right)=-2\left(-\frac{\sqrt{3}}{2}\right)=\sqrt{3}
$$

 2.14e8 it i玉redreisessction of $x$. This notation (using $y$ to be two functions, $y(t)$ and $y(x))$ is actually the accepted standard, so you should be able to understand it.

## Exercises - Stage 1

2.10.3.1. Solution. We are given that one speaker produces 3 dB . So if $P$ is the power of one speaker,

$$
3=V(P)=10 \log _{10}\left(\frac{P}{S}\right)
$$

So, for ten speakers:

$$
\begin{aligned}
V(10 P) & =10 \log _{10}\left(\frac{10 P}{S}\right)=10 \log _{10}\left(\frac{P}{S}\right)+10 \log _{10}(10) \\
& =3+10(1)=13 \mathrm{~dB}
\end{aligned}
$$

and for one hundred speakers:

$$
\begin{aligned}
V(100 P) & =10 \log _{10}\left(\frac{100 P}{S}\right)=10 \log _{10}\left(\frac{P}{S}\right)+10 \log _{10}(100) \\
& =3+10(2)=23 \mathrm{~dB}
\end{aligned}
$$

2.10.3.2. Solution. The investment doubles when it hits $\$ 2000$. So, we find the value of $t$ that gives $A(t)=2000$ :

$$
\begin{aligned}
2000 & =A(t) \\
2000 & =1000 e^{t / 20} \\
2 & =e^{t / 20} \\
\log 2 & =\frac{t}{20} \\
20 \log 2 & =t
\end{aligned}
$$

2.10.3.3. Solution. From our logarithm rules, we know that when $y$ is positive, $\log \left(y^{2}\right)=2 \log y$. However, the expression $\cos x$ does not always take on positive values, so (a) is not correct. (For instance, when $x=\pi, \log \left(\cos ^{2} x\right)=\log \left(\cos ^{2} \pi\right)=$ $\log \left((-1)^{2}\right)=\log (1)=0$, while $2 \log (\cos \pi)=2 \log (-1)$, which does not exist.)
Because $\cos ^{2} x$ is never negative, we notice that $\cos ^{2} x=|\cos x|^{2}$. When $\cos x$ is nonzero, $|\cos x|$ is positive, so our logarithm rules tell us $\log \left(|\cos x|^{2}\right)=2 \log |\cos x|$. When $\cos x$ is exactly zero, then both $\log \left(\cos ^{2} x\right)$ and $2 \log |\cos x|$ do not exist. So, $\log \left(\cos ^{2} x\right)=2 \log |\cos x|$.

## Exercises - Stage 2

2.10.3.4. Solution.

- Solution 1: Using the chain rule, $\frac{\mathrm{d}}{\mathrm{d} x}\{\log (10 x)\}=\frac{1}{10 x} \cdot 10=\frac{1}{x}$.
- Solution 2: Simplifying, $\frac{\mathrm{d}}{\mathrm{d} x}\{\log (10 x)\}=\frac{\mathrm{d}}{\mathrm{d} x}\{\log (10)+\log x\}=0+\frac{1}{x}=\frac{1}{x}$.


### 2.10.3.5. Solution.

- Solution 1: Using the chain rule, $\frac{\mathrm{d}}{\mathrm{d} x}\left\{\log \left(x^{2}\right)\right\}=\frac{1}{x^{2}} \cdot 2 x=\frac{2}{x}$.
- Solution 2: Simplifying, $\frac{\mathrm{d}}{\mathrm{d} x}\left\{\log \left(x^{2}\right)\right\}=\frac{\mathrm{d}}{\mathrm{d} x}\{2 \log (x)\}=\frac{2}{x}$.
2.10.3.6. Solution. Don't be fooled by a common mistake: $\log \left(x^{2}+x\right)$ is not the same as $\log \left(x^{2}\right)+\log x$. We differentiate using the chain rule: $\frac{\mathrm{d}}{\mathrm{d} x}\left\{\log \left(x^{2}+x\right)\right\}=$ $\frac{1}{x^{2}+x} \cdot(2 x+1)=\frac{2 x+1}{x^{2}+x}$.
2.10.3.7. Solution. We know the derivative of the natural logarithm (base e), so we use the base-change formula:

$$
f(x)=\log _{10} x=\frac{\log x}{\log 10}
$$

Since $\log 10$ is a constant:

$$
f^{\prime}(x)=\frac{1}{x \log 10}
$$

### 2.10.3.8. *. Solution.

- Solution 1: Using the quotient rule,

$$
y^{\prime}=\frac{x^{3} \frac{1}{x}-(\log x) \cdot 3 x^{2}}{x^{6}}=\frac{x^{2}-3 x^{2} \log x}{x^{6}}=\frac{1-3 \log x}{x^{4}} .
$$

- Solution 2: Using the product rule with $y=\log x \cdot x^{-3}$,

$$
y^{\prime}=\frac{1}{x} x^{-3}+\log x \cdot(-3) x^{-4}=x^{-4}(1-3 \log x)
$$

2.10.3.9. Solution. Using the chain rule,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} \theta} \log (\sec \theta) & =\frac{1}{\sec \theta} \cdot(\sec \theta \cdot \tan \theta) \\
& =\tan \theta
\end{aligned}
$$

Remark: the domain of the function $\log (\sec \theta)$ is those values of $\theta$ for which $\sec \theta$ is positive: so, the intervals $\left(\left(2 n-\frac{1}{2}\right) \pi,\left(2 n+\frac{1}{2}\right) \pi\right)$ where $n$ is any integer. Certainly the tangent function has a larger domain than this, but outside the domain of $\log (\sec \theta), \tan \theta$ is not the derivative of $\log (\sec \theta)$.
2.10.3.10. Solution. Let's start in with the chain rule.

$$
f^{\prime}(x)=e^{\cos (\log x)} \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\{\cos (\log x)\}
$$

We'll need the chain rule again:

$$
\begin{aligned}
& =e^{\cos (\log x)}(-\sin (\log x)) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\{\log x\} \\
& =e^{\cos (\log x)}(-\sin (\log x)) \cdot \frac{1}{x} \\
& =\frac{-e^{\cos (\log x)} \sin (\log x)}{x}
\end{aligned}
$$

Remark: Although we have a logarithm in the exponent, we can't cancel. The expression $e^{\cos (\log x)}$ is not the same as the expression $x^{\cos x}$, or $\cos x$.

### 2.10.3.11. *. Solution.

$$
y=\log \left(x^{2}+\sqrt{x^{4}+1}\right)
$$

So, we'll need the chain rule:

$$
\begin{aligned}
y^{\prime} & =\frac{\frac{\mathrm{d}}{\mathrm{~d} x}\left\{x^{2}+\sqrt{x^{4}+1}\right\}}{x^{2}+\sqrt{x^{4}+1}} \\
& =\frac{2 x+\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\sqrt{x^{4}+1}\right\}}{x^{2}+\sqrt{x^{4}+1}}
\end{aligned}
$$

We need the chain rule again:

$$
\begin{aligned}
& =\frac{2 x+\frac{\frac{\mathrm{d}}{\mathrm{~d} x}\left\{x^{4}+1\right\}}{2 \sqrt{x^{4}+1}}}{x^{2}+\sqrt{x^{4}+1}} \\
& =\frac{2 x+\frac{4 x^{3}}{2 \sqrt{x^{4}+1}}}{x^{2}+\sqrt{x^{4}+1}}
\end{aligned}
$$

2.10.3.12. *. Solution. This requires us to apply the chain rule twice.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{\sqrt{-\log (\cos x)}\} & =\frac{1}{2 \sqrt{-\log (\cos x)}} \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\{-\log (\cos x)\} \\
& =-\frac{1}{2 \sqrt{-\log (\cos x)}} \cdot \frac{1}{\cos x} \frac{\mathrm{~d}}{\mathrm{~d} x}\{\cos x\} \\
& =-\frac{1}{2 \sqrt{-\log (\cos x)}} \cdot \frac{1}{\cos x} \cdot(-\sin x) \\
& =\frac{\tan x}{2 \sqrt{-\log (\cos x)}}
\end{aligned}
$$

Remark: it looks strange to see a negative sign in the argument of a square root. Since the cosine function always gives values that are at most $1, \log (\cos x)$ is always negative or zero over its domain. So, $\sqrt{\log (\cos x)}$ is only defined for the points where $\cos x=1$ (and so $\log (\cos x)=0$-this isn't a very interesting function! In contrast, $-\log (\cos x)$ is always positive or zero over its domain - and therefore we can always take its square root.
2.10.3.13. *. Solution. Under the chain rule, $\frac{\mathrm{d}}{\mathrm{d} x} \log f(x)=\frac{1}{f(x)} f^{\prime}(x)$. So

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\log \left(x+\sqrt{x^{2}+4}\right)\right\} & =\frac{1}{x+\sqrt{x^{2}+4}} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{x+\sqrt{x^{2}+4}\right\} \\
& =\frac{1}{x+\sqrt{x^{2}+4}} \cdot\left(1+\frac{2 x}{2 \sqrt{x^{2}+4}}\right) \\
& =\frac{1}{x+\sqrt{x^{2}+4}} \cdot\left(\frac{2 \sqrt{x^{2}+4}+2 x}{2 \sqrt{x^{2}+4}}\right) \\
& =\frac{1}{\sqrt{x^{2}+4}}
\end{aligned}
$$

2.10.3.14. *. Solution. Using the chain rule,

$$
\begin{aligned}
g^{\prime}(x) & =\frac{\frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{x^{2}}+\sqrt{1+x^{4}}\right\}}{e^{x^{2}}+\sqrt{1+x^{4}}} \\
& =\frac{2 x e^{x^{2}}+\frac{4 x^{3}}{2 \sqrt{1+x^{4}}}}{e^{x^{2}}+\sqrt{1+x^{4}}}\left(\frac{\sqrt{1+x^{4}}}{\sqrt{1+x^{4}}}\right) \\
& =\frac{2 x e^{x^{2}} \sqrt{1+x^{4}}+2 x^{3}}{e^{x^{2}} \sqrt{1+x^{4}}+1+x^{4}}
\end{aligned}
$$

2.10.3.15. *. Solution. Using logarithm rules makes this an easier problem:

$$
\begin{aligned}
g(x) & =\log (2 x-1)-\log (2 x+1) \\
\text { So, } \quad g^{\prime}(x) & =\frac{2}{2 x-1}-\frac{2}{2 x+1} \\
\text { and } \quad g^{\prime}(1) & =\frac{2}{1}-\frac{2}{3}=\frac{4}{3}
\end{aligned}
$$

2.10.3.16. Solution. We begin by simplifying:

$$
\begin{aligned}
f(x) & =\log \left(\sqrt{\frac{\left(x^{2}+5\right)^{3}}{x^{4}+10}}\right) \\
& =\log \left(\left(\frac{\left(x^{2}+5\right)^{3}}{x^{4}+10}\right)^{1 / 2}\right) \\
& =\frac{1}{2} \log \left(\frac{\left(x^{2}+5\right)^{3}}{x^{4}+10}\right) \\
& =\frac{1}{2}\left[\log \left(\left(x^{2}+5\right)^{3}\right)-\log \left(x^{4}+10\right)\right] \\
& =\frac{1}{2}\left[3 \log \left(\left(x^{2}+5\right)\right)-\log \left(x^{4}+10\right)\right]
\end{aligned}
$$

Now, we differentiate using the chain rule:

$$
f^{\prime}(x)=\frac{1}{2}\left[3 \frac{2 x}{x^{2}+5}-\frac{4 x^{3}}{x^{4}+10}\right]
$$

$$
=\frac{3 x}{x^{2}+5}-\frac{2 x^{3}}{x^{4}+10}
$$

Remark: it is a common mistake to write $\log \left(x^{2}+4\right)$ as $\log \left(x^{2}\right)+\log (4)$. These expressions are not equivalent!
2.10.3.17. Solution. We use the chain rule twice, followed by the product rule:

$$
\begin{aligned}
f^{\prime}(x) & =\frac{1}{g(x h(x))} \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\{g(x h(x))\} \\
& =\frac{1}{g(x h(x))} \cdot g^{\prime}(x h(x)) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\{x h(x)\} \\
& =\frac{1}{g(x h(x))} \cdot g^{\prime}(x h(x)) \cdot\left[h(x)+x h^{\prime}(x)\right]
\end{aligned}
$$

In particular, when $x=2$ :

$$
\begin{aligned}
f^{\prime}(2) & =\frac{1}{g(2 h(2))} \cdot g^{\prime}(2 h(2)) \cdot\left[h(2)+2 h^{\prime}(2)\right] \\
& =\frac{g^{\prime}(4)}{g(4)}[2+2 \times 3]=\frac{5}{3}[2+2 \times 3] \\
& =\frac{40}{3}
\end{aligned}
$$

2.10.3.18. *. Solution. In the text, we saw that $\frac{\mathrm{d}}{\mathrm{d} x}\left\{a^{x}\right\}=a^{x} \log a$ for any constant $a$. So, $\frac{\mathrm{d}}{\mathrm{d} x}\left\{\pi^{x}\right\}=\pi^{x} \log \pi$.
By the power rule, $\frac{\mathrm{d}}{\mathrm{d} x}\left\{x^{\pi}\right\}=\pi x^{\pi-1}$.
Therefore, $g^{\prime}(x)=\pi^{x} \log \pi+\pi x^{\pi-1}$.
Remark: we had to use two different rules for the two different terms in $g(x)$. Although the functions $\pi^{x}$ and $x^{\pi}$ look superficially the same, they behave differently, as do their derivatives. A function of the form (constant) ${ }^{x}$ is an exponential function and not eligible for the power rule, while a function of the form $x^{\text {constant }}$ is exactly the class of function the power rule applies to.
2.10.3.19. Solution. We have the power rule to tell us the derivative of functions of the form $x^{n}$, where $n$ is a constant. However, here our exponent is not a constant. Similarly, in this section we learned the derivative of functions of the form $a^{x}$, where $a$ is a constant, but again, our base is not a constant! Although the result $\frac{\mathrm{d}}{\mathrm{d} x} a^{x}=a^{x} \log a$ is not what we need, the method used to differentiate $a^{x}$ will tell us the derivative of $x^{x}$.
We'll set $g(x)=\log \left(x^{x}\right)$, because now we can use logarithm rules to simplify:

$$
g(x)=\log (f(x))=x \log x
$$

Now, we can use the product rule to differentiate the right side, and the chain rule to differentiate $\log (f(x))$ :

$$
g^{\prime}(x)=\frac{f^{\prime}(x)}{f(x)}=\log x+x \frac{1}{x}=\log x+1
$$

Finally, we solve for $f^{\prime}(x)$ :

$$
f^{\prime}(x)=f(x)(\log x+1)=x^{x}(\log x+1)
$$

2.10.3.20. *. Solution. In Question 2.10.3.19, we saw $\frac{\mathrm{d}}{\mathrm{d} x}\left\{x^{x}\right\}=x^{x}(\log x+1)$. Using the base-change formula, $\log _{10}(x)=\frac{\log x}{\log 10}$. Since $\log _{10}$ is a constant,

$$
\begin{aligned}
f^{\prime}(x) & =\frac{\mathrm{d}}{\mathrm{~d} x}\left\{x^{x}+\frac{\log x}{\log 10}\right\} \\
& =x^{x}(\log x+1)+\frac{1}{x \log 10}
\end{aligned}
$$

2.10.3.21. Solution. Rather than set in with a terrible chain rule problem, we'll use logarithmic differentiation. Instead of differentiating $f(x)$, we differentiate a new function $\log (f(x))$, after simplifying.

$$
\begin{aligned}
\log (f(x)) & =\log \sqrt[4]{\frac{\left(x^{4}+12\right)\left(x^{4}-x^{2}+2\right)}{x^{3}}} \\
& =\frac{1}{4} \log \left(\frac{\left(x^{4}+12\right)\left(x^{4}-x^{2}+2\right)}{x^{3}}\right) \\
& =\frac{1}{4}\left(\log \left(x^{4}+12\right)+\log \left(x^{4}-x^{2}+2\right)-3 \log x\right)
\end{aligned}
$$

Now that we've simplified, we can efficiently differentiate both sides. It is important to remember that we aren't differentiating $f(x)$ directly-we're differentiating $\log (f(x))$.

$$
\frac{f^{\prime}(x)}{f(x)}=\frac{1}{4}\left(\frac{4 x^{3}}{x^{4}+12}+\frac{4 x^{3}-2 x}{x^{4}-x^{2}+2}-\frac{3}{x}\right)
$$

Our final step is to solve for $f^{\prime}(x)$ :

$$
\begin{aligned}
f^{\prime}(x) & =f(x) \frac{1}{4}\left(\frac{4 x^{3}}{x^{4}+12}+\frac{4 x^{3}-2 x}{x^{4}-x^{2}+2}-\frac{3}{x}\right) \\
& =\frac{1}{4}\left(\sqrt[4]{\frac{\left(x^{4}+12\right)\left(x^{4}-x^{2}+2\right)}{x^{3}}}\right)\left(\frac{4 x^{3}}{x^{4}+12}+\frac{4 x^{3}-2 x}{x^{4}-x^{2}+2}-\frac{3}{x}\right)
\end{aligned}
$$

It was possible to differentiate this function without logarithms, but the logarithms make it more efficient.
2.10.3.22. Solution. It's possible to do this using the product rule a number of times, but it's easier to use logarithmic differentiation. Set

$$
g(x)=\log (f(x))=\log \left[(x+1)\left(x^{2}+1\right)^{2}\left(x^{3}+1\right)^{3}\left(x^{4}+1\right)^{4}\left(x^{5}+1\right)^{5}\right]
$$

Now we can use logarithm rules to change $g(x)$ into a form that is friendlier to differentiate:

$$
\begin{aligned}
=\log (x+1)+\log \left(x^{2}+1\right)^{2} & +\log \left(x^{3}+1\right)^{3}+\log \left(x^{4}+1\right)^{4} \\
& +\log \left(x^{5}+1\right)^{5} \\
=\log (x+1)+2 \log \left(x^{2}+1\right) & +3 \log \left(x^{3}+1\right)+4 \log \left(x^{4}+1\right) \\
& +5 \log \left(x^{5}+1\right)
\end{aligned}
$$

Now, we differentiate $g(x)$ using the chain rule:

$$
g^{\prime}(x)=\frac{f^{\prime}(x)}{f(x)}=\frac{1}{x+1}+\frac{4 x}{x^{2}+1}+\frac{9 x^{2}}{x^{3}+1}+\frac{16 x^{3}}{x^{4}+1}+\frac{25 x^{4}}{x^{5}+1}
$$

Finally, we solve for $f^{\prime}(x)$ :

$$
\begin{aligned}
f^{\prime}(x)= & f(x)\left[\frac{1}{x+1}+\frac{4 x}{x^{2}+1}+\frac{9 x^{2}}{x^{3}+1}+\frac{16 x^{3}}{x^{4}+1}+\frac{25 x^{4}}{x^{5}+1}\right] \\
= & (x+1)\left(x^{2}+1\right)^{2}\left(x^{3}+1\right)^{3}\left(x^{4}+1\right)^{4}\left(x^{5}+1\right)^{5} \\
& \cdot\left[\frac{1}{x+1}+\frac{4 x}{x^{2}+1}+\frac{9 x^{2}}{x^{3}+1}+\frac{16 x^{3}}{x^{4}+1}+\frac{25 x^{4}}{x^{5}+1}\right]
\end{aligned}
$$

2.10.3.23. Solution. We could do this with quotient and product rules, but it would be pretty painful. Insteady, let's use a logarithm.

$$
\begin{aligned}
f(x) & =\left(\frac{5 x^{2}+10 x+15}{3 x^{4}+4 x^{3}+5}\right)\left(\frac{1}{10(x+1)}\right) \\
& =\left(\frac{x^{2}+2 x+3}{3 x^{4}+4 x^{3}+5}\right)\left(\frac{1}{2(x+1)}\right) \\
\log (f(x)) & =\log \left[\left(\frac{x^{2}+2 x+3}{3 x^{4}+4 x^{3}+5}\right)\left(\frac{1}{2(x+1)}\right)\right] \\
& =\log \left(\frac{x^{2}+2 x+3}{3 x^{4}+4 x^{3}+5}\right)+\log \left(\frac{1}{2(x+1)}\right) \\
& =\log \left(x^{2}+2 x+3\right)-\log \left(3 x^{4}+4 x^{3}+5\right)-\log (x+1)-\log (2)
\end{aligned}
$$

Now we have a function that we can differentiate more cleanly than our original function.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{\log (f(x))\}= & \frac{\mathrm{d}}{\mathrm{~d} x}\left\{\log \left(x^{2}+2 x+3\right)-\log \left(3 x^{4}+4 x^{3}+5\right)\right. \\
& -\log (x+1)-\log (2)\} \\
\frac{f^{\prime}(x)}{f(x)}= & \frac{2 x+2}{x^{2}+2 x+3}-\frac{12 x^{3}+12 x^{2}}{3 x^{4}+4 x^{3}+5}-\frac{1}{x+1}
\end{aligned}
$$

$$
=\frac{2(x+1)}{x^{2}+2 x+3}-\frac{12 x^{2}(x+1)}{3 x^{4}+4 x^{3}+5}-\frac{1}{x+1}
$$

Finally, we solve for $f(x)$ :

$$
\begin{aligned}
& f^{\prime}(x)=f(x)\left(\frac{2(x+1)}{x^{2}+2 x+3}-\frac{12 x^{2}(x+1)}{3 x^{4}+4 x^{3}+5}-\frac{1}{x+1}\right) \\
& =\left(\frac{x^{2}+2 x+3}{3 x^{4}+4 x^{3}+5}\right)\left(\frac{1}{2(x+1)}\right)\left(\frac{2(x+1)}{x^{2}+2 x+3}-\frac{12 x^{2}(x+1)}{3 x^{4}+4 x^{3}+5}-\frac{1}{x+1}\right) \\
& =\left(\frac{x^{2}+2 x+3}{3 x^{4}+4 x^{3}+5}\right)\left(\frac{1}{x^{2}+2 x+3}-\frac{6 x^{2}}{3 x^{4}+4 x^{3}+5}-\frac{1}{2(x+1)^{2}}\right)
\end{aligned}
$$

2.10.3.24. *. Solution. Since $f(x)$ has the form of a function raised to a functional power, we will use logarithmic differentiation.

$$
\log (f(x))=\log \left((\cos x)^{\sin x}\right)=\sin x \cdot \log (\cos x)
$$

Logarithm rules allowed us to simplify. Now, we differentiate both sides of this equation:

$$
\begin{aligned}
\frac{f^{\prime}(x)}{f(x)} & =(\cos x) \log (\cos x)+\sin x \cdot \frac{-\sin x}{\cos x} \\
& =(\cos x) \log (\cos x)-\sin x \tan x
\end{aligned}
$$

Finally, we solve for $f^{\prime}(x)$ :

$$
\begin{aligned}
f^{\prime}(x) & =f(x)[(\cos x) \log (\cos x)-\sin x \tan x] \\
& =(\cos x)^{\sin x}[(\cos x) \log (\cos x)-\sin x \tan x]
\end{aligned}
$$

Remark: negative numbers behave in a complicated manner when they are the base of an exponential expression. For example, the expression $(-1)^{x}$ is defined when $x$ is the reciprocal of an odd number (like $x=\frac{1}{5}$ or $x=\frac{1}{7}$ ), but not when $x$ is the reciprocal of an even number (like $x=\frac{1}{2}$ ). Since the domain of $f(x)$ was restricted to $\left(0, \frac{\pi}{2}\right), \cos x$ is always positive, and we avoid these complications.
2.10.3.25. *. Solution. Since $f(x)$ has the form of a function raised to a functional power, we will use logarithmic differentiation. We take the logarithm of the function, and make use of logarithm rules:

$$
\log \left((\tan x)^{x}\right)=x \log (\tan x)
$$

Now, we can differentiate:

$$
\begin{aligned}
\frac{\frac{d}{d x}\left\{(\tan x)^{x}\right\}}{(\tan x)^{x}} & =\log (\tan x)+x \cdot \frac{\sec ^{2} x}{\tan x} \\
& =\log (\tan x)+\frac{x}{\sin x \cos x}
\end{aligned}
$$

Finally, we solve for the derivative we want, $\frac{\mathrm{d}}{\mathrm{d} x}\left\{(\tan x)^{x}\right\}$ :

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{(\tan x)^{x}\right\}=(\tan x)^{x}\left(\log (\tan x)+\frac{x}{\sin x \cos x}\right)
$$

Remark: the restricted domain $(0, \pi / 2)$ ensures that $\tan x$ is a positive number, so we avoid the problems that arise by raising a negative number to a variety of powers.
2.10.3.26. *. Solution. We use logarithmic differentiation.

$$
\log f(x)=\log \left(x^{2}+1\right) \cdot\left(x^{2}+1\right)
$$

We differentiate both sides to obtain:

$$
\begin{aligned}
\frac{f^{\prime}(x)}{f(x)} & =\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\log \left(x^{2}+1\right) \cdot\left(x^{2}+1\right)\right\} \\
& =\frac{2 x}{x^{2}+1}\left(x^{2}+1\right)+2 x \log \left(x^{2}+1\right) \\
& =2 x\left(1+\log \left(x^{2}+1\right)\right)
\end{aligned}
$$

Now, we solve for $f^{\prime}(x)$ :

$$
\begin{aligned}
f^{\prime}(x) & =f(x) \cdot 2 x\left(1+\log \left(x^{2}+1\right)\right) \\
& =\left(x^{2}+1\right)^{x^{2}+1} \cdot 2 x\left(1+\log \left(x^{2}+1\right)\right)
\end{aligned}
$$

2.10.3.27. *. Solution. We use logarithmic differentiation: we modify our function to consider

$$
\log f(x)=\log \left(x^{2}+1\right) \cdot \sin x
$$

We differentiate using the product and chain rules:

$$
\frac{f^{\prime}(x)}{f(x)}=\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\log \left(x^{2}+1\right) \cdot \sin x\right\}=\cos x \cdot \log \left(x^{2}+1\right)+\frac{2 x \sin x}{x^{2}+1}
$$

Finally, we solve for $f^{\prime}(x)$

$$
\begin{aligned}
f^{\prime}(x) & =f(x) \cdot\left(\cos x \cdot \log \left(x^{2}+1\right)+\frac{2 x \sin x}{x^{2}+1}\right) \\
& =\left(x^{2}+1\right)^{\sin (x)} \cdot\left(\cos x \cdot \log \left(x^{2}+1\right)+\frac{2 x \sin x}{x^{2}+1}\right)
\end{aligned}
$$

2.10.3.28. *. Solution. We use logarithmic differentiation; so we modify our function to consider

$$
\log f(x)=\log (x) \cdot \cos ^{3}(x)
$$

Differentiating, we find:

$$
\begin{aligned}
\frac{f^{\prime}(x)}{f(x)} & =\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\log (x) \cdot \cos ^{3}(x)\right\} \\
& =3 \cos ^{2}(x) \cdot(-\sin (x)) \cdot \log (x)+\frac{\cos ^{3}(x)}{x}
\end{aligned}
$$

Finally, we solve for $f^{\prime}(x)$ :

$$
\begin{aligned}
f^{\prime}(x) & =f(x) \cdot\left(-3 \cos ^{2}(x) \sin (x) \log (x)+\frac{\cos ^{3}(x)}{x}\right) \\
& =x^{\cos ^{3}(x)} \cdot\left(-3 \cos ^{2}(x) \sin (x) \log (x)+\frac{\cos ^{3}(x)}{x}\right)
\end{aligned}
$$

Remark: negative numbers behave in a complicated manner when they are the base of an exponential expression. For example, the expression $(-1)^{x}$ is defined when $x$ is the reciprocal of an odd number (like $x=\frac{1}{5}$ or $x=\frac{1}{7}$ ), but not when $x$ is the reciprocal of an even number (like $x=\frac{1}{2}$ ). Since the domain of $f(x)$ was restricted so that $x$ is always positive, we avoid these complications.
2.10.3.29. *. Solution. We use logarithmic differentiation. So, we modify our function and consider

$$
\log f(x)=\left(x^{2}-3\right) \cdot \log (3+\sin (x))
$$

We differentiate:

$$
\begin{aligned}
\frac{f^{\prime}(x)}{f(x)} & =\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\left(x^{2}-3\right) \cdot \log (3+\sin (x))\right\} \\
& =2 x \log (3+\sin (x))+\left(x^{2}-3\right) \frac{\cos (x)}{3+\sin (x)}
\end{aligned}
$$

Finally, we solve for $f^{\prime}(x)$ :

$$
\begin{aligned}
f^{\prime}(x) & =f(x) \cdot\left[2 x \log (3+\sin (x))+\frac{\left(x^{2}-3\right) \cos (x)}{3+\sin (x)}\right] \\
& =(3+\sin (x))^{x^{2}-3} \cdot\left[2 x \log (3+\sin (x))+\frac{\left(x^{2}-3\right) \cos (x)}{3+\sin (x)}\right]
\end{aligned}
$$

## Exercises - Stage 3

2.10.3.30. Solution. We will use logarithmic differentiation. First, we take the logarithm of our function, so we can use logarithm rules.

$$
\log \left([f(x)]^{g(x)}\right)=g(x) \log (f(x))
$$

Now, we differentiate. On the left side we use the chain rule, and on the right side we use product and chain rules.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\log \left([f(x)]^{g(x)}\right)\right\} & =\frac{\mathrm{d}}{\mathrm{~d} x}\{g(x) \log (f(x))\} \\
\frac{\frac{\mathrm{d}}{\mathrm{~d} x}\left\{[f(x)]^{g(x)}\right\}}{[f(x)]^{g(x)}} & =g^{\prime}(x) \log (f(x))+g(x) \cdot \frac{f^{\prime}(x)}{f(x)}
\end{aligned}
$$

Finally, we solve for the derivative of our original function.

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{[f(x)]^{g(x)}\right\}=[f(x)]^{g(x)}\left(g^{\prime}(x) \log (f(x))+g(x) \cdot \frac{f^{\prime}(x)}{f(x)}\right)
$$

Remark: in this section, we have differentiated problems of this type several timesfor example, Questions 2.10.3.24
through 2.10.3.29.
2.10.3.31. Solution. Let $g(x):=\log (f(x))$. Notice $g^{\prime}(x)=\frac{f^{\prime}(x)}{f(x)}$.

In order to show that the two curves have horizontal tangent lines at the same values of $x$, we will show two things: first, that if $f(x)$ has a horizontal tangent line at some value of $x$, then also $g(x)$ has a horizontal tangent line at that value of $x$. Second, we will show that if $g(x)$ has a horizontal tangent line at some value of $x$, then also $f(x)$ has a horizontal tangent line at that value of $x$.
Suppose $f(x)$ has a horizontal tangent line where $x=x_{0}$ for some point $x_{0}$. This means $f^{\prime}\left(x_{0}\right)=0$. Then $g^{\prime}\left(x_{0}\right)=\frac{f^{\prime}\left(x_{0}\right)}{f\left(x_{0}\right)}$. Since $f\left(x_{0}\right) \neq 0, \frac{f^{\prime}\left(x_{0}\right)}{f\left(x_{0}\right)}=\frac{0}{f\left(x_{0}\right)}=0$, so $g(x)$ also has a horizontal tangent line when $x=x_{0}$. This shows that whenever $f$ has a horizontal tangent line, $g$ has one too.
Now suppose $g(x)$ has a horizontal tangent line where $x=x_{0}$ for some point $x_{0}$. This means $g^{\prime}\left(x_{0}\right)=0$. Then $g^{\prime}\left(x_{0}\right)=\frac{f^{\prime}\left(x_{0}\right)}{f\left(x_{0}\right)}=0$, so $f^{\prime}\left(x_{0}\right)$ exists and is equal to zero. Therefore, $f(x)$ also has a horizontal tangent line when $x=x_{0}$. This shows that whenever $g$ has a horizontal tangent line, $f$ has one too.
Remark: if we were not told that $f(x)$ gives only positive numbers, it would not necessarily be true that $f(x)$ and $\log (f(x))$ have horizontal tangent lines at the same values of $x$. If $f(x)$ had a horizontal tangent line at an $x$-value where $f(x)$ were negative, then $\log (f(x))$ would not exist there, let alone have a horizontal tangent line.

### 2.11 • Implicit Differentiation

### 2.11.2 • Exercises

## Exercises - Stage 1

2.11.2.1. Solution. We use the power rule (a) and the chain rule (b): the power rule tells us to "bring down the 2", and the chain rule tells us to multiply by $y^{\prime}$. There is no need for the quotient rule here, as there are no quotients. Exponential functions have the form (constant) function $^{\text {, but our function has the form }}$ (function) ${ }^{\text {constant }}$, so we did not use (d).
2.11.2.2. Solution. At $(0,4)$ and $(0,-4)$, the curve looks to be horizontal, if you zoom in: a tangent line here would have derivative zero. At the origin, the curve looks like its tangent line is vertical, so $\frac{\mathrm{d} y}{\mathrm{~d} x}$ does not exist.

2.11.2.3. Solution. (a) No. A function must pass the vertical line test: one input cannot result in two (or more) outputs. Since one value of $x$ sometimes corresponds to two values of $y$ (for example, when $x=\pi / 4, y$ is $\pm 1 / \sqrt{2}$ ), there is no function $f(x)$ so that $y=f(x)$ captures every point on the circle.
Remark: $y= \pm \sqrt{1-x^{2}}$ does capture every point on the unit circle. However, since one input $x$ sometimes results in two outputs $y$, this expression is not a function.
(b) No, for the same reasons as (a). If $f^{\prime}(x)$ is a function, then it can give at most one slope corresponding to one value of $x$. Since one value of $x$ can correspond to two points on the circle with different slopes, $f^{\prime}(x)$ cannot give the slope of every point on the circle. For example, fix any $0<a<1$. There are two points on the circle with $x$-coordinate equal to $a$. At the upper one, the slope is strictly negative. At the lower one, the slope is strictly positive.
(c) We differentiate:

$$
2 x+2 y \frac{\mathrm{~d} y}{\mathrm{~d} x}=0
$$

and solve for $\frac{d y}{d x}$

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=-\frac{x}{y}
$$

But there is a $y$ in the right-hand side of this equation, and it's not clear how to get it out. Our answer in (b) tells us that, actually, we can't get it out, if we want
the right-hand side to be a function of $x$. The derivative cannot be expressed as a function of $x$, because one value of $x$ corresponds to multiple points on the circle. Remark: since $y= \pm \sqrt{1-x^{2}}$, we could try writing

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=-\frac{x}{y}= \pm \frac{x}{\sqrt{1-x^{2}}}
$$

but this is not a function of $x$. Again, in a function, one input leads to at most one output, but here one value of $x$ will usually lead to two values of $\frac{\mathrm{d} y}{\mathrm{~d} x}$.

## Exercises - Stage 2

2.11.2.4. *. Solution. Remember that $y$ is a function of $x$. We begin with implicit differentiation.

$$
\begin{aligned}
x y+e^{x}+e^{y} & =1 \\
y+x \frac{\mathrm{~d} y}{\mathrm{~d} x}+e^{x}+e^{y} \frac{\mathrm{~d} y}{\mathrm{~d} x} & =0
\end{aligned}
$$

Now, we solve for $\frac{\mathrm{d} y}{\mathrm{~d} x}$.

$$
\begin{aligned}
x \frac{\mathrm{~d} y}{\mathrm{~d} x}+e^{y} \frac{\mathrm{~d} y}{\mathrm{~d} x} & =-\left(e^{x}+y\right) \\
\left(x+e^{y}\right) \frac{\mathrm{d} y}{\mathrm{~d} x} & =-\left(e^{x}+y\right) \\
\frac{\mathrm{d} y}{\mathrm{~d} x} & =-\frac{e^{x}+y}{e^{y}+x}
\end{aligned}
$$

2.11.2.5. *. Solution. Differentiate both sides of the equation with respect to $x$ :

$$
e^{y} \frac{\mathrm{~d} y}{\mathrm{~d} x}=x \cdot 2 y \frac{\mathrm{~d} y}{\mathrm{~d} x}+y^{2}+1
$$

Now, get the derivative on one side and solve

$$
\begin{aligned}
e^{y} \frac{\mathrm{~d} y}{\mathrm{~d} x}-2 x y \frac{\mathrm{~d} y}{\mathrm{~d} x} & =y^{2}+1 \\
\frac{\mathrm{~d} y}{\mathrm{~d} x}\left(e^{y}-2 x y\right) & =y^{2}+1 \\
\frac{\mathrm{~d} y}{\mathrm{~d} x} & =\frac{y^{2}+1}{e^{y}-2 x y}
\end{aligned}
$$

### 2.11.2.6. *. Solution.

- First we find the $x$-coordinates where $y=1$.

$$
\begin{aligned}
x^{2} \tan \left(\frac{\pi}{4}\right)+2 x \log (1) & =16 \\
x^{2} \cdot 1+2 x \cdot 0 & =16
\end{aligned}
$$

$$
x^{2}=16
$$

So $x= \pm 4$.

- Now we use implicit differentiation to get $y^{\prime}$ in terms of $x, y$ :

$$
\begin{aligned}
x^{2} \tan (\pi y / 4)+2 x \log (y) & =16 \\
2 x \tan (\pi y / 4)+x^{2} \frac{\pi}{4} \sec ^{2}(\pi y / 4) \cdot y^{\prime}+2 \log (y)+\frac{2 x}{y} \cdot y^{\prime} & =0 .
\end{aligned}
$$

- Now set $y=1$ and use $\tan (\pi / 4)=1, \sec (\pi / 4)=\sqrt{2}$ to get

$$
\begin{aligned}
2 x \tan (\pi / 4)+x^{2} \frac{\pi}{4} \sec ^{2}(\pi / 4) y^{\prime}+2 \log (1)+2 x \cdot y^{\prime} & =0 \\
2 x+\frac{\pi}{2} x^{2} y^{\prime}+2 x y^{\prime} & =0 \\
y^{\prime} & =-\frac{2 x}{x^{2} \pi / 2+2 x}
\end{aligned}=-\frac{4}{\pi x+4}
$$

- So at $(x, y)=(4,1)$ we have $y^{\prime}=-\frac{4}{4 \pi+4}=-\frac{1}{\pi+1}$
- and at $(x, y)=(-4,1)$ we have $y^{\prime}=\frac{1}{\pi-1}$
2.11.2.7. *. Solution. Differentiate the equation and solve:

$$
\begin{aligned}
3 x^{2}+4 y^{3} \frac{\mathrm{~d} y}{\mathrm{~d} x} & =-\sin \left(x^{2}+y\right) \cdot\left(2 x+\frac{\mathrm{d} y}{\mathrm{~d} x}\right) \\
\frac{\mathrm{d} y}{\mathrm{~d} x} & =-\frac{2 x \sin \left(x^{2}+y\right)+3 x^{2}}{4 y^{3}+\sin \left(x^{2}+y\right)}
\end{aligned}
$$

### 2.11.2.8. *. Solution.

- First we find the $x$-coordinates where $y=0$.

$$
\begin{aligned}
x^{2} e^{0}+4 x \cos (0) & =5 \\
x^{2}+4 x-5 & =0 \\
(x+5)(x-1) & =0
\end{aligned}
$$

So $x=1,-5$.

- Now we use implicit differentiation to get $y^{\prime}$ in terms of $x, y$. Differentiate both sides of

$$
x^{2} e^{y}+4 x \cos (y)=5
$$

to get

$$
x^{2} \cdot e^{y} \cdot y^{\prime}+2 x e^{y}+4 x(-\sin (y)) \cdot y^{\prime}+4 \cos (y)=0
$$

- Now set $y=0$ to get

$$
\begin{aligned}
x^{2} \cdot e^{0} \cdot y^{\prime}+2 x e^{0}+4 x(-\sin (0)) \cdot y^{\prime}+4 \cos (0) & =0 \\
x^{2} y^{\prime}+2 x+4 & =0 \\
y^{\prime} & =-\frac{4+2 x}{x^{2}} .
\end{aligned}
$$

- So at $(x, y)=(1,0)$ we have $y^{\prime}=-6$,
- and at $(x, y)=(-5,0)$ we have $y^{\prime}=\frac{6}{25}$.
2.11.2.9. *. Solution. Differentiate the equation and solve:

$$
\begin{aligned}
2 x+2 y \frac{\mathrm{~d} y}{\mathrm{~d} x} & =\cos (x+y) \cdot\left(1+\frac{\mathrm{d} y}{\mathrm{~d} x}\right) \\
\frac{\mathrm{d} y}{\mathrm{~d} x} & =\frac{\cos (x+y)-2 x}{2 y-\cos (x+y)}
\end{aligned}
$$

### 2.11.2.10. *. Solution.

- First we find the $x$-coordinates where $y=0$.

$$
\begin{aligned}
x^{2} \cos (0)+2 x e^{0} & =8 \\
x^{2}+2 x-8 & =0 \\
(x+4)(x-2) & =0
\end{aligned}
$$

So $x=2,-4$.

- Now we use implicit differentiation to get $y^{\prime}$ in terms of $x, y$. Differentiate both sides of

$$
x^{2} \cos (y)+2 x e^{y}=8
$$

to get

$$
x^{2} \cdot(-\sin y) \cdot y^{\prime}+2 x \cos y+2 x e^{y} \cdot y^{\prime}+2 e^{y}=0
$$

- Now set $y=0$ to get

$$
\begin{aligned}
x^{2} \cdot(-\sin 0) \cdot y^{\prime}+2 x \cos 0+2 x e^{0} \cdot y^{\prime}+2 e^{0} & =0 \\
0+2 x+2 x y^{\prime}+2 & =0 \\
y^{\prime} & =-\frac{2+2 x}{2 x} \\
& =-\frac{1+x}{x}
\end{aligned}
$$

- So at $(x, y)=(2,0)$ we have $y^{\prime}=-\frac{3}{2}$,
- and at $(x, y)=(-4,0)$ we have $y^{\prime}=-\frac{3}{4}$.
2.11.2.11. Solution. The question asks at which points on the ellipse $\frac{\mathrm{d} y}{\mathrm{~d} x}=1$. So, we begin by differentiating, implicitly:

$$
2 x+6 y \frac{\mathrm{~d} y}{\mathrm{~d} x}=0
$$

We could solve for $\frac{\mathrm{d} y}{\mathrm{~d} x}$ at this point, but it's not necessary. We want to know when $\frac{\mathrm{d} y}{\mathrm{~d} x}$ is equal to one:

$$
\begin{aligned}
2 x+6 y(1) & =0 \\
x & =-3 y
\end{aligned}
$$

That is, $\frac{\mathrm{d} y}{\mathrm{~d} x}=1$ at those points along the ellipse where $x=-3 y$. We plug this into the equation of the ellipse to find the coordinates of these points.

$$
\begin{aligned}
(-3 y)^{2}+3 y^{2} & =1 \\
12 y^{2} & =1 \\
y= \pm \frac{1}{\sqrt{12}}= \pm \frac{1}{2 \sqrt{3}} & =2 .
\end{aligned}
$$

So, the points along the ellipse where the tangent line is parallel to the line $y=x$ occur when $y=\frac{1}{2 \sqrt{3}}$ and $x=-3 y$, and when $y=\frac{-1}{2 \sqrt{3}}$ and $x=-3 y$. That is, the points $\left(\frac{-\sqrt{3}}{2}, \frac{1}{2 \sqrt{3}}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{-1}{2 \sqrt{3}}\right)$.
2.11.2.12. *. Solution. First, we differentiate implicitly with respect to $x$.

$$
\begin{aligned}
\sqrt{x y} & =x^{2} y-2 \\
\frac{1}{2 \sqrt{x y}} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\{x y\} & =(2 x) y+x^{2} \frac{\mathrm{~d} y}{\mathrm{~d} x} \\
\frac{y+x \frac{\mathrm{~d} y}{\mathrm{~d} x}}{2 \sqrt{x y}} & =2 x y+x^{2} \frac{\mathrm{~d} y}{\mathrm{~d} x}
\end{aligned}
$$

Now, we plug in $x=1, y=4$, and solve for $\frac{\mathrm{d} y}{\mathrm{~d} x}$ :

$$
\begin{aligned}
\frac{4+\frac{\mathrm{d} y}{\mathrm{~d} x}}{4} & =8+\frac{\mathrm{d} y}{\mathrm{~d} x} \\
\frac{\mathrm{~d} y}{\mathrm{~d} x} & =-\frac{28}{3}
\end{aligned}
$$

2.11.2.13. *. Solution. Implicitly differentiating $x^{2} y(x)^{2}+x \sin (y(x))=4$ with
respect to $x$ gives

$$
2 x y^{2}+2 x^{2} y y^{\prime}+\sin y+x y^{\prime} \cos y=0
$$

Then we gather the terms containing $y^{\prime}$ on one side, so we can solve for $y^{\prime}$ :

$$
\begin{gathered}
2 x^{2} y y^{\prime}+x y^{\prime} \cos y=-2 x y^{2}-\sin y \\
y^{\prime}\left(2 x^{2} y+x \cos y\right)=-2 x y^{2}-\sin y \\
y^{\prime}=-\frac{2 x y^{2}+\sin y}{2 x^{2} y+x \cos y}
\end{gathered}
$$

## Exercises - Stage 3

2.11.2.14. *. Solution.

- First we find the $x$-ordinates where $y=0$.

$$
\begin{aligned}
x^{2}+(1) e^{0} & =5 \\
x^{2}+1 & =5 \\
x^{2} & =4
\end{aligned}
$$

So $x=2,-2$.

- Now we use implicit differentiation to get $y^{\prime}$ in terms of $x, y$ :

$$
2 x+(y+1) e^{y} \frac{\mathrm{~d} y}{\mathrm{~d} x}+e^{y} \frac{\mathrm{~d} y}{\mathrm{~d} x}=0
$$

- Now set $y=0$ to get

$$
\begin{aligned}
2 x+(0+1) e^{0} \frac{\mathrm{~d} y}{\mathrm{~d} x}+e^{0} \frac{\mathrm{~d} y}{\mathrm{~d} x} & =0 \\
2 x+\frac{\mathrm{d} y}{\mathrm{~d} x}+\frac{\mathrm{d} y}{\mathrm{~d} x} & =0 \\
2 x & =-2 \frac{\mathrm{~d} y}{\mathrm{~d} x} \\
x & =-\frac{\mathrm{d} y}{\mathrm{~d} x}
\end{aligned}
$$

- So at $(x, y)=(2,0)$ we have $y^{\prime}=-2$,
- and at $(x, y)=(-2,0)$ we have $y^{\prime}=2$.
2.11.2.15. Solution. The slope of the tangent line is, of course, given by the derivative, so let's start by finding $\frac{\mathrm{d} y}{\mathrm{~d} x}$ of both shapes.

For the circle, we differentiate implicitly

$$
2 x+2 y \frac{\mathrm{~d} y}{\mathrm{~d} x}=0
$$

and solve for $\frac{d y}{d x}$

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=-\frac{x}{y}
$$

For the ellipse, we also differentiate implicitly:

$$
2 x+6 y \frac{\mathrm{~d} y}{\mathrm{~d} x}=0
$$

and solve for $\frac{\mathrm{d} y}{\mathrm{~d} x}$

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=-\frac{x}{3 y}
$$

What we want is a value of $x$ where both derivatives are equal. However, they might have different values of $y$, so let's let $y_{1}$ be the $y$-values associated with $x$ on the circle, and let $y_{2}$ be the $y$-values associated with $x$ on the ellipse. That is, $x^{2}+y_{1}^{2}=1$ and $x^{2}+3 y_{2}^{2}=1$. For the slopes at $\left(x, y_{1}\right)$ on the circle and $\left(x, y_{2}\right)$ on the ellipse to be equal, we need:

$$
\begin{aligned}
-\frac{x}{y_{1}} & =-\frac{x}{3 y_{2}} \\
x\left(\frac{1}{y_{1}}-\frac{1}{3 y_{2}}\right) & =0
\end{aligned}
$$

So $x=0$ or $y_{1}=3 y_{2}$. Let's think about which $x$-values will have a $y$-coordinate of the circle be three times as large as a $y$-coordinate of the ellipse. If $y_{1}=3 y_{2},\left(x, y_{1}\right)$ is on the circle, and $\left(x, y_{2}\right)$ is on the ellipse, then $x^{2}+y_{1}^{2}=x^{2}+\left(3 y_{2}\right)^{2}=1$ and $x^{2}+3 y_{2}^{2}=1$. In this case:

$$
\begin{aligned}
x^{2}+9 y_{2}^{2} & =x^{2}+3 y_{2}^{2} \\
9 y_{2}^{2} & =3 y_{2}^{2} \\
y_{2} & =0 \\
x & = \pm 1
\end{aligned}
$$

We need to be a tiny bit careful here: when $y=0, y^{\prime}$ is not defined for either curve. For both curves, when $y=0$, the tangent lines are vertical (and so have no real-valued slope!). Two vertical lines are indeed parallel.
So, for $x=0$ and for $x= \pm 1$, the two curves have parallel tangent lines.


### 2.12 • Inverse Trigonometric Functions

### 2.12.2 • Exercises

## Exercises - Stage 1

2.12.2.1. Solution. (a) We can plug any number into the cosine function, and it will return a number in $[-1,1]$. The domain of $\arcsin x$ is $[-1,1]$, so any number we plug into cosine will give us a valid number to plug into arcsine. So, the domain of $f(x)$ is all real numbers.
(b) We can plug any number into the cosine function, and it will return a number in $[-1,1]$. The domain of $\operatorname{arccsc} x$ is $(-\infty,-1] \cup[1, \infty)$, so in order to have a valid number to plug into arccosecant, we need $\cos x= \pm 1$. That is, the domain of $g(x)$ is all values $x=n \pi$ for some integer $n$.
(c) The domain of arccosine is $[-1,1]$. The domain of sine is all real numbers, so no matter what number arccosine spits out, we can safely plug it into sine. So, the domain of $h(x)$ is $[-1,1]$.
2.12.2.2. Solution. False: $\cos t=1$ for infinitely many values of $t$; arccosine gives only the single value $t=0$ for which $\cos t=1$ and $0 \leq t \leq \pi$. The particle does not start moving until $t=10$, so $t=0$ is not in the domain of the function describing its motion.
The particle will have height 1 at time $2 \pi n$, for any integer $n \geq 2$.
2.12.2.3. Solution. First, we restrict the domain of $f$ to force it to be one-toone. There are many intervals we could choose over which $f$ is one-to-one, but the question asks us to contain $x=0$ and be as large as possible; this leaves us with the following restricted function:


The inverse of a function swaps the role of the input and output; so if the graph of $y=f(x)$ contains the point $(a, b)$, then the graph of $Y=f^{-1}(X)$ contains the point $(b, a)$. That is, the graph of $Y=f^{-1}(X)$ is the graph of $y=f(x)$ with the $x$-coordinates and $y$-coordinates swapped. (So, since $y=f(x)$ crosses the $y$-axis at $y=1$, then $Y=f^{-1}(X)$ crosses the $X$-axis at $X=1$.) This swapping is equivalent to reflecting the curve $y=f(x)$ over the line $y=x$.


Remark: while you're getting accustomed to inverse functions, it is sometimes clearer to consider $y=f(x)$ and $Y=f^{-1}(X)$ : using slightly different notations
for $x$ (the input of $f$, hence the output of $f^{-1}$ ) and $X$ (the input of $f^{-1}$, which comes from the output of $f$ ). However, the convention is to use $x$ for the inputs of both functions, and $y$ as the outputs of both functions, as is written on the graph above.
2.12.2.4. Solution. The tangent line is horizontal when $0=y^{\prime}=a-\sin x$. That is, when $a=\sin x$.

- If $|a|>1$, then there is no value of $x$ for which $a=\sin x$, so the curve has no horizontal tangent lines.
- If $|a|=1$, then there are infinitely many solutions to $a=\sin x$, but only one solution in the interval $[-\pi, \pi]: x=\arcsin (a)=\arcsin ( \pm 1)= \pm \frac{\pi}{2}$. Then the values of $x$ for which $a=\sin x$ are $x=2 \pi n+a \frac{\pi}{2}$ for any integer $n$.
- If $|a|<1$, then there are infinitely many solutions to $a=\sin x$. The solution in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is given by $x=\arcsin (a)$. The other solution in the interval $(-\pi, \pi)$ is given by $x=\pi-\arcsin (a)$, as shown in the unit circles below.



So, the values of $x$ for which $x=\sin a$ are $x=2 \pi n+\arcsin (a)$ and $x=$ $2 \pi n+\pi-\arcsin (a)$ for any integer $n$.

Remark: when $a=1$, then

$$
2 \pi n+\arcsin (a)=2 \pi n+\frac{\pi}{2}=2 \pi n+\pi-\left(\frac{\pi}{2}\right)=2 \pi n+\pi-\arcsin (a)
$$

Similarly, when $a=-1$,

$$
\begin{aligned}
2 \pi n+\arcsin (a) & =2 \pi n-\frac{\pi}{2}=2 \pi(n-1)+\pi-\left(-\frac{\pi}{2}\right) \\
& =2 \pi(n-1)+\pi-\arcsin (a)
\end{aligned}
$$

So, if we try to use the descriptions in the third bullet point to describe points where the tangent line is horizontal when $|a|=1$, we get the correct points but each point is listed twice. This is why we separated the case $|a|=1$ from the case $|a|<1$.
2.12.2.5. Solution. The function $\arcsin x$ is only defined for $|x| \leq 1$, and the function $\operatorname{arccsc} x$ is only defined for $|x| \geq 1$, so $f(x)$ has domain $|x|=1$. That is, $x= \pm 1$.
In order for $f(x)$ to be differentiable at a point, it must exist in an open interval around that point. (See Definition 2.2.1.) Since our function does not exist over any open interval, $f(x)$ is not differentiable anywhere.
So, actually, $f(x)$ is a pretty boring function, which we can entirely describe as: $f(-1)=-\pi$ and $f(1)=\pi$.

## Exercises - Stage 2

2.12.2.6. Solution. Using the chain rule,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\arcsin \left(\frac{x}{3}\right)\right\} & =\frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^{2}}} \cdot \frac{1}{3} \\
& =\frac{1}{3 \sqrt{1-\frac{x^{2}}{9}}} \\
& =\frac{1}{\sqrt{9-x^{2}}}
\end{aligned}
$$

Since the domain of arcsine is $[-1,1]$, and we are plugging in $\frac{x}{3}$ to arcsine, the values of $x$ that we can plug in are those that satisfy $-1 \leq \frac{x}{3} \leq 1$, or $-3 \leq x \leq 3$. So the domain of $f$ is $[-3,3]$.
2.12.2.7. Solution. Using the quotient rule,

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\left\{\frac{\arccos t}{t^{2}-1}\right\}=\frac{\left(t^{2}-1\right)\left(\frac{-1}{\sqrt{1-t^{2}}}\right)-(\arccos t)(2 t)}{\left(t^{2}-1\right)^{2}}
$$

The domain of arccosine is $[-1,1]$, and since $t^{2}-1$ is in the denominator, the domain of $f$ requires $t^{2}-1 \neq 0$, that is, $t \neq \pm 1$. So the domain of $f(t)$ is $(-1,1)$.
2.12.2.8. Solution. The domain of $\operatorname{arcsec} x$ is $|x| \geq 1$ : that is, we can plug into arcsecant only values with absolute value greater than or equal to one. Since $-x^{2}-2 \leq-2$, every real value of $x$ gives us an acceptable value to plug into arcsecant. So, the domain of $f(x)$ is all real numbers.
To differentiate, we use the chain rule. Remember $\frac{\mathrm{d}}{\mathrm{d} x}\{\operatorname{arcsec} x\}=\frac{1}{|x| \sqrt{x^{2}-1}}$.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\operatorname{arcsec}\left(-x^{2}-2\right)\right\} & =\frac{1}{\left|-x^{2}-2\right| \sqrt{\left(-x^{2}-2\right)^{2}-1}} \cdot(-2 x) \\
& =\frac{-2 x}{\left(x^{2}+2\right) \sqrt{x^{4}+4 x+3}}
\end{aligned}
$$

2.12.2.9. Solution. We use the chain rule, remembering that $a$ is a constant.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{1}{a} \arctan \left(\frac{x}{a}\right)\right\} & =\frac{1}{a} \cdot \frac{1}{1+\left(\frac{x}{a}\right)^{2}} \cdot \frac{1}{a} \\
& =\frac{1}{a^{2}+x^{2}}
\end{aligned}
$$

The domain of arctangent is all real numbers, so the domain of $f(x)$ is also all real numbers.
2.12.2.10. Solution. We differentiate using the product and chain rules.

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{x \arcsin x+\sqrt{1-x^{2}}\right\}=\arcsin x+\frac{x}{\sqrt{1-x^{2}}}+\frac{-2 x}{2 \sqrt{1-x^{2}}}
$$

$$
=\arcsin x
$$

The domain of $\arcsin x$ is $[-1,1]$, and the domain of $\sqrt{1-x^{2}}$ is all values of $x$ so that $1-x^{2} \geq 0$, so $x$ in $[-1,1]$. Therefore, the domain of $f(x)$ is $[-1,1]$.
2.12.2.11. Solution. We differentiate using the chain rule:

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\arctan \left(x^{2}\right)\right\}=\frac{2 x}{1+x^{4}}
$$

This is zero exactly when $x=0$.
2.12.2.12. Solution. Using formulas you should memorize from this section,

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\{\arcsin x+\arccos x\}=\frac{1}{\sqrt{1-x^{2}}}+\frac{-1}{\sqrt{1-x^{2}}}=0
$$

Remark: the only functions with derivative equal to zero everywhere are constant functions, so $\arcsin x+\arccos x$ should be a constant. Since $\sin \theta=\cos \left(\frac{\pi}{2}-\theta\right)$, we can set

$$
\sin \theta=x \quad \cos \left(\frac{\pi}{2}-\theta\right)=x
$$

where $x$ and $\theta$ are the same in both expressions, and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then

$$
\arcsin x=\theta \quad \arccos x=\frac{\pi}{2}-\theta
$$

We note here that arcsine is the inverse of the sine function restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. So, since we restricted $\theta$ to this domain, $\sin \theta=x$ really does imply $\arcsin x=\theta$. (For an example of why this matters, note $\sin (2 \pi)=0$, but $\arcsin (0)=0 \neq 2 \pi$.) Similarly, arccosine is the inverse of the cosine function restricted to $[0, \pi]$. Since $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, then $0 \leq\left(\frac{\pi}{2}-\theta\right) \leq \pi$, so $\cos \left(\frac{\pi}{2}-\theta\right)=x$ really does imply $\arccos x=\frac{\pi}{2}-\theta$. So,

$$
\arcsin x+\arccos x=\theta+\frac{\pi}{2}-\theta=\frac{\pi}{2}
$$

which means the derivative we were calculating was actually just $\frac{\mathrm{d}}{\mathrm{d} x}\left\{\frac{\pi}{2}\right\}=0$.
2.12.2.13. *. Solution. Using the chain rule,

$$
y^{\prime}=\frac{-\frac{1}{x^{2}}}{\sqrt{1-\left(\frac{1}{x}\right)^{2}}}=\frac{-1}{x^{2} \sqrt{1-\frac{1}{x^{2}}}}
$$

2.12.2.14. *. Solution. Using the chain rule,

$$
y^{\prime}=\frac{-\frac{1}{x^{2}}}{1+\left(\frac{1}{x}\right)^{2}}=\frac{-1}{x^{2}+1} .
$$

2.12.2.15. *. Solution. Using the product rule:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\left(1+x^{2}\right) \arctan x\right\} & =2 x \arctan x+\left(1+x^{2}\right) \frac{1}{1+x^{2}} \\
& =2 x \arctan x+1
\end{aligned}
$$

2.12.2.16. Solution. Let $\theta=\arctan x$. Then $\theta$ is the angle of a right triangle that gives $\tan \theta=x$. In particular, the ratio of the opposite side to the adjacent side is $x$. So, we have a triangle that looks like this:

where the length of the hypotenuse came from the Pythagorean Theorem. Now,

$$
\sin (\arctan x)=\sin \theta=\frac{\text { opp }}{\text { hyp }}=\frac{x}{\sqrt{x^{2}+1}}
$$

From here, we differentiate using the quotient rule:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{x}{\sqrt{x^{2}+1}}\right\} & =\frac{\sqrt{x^{2}+1}-x \frac{2 x}{2 \sqrt{x^{2}+1}}}{x^{2}+1} \\
& =\left(\frac{\sqrt{x^{2}+1}-\frac{x^{2}}{\sqrt{x^{2}+1}}}{x^{2}+1}\right) \cdot \frac{\sqrt{x^{2}+1}}{\sqrt{x^{2}+1}} \\
& =\frac{\left(x^{2}+1\right)-x^{2}}{\left(x^{2}+1\right)^{3 / 2}} \\
& =\frac{1}{\left(x^{2}+1\right)^{3 / 2}}=\left(x^{2}+1\right)^{-3 / 2}
\end{aligned}
$$

Remark: another strategy is to differentiate first, using the chain rule, then draw a triangle to simplify the resulting expression $\frac{\mathrm{d}}{\mathrm{d} x}\{\sin (\arctan x)\}=\frac{\cos (\arctan x)}{1+x^{2}}$.
2.12.2.17. Solution. Let $\theta=\arcsin x$. Then $\theta$ is the angle of a right triangle that gives $\sin \theta=x$. In particular, the ratio of the opposite side to the hypotenuse is $x$. So, we have a triangle that looks like this:

where the length of the adjacent side came from the Pythagorean Theorem. Now,

$$
\cot (\arcsin x)=\cot \theta=\frac{\mathrm{adj}}{\mathrm{opp}}=\frac{\sqrt{1-x^{2}}}{x}
$$

From here, we differentiate using the quotient rule:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{\sqrt{1-x^{2}}}{x}\right\} & =\frac{x \frac{-2 x}{2 \sqrt{1-x^{2}}}-\sqrt{1-x^{2}}}{x^{2}} \\
& =\frac{-x^{2}-\left(1-x^{2}\right)}{x^{2} \sqrt{1-x^{2}}} \\
& =\frac{-1}{x^{2} \sqrt{1-x^{2}}}
\end{aligned}
$$

Remark: another strategy is to differentiate first, using the chain rule, then draw a triangle to simplify the resulting expression $\frac{\mathrm{d}}{\mathrm{d} x}\{\cot (\arcsin x)\}=\frac{-\csc ^{2}(\arcsin x)}{\sqrt{1-x^{2}}}$.
2.12.2.18. *. Solution. The line $y=2 x+9$ has slope 2 , so we must find all values of $x$ between -1 and $1(\arcsin x$ is only defined for these values of $x)$ for which $\frac{\mathrm{d}}{\mathrm{d} x}\{\arcsin x\}=2$. Evaluating the derivative:

$$
\begin{aligned}
y & =\arcsin x \\
2=y^{\prime} & =\frac{1}{\sqrt{1-x^{2}}} \\
4 & =\frac{1}{1-x^{2}} \\
\frac{1}{4} & =1-x^{2} \\
x^{2} & =\frac{3}{4} \\
x & = \pm \frac{\sqrt{3}}{2} \\
(x, y) & = \pm\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)
\end{aligned}
$$

2.12.2.19. Solution. We differentiate using the chain rule:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{\arctan (\csc x)\} & =\frac{1}{1+\csc ^{2} x} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\{\csc x\} \\
& =\frac{-\csc x \cot x}{1+\csc ^{2} x} \\
& =\frac{-\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}}{1+\left(\frac{1}{\sin x}\right)^{2}} \\
& =\frac{-\cos x}{\sin ^{2} x+1}
\end{aligned}
$$

So if $f^{\prime}(x)=0$, then $\cos x=0$, and this happens when $x=\frac{(2 n+1) \pi}{2}$ for any integer $n$. We should check that these points are in the domain of $f$. Arctangent is defined for all real numbers, so we only need to check the domain of cosecant; when $x=\frac{(2 n+1) \pi}{2}$, then $\sin x= \pm 1 \neq 0$, so $\csc x=\frac{1}{\sin x}$ exists.

## Exercises - Stage 3

2.12.2.20. *. Solution. Since $g(y)=f^{-1}(y)$,

$$
f(g(y))=f\left(f^{-1}(y)\right)=y
$$

Now, we can differentiate with respect to $y$ using the chain rule.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} y}\{f(g(y))\} & =\frac{\mathrm{d}}{\mathrm{~d} y}\{y\} \\
f^{\prime}(g(y)) \cdot g^{\prime}(y) & =1 \\
g^{\prime}(y) & =\frac{1}{f^{\prime}(g(y))}=\frac{1}{1-\sin g(y)}
\end{aligned}
$$

2.12.2.21. *. Solution. Write $g(y)=f^{-1}(y)$. Then $g(f(x))=x$, so differentiating both sides (using the chain rule), we see

$$
g^{\prime}(f(x)) \cdot f^{\prime}(x)=1
$$

What we want is $g^{\prime}(\pi-1)$, so we need to figure out which value of $x$ gives $f(x)=$ $\pi-1$. A little trial and error leads us to $x=\frac{\pi}{2}$.

$$
g^{\prime}(\pi-1) \cdot f^{\prime}\left(\frac{\pi}{2}\right)=1
$$

Since $f^{\prime}(x)=2-\cos (x), f^{\prime}\left(\frac{\pi}{2}\right)=2-0=2$ :

$$
\begin{gathered}
g^{\prime}(\pi-1) \cdot 2=1 \\
g^{\prime}(\pi-1)=\frac{1}{2}
\end{gathered}
$$

2.12.2.22. *. Solution. Write $g(y)=f^{-1}(y)$. Then $g(f(x))=x$, so differentiating both sides (using the chain rule), we see

$$
g^{\prime}(f(x)) f^{\prime}(x)=1
$$

What we want is $g^{\prime}(e+1)$, so we need to figure out which value of $x$ gives $f(x)=e+1$. A little trial and error leads us to $x=1$.

$$
\begin{aligned}
g^{\prime}(f(1)) f^{\prime}(1) & =1 \\
g^{\prime}(e+1) \cdot f^{\prime}(1) & =1 \\
g^{\prime}(e+1) & =\frac{1}{f^{\prime}(1)}
\end{aligned}
$$

It remains only to note that $f^{\prime}(x)=e^{x}+1$, so $f^{\prime}(1)=e+1$

$$
g^{\prime}(e+1)=\frac{1}{e+1}
$$

2.12.2.23. Solution. We use logarithmic differentiation, our standard method of differentiating an expression of the form (function) function.

$$
\begin{aligned}
f(x) & =[\sin x+2]^{\operatorname{arcsec} x} \\
\log (f(x)) & =\operatorname{arcsec} x \cdot \log [\sin x+2] \\
\frac{f^{\prime}(x)}{f(x)} & =\frac{1}{|x| \sqrt{x^{2}-1}} \log [\sin x+2]+\operatorname{arcsec} x \cdot \frac{\cos x}{\sin x+2} \\
f^{\prime}(x) & =[\sin x+2]^{\operatorname{arcsec} x}\left(\frac{\log [\sin x+2]}{|x| \sqrt{x^{2}-1}}+\frac{\operatorname{arcsec} x \cdot \cos x}{\sin x+2}\right)
\end{aligned}
$$

The domain of $\operatorname{arcsec} x$ is $|x| \geq 1$. For any $x, \sin x+2$ is positive, and a positive number can be raised to any power. (Recall negative numbers cannot be raised to any power-for example, $(-1)^{1 / 2}=\sqrt{-1}$ is not a real number.) So, the domain of $f(x)$ is $|x| \geq 1$.
2.12.2.24. Solution. The function $\frac{1}{\sqrt{x^{2}-1}}$ exists only for those values of $x$ with $x^{2}-1>0$ : that is, the domain of $\frac{1}{\sqrt{x^{2}-1}}$ is $|x|>1$. However, the domain of arcsine is $|x| \leq 1$. So, there is not one single value of $x$ where $\arcsin x$ and $\frac{1}{\sqrt{x^{2}-1}}$ are both defined.
If the derivative of $\arcsin (x)$ were given by $\frac{1}{\sqrt{x^{2}-1}}$, then the derivative of $\arcsin (x)$ would not exist anywhere, so we would probably just write "derivative does not exist," instead of making up a function with a mismatched domain. Also, the function $f(x)=\arcsin (x)$ is a smooth curve-its derivative exists at every point strictly inside its domain. (Remember not all curves are like this: for instance, $g(x)=|x|$ does not have a derivative at $x=0$, but $x=0$ is strictly inside its
domain.) So, it's a pretty good bet that the derivative of arcsine is not $\frac{1}{\sqrt{x^{2}-1}}$.
2.12.2.25. Solution. This limit represents the derivative computed at $x=1$ of the function $f(x)=\arctan x$. To see this, simply use the definition of the derivative at $a=1$ :

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{~d} x}\{f(x)\}\right|_{a} & =\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a} \\
\left.\frac{\mathrm{~d}}{\mathrm{~d} x}\{\arctan x\}\right|_{1} & =\lim _{x \rightarrow 1} \frac{\arctan x-\arctan 1}{x-1} \\
& =\lim _{x \rightarrow 1} \frac{\arctan x-\frac{\pi}{4}}{x-1} \\
& =\lim _{x \rightarrow 1}\left((x-1)^{-1}\left(\arctan x-\frac{\pi}{4}\right)\right)
\end{aligned}
$$

Since the derivative of $f(x)$ is $\frac{1}{1+x^{2}}$, its value at $x=1$ is exactly $\frac{1}{2}$.
2.12.2.26. Solution. First, let's interpret the given information: when the input of our function is $2 x+1$ for some $x$, then its output is $\frac{5 x-9}{3 x+7}$, for that same $x$. We're asked to evaluate $f^{-1}(7)$, which is the number $y$ with the property that $f(y)=7$. If the output of our function is 7 , that means

$$
7=\frac{5 x-9}{3 x+7}
$$

and so

$$
\begin{aligned}
7(3 x+7) & =5 x-9 \\
x & =-\frac{29}{8}
\end{aligned}
$$

So, when $x=-\frac{29}{8}$, our equation $f(2 x+1)=\frac{5 x-9}{3 x+7}$ becomes:

$$
f\left(2 \cdot \frac{-29}{8}+1\right)=\frac{5 \cdot \frac{-29}{8}-9}{3 \cdot \frac{-29}{8}+7}
$$

Or, equivalently:

$$
f\left(-\frac{25}{4}\right)=7
$$

Therefore, $f^{-1}(7)=-\frac{25}{4}$.
2.12.2.27. Solution. If $f^{-1}(y)=0$, that means $f(0)=y$. So, we want to find out what we plug into $f^{-1}$ to get 0 . Since we only know $f^{-1}$ in terms of a variable
$x$, let's figure out what $x$ gives us an output of 0 :

$$
\begin{aligned}
\frac{2 x+3}{x+1} & =0 \\
2 x+3 & =0 \\
x & =-\frac{3}{2}
\end{aligned}
$$

Now, the equation $f^{-1}(4 x-1)=\frac{2 x+3}{x+1}$ with $x=\frac{-3}{2}$ tells us:

$$
f^{-1}\left(4 \cdot \frac{-3}{2}-1\right)=\frac{2 \cdot \frac{-3}{2}+3}{\frac{-3}{2}+1}
$$

Or, equivalently:

$$
f^{-1}(-7)=0
$$

Therefore, $f(0)=-7$.

### 2.12.2.28. Solution.

- Solution 1: We begin by differentiating implicitly. Following the usual convention, we use $y^{\prime}$ to mean $y^{\prime}(x)$. We start with

$$
\arcsin (x+2 y)=x^{2}+y^{2}
$$

Using the chain rule,

$$
\begin{aligned}
\frac{1+2 y^{\prime}}{\sqrt{1-(x+2 y)^{2}}} & =2 x+2 y y^{\prime} \\
\frac{1}{\sqrt{1-(x+2 y)^{2}}}+\frac{2 y^{\prime}}{\sqrt{1-(x+2 y)^{2}}} & =2 x+2 y y^{\prime} \\
\frac{2 y^{\prime}}{\sqrt{1-(x+2 y)^{2}}}-2 y y^{\prime} & =2 x-\frac{1}{\sqrt{1-(x+2 y)^{2}}} \\
y^{\prime}\left(\frac{2}{\sqrt{1-(x+2 y)^{2}}}-2 y\right) & =2 x-\frac{1}{\sqrt{1-(x+2 y)^{2}}}
\end{aligned}
$$

Finally, solving for $y^{\prime}$ gives

$$
\begin{aligned}
y^{\prime} & =\frac{2 x-\frac{1}{\sqrt{1-(x+2 y)^{2}}}}{\frac{2}{\sqrt{1-(x+2 y)^{2}}}-2 y}\left(\frac{\sqrt{1-(x+2 y)^{2}}}{\sqrt{1-(x+2 y)^{2}}}\right) \\
y^{\prime} & =\frac{2 x \sqrt{1-(x+2 y)^{2}}-1}{2-2 y \sqrt{1-(x+2 y)^{2}}}
\end{aligned}
$$

- Solution 2: We begin by taking the sine of both sides of the equation.

$$
\arcsin (x+2 y)=x^{2}+y^{2}
$$

$$
x+2 y=\sin \left(x^{2}+y^{2}\right)
$$

Now, we differentiate implicitly.

$$
\begin{aligned}
1+2 y^{\prime} & =\cos \left(x^{2}+y^{2}\right) \cdot\left(2 x+2 y y^{\prime}\right) \\
1+2 y^{\prime} & =2 x \cos \left(x^{2}+y^{2}\right)+2 y y^{\prime} \cos \left(x^{2}+y^{2}\right) \\
2 y^{\prime}-2 y y^{\prime} \cos \left(x^{2}+y^{2}\right) & =2 x \cos \left(x^{2}+y^{2}\right)-1 \\
y^{\prime}\left(2-2 y \cos \left(x^{2}+y^{2}\right)\right) & =2 x \cos \left(x^{2}+y^{2}\right)-1 \\
y^{\prime} & =\frac{2 x \cos \left(x^{2}+y^{2}\right)-1}{2-2 y \cos \left(x^{2}+y^{2}\right)}
\end{aligned}
$$

- We used two different methods, and got two answers that look pretty different. However, the answers ought to be equivalent. To see this, we remember that for all values of $x$ and $y$ that we care about (those pairs $(x, y)$ in the domain of our curve), the equality

$$
\arcsin (x+2 y)=x^{2}+y^{2}
$$

holds. Drawing a triangle:

where the adjacent side (in red) come from the Pythagorean Theorem. Then, $\cos \left(x^{2}+y^{2}\right)=\sqrt{1-(x+2 y)^{2}}$, so using our second solution:

$$
\begin{aligned}
y^{\prime} & =\frac{2 x \cos \left(x^{2}+y^{2}\right)-1}{2-2 y \cos \left(x^{2}+y^{2}\right)} \\
& =\frac{2 x \sqrt{1-(x+2 y)^{2}}-1}{2-2 y \sqrt{1-(x+2 y)^{2}}}
\end{aligned}
$$

which is exactly the answer from our first solution.

### 2.13 . The Mean Value Theorem

### 2.13.5 •xercises

## Exercises - Stage 1

2.13.5.1. Solution. We know the top speed of the caribou, so we can use this to give the minimum possible number of hours the caribou spent travelling during its migration. If the caribou travels at 70 kph , it will take $(5000 \mathrm{~km})\left(\frac{1 \mathrm{hr}}{70 \mathrm{~km}}\right) \approx 71.4 \mathrm{hrs}$ to travel 5000 kilometres. Probably the caribou wasn't sprinting the whole time, so probably it took it longer than that, but we can only say for sure that the caribou spent at least about 71.4 hours migrating.
2.13.5.2. Solution. If $f(x)$ is the position of the crane at time $x$, measured in hours, then (if we let $x=0$ be the beginning of the day) we know that $f(24)-f(0)=$ 240. Since $f(x)$ is the position of the bird, $f(x)$ is continuous and differentiable. So, the MVT says there is a $c$ in $(0,24)$ such that $f^{\prime}(x)=\frac{f(24)-f(0)}{24-0}=\frac{240}{24}=10$. That is, at some point $c$ during the day, the speed of the crane was exactly 10 kph .
2.13.5.3. Solution. The MVT guarantees there is some point $c$ strictly between $a$ and $b$ where the tangent line to $f(x)$ at $x=c$ has the same slope as the secant line of $f(x)$ from $x=a$ to $x=b$. So, let's start by drawing in the secant line.


What we're looking for is a point on the curve where the tangent line is parallel to this secant line. In fact, there are two.


So, either of the two values $c_{1}$ and $c_{2}$ marked below can serve as the point guaranteed by the MVT:

2.13.5.4. Solution. Since $f(x)$ is differentiable for all $x \in(0,10)$, then $f(x)$ is also continuous for all $x \in(0,10)$. If $f(x)$ were continuous on the closed interval $[0,10]$, then the MVT would guarantee $f^{\prime}(x)=\frac{f(10)-f(0)}{10-0}=1$ for some $c \in$ $(0,10)$; however, this is not the case. So, it must be that $f(x)$ is continuous for all $x \in(0,10)$, but not for all $x \in[0,10]$.
Since $f^{\prime}(c)=0$ for $c \in(0,10)$, that means $f$ is constant on that interval. So, $f(x)$ is a function like this:

where the height of the constant function can be anything.
So, one possible answer is $f(x)=\left\{\begin{array}{ll}0 & x \neq 10 \\ 10 & x=10\end{array}\right.$.
2.13.5.5. Solution. (a) No such function is possible: Rolle's Theorem guarantees $f^{\prime}(c)=0$ for at least one point $c \in(1,2)$.
For the other functions, examples are below, but many answers are possible.

(c)


2.13.5.6. Solution. The function $f(x)$ is continuous over all real numbers, but it is only differentiable when $x \neq 0$. So, if we want to apply the MVT, our interval must consist of only positive numbers or only negative numbers: the interval $(-4,13)$ is not valid.
It is possible to use the mean value theorem to prove what we want: if $a=1$ and $b=144$, then $f(x)$ is differentiable over the interval $(1,144)$ (since 0 is not contained in that interval), and $f(x)$ is continuous everywhere, so by the mean value theorem there exists some point $c$ where $f^{\prime}(x)=\frac{\sqrt{|144|}-\sqrt{|1|}}{144-1}=\frac{11}{143}=\frac{1}{13}$.
That being said, an easier way to prove that a point exists is to simply find itwithout using the MVT. When $x>0, f(x)=\sqrt{x}$, so $f^{\prime}(x)=\frac{1}{2 \sqrt{x}}$. Then $f^{\prime}\left(\frac{169}{4}\right)=\frac{1}{13}$.

## Exercises - Stage 2

2.13.5.7. *. Solution. We note that $f(0)=f(2 \pi)=0$. Then using the Mean Value Theorem (note that the function is differentiable for all real numbers), we conclude that there exists $c$ in $(0,2 \pi)$ such that

$$
f^{\prime}(c)=\frac{f(2 \pi)-f(0)}{2 \pi-0}=0
$$

2.13.5.8. *. Solution. We note that $f(0)=f(1)=0$. Then using the Mean Value Theorem (note that the function is differentiable for all real numbers), we get that there exists $c \in(0,1)$ such that

$$
f^{\prime}(c)=\frac{f(1)-f(0)}{1-0}=0
$$

2.13.5.9. *. Solution. We note that $f(0)=f(2 \pi)=\sqrt{3}+\pi^{2}$. Then using the Mean Value Theorem (note that the function is differentiable for all real numbers since $3+\sin x>0)$, we get that there exists $c \in(0,2 \pi)$ such that

$$
f^{\prime}(c)=\frac{f(2 \pi)-f(0)}{2 \pi-0}=0
$$

2.13.5.10. *. Solution. We note that $f(0)=0$ and $f(\pi / 4)=0$. Then using the Mean Value Theorem (note that the function is differentiable for all real numbers), we get that there exists $c \in(0, \pi / 4)$ such that

$$
f^{\prime}(c)=\frac{f(\pi / 4)-f(0)}{\pi / 4-0}=0
$$

2.13.5.11. Solution. By inspection, we see $x=0$ is a root of $f(x)$. The question now is whether there could possibly be other roots. Since $f(x)$ is differentiable over all real numbers, if there is another root $a$, then by Rolle's Theorem, $f^{\prime}(c)=0$ for some $c$ strictly between 0 and $a$. However, $f^{\prime}(x)=3-\cos x$ is never zero. So, there is no second root: $f(x)$ has precisely one root.
2.13.5.12. Solution. The function $f(x)$ is continuous and differentiable over all real numbers. If $a$ and $b$ are distinct roots of $f(x)$, then $f^{\prime}(c)=0$ for some $c$ strictly between $a$ and $b$ (Rolle's Theorem). So, let's think about $f^{\prime}(x)$.

$$
f^{\prime}(x)=(4 x+1)^{3}+1
$$

This is simple enough that we can find its zero explicitly:

$$
\begin{aligned}
(4 x+1)^{3}+1=0 & \Leftrightarrow(4 x+1)^{3}=-1 \quad \Leftrightarrow \quad 4 x+1=-1 \\
& \Leftrightarrow 4 x=-2
\end{aligned}
$$

Hence $f^{\prime}(c)=0$ only when $c=\frac{-1}{2}$. That the derivative only has a single zero is very useful. It means (via Rolle's theorem) that if $f(x)$ has distinct roots $a$ and $b$ with $a<b$, then we must have $a<\frac{-1}{2}<b$. This also means that $f(x)$ cannot have 3 distinct roots $a, b$ and (say) $q$ with $a<b<q$, because then Rolle's theorem would imply that $f^{\prime}(x)$ would have two zeros - one between $a$ and $b$ and another between $b$ and $q$.
We've learned that $f(x)$ has at most two roots, and we've learned something about where those roots can exist, if there are indeed two of them. But that means $f(x)$ could have 0 , 1 , or 2 roots.
It's not easy to find a root of $f(x)$ by inspection. But we can get a good enough picture of the graph of $y=f(x)$ to tell exactly how many roots there are, just by exploiting the following properties of $f(x)$ and $f^{\prime}(x)$.

- As $x$ tends to $\pm \infty, f(x)$ tends to $+\infty$.
- The derivative $f^{\prime}(x)=(4 x+1)^{3}+1$ is negative for $x<-\frac{1}{2}$ and is positive for $x>-\frac{1}{2}$. That is, $f(x)$ is decreasing for $x<-\frac{1}{2}$ and increasing for $x>-\frac{1}{2}$.
- $f\left(-\frac{1}{2}\right)=\frac{1}{16}-\frac{1}{2}<0$.

This means the function must look something like the graph below:

except we are unsure of the locations of the $x$-intercepts. At present we only know that one is to the left of $x=1 / 2$ and one is to the right.
Note what happens to $f(x)$ as $x$ increases from strongly negative values to strongly positive values.

- When $x$ is large and negative, $f(x)>0$.
- As $x$ increases, $f(x)$ decreases continuously until $x=-\frac{1}{2}$, where $f(x)<0$. In particular, since $f(-1)=\frac{65}{16}>0$ and $f\left(-\frac{1}{2}\right)<0$ and $f(x)$ is continuous, the intermediate value theorem guarantees that $f(x)$ takes the value zero for some $x$ between $-\frac{1}{2}$ and -1 . More descriptively put, as $x$ increases from hugely negative numbers to $-\frac{1}{2}, f(x)$ passes through zero exactly once.
- As $x$ increases beyond $-\frac{1}{2}, f(x)$ increases continuously, starting negative and becoming very large and positive when $x$ becomes large and positive. In particular, since $f(0)=\frac{1}{16}>0$ and $f\left(-\frac{1}{2}\right)<0$ and $f(x)$ is continuous, the intermediate value theorem guarantees that $f(x)$ takes the value zero for some $x$ between $-\frac{1}{2}$ and 0 . So, as $x$ increases from $-\frac{1}{2}$ to near $+\infty, f(x)$ again passes through zero exactly once.

So $f(x)$ must have exactly two roots, one with $x<-\frac{1}{2}$ and one with $x>-\frac{1}{2}$.

### 2.13.5.13. Solution.

- We can see by inspection that $f(0)=0$, so there is at least one root. We have to determine how many any other roots there are, if any.
- The function $f(x)$ is the sum of the two terms $x^{3}$ and $\sin \left(x^{5}\right)$. A number $x$ is a root of $f(x)$ if and only if the two terms cancel each other exactly for that value of $x$. That is, $x$ is a root of $f$ if and only if $x^{3}=-\sin \left(x^{5}\right)$. To develope some intuition in our hunt for other roots, we sketch, in the same figure, the graphs $y=x^{3}$ and $y=-\sin \left(x^{5}\right)$. Then the roots of $f(x)$ are precisely the $x$ 's where the two curves $y=x^{3}$ and $y=-\sin \left(x^{5}\right)$ intersect.

- Looking at the sketch, we see that the two curves cannot possibly intersect at any point having $|x|>1$ - if $|x|>1$, then $\left|x^{3}\right|>1$ but $\left|-\sin \left(x^{5}\right)\right| \leq 1$ and we cannot possibly have $x^{3}=-\sin \left(x^{5}\right)$. That is, all roots of $f(x)$ are in the interval $[-1,1]$.
- From the sketch, we would probably guess that $y=x^{3}$ and $y=-\sin \left(x^{5}\right)$ cross only at $x=0$. We can use Rolle's Theorem to verify that that is indeed the case.
- If there is a root $a \neq 0$, then by Rolle's Theorem (since $f(x)$ is continuous and differentiable for all real numbers $x) f^{\prime}(c)=0$ for some $c$ strictly between 0 and $a$. In particular, since we already know any roots $a$ will be between -1 and 1 , if $f(x)$ has two roots then $f^{\prime}(c)=0$ for some $c \in(-1,0) \cup(0,1)$.
- The derivative of $f$ is

$$
f^{\prime}(x)=3 x^{2}+5 x^{4} \cos \left(x^{5}\right)=x^{2}\left(3+5 x^{2} \cos \left(x^{5}\right)\right)
$$

So, if $f^{\prime}(x)=0$, then $x=0$ or $\left(3+5 x^{2} \cos \left(x^{5}\right)\right)=0$. If $x \in(-1,0) \cup(0,1)$, then

$$
\begin{aligned}
|x|<1 & \Longrightarrow\left|x^{5}\right|<1<\frac{\pi}{2} \Longrightarrow \cos \left(x^{5}\right)>0 \\
& \Longrightarrow 3+5 x^{2} \cos \left(x^{5}\right)>3 \\
& \Longrightarrow f^{\prime}(x) \neq 0
\end{aligned}
$$

That is, there is no $c \in(-1,0) \cup(0,1)$ with $f^{\prime}(c)=0$. Therefore, following our last bullet point, $f(x)$ has only one root.

Note here that $f^{\prime}(x)$ has many zeroes - infinitely many, in fact. However, $x=0$ is the only root of $f^{\prime}(x)$ in the interval $(-1,1)$.
2.13.5.14. Solution. We are to find the number of positive solutions to the equation $e^{x}=4 \cos (2 x)$. The figure below contains the graphs of $y=e^{x}$ and $y=4 \cos (2 x)$. The solutions to $e^{x}=4 \cos (2 x)$ are precisely the $x$ 's where $y=e^{x}$ and $y=4 \cos (2 x)$ cross.


It sure looks like there is exactly one crossing with $x \geq 0$ and that one crossing is somewhere between $x=0$ and $x=1$. Indeed since $\left[e^{x}-4 \cos (2 x)\right]_{x=0}=-3<0$ and $\left[e^{x}-4 \cos (2 x)\right]_{x=1}>e>0$ and $f(x)=e^{x}-4 \cos (2 x)$ is continuous, the intermediate value theorem guarantees that there is at least one root with $0<x<1$.
We still have to show that there is no second root - even if our graphs are not accurate.
Recall that the range of the cosine function is $[-1,1]$. If $e^{x}=4 \cos (2 x)$, then $e^{x} \leq 4$, so $x \leq \log (4) \approx 1.39$. So, we only need to search for roots of $f(x)$ on the interval $(0,1.4)$ : we are guaranteed there are no roots elsewhere. Over this interval, $2 x \in(0,2.8)$, so $\sin (2 x)>0$, and thus $f^{\prime}(x)=e^{x}+8 \sin (2 x)>0$. Since $f^{\prime}(x)$ has no roots in $(0,1.4)$, we conclude by Rolle's Theorem that $f(x)$ has at most one root in $(0,1.4)$ (and so at most one positive root total). Since we've already found that a root of $f(x)$ exists in $(0,1)$, we conclude $e^{x}=4 \cos (2 x)$ has precisely one positive-valued solution.
2.13.5.15. *. Solution. 2.13.5.15.a

$$
f^{\prime}(x)=15 x^{4}-30 x^{2}+15=15\left(x^{4}-2 x^{2}+1\right)=15\left(x^{2}-1\right)^{2} \geq 0
$$

The derivative is nonnegative everywhere. The only values of $x$ for which $f^{\prime}(x)=0$ are 1 and -1 , so $f^{\prime}(x)>0$ for every $x$ in $(-1,1)$.
2.13.5.15.b If $f(x)$ has two roots $a$ and $b$ in $[-1,1]$, then by Rolle's Theorem, $f^{\prime}(c)=$ 0 for some $x$ strictly between $a$ and $b$. But since $a$ and $b$ are in $[-1,1]$, and $c$ is between $a$ and $b$, that means $c$ is in $(-1,1)$; however, we know for every $c$ in $(-1,1)$, $f^{\prime}(c)>0$, so this can't happen. Therefore, $f(x)$ does not have two roots $a$ and $b$ in $[-1,1]$. This means $f(x)$ has at most one root in $[-1,1]$.
2.13.5.16. *. Solution. Write $f(x)=e^{x}$. Since $f(x)$ is continuous and differentiable, the Mean Value Theorem asserts that there exists some $c$ between 0 and $T$
such that

$$
f^{\prime}(c)=\frac{f(T)-f(0)}{T-0}
$$

The problem asks us to find this value of $c$. Solving:

$$
\begin{aligned}
e^{c} & =\frac{e^{T}-e^{0}}{T} \\
e^{c} & =\frac{e^{T}-1}{T} \\
c & =\log \left(\frac{e^{T}-1}{T}\right)
\end{aligned}
$$

2.13.5.17. Solution. The domains of $\operatorname{arcsec} x$ and $C-\operatorname{arccsc} x$ are the same: $|x| \geq 1$. Define $f(x)=\operatorname{arcsec} x+\operatorname{arccsc} x$, and note the domain of $f(x)$ is also $|x| \geq 1$. Using Theorem 2.12.8,

$$
f^{\prime}(x)=\frac{\mathrm{d}}{\mathrm{~d} x} \operatorname{arcsec} x+\frac{\mathrm{d}}{\mathrm{~d} x} \operatorname{arccsc}=\frac{1}{|x| \sqrt{x^{2}-1}}+\frac{-1}{|x| \sqrt{x^{2}-1}}=0 .
$$

By Corollary 2.13.12, this means that $f(x)$ is constant on any interval in $|x| \geq 1$. So $f(x)$ is a constant, call it $C_{+}$, on $x \geq 1$, and $f(x)$ is also a constant, call it $C_{-}$, on $x \leq-1$.
In order to find $C_{+}$, we find $f(1)$, because we know angles for which the secant and cosecant are $x=1$.

$$
\begin{array}{r}
\cos (0)=1 \Longrightarrow \sec (0)=\frac{1}{1}=1 \Longrightarrow \operatorname{arcsec}(1)=0 \\
\sin \left(\frac{\pi}{2}\right)=1 \Longrightarrow \csc \left(\frac{\pi}{2}\right)=\frac{1}{1}=1 \Longrightarrow \operatorname{arccsc}(1)=\frac{\pi}{2}
\end{array}
$$

So

$$
C_{+}=\operatorname{arcsec}(1)+\operatorname{arccsc}(1)=\frac{\pi}{2}
$$

In order to find $C_{-}$, we find $f(-1)$, because we know angles for which the secant and cosecant are $x=-1$.

$$
\begin{aligned}
\cos (\pi) & =-1 \Longrightarrow \quad \sec (\pi)=\frac{1}{-1}=-1 \Longrightarrow \operatorname{arcsec}(-1)=\pi \\
\sin \left(-\frac{\pi}{2}\right) & =-1 \Longrightarrow \csc \left(-\frac{\pi}{2}\right)=\frac{1}{-1}=-1 \Longrightarrow \operatorname{arccsc}(-1)=-\frac{\pi}{2}
\end{aligned}
$$

So

$$
C_{-}=\operatorname{arcsec}(-1)+\operatorname{arccsc}(-1)=\frac{\pi}{2}
$$

This shows that $f(x)=\operatorname{arcsec} x+\operatorname{arccsc} x=\frac{\pi}{2}$ for all $|x| \geq 1$ and $\operatorname{arcsec} x=$ $\frac{\pi}{2}-\operatorname{arccsc} x$ for all $|x| \geq 1$.

## Exercises - Stage 3

2.13.5.18. *. Solution. Since $e^{-f(x)}$ is always positive (regardless of the value
of $f(x))$,

$$
f^{\prime}(x)=\frac{1}{1+e^{-f(x)}}<\frac{1}{1+0}=1
$$

for every $x$.
Since $f^{\prime}(x)$ exists for every $x$, we see that $f$ is differentiable, so the Mean Value Theorem applies. If $f(100)$ is greater than or equal to 100 , then by the Mean Value Theorem, there would have to be some $c$ between 0 and 100 such that

$$
f^{\prime}(c)=\frac{f(100)-f(0)}{100} \geq \frac{100}{100}=1
$$

Since $f^{\prime}(x)<1$ for every $x$, there is no value of $c$ as described. Therefore, it is not possible that $f(100) \geq 100$. So, $f(100)<100$.
2.13.5.19. Solution. If $2 x+\sin x$ is one-to-one over an interval, it never takes the same value for two distinct numbers in that interval. By Rolle's Theorem, if $f(a)=f(b)$ for distinct $a$ and $b$, then $f^{\prime}(c)=0$ for some $c$ between $a$ and $b$. However, $f^{\prime}(x)=2+\cos x$, which is never zero. In fact, $f^{\prime}(x) \geq 1$ for all $x$, so $f(x)$ is strictly increasing over its entire domain. Therefore, our function $f$ never takes the same value twice, so it is one-to-one over all the real numbers, $(-\infty, \infty)$.
When we define the inverse function $f^{-1}(x)$, the domain of $f$ is the range of $f^{-1}$, and vice-versa. In general, we might have to restrict the domain of $f$ (and hence the range of $f^{-1}$ ) to an interval where $f$ is one-to-one, but in our case, this isn't necessary. So, the range of $f^{-1}$ is $(-\infty, \infty)$ and the domain of $f^{-1}$ is the range of $f:(-\infty, \infty)$.
2.13.5.20. Solution. If $f(x)=\frac{x}{2}+\sin x$ is one-to-one over an interval, it never takes the same value for two distinct numbers in that interval. By Rolle's Theorem, if $f(a)=f(b)$ for distinct $a$ and $b$, then $f^{\prime}(c)=0$ for some $c$ between $a$ and $b$. Since $f^{\prime}(x)=\frac{1}{2}+\cos x, f^{\prime}(x)=0$ when $x=2 n \pi \pm \frac{2 \pi}{3}$ for some integer $n$. So, in particular, if $a$ and $b$ are distinct numbers in the interval $\left[-\frac{2 \pi}{3}, \frac{2 \pi}{3}\right]$, then for every $c$ strictly between $a$ and $b, f^{\prime}(c) \neq 0$, so by Rolle's Theorem $f(a) \neq f(b)$. Therefore $f(x)$ is one-to-one on the interval $\left[-\frac{2 \pi}{3}, \frac{2 \pi}{3}\right]$.
We should also show that the interval $\left[-\frac{2 \pi}{3}, \frac{2 \pi}{3}\right]$ cannot be extended to a larger interval over which $f(x)$ is still one-to-one. Consider the derivative $f^{\prime}(x)=\frac{1}{2}+\cos x$. For all $-\frac{2 \pi}{3}<x<\frac{2 \pi}{3}$, we have $\cos x>-\frac{1}{2}$ (sketch the graph of $\cos x$ yourself) so that $f^{\prime}(x)>0$ and $f(x)$ is increasing. But at $x=\frac{2 \pi}{3}, f^{\prime}(x)=0$, and then for $x$ a bit bigger than $\frac{2 \pi}{3}$ we have $\cos x<-\frac{1}{2}$ so that $f^{\prime}(x)<0$ and $f(x)$ is decreasing. So the graph "reverses direction", and $f(x)$ repeats values. (See the graph of $y=f(x)$ below.) The same is true for $x$ a little smaller than $-\frac{2 \pi}{3}$.


When we define the inverse function $f^{-1}(x)$, first we restrict $f$ to $\left[-\frac{2 \pi}{3}, \frac{2 \pi}{3}\right]$. Then the range of $f^{-1}$ is also $\left[-\frac{2 \pi}{3}, \frac{2 \pi}{3}\right]$. The domain of $f^{-1}$ is the range of $f$ over this interval, so $\left[-\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right),\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)\right]$.
2.13.5.21. Solution. Define $h(x)=f(x)-g(x)$, and notice $h(a)=f(a)-g(a)<$ 0 and $h(b)=f(b)-g(b)>0$. Since $h$ is the difference of two functions that are continuous over $[a, b]$ and differentiable over $(a, b)$, also $h$ is continuous over $[a, b]$ and differentiable over $(a, b)$. So, by the Mean Value Theorem, there exists some $c \in(a, b)$ with

$$
h^{\prime}(c)=\frac{h(b)-h(a)}{b-a}
$$

Since $(a, b)$ is an interval, $b>a$, so the denominator of the above expression is positive; since $h(b)>0>h(a)$, also the numerator of the above expression is positive. So, $h^{\prime}(c)>0$ for some $c \in(a, b)$. Since $h^{\prime}(c)=f^{\prime}(c)-g^{\prime}(c)$, we conclude $f^{\prime}(c)>g^{\prime}(c)$ for some $c \in(a, b)$.
2.13.5.22. Solution. Since $f(x)$ is differentiable over all real numbers, it is also continuous over all real numbers. We claim that $f(x)$ cannot have four or more distinct roots. For every two distinct roots $a<b$, Rolle's Theorem tells us there is a $c \in(a, b)$ such that $f^{\prime}(c)=0$ : that is, $c$ is a root of $f^{\prime}$. Since $f^{\prime}$ has only two distinct roots, $f$ can have at most three distinct roots.

2.13.5.23. Solution. We are asked to find the number of solutions to the equation $x^{2}+5 x+1=-\sin x$. The figure below contains the graphs of $y=x^{2}+5 x+1$ and $y=-\sin x$. The solutions to $x^{2}+5 x+1=-\sin x$ are precisely the $x$ 's where $y=x^{2}+5 x+1$ and $y=-\sin x$ cross.


From the figure, it sure looks like there are two crossings. Since the function $-\sin x$ has range $[-1,1]$, if the two functions cross, then also $-1 \leq x^{2}+5 x+1 \leq 1$. This portion of the quadratic function is highlighted in blue in the figure. The $x$ coordinates of the end points of the blue arcs are found by solving $x^{2}+5 x+1= \pm 1$, i.e. $x=0,-5$, and (using the quadratic equation) $x=\frac{-5 \pm \sqrt{17}}{2}$.

We are now in a position to exploit the intuition that we have built using the
above figure to write a concise argument showing that $f(x)$ has exactly two roots. Remember that, in general, if we want to show that a function has $n$ roots, we need to show that there exist $n$ distinct roots somewhere, and that there do not exist $n+1$ distinct roots. This argument is given below, in blue text.

- $f(x)$ is continuous over all real numbers
- $f(x)=0$ only when $-\sin x=x^{2}+5 x+1$, which only happens when $\mid x^{2}+$ $5 x+1 \mid \leq 1$. Thus, $f(x)$ only has roots in the intervals $\left[-5, \frac{-5-\sqrt{17}}{2}\right]$ and $\left[\frac{-5+\sqrt{17}}{2}, 0\right]$.
- $f(-5)=\sin (-5)+1>0$, and $f\left(\frac{-5-\sqrt{17}}{2}\right)=\sin \left(\frac{-5-\sqrt{17}}{2}\right)-1<0$. So, by the IVT, $f(c)=0$ for some $c \in\left(-5, \frac{-5-\sqrt{17}}{2}\right)$.
- $f(0)=1>0$, and $f\left(\frac{-5+\sqrt{17}}{2}\right)=\sin \left(\frac{-5+\sqrt{17}}{2}\right)-1<0$. So, by the IVT, $f(c)=0$ for some $c \in\left(\frac{-5+\sqrt{17}}{2}, 0\right)$.
- $f^{\prime}(x)=\cos x+2 x+5$. If $f^{\prime}(x)=0$, then $2 x+5=-\cos x$, so $|2 x+5| \leq 1$. So, the only interval that can contain roots of $f^{\prime}(x)$ is $[-3,-2]$.
- Suppose $f(x)$ has more than two roots. Then it has two roots in the interval $\left[-5, \frac{-5-\sqrt{17}}{2}\right]$ OR it has two roots in the interval $\left[\frac{-5+\sqrt{17}}{2}, 0\right]$. Since $f(x)$ is differentiable for all real numbers, Rolle's Theorem tells us that $f^{\prime}(x)$ has a root in $\left(-5, \frac{-5-\sqrt{17}}{2}\right)$ or in $\left(\frac{-5+\sqrt{17}}{2}, 0\right)$. However, since all roots of $f^{\prime}(x)$ are in the interval $[-3,-2]$, and this interval shares no points with $\left(-5, \frac{-5-\sqrt{17}}{2}\right)$ or $\left(\frac{-5+\sqrt{17}}{2}, 0\right)$, this cannot be the case. Therefore $f(x)$ does not have more than two roots.

- Since $f(x)$ has at least two roots, and not more than two roots, $f(x)$ has exactly two roots.


## $2.14 \cdot$ Higher Order Derivatives

### 2.14.2 • Exercises

## Exercises - Stage 1

2.14.2.1. Solution. The derivative of $e^{x}$ is $e^{x}$ : taking derivatives leaves the function unchanged, even if we do it 180 times. So $f^{(180)}=e^{x}$.
2.14.2.2. Solution. Since $f^{\prime}(x)>0$ over $(a, b)$, we know from Corollary 2.13 .12 that $f(x)$ is increasing over $(a, b)$, so 2.14.2.2.ii holds. Since $f^{\prime \prime}(x)>0$, and $f^{\prime \prime}(x)$ is the derivative of $f^{\prime}(x)$, by the same reasoning we see that $f^{\prime}(x)$ is increasing. Since $f^{\prime}(x)$ is the rate at which $f(x)$ is increasing, that means that the rate at which $f^{\prime}(x)$ is increasing is itself increasing: this is, 2.14.2.2.iv holds (and not 2.14.2.2.iii).
There is no reason to think 2.14.2.2.i or 2.14.2.2.v holds, but to be thorough we will give an example showing that they do not need to be true. If $f(x)=x^{2}-10$ and $(a, b)=(0,1)$, then $f^{\prime}(x)=2 x>0$ over $(0,1)$, and $f^{\prime \prime}(x)=2>0$ everywhere, but $f(x)<0$ for all $x \in(0,1)$, so 2.14.2.2.i does not hold. Also, $f^{\prime \prime \prime}(x)=0$ everywhere, so 2.14.2.2.v does not hold either.
2.14.2.3. Solution. Every time we differentiate $f(x)$, the constant out front gets multiplied by an ever-decreasing constant, while the power decreases by one. As in Example 2.14.2, $\frac{\mathrm{d}^{15}}{\mathrm{~d} x^{15}} a x^{15}=a \cdot 15$ !. So, if $a \cdot 15!=3$, then $a=\frac{3}{15!}$.
2.14.2.4. Solution. The derivative $\frac{\mathrm{d} y}{\mathrm{~d} x}$ is $\frac{11}{4}$ only at the point $(1,3)$ : it is not constantly $\frac{11}{4}$, so it is wrong to differentiate the constant $\frac{11}{4}$ to find $\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}$. Below is a correct solution.

$$
-28 x+2 y+2 x y^{\prime}+2 y y^{\prime}=0
$$

Plugging in $x=1, y=3$ :

$$
\begin{aligned}
-28+6+2 y^{\prime}+6 y^{\prime} & =0 \\
y^{\prime} & =\frac{11}{4} \quad \text { at the point }(1,3)
\end{aligned}
$$

Differentiating the equation $-28 x+2 y+2 x y^{\prime}+2 y y^{\prime}=0$ :

$$
\begin{aligned}
-28+2 y^{\prime}+2 y^{\prime}+2 x y^{\prime \prime}+2 y^{\prime} y^{\prime}+2 y y^{\prime \prime} & =0 \\
4 y^{\prime}+2\left(y^{\prime}\right)^{2}+2 x y^{\prime \prime}+2 y y^{\prime \prime} & =28
\end{aligned}
$$

At the point $(1,3), y^{\prime}=\frac{11}{4}$. Plugging in:

$$
\begin{aligned}
4\left(\frac{11}{4}\right)+2\left(\frac{11}{4}\right)^{2}+2(1) y^{\prime \prime}+2(3) y^{\prime \prime} & =28 \\
y^{\prime \prime} & =\frac{15}{64}
\end{aligned}
$$

## Exercises - Stage 2

2.14.2.5. Solution.

$$
\begin{aligned}
f(x) & =x \log x-x \\
f^{\prime}(x) & =\log x+x \cdot \frac{1}{x}-1 \\
& =\log x \\
f^{\prime \prime}(x) & =\frac{1}{x}
\end{aligned}
$$

### 2.14.2.6. Solution.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{\arctan x\} & =\frac{1}{1+x^{2}} \\
\frac{\mathrm{~d}}{\mathrm{~d} x}\left\{\frac{1}{1+x^{2}}\right\} & =\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\left(1+x^{2}\right)^{-1}\right\} \\
& =(-1)\left(1+x^{2}\right)^{-2}(2 x) \\
& =\frac{-2 x}{\left(1+x^{2}\right)^{2}}
\end{aligned}
$$

2.14.2.7. Solution. We use implicit differentiation, twice.

$$
\begin{aligned}
2 x+2 y y^{\prime} & =0 \\
2+(2 y) y^{\prime \prime}+\left(2 y^{\prime}\right) y^{\prime} & =0 \\
y^{\prime \prime} & =-\frac{\left(y^{\prime}\right)^{2}+1}{y}
\end{aligned}
$$

So, we need an expression for $y^{\prime}$. We use the equation $2 x+2 y y^{\prime}=0$ to conclude $y^{\prime}=-\frac{x}{y}$ :

$$
\begin{aligned}
y^{\prime \prime} & =-\frac{\left(-\frac{x}{y}\right)^{2}+1}{y} \\
& =-\frac{\frac{x^{2}}{y^{2}}+1}{y} \\
& =-\frac{x^{2}+y^{2}}{y^{3}} \\
& =-\frac{1}{y^{3}}
\end{aligned}
$$

2.14.2.8. Solution. The question asks for $s^{\prime \prime}(1)$. We start our differentiation using the quotient rule:

$$
\begin{aligned}
s^{\prime}(t) & =\frac{e^{t}\left(t^{2}+1\right)-e^{t}(2 t)}{\left(t^{2}+1\right)^{2}} \\
& =\frac{e^{t}\left(t^{2}-2 t+1\right)}{\left(t^{2}+1\right)^{2}}
\end{aligned}
$$

Using the quotient rule again,

$$
\begin{aligned}
& s^{\prime \prime}(t)=\frac{\left(t^{2}+1\right)^{2} \frac{\mathrm{~d}}{\mathrm{~d} t}\left\{e^{t}\left(t^{2}-2 t+1\right)\right\}-e^{t}\left(t^{2}-2 t+1\right) \frac{\mathrm{d}}{\mathrm{~d} t}\left\{\left(t^{2}+1\right)^{2}\right\}}{\left(t^{2}+1\right)^{4}} \\
& =\frac{\left(t^{2}+1\right)^{2} \cdot\left[e^{t}(2 t-2)+e^{t}\left(t^{2}-2+1\right)\right]-e^{t}\left(t^{2}-2 t+1\right) \cdot 2\left(t^{2}+1\right)(2 t)}{\left(t^{2}+1\right)^{4}} \\
& =\frac{e^{t}\left(t^{2}+1\right)^{2}\left(t^{2}-1\right)-4 t e^{t}(t-1)^{2}\left(t^{2}+1\right)}{\left(t^{2}+1\right)^{4}}
\end{aligned}
$$

so that

$$
s^{\prime \prime}(1)=0
$$

2.14.2.9. Solution. We differentiate using the chain rule.

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\log \left(5 x^{2}-12\right)\right\}=\frac{10 x}{5 x^{2}-12}
$$

Using the quotient rule:

$$
\begin{aligned}
\frac{\mathrm{d}^{2}}{\mathrm{~d} x^{2}}\left\{\log \left(5 x^{2}-12\right)\right\} & =\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{10 x}{5 x^{2}-12}\right\} \\
& =\frac{\left(5 x^{2}-12\right)(10)-10 x(10 x)}{\left(5 x^{2}-12\right)^{2}} \\
& =\frac{-10\left(5 x^{2}+12\right)}{\left(5 x^{2}-12\right)^{2}}
\end{aligned}
$$

Using the quotient rule one last time:

$$
\begin{aligned}
\frac{\mathrm{d}^{3}}{\mathrm{~d} x^{3}} & \left\{\log \left(5 x^{2}-12\right)\right\}=\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{-10\left(5 x^{2}+12\right)}{\left(5 x^{2}-12\right)^{2}}\right\} \\
& =\frac{\left(5 x^{2}-12\right)^{2}(-10)(10 x)+10\left(5 x^{2}+12\right)(2)\left(5 x^{2}-12\right)(10 x)}{\left(5 x^{2}-12\right)^{4}} \\
& =\frac{\left(5 x^{2}-12\right)(-100 x)+(200 x)\left(5 x^{2}+12\right)}{\left(5 x^{2}-12\right)^{3}} \\
& =\frac{100 x\left(-5 x^{2}+12+10 x^{2}+24\right)}{\left(5 x^{2}-12\right)^{3}} \\
& =\frac{100 x\left(5 x^{2}+36\right)}{\left(5 x^{2}-12\right)^{3}}
\end{aligned}
$$

2.14.2.10. Solution. The velocity of the particle is given by $h^{\prime}(t)=\sin t$. Note $0<1<\pi$, so $h^{\prime}(1)>0$-the particle is rising (moving in the positive direction, in this case "up"). The acceleration of the particle is $h^{\prime \prime}(t)=\cos t$. Since $0<1<\frac{\pi}{2}$, $h^{\prime \prime}(t)>0$, so $h^{\prime}(t)$ is increasing: the particle is moving up, and it's doing so at an increasing rate. So, the particle is speeding up.
2.14.2.11. Solution. For this problem, remember that velocity has a sign indicating direction, while speed does not.
The velocity of the particle is given by $h^{\prime}(t)=3 t^{2}-2 t-5$. At $t=1$, the velocity of the particle is -4 , so the particle is moving downwards with a speed of 4 units per second. The acceleration of the particle is $h^{\prime \prime}(t)=6 t-2$, so when $t=1$, the acceleration is (positive) 4 units per second per second. That means the velocity (currently -4 units per second) is becoming a bigger number-since the velocity is negative, a bigger number is closer to zero, so the speed of the particle is getting smaller. (For instance, a velocity of -3 represents a slower motion than a velocity of -4 .) So, the particle is slowing down at $t=1$.

### 2.14.2.12. Solution.

$$
x^{2}+x+y=\sin (x y)
$$

We differentiate implicitly. For ease of notation, we write $y^{\prime}$ for $\frac{\mathrm{d} y}{\mathrm{~d} x}$.

$$
2 x+1+y^{\prime}=\cos (x y)\left(y+x y^{\prime}\right)
$$

We're interested in $y^{\prime \prime}$, so we implicitly differentiate again.

$$
2+y^{\prime \prime}=-\sin (x y)\left(y+x y^{\prime}\right)^{2}+\cos (x y)\left(2 y^{\prime}+x y^{\prime \prime}\right)
$$

We want to know what $y^{\prime \prime}$ is when $x=y=0$. Plugging these in yields the following:

$$
2+y^{\prime \prime}=2 y^{\prime}
$$

So, we need to know what $y^{\prime}$ is when $x=y=0$. We can get this from the equation $2 x+1+y^{\prime}=\cos (x y)\left(y+x y^{\prime}\right)$, which becomes $1+y^{\prime}=0$ when $x=y=0$. So, at the origin, $y^{\prime}=-1$, and

$$
\begin{aligned}
2+y^{\prime \prime} & =2(-1) \\
y^{\prime \prime} & =-4
\end{aligned}
$$

Remark: a common mistake is to stop at the equation $2 x+1+y^{\prime}=\cos (x y)\left(y+x y^{\prime}\right)$, plug in $x=y=0$, find $y^{\prime}=-1$, and decide $y^{\prime \prime}=\frac{\mathrm{d}}{\mathrm{d} x}\{-1\}=0$. This is due to a slight sloppiness in the usual notation. When we wrote $y^{\prime}=1$, what we meant is that at the point $(0,0), \frac{\mathrm{d} y}{\mathrm{~d} x}=-1$. More properly written: $\left.\frac{\mathrm{d} y}{\mathrm{~d} x}\right|_{x=0, y=0}=-1$. This is not the same as saying $y^{\prime}=1$ everywhere (in which case, indeed, $y^{\prime \prime}$ would be 0 everywhere).
2.14.2.13. Solution. For (a) and (b), notice the following:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x} \sin x & =\cos x \\
\frac{\mathrm{~d}}{\mathrm{~d} x} \cos x & =-\sin x
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{-\sin x\} & =-\cos x \\
\frac{\mathrm{~d}}{\mathrm{~d} x}\{-\cos x\} & =\sin x \\
\frac{\mathrm{~d}}{\mathrm{~d} x} \sin x & =\cos x
\end{aligned}
$$

The fourth derivative is $\sin x$ is $\sin x$, and the fourth derivative of $\cos x$ is $\cos x$, so (a) and (b) are true.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x} \tan x & =\sec ^{2} x \\
\frac{\mathrm{~d}}{\mathrm{~d} x} \sec ^{2} x & =2 \sec x(\sec x \tan x)=2 \sec ^{2} x \tan x \\
\frac{\mathrm{~d}}{\mathrm{~d} x}\left\{2 \sec ^{2} x \tan x\right\} & =(4 \sec x \cdot \sec x \tan x) \tan x+2 \sec ^{2} x \sec ^{2} x \\
& =4 \sec ^{2} x \tan ^{2} x+2 \sec ^{4} x
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} x}\left\{4 \sec ^{2} x \tan ^{2} x+2 \sec ^{4} x\right\} \\
& = \\
& =(8 \sec x \cdot \sec x \tan x) \tan ^{2} x+4 \sec ^{2} x\left(2 \tan x \cdot \sec ^{2} x\right) \\
& \quad+8 \sec ^{3} x \cdot \sec x \tan x \\
& =
\end{aligned}
$$

So, $\frac{\mathrm{d}^{4}}{\mathrm{~d} x^{4}} \tan x=8 \sec ^{2} x \tan ^{3} x+16 \sec ^{4} x \tan x$. It certainly seems like this is not the same as $\tan x$, but remember that sometimes trig identities can fool you: $\tan ^{2} x+1=$ $\sec ^{2} x$, and so on. So, to be absolutely sure that these are not equal, we need to find a value of $x$ so that the output of one is not the same as the output of the other. When $x=\frac{\pi}{4}$ :

$$
\begin{align*}
8 \sec ^{2} x \tan ^{3} x+16 \sec ^{4} x \tan x & =8(\sqrt{2})^{2}(1)^{3}+16(\sqrt{2})^{4}  \tag{1}\\
& =80 \neq 1=\tan x
\end{align*}
$$

So, (c) is false.

## Exercises - Stage 3

2.14.2.14. Solution. Since $f^{\prime}(x)<0$, we need a decreasing function. This only applies to (ii), (iii), and (v). Since $f^{\prime \prime}(x)>0$, that means $f^{\prime}(x)$ is increasing, so the slope of the function must be increasing. In (v), the slope is constant, so $f^{\prime \prime}(x)=0-$ therefore, it's not (v). In (iii), the slope is decreasing, because near $a$ the curve is quite flat $\left(f^{\prime}(x)\right.$ near zero) but near $b$ the curve is very steeply decreasing $\left(f^{\prime}(x)\right.$ is a large negative number), so (iii) has a negative second derivative. By contrast, in (ii), the line starts out as steeply decreasing $\left(f^{\prime}(x)\right.$ is a strongly negative number)
and becomes flatter and flatter $\left(f^{\prime}(x)\right.$ nears 0$)$, so $f^{\prime}(x)$ is increasing-in other words, $f^{\prime \prime}(x)>0$. So, (ii) is the only curve that has $f^{\prime}(x)<0$ and $f^{\prime \prime}(x)>0$.

### 2.14.2.15. Solution. We differentiate a few time to find the pattern.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{2^{x}\right\} & =2^{x} \log 2 \\
\frac{\mathrm{~d}^{2}}{\mathrm{~d} x^{2}}\left\{2^{x}\right\} & =2^{x} \log 2 \cdot \log 2=2^{x}(\log 2)^{2} \\
\frac{\mathrm{~d}^{3}}{\mathrm{~d} x^{3}}\left\{2^{x}\right\} & =2^{x}(\log 2)^{2} \cdot \log 2=2^{x}(\log 2)^{3}
\end{aligned}
$$

Every time we differentiate, we multiply the original function by another factor of $\log 2$. So, the $n$th derivative is given by:

$$
\frac{\mathrm{d}^{n}}{\mathrm{~d} x^{n}}\left\{2^{x}\right\}=2^{x}(\log 2)^{n}
$$

2.14.2.16. Solution. We differentiate using the power rule.

$$
\begin{aligned}
\frac{\mathrm{d} f}{\mathrm{~d} x} & =3 a x^{2}+2 b x+c \\
\frac{\mathrm{~d}^{2} f}{\mathrm{~d} x^{2}} & =6 a x+2 b \\
\frac{\mathrm{~d}^{3} f}{\mathrm{~d} x^{3}} & =6 a \\
\frac{\mathrm{~d}^{4} f}{\mathrm{~d} x^{4}} & =0
\end{aligned}
$$

In the above work, remember that $a, b, c$, and $d$ are all constants. Since they are nonzero constants, $\frac{\mathrm{d}^{3} f}{\mathrm{~d} x^{3}}=6 a \neq 0$. So, the fourth derivative is the first derivative to be identically zero: $n=4$.
2.14.2.17. *. Solution. 2.14.2.17.a Using the chain rule for $f(x)$ :

$$
\begin{aligned}
f^{\prime}(x) & =(1+2 x) e^{x+x^{2}} \\
f^{\prime \prime}(x) & =(1+2 x)(1+2 x) e^{x+x^{2}}+(2) e^{x+x^{2}}=\left(4 x^{2}+4 x+3\right) e^{x+x^{2}} \\
h^{\prime}(x) & =1+3 x \\
h^{\prime \prime}(x) & =3
\end{aligned}
$$

- 2.14.2.17.b $f(0)=h(0)=1 ; f^{\prime}(0)=h^{\prime}(0)=1 ; f^{\prime \prime}(0)=h^{\prime \prime}(0)=3$
- 2.14.2.17.c $f$ and $h$ "start at the same place", since $f(0)=h(0)$. If it were clear that $f^{\prime}(x)$ were greater than $h^{\prime}(x)$ for $x>0$, then we would know that $f$ grows faster than $h$, so we could conclude that $f(x)>h(x)$, as desired. Unfortunately, it is not obvious whether $(1+2 x) e^{x+x^{2}}$ is always greater than $1+3 x$ for positive $x$. So, we look to the second derivative. $f^{\prime}(0)=h^{\prime}(0)$,
and $f^{\prime \prime}(x)=\left(4 x^{2}+4 x+3\right) e^{x+x^{2}}>3 e^{x+x^{2}}>3=h^{\prime \prime}(x)$ when $x>0$. Since $f^{\prime}(0)=h^{\prime}(0)$, and since $f^{\prime}$ grows faster than $h^{\prime}$ for positive $x$, we conclude $f^{\prime}(x)>h^{\prime}(x)$ for all positive $x$. Now we can conclude that (since $f(0)=h(0)$ and $f$ grows faster than $h$ when $x>0)$ also $f(x)>h(x)$ for all positive $x$.


### 2.14.2.18. *. Solution.

a We differentiate implicitly.

$$
\begin{aligned}
x^{3} y(x)+y(x)^{3} & =10 x \\
3 x^{2} y(x)+x^{3} y^{\prime}(x)+3 y(x)^{2} y^{\prime}(x) & =10
\end{aligned}
$$

Subbing in $x=1$ and $y(1)=2$ gives

$$
\begin{aligned}
(3)(1)(2)+(1) y^{\prime}(1)+(3)(4) y^{\prime}(1) & =10 \\
13 y^{\prime}(1) & =4 \\
y^{\prime}(1) & =\frac{4}{13}
\end{aligned}
$$

b From part 2.14.2.18.a, the slope of the curve at $x=1, y=2$ is $\frac{4}{13}$, so the curve is increasing, but fairly slowly. The angle of the tangent line is $\tan ^{-1}\left(\frac{4}{13}\right) \approx 17^{\circ}$. We are also told that $y^{\prime \prime}(1)<0$. So the slope of the curve is decreasing as $x$ passes through 1 . That is, the line is more steeply increasing to the left of $x=1$, and its slope is decreasing (getting less steep, then possibly the slope even becomes negative) as we move past $x=1$.

2.14.2.19. Solution. 2.14.2.19.a Using the product rule,

$$
\begin{aligned}
g^{\prime \prime}(x) & =\left[f^{\prime}(x)+f^{\prime \prime}(x)\right] e^{x}+\left[f(x)+f^{\prime}(x)\right] e^{x} \\
& =\left[f(x)+2 f^{\prime}(x)+f^{\prime \prime}(x)\right] e^{x}
\end{aligned}
$$

2.14.2.19.b Using the product rule and our answer from 2.14.2.19.a,

$$
\begin{aligned}
g^{\prime \prime \prime}(x) & =\left[f^{\prime}(x)+2 f^{\prime \prime}(x)+f^{\prime \prime \prime}(x)\right] e^{x}+\left[f(x)+2 f^{\prime}(x)+f^{\prime \prime}(x)\right] e^{x} \\
& =\left[f(x)+3 f^{\prime}(x)+3 f^{\prime \prime}(x)+f^{\prime \prime \prime}(x)\right] e^{x}
\end{aligned}
$$

2.14.2.19.c We notice that the coefficients of the derivatives of $f$ correspond to the entries in the rows of Pascal's Triangle.


Pascal's Triangle

- In the first derivative of $g$, the coefficients of $f$ and $f^{\prime}$ correspond to the entries in the second row of Pascal's Triangle.
- In the second derivative of $g$, the coefficients of $f, f^{\prime}$, and $f^{\prime \prime}$ correspond to the entries in the third row of Pascal's Triangle.
- In the third derivative of $g$, the coefficients of $f, f^{\prime}, f^{\prime \prime}$, and $f^{\prime \prime \prime}$ correspond to the entries in the fourth row of Pascal's Triangle.
- We guess that, in the fourth derivative of $g$, the coefficients of $f, f^{\prime}, f^{\prime \prime}, f^{\prime \prime \prime}$, and $f^{(4)}$ will correspond to the entries in the fifth row of Pascal's Triangle.

That is, we guess

$$
g^{(4)}(x)=\left[f(x)+4 f^{\prime}(x)+6 f^{\prime \prime}(x)+4 f^{\prime \prime \prime}(x)+f^{(4)}(x)\right] e^{x}
$$

This is verified by differentiating our answer from 2.14.2.19.a using the product rule:

$$
\begin{aligned}
g^{\prime \prime \prime}(x)= & {\left[f(x)+3 f^{\prime}(x)+3 f^{\prime \prime}(x)+f^{\prime \prime \prime}(x)\right] e^{x} } \\
g^{(4)}(x)= & {\left[f^{\prime}(x)+3 f^{\prime \prime}(x)+3 f^{\prime \prime \prime}(x)+f^{(4)}(x)\right] e^{x} } \\
& \quad+\left[f(x)+3 f^{\prime}(x)+3 f^{\prime \prime}(x)+f^{\prime \prime \prime}(x)\right] e^{x} \\
= & {\left[f(x)+4 f^{\prime}(x)+6 f^{\prime \prime}(x)+4 f^{\prime \prime \prime}(x)+f^{(4)}(x)\right] e^{x} . }
\end{aligned}
$$

2.14.2.20. Solution. Since $f(x)$ is differentiable over all real numbers, it is also continuous over all real numbers. Similarly, $f^{\prime}(x)$ is differentiable over all real numbers, so it is also continuous over all real numbers, and so on for the first $n$ derivatives of $f(x)$.
Rolle's Theorem tells us that if $a$ and $b$ are distinct roots of a function $g$, then $g^{\prime}(x)=0$ for some $c$ in $(a, b)$. That is, $g^{\prime}$ has a root strictly between $a$ and $b$. Expanding this idea, if $g$ has $m+1$ distinct roots, then $g^{\prime}$ must have at least $m$ distinct roots, as in the sketch below.


So, if $f^{(n)}(x)$ has only $m$ roots, then $f^{(n-1)}(x)$ has at most $m+1$ roots. Similarly, since $f^{(n-1)}(x)$ has at most $m+1$ roots, $f^{(n-2)}(x)$ has at most $m+2$ roots. Continuing in this way, we see $f(x)=f^{(n-n)}(x)$ has at most $m+n$ distinct roots.

### 2.14.2.21. Solution.

- Let's begin by noticing that the domain of $f(x)$ is $(-1, \infty)$.
- By inspection, $f(0)=0$, so $f(x)$ has at least one root.
- If $x \in(-1,0)$, then $(x+1)$ is positive, $\log (x+1)$ is negative, $\sin (x)$ is negative, and $-x^{2}$ is negative. Therefore, if $x<0$ is in the domain of $f$, then $f(x)<0$. So, $f(x)$ has no negative roots. We focus our attention on the case $x>0$.
- $f^{\prime}(x)=1-2 x+\log (x+1)+\cos x$. We would like to know how many positive roots $f^{\prime}(x)$ has, but it isn't obvious. So, let's differentiate again.
- $f^{\prime \prime}(x)=-2+\frac{1}{x+1}-\sin x$. When $x>0, \frac{1}{x+1}<1$, so $f^{\prime \prime}(x)<-1-\sin (x) \leq 0$, so $f^{\prime \prime}(x)$ has no positive roots. Since $f^{\prime}(x)$ is continuous and differentiable over $(0, \infty)$, and since $f^{\prime \prime}(x) \neq 0$ for all $x \in(0, \infty)$, by Rolle's Theorem, $f^{\prime}(x)$ has at most one root in $[0, \infty)$.
- Since $f(x)$ is continuous and differentiable over $[0, \infty)$, and $f^{\prime}(x)$ has at most one root in $(0, \infty)$, by Rolle's Theorem $f(x)$ has at most two distinct roots in $[0, \infty)$. (Otherwise, $f(a)=f(b)=f(c)=0$ for some values $0 \leq a<b<c$, so $f^{\prime}(d)=f^{\prime}(e)=0$ for some $d \in(a, b)$ and some $e \in(b, c)$, but since $f^{\prime}(x)$ has at most one root, this is impossible.)
- We know $f(0)=0$, so the remaining question is whether or not $f(x)$ has a second root (which would have to be positive). As usual, we can show another root exists using the intermediate value theorem. We see that for large values of $x, f(x)$ is negative, for example:

$$
f(4)=5 \log 5+\sin (4)-(4)^{2}<5 \log \left(e^{2}\right)+1-16=11-16<0
$$

For positive values of $x$ closer to zero, we hope to find a positive value of $f(x)$. However, it's quite difficult to get a number $c$ that obviously gives $f(c)>0$. It suffices to observe that $f(0)=0$ and $f^{\prime}(0)=2>0$. From the definition of the derivative, we can conclude $f(x)>0$ for some $x>0$. (If it is not true that $f(x)>0$ for some $x>0$, then $f(x) \leq 0$ for all $x>0$. The definition of the derivative tells us that [since $f^{\prime}(0)$ exists] $f^{\prime}(0)=\lim _{h \rightarrow 0^{+}} \frac{f(h)-f(0)}{h}=$ $\lim _{h \rightarrow 0^{+}} \frac{f(h)}{h}$; the denominator is positive, so if the numerator were always less than or equal to zero, the limit would be less than or equal to zero as well. However, the derivative is positive, so $f(x)>0$ for some $x>0$.) Therefore, $f(x)$ has a second root, so $f(x)$ has precisely two roots.
2.14.2.22. *. Solution. 2.14 .2 .22 .a In order to make $f(x)$ a little more tractable, let's change the format. Since $|x|=\left\{\begin{array}{rr}x & x \geq 0 \\ -x & x<0\end{array}\right.$, then:

$$
f(x)=\left\{\begin{array}{rl}
-x^{2} & x<0 \\
x^{2} & x \geq 0
\end{array}\right.
$$

Now, we turn to the definition of the derivative to figure out whether $f^{\prime}(0)$ exists.

$$
f^{\prime}(0)=\lim _{h \rightarrow 0} \frac{f(0+h)-f(0)}{h}=\lim _{h \rightarrow 0} \frac{f(h)-0}{h}=\lim _{h \rightarrow 0} \frac{f(h)}{h} \quad \text { if it exists. }
$$

Since $f$ looks different to the left and right of 0 , in order to evaluate this limit, we look at the corresponding one-sided limits. Note that when $h$ approaches 0 from the right, $h>0$ so $f(h)=h^{2}$. By contrast, when $h$ approaches 0 from the left, $h<0$ so $f(h)=-h^{2}$.

$$
\begin{aligned}
\lim _{h \rightarrow 0^{+}} \frac{f(h)}{h} & =\lim _{h \rightarrow 0^{+}} \frac{h^{2}}{h}=\lim _{h \rightarrow 0^{+}} h=0 \\
\lim _{h \rightarrow 0^{-}} \frac{f(h)}{h} & =\lim _{h \rightarrow 0^{-}} \frac{-h^{2}}{h}=\lim _{h \rightarrow 0^{-}}-h=0
\end{aligned}
$$

Since both one-sided limits exist and are equal to 0 ,

$$
\lim _{h \rightarrow 0} \frac{f(0+h)-f(0)}{h}=0
$$

and so $f$ is differentiable at $x=0$ and $f^{\prime}(0)=0$.
2.14.2.22.b From 2.14.2.22.a, $f^{\prime}(0)=0$ and

$$
f(x)=\left\{\begin{array}{rl}
-x^{2} & x<0 \\
x^{2} & x \geq 0
\end{array}\right.
$$

So,

$$
f^{\prime}(x)=\left\{\begin{array}{rl}
-2 x & x<0 \\
2 x & x \geq 0
\end{array}\right.
$$

Then, we know the second derivative of $f$ everywhere except at $x=0$ :

$$
f^{\prime \prime}(x)=\left\{\begin{array}{cc}
-2 & x<0 \\
? ? & x=0 \\
2 & x>0
\end{array}\right.
$$

So, whenever $x \neq 0, f^{\prime \prime}(x)$ exists. To investigate the differentiability of $f^{\prime}(x)$ when $x=0$, again we turn to the definition of a derivative. If

$$
\lim _{h \rightarrow 0} \frac{f^{\prime}(0+h)-f^{\prime}(0)}{h}
$$

exists, then $f^{\prime \prime}(0)$ exists.

$$
\lim _{h \rightarrow 0} \frac{f^{\prime}(0+h)-f^{\prime}(0)}{h}=\lim _{h \rightarrow 0} \frac{f^{\prime}(h)-0}{h}=\lim _{h \rightarrow 0} \frac{f^{\prime}(h)}{h}
$$

Since $f(h)$ behaves differently when $h$ is greater than or less than zero, we look at the one-sided limits.

$$
\begin{aligned}
\lim _{h \rightarrow 0^{+}} \frac{f^{\prime}(h)}{h} & =\lim _{h \rightarrow 0^{+}} \frac{2 h}{h}=2 \\
\lim _{h \rightarrow 0^{-}} \frac{f^{\prime}(h)}{h} & =\lim _{h \rightarrow 0^{-}} \frac{-2 h}{h}=-2
\end{aligned}
$$

Since the one-sided limits do not agree,

$$
\lim _{h \rightarrow 0} \frac{f^{\prime}(0+h)-f^{\prime}(0)}{h}=D N E
$$

So, $f^{\prime \prime}(0)$ does not exist. Now we have a complete picture of $f^{\prime \prime}(x)$ :

$$
f^{\prime \prime}(x)= \begin{cases}-2 & x<0 \\ D N E & x=0 \\ 2 & x>0\end{cases}
$$

## 3 • Applications of derivatives

## 3.1 • Velocity and Acceleration

### 3.1.2 • Exercises

## Exercises - Stage 1

3.1.2.1. Solution. False. The acceleration of the ball is given by $h^{\prime \prime}(t)=-9.8$. This is constant throughout its trajectory (and is due to gravity).
Remark: the velocity of the ball at $t=2$ is zero, since $h^{\prime}(2)=-9.8(2)+19.6=0$, but the velocity is only zero for an instant. Since the velocity is changing, the acceleration is nonzero.
3.1.2.2. Solution. The acceleration is constant, which means the rate of change of the velocity is constant. So, since it took 10 seconds for the velocity to increase by 1 metre per second (from $1 \frac{\mathrm{~m}}{\mathrm{~s}}$ to $2 \frac{\mathrm{~m}}{\mathrm{~s}}$ ), then it always takes 10 seconds for the velocity to increase by 1 metre per second.
So, it takes 10 seconds to accelerate from $2 \frac{\mathrm{~m}}{\mathrm{~s}}$ to $3 \frac{\mathrm{~m}}{\mathrm{~s}}$. To accelerate from $3 \frac{\mathrm{~m}}{\mathrm{~s}}$ to $13 \frac{\mathrm{~m}}{\mathrm{~s}}$ (that is, to change its velocity by 10 metres per second), it takes $10 \times 10=100$ seconds.
3.1.2.3. Solution. Let $v(a)=s^{\prime}(a)$ be the velocity of the particle. If $s^{\prime \prime}(a)>0$, then $v^{\prime}(a)>0-$ so the velocity of the particle is increasing. However, that does not mean that its speed (the absolute value of velocity) is increasing as well. For example, if a velocity is increasing from -4 kph to -3 kph , the speed is decreasing from 4 kph to 3 kph . So, the statement is false in general.
Contrast this to Question 3.1.2.4.
3.1.2.4. Solution. Since $s^{\prime}(a)>0,\left|s^{\prime}(a)\right|=s^{\prime}(a)$ : that is, the speed and velocity of the particle are the same. (This means the particle is moving in the positive direction.) If $s^{\prime \prime}(a)>0$, then the velocity (and hence speed) of the particle is increasing. So, the statement is true.

## Exercises - Stage 2

3.1.2.5. Solution. From Example 3.1.2, we know that an object falling from rest on the Earth is subject to the acceleration due to gravity, $9.8 \frac{\mathrm{~m}}{\mathrm{~s}^{2}}$. So, if $h(t)$ is the height of the flower pot $t$ seconds after it rolls out the window, then $h^{\prime \prime}(t)=-9.8$. (We make the acceleration negative, since the measure "height" has "up" as the positive direction, while gravity pulls the pot in the negative direction, "down.") Then $h^{\prime}(t)$ is a function whose derivative is the constant -9.8 and with $h^{\prime}(0)=0$ (since the object fell, instead of being thrown up or down), so $h^{\prime}(t)=-9.8 t$.
What we want to know is $h^{\prime}(t)$ at the time $t$ when the pot hits the ground. We don't know yet exactly what time that happens, so we go a little farther and find an expression for $h(t)$. The function $h(t)$ has derivative $-9.8 t$ and $h(0)=10$, so (again following the ideas in Example 3.1.2)

$$
h(t)=\frac{-9.8}{2} t^{2}+10
$$

Now, we can find the time when the pot hits the ground: it is the time when $h(t)=0$ (and $t>0$ ).

$$
\begin{aligned}
0 & =\frac{-9.8}{2} t^{2}+10 \\
\frac{9.8}{2} t^{2} & =10 \\
t^{2} & =\frac{20}{9.8} \\
t & =+\sqrt{\frac{20}{9.8}} \approx 1.4 \mathrm{sec}
\end{aligned}
$$

The velocity of the pot at this time is

$$
h^{\prime}\left(\sqrt{\frac{20}{9.8}}\right)=-9.8\left(\sqrt{\frac{20}{9.8}}\right)=-\sqrt{20 \cdot 9.8}=-14 \frac{\mathrm{~m}}{\mathrm{~s}}
$$

So, the pot is falling at 14 metres per second, just as it hits the ground.

### 3.1.2.6. Solution. (a)

- Let $s(t)$ be the distance the stone has fallen $t$ seconds after dropping it. Since the acceleration due to gravity is $9.8 \frac{\mathrm{~m}}{\mathrm{~s}^{2}}, s^{\prime \prime}(t)=9.8$. (We don't make this negative, because $s(t)$ measures how far the stone has fallen, which means the positive direction in our coordinate system is "down", which is exactly the way gravity is pulling.)
- Then $s^{\prime}(t)$ has a constant derivative of 9.8 , so $s^{\prime}(t)=9.8 t+c$ for some constant $c$. Notice $s^{\prime}(0)=c$, so $c$ is the velocity of the stone at the very instant you dropped it, which is zero. Therefore, $s^{\prime}(t)=9.8 t$.
- So, $s(t)$ is a function with derivative $9.8 t$. It's not too hard to figure out by guessing and checking that $s(t)=\frac{9.8}{2} t^{2}+d$ for some constant $d$. Notice $s(0)=d$, so $d$ is the distance the rock has travelled at the instant you dropped it, which is zero. So, $s(t)=\frac{9.8}{2} t^{2}=4.9 t^{2}$.
Remark: this is exactly the formula found in Example 3.1.2. You may, in general, use that formula without proof, but you need to know where it comes from and be able to apply it in other circumstances where it might be slightly different-like part (b) below.
- The rock falls for $x$ seconds, so the distance fallen is

$$
4.9 x^{2}
$$

Remark: this is a decent (if imperfect) way to figure out how deep a well is, or how tall a cliff is, when you're out and about. Drop a rock, square the time, multiply by 5 .
(b) We'll go through a similar process as before.

Again, let $s(t)$ be the distance the rock has fallen $t$ seconds after it is let go. Then $s^{\prime \prime}(t)=9.8$, so $s^{\prime}(t)=9.8 t+c$. In this case, since the initial speed of the rock is 1 metre per second, $1=s^{\prime}(0)=c$, so $s^{\prime}(t)=9.8 t+1$.
Then, $s(t)$ is a function whose derivative is $9.8 t+1$, so $s(t)=\frac{9.8}{2} t^{2}+t+d$ for some constant $d$. Since $0=s(0)=d$, we see $s(t)=4.9 t^{2}+t$.
So, if the rock falls for $x$ seconds, the distance fallen is

$$
4.9 x^{2}+x
$$

Remark: This means there is an error of $x$ metres in your estimation of the depth of the well.
3.1.2.7. Solution. Let $s(t)$ be the distance your keys have travelled since they left your hand. The rate at which they are travelling, $s^{\prime}(t)$, is decreasing by 0.25 metres per second. That is, $s^{\prime \prime}(t)=-0.25$. Therefore, $s^{\prime}(t)=-0.25 t+c$ for some constant $c$. Since $c=s^{\prime}(0)=2$, we see

$$
s^{\prime}(t)=2-0.25 t
$$

Then $s(t)$ has $2-0.25 t$ as its derivative, so $s(t)=2 t-\frac{1}{8} t^{2}+d$ for some constant $d$. At time $t=0$, the keys have not yet gone anywhere, so $0=s(0)=d$. Therefore,

$$
s(t)=2 t-\frac{1}{8} t^{2}
$$

The keys reach your friend when $s(t)=2$ and $t>0$. That is:

$$
\begin{aligned}
& 2=2 t-\frac{1}{8} t^{2} \\
& 0=\frac{1}{8} t^{2}-2 t+2 \\
& t=8 \pm 4 \sqrt{3}
\end{aligned}
$$

We need to figure out which of these values of $t$ is really the time when the keys reach your friend. The keys travel this way from $t=0$ to the time they reach your friend. (Then $s(t)$ no longer describes their motion.) So, we need to find the first value of $t$ that is positive with $s(t)=2$. Since $8-4 \sqrt{3}>0$, this is the first time $s(t)=2$ and $t>0$. So, the keys take

$$
8-4 \sqrt{3} \approx 1 \text { second }
$$

to reach your friend.
3.1.2.8. Solution. We proceed with the technique of Example 3.1.3 in mind.

Let $v(t)$ be the velocity (in kph ) of the car at time $t$, where $t$ is measured in hours and $t=0$ is the instant the brakes are applied. Then $v(0)=100$ and $v^{\prime}(t)=-50000$. Since $v^{\prime}(t)$ is constant, $v(t)$ is a line with slope -50000 and intercept $(0,100)$, so

$$
v(t)=100-50000 t
$$

The car comes to a complete stop when $v(t)=0$, which occurs at $t=\frac{100}{50000}=\frac{1}{500}$ hours. This is a confusing measure, so we convert it to seconds:

$$
\left(\frac{1}{500} \mathrm{hrs}\right)\left(\frac{3600 \mathrm{sec}}{1 \mathrm{hr}}\right)=7.2 \mathrm{sec}
$$

3.1.2.9. Solution. Suppose the deceleration provided by the brakes is $d \frac{\mathrm{~km}}{\mathrm{hr}^{2}}$. Then if $v(t)$ is the velocity of the car, $v(t)=120-d t$ (at $t=0$, the velocity is 120 , and it decreases by $d \mathrm{kph}$ per hour). The car stops when $0=v(t)$, so $t=\frac{120}{d}$ hours. Let $s(t)$ be the distance the car has travelled $t$ hours after applying the brakes.

Then $s^{\prime}(t)=v(t)$, so $s(t)=120 t-\frac{d}{2} t^{2}+c$ for some constant $c$. Since $0=s(0)=c$,

$$
s(t)=120 t-\frac{d}{2} t^{2}
$$

The car needs to stop in 100 metres, which is $\frac{1}{10}$ kilometres. We already found that the stopping time is $t=\frac{120}{d}$. So:

$$
\begin{aligned}
& \frac{1}{10}=s\left(\frac{120}{d}\right) \\
& \frac{1}{10}=120\left(\frac{120}{d}\right)-\frac{d}{2}\left(\frac{120}{d}\right)^{2}
\end{aligned}
$$

Multiplying both sides by $d$ :

$$
\begin{aligned}
\frac{d}{10} & =120^{2}-\frac{120^{2}}{2} \\
d & =5 \cdot 120^{2}
\end{aligned}
$$

So, the brakes need to apply 72000 kph per hour of deceleration.
3.1.2.10. Solution. Since your deceleration is constant, your speed decreases smoothly from 100 kph to 0 kph . So, one second before your stop, you only have $\frac{1}{7}$ of our speed left: you're going $\frac{100}{7} \mathrm{kph}$.
A less direct way to solve this problem is to note that $v(t)=100-d t$ is the velocity of car $t$ hours after braking, if $d$ is its deceleration. Since it stops in 7 seconds (or $\frac{7}{3600}$ hours), $0=v\left(\frac{7}{3600}\right)=100-\frac{7}{3600} d$, so $d=\frac{360000}{7}$. Then

$$
v\left(\frac{6}{3600}\right)=100-\left(\frac{360000}{7}\right)\left(\frac{6}{3600}\right)=100-\frac{6}{7} \cdot 100=\frac{100}{7} \mathrm{kph}
$$

3.1.2.11. Solution. If the acceleration was constant, then it was

$$
\frac{17500 \mathrm{mph}}{\frac{8.5}{60} \mathrm{hr}} \approx 123500 \frac{\mathrm{miles}}{\mathrm{hr}^{2}}
$$

So, the velocity $t$ hours from liftoff is

$$
v(t)=123500 t
$$

Therefore, the position of the shuttle $t$ hours from liftoff (taking $s(0)=0$ to be its initial position) is

$$
s(t)=\frac{123500}{2} t^{2}=61750 t^{2}
$$

So, after $\frac{8.5}{60}$ hours, the shuttle has travelled

$$
s\left(\frac{8.5}{60}\right)=(61750)\left(\frac{8.5}{60}\right)^{2} \approx 1240 \text { miles }
$$

or a little less than 2000 kilometres.
3.1.2.12. Solution. We know that the acceleration of the ball will be constant. If the height of the ball is given by $h(t)$ while it is in the air, $h^{\prime \prime}(t)=-9.8$. (The negative indicates that the velocity is decreasing: the ball starts at its largest velocity, moving in the positive direction, then the velocity decreases to zero and then to a negative number as the ball falls.) As in Example 3.1.2, we need a function $h(t)$ with $h^{\prime \prime}(t)=-9.8$. Since this is a constant, $h^{\prime}(t)$ is a line with slope -9.8 , so it has the form

$$
h^{\prime}(t)=-9.8 t+a
$$

for some constant $a$. Notice when $t=0, h^{\prime}(0)=a$, so in fact $a$ is the initial velocity of the ball-the quantity we want to solve for.
Again, as in Example 3.1.2, we need a function $h(t)$ with $h^{\prime}(t)=-9.8 t+a$. Such a function must have the form

$$
h(t)=-4.9 t^{2}+a t+b
$$

for some constant $b$. You can find this by guessing and checking, or simply remember it from the text. (In Section 4.1, you'll learn more about figuring out which functions have a particular derivative.) Notice when $t=0, h(0)=b$, so $b$ is the initial height of the baseball, which is 0 .
So, $h(t)=-4.9 t^{2}+a t=t(-4.9 t+a)$. The baseball is at height zero when it is pitched $(t=0)$ and when it hits the ground (which we want to be $t=10$ ). So, we want $(-4.9)(10)+a=0$. That is, $a=49$. So, the initial pitch should be at 49 metres per second.
Incidentally, this is on par with the fastest pitch in baseball, as recorded by Guiness World Records: https://www.guinnessworldrecords.com/world-records/ fastest-baseball-pitch-(male)
3.1.2.13. Solution. The acceleration of a falling object due to gravity is 9.8 metres per second squared. So, the object's velocity $t$ seconds after being dropped is

$$
v(t)=9.8 t \frac{\mathrm{~m}}{\mathrm{~s}}
$$

We want $v(t)$ to be the speed of the peregrine's dive, so we should convert that to metres per second:

$$
325 \frac{\mathrm{~km}}{\mathrm{hr}} \cdot\left(\frac{1000 \mathrm{~m}}{1 \mathrm{~km}}\right)\left(\frac{1 \mathrm{hr}}{3600 \mathrm{sec}}\right)=\frac{1625}{18} \frac{\mathrm{~m}}{\mathrm{~s}}
$$

The stone will reach this velocity when

$$
9.8 t=\frac{1625}{18} \quad \Rightarrow \quad t=\frac{1625}{18(9.8)}
$$

What is left to figure out is how far the stone will fall in this time. The position of the stone $s(t)$ has derivative $9.8 t$, so

$$
s(t)=4.9 t^{2}
$$

if we take $s(0)=0$. So, if the stone falls for $\frac{1625}{18(9.8)}$ seconds, in that time it travels

$$
s\left(\frac{1625}{18(9.8)}\right)=4.9\left(\frac{1625}{18(9.8)}\right)^{2} \approx 416 \mathrm{~m}
$$

So, you would have to drop a stone from about 416 metres for it to fall as fast as the falcon.
3.1.2.14. Solution. Since gravity alone brings your cannon ball down, its acceleration is a constant $-9.8 \frac{\mathrm{~m}}{\mathrm{~s}^{2}}$. So, $v(t)=v_{0}-9.8 t$ and thus its height is given by $s(t)=v_{0} t-4.9 t^{2}$ (if we set $s(0)=0$ ).
We want to know what value of $v_{0}$ makes the maximum height 100 metres. The maximum height is reached when $v(t)=0$, which is at time $t=\frac{v_{0}}{9.8}$. So, we solve:

$$
\begin{aligned}
100 & =s\left(\frac{v_{0}}{9.8}\right) \\
100 & =v_{0}\left(\frac{v_{0}}{9.8}\right)-4.9\left(\frac{v_{0}}{9.8}\right)^{2} \\
100 & =\left(\frac{1}{9.8}-\frac{4.9}{9.8^{2}}\right) v_{0}^{2} \\
100 & =\frac{1}{2 \cdot 9.8} v_{0}^{2} \\
v_{0}^{2} & =1960 \\
v_{0} & =\sqrt{1960} \approx 44 \frac{\mathrm{~m}}{\mathrm{~s}}
\end{aligned}
$$

where we choose the positive square root because $v_{0}$ must be positive for the cannon ball to get off the ground.
3.1.2.15. Solution. The derivative of acceleration is constant, so the acceleration $a(t)$ has the form $m t+b$. We know $a(0)=-50000$ and $a\left(\frac{3}{3600}\right)=-60000$ (where we note that 3 seconds is $\frac{3}{3600}$ hours). So, the slope of $a(t)$ is $\frac{-60000+50000}{\frac{3}{3600}}=-12000000$, which leads us to

$$
a(t)=-50000-(12000000) t
$$

where $t$ is measured in hours.
Since $v^{\prime}(t)=a(t)=-50000-(12000000) t$, we see

$$
v(t)=\frac{-12000000}{2} t^{2}-50000 t+c=-6000000 t^{2}-50000 t+c
$$

for some constant $c$. Since $120=v(0)=c$ :

$$
v(t)=-6000000 t^{2}-50000 t+120
$$

Then after three seconds of braking,

$$
\begin{aligned}
v\left(\frac{3}{3600}\right) & =-6000000\left(\frac{3}{3600}\right)^{2}-50000\left(\frac{3}{3600}\right)+120 \\
& =-\frac{25}{6}-\frac{125}{3}+120
\end{aligned}
$$

$$
\approx 74.2 \mathrm{kph}
$$

Remark: When acceleration is constant, the position function is a quadratic function, but we don't want you to get the idea that position functions are always quadratic functions-in the example you just did, it was the velocity function that was quadratic. Position, velocity, and acceleration functions don't have to be polynomial at all-it's only in this section, where we're dealing with the simplest cases, that they seem that way.

## Exercises - Stage 3

3.1.2.16. Solution. Different forces are acting on you (1) after you jump but before you land on the trampoline, and (2) while you are falling into the trampoline. In both instances, the acceleration is constant, so both height functions are quadratic, of the form $\frac{a}{2} t^{2}+v t+h$, where $a$ is the acceleration, $v$ is the velocity when $t=0$, and $h$ is the initial height.

- Let's consider (1) first, the time during your jump before your feet touch the trampoline. Let $t=0$ be the moment you jump, and let the rim of the trampoline be height 0 . Then, since your initial velocity was (positive) 1 meter per second, your height is given by

$$
h_{1}(t)=\frac{-9.8}{2} t^{2}+t=t\left(\frac{-9.8}{2} t+1\right)
$$

Notice that, because your acceleration is working against your positive velocity, it has a negative sign.

- We'll need to know your velocity when your feet first touch the trampoline on your fall. The time your feet first first touch the trampoline after your jump is precisely when $h_{1}(t)=0$ and $t>0$. That is, when $t=\frac{2}{9.8}$. Now, since $h^{\prime}(t)=-9.8 t+1, h^{\prime}\left(\frac{2}{9.8}\right)=-9.8\left(\frac{2}{9.8}\right)+1=-1$. So, you are descending at a rate of 1 metre per second at the instant your feet touch the trampoline.
Remark: it is not only coincidence that this was your initial speed. Think about the symmetries of parabolas, and conservation of energy.
- Now we need to think about your height as the trampoline is slowing your fall. One thing to remember about our general equation $\frac{a}{2} t^{2}+v t+h$ is that $v$ is the velocity when $t=0$. But, you don't hit the trampoline at $t=0$, you hit it at $t=\frac{2}{9.8}$. In order to keep things simple, let's use a different time scale for this second part of your journey. Let's let $h_{2}(T)$ be your height at time $T$, from the moment your feet touch the trampoline skin $(T=0)$ to the bottom of your fall. Now, we can use the fact that your initial velocity is -1 metres per second (negative, since your height is decreasing) and your acceleration is 4.9 metres per second per second (positive, since your velocity is increasing from a negative number to zero):

$$
h_{2}(T)=\frac{4.9}{2} T^{2}-T
$$

where still the height of the rim of the trampoline is taken to be zero.
Remark: if it seems very confusing that your free-falling acceleration is negative, while your acceleration in the trampoline is positive, remember that gravity is pushing you down, but the trampoline is pushing you up.

- How long were you falling in the trampoline? The equation $h_{2}(T)$ tells you your height only as long as the trampoline is slowing your fall. You reach the bottom of your fall when your velocity is zero.

$$
h_{2}^{\prime}(T)=4.9 T-1
$$

so you reach the bottom of your fall at $T=\frac{1}{4.9}$. Be careful: this is $\frac{1}{4.9}$ seconds after you entered the trampoline, not after the peak of your fall, or after you jumped.

- The last piece of the puzzle is how long it took you to fall from the peak of your jump to the surface of the trampoline. We know the equation of your motion during that time: $h_{1}(t)=\frac{-9.8}{2} t^{2}+t$. You reached the peak when your velocity was zero:

$$
h_{1}^{\prime}(t)=-9.8 t+1=0 \quad \Rightarrow \quad t=\frac{1}{9.8}
$$

So, you fell from your peak at $t=\frac{1}{9.8}$ and reached the level of the trampoline rim at $t=\frac{2}{1.98}$, which means the fall took $\frac{1}{9.8}$ seconds.
Remark: by the symmetry mentioned early, the time it took to fall from the peak of your jump to the surface of the trampoline is the same as one-half the time from the moment you jumped off the rim to the moment you're back on the surface of the trampoline.

- So, your time falling from the peak of your jump to its bottom was $\frac{1}{9.8}+\frac{1}{4.9} \approx$ 0.3 seconds.
3.1.2.17. Solution. Let $v(t)$ be the velocity of the object. From the given information:
- $v(0)$ is some value, call it $v_{0}$,
- $v(1)=2 v_{0}$ (since the speed doubled in the first second),
- $v(2)=2(2) v_{0}$ (since the speed doubled in the second second),
- $v(3)=2(2)(2) v_{0}$, and so on.

So, for general $t$ :

$$
v(t)=2^{t} v(0)
$$

To find its acceleration, we simply differentiate. Recall $\frac{\mathrm{d}}{\mathrm{d} x}\left\{2^{x}\right\}=2^{x} \log 2$, where $\log$ denotes logarithm base $e$.

$$
a(t)=2^{t} v_{0} \log 2
$$

Remark: we can also write $a(t)=v(t) \log 2$. The acceleration doubles every second as well.

## 3.2 - Related Rates

### 3.2.2 • Exercises

## Exercises - Stage 1

3.2.2.1. Solution. We have an equation relating $P$ and $Q$ :

$$
P=Q^{3}
$$

We differentiate implicitly with respect to a third variable, $t$ :

$$
\frac{\mathrm{d} P}{\mathrm{~d} t}=3 Q^{2} \cdot \frac{\mathrm{~d} Q}{\mathrm{~d} t}
$$

If we know two of the three quantities $\frac{\mathrm{d} P}{\mathrm{~d} t}, Q$, and $\frac{\mathrm{d} Q}{\mathrm{~d} t}$, then we can find the third. Therefore, ii is a question we can solve. If we know $P$, then we also know $Q$ (it's just the cube root of $P$ ), so also we can solve iv. However, if we know neither $P$ nor $Q$, then we can't find $\frac{\mathrm{d} P}{\mathrm{~d} t}$ based only off $\frac{\mathrm{d} Q}{\mathrm{~d} t}$, and we can't find $\frac{\mathrm{d} Q}{\mathrm{~d} t}$ based only off $\frac{\mathrm{d} P}{\mathrm{~d} t}$. So we can't solve i or iii.

## Exercises - Stage 2

3.2.2.2. *. Solution. Suppose that at time $t$, the point is at $(x(t), y(t))$. Then $x(t)^{2}+y(t)^{2}=1$ so that $2 x(t) x^{\prime}(t)+2 y(t) y^{\prime}(t)=0$. We are told that at some time $t_{0}, x\left(t_{0}\right)=2 / \sqrt{5}, y\left(t_{0}\right)=1 / \sqrt{5}$ and $y^{\prime}\left(t_{0}\right)=3$. Then

$$
\begin{aligned}
2 x\left(t_{0}\right) x^{\prime}\left(t_{0}\right)+2 y\left(t_{0}\right) y^{\prime}\left(t_{0}\right) & =0 & \Rightarrow \\
2\left(\frac{2}{\sqrt{5}}\right) x^{\prime}(t)+2\left(\frac{1}{\sqrt{5}}\right)(3) & =0 & \Rightarrow \\
x^{\prime}\left(t_{0}\right) & =-\frac{3}{2} &
\end{aligned}
$$

3.2.2.3. *. Solution. The instantaneous percentage rate of change for $R$ is

$$
\begin{array}{rlr}
100 \frac{R^{\prime}}{R} & =100 \frac{(P Q)^{\prime}}{P Q} & \mathrm{R}=\mathrm{PQ} \\
& =100 \frac{P^{\prime} Q+P Q^{\prime}}{P Q} & \text { product rule } \\
& =100\left[\frac{P^{\prime}}{P}+\frac{Q^{\prime}}{Q}\right] & \text { simplify } \\
& =100[0.08-0.02]=6 \% &
\end{array}
$$

3.2.2.4. *. Solution. 3.2.2.4.a By the quotient rule, $F^{\prime}=\frac{P^{\prime} Q-P Q^{\prime}}{Q^{2}}$. At the moment in question, $F^{\prime}=\frac{5 \times 5-25 \times 1}{5^{2}}=0$.
3.2.2.4.b We are told that, at the second moment in time, $P^{\prime}=0.1 P$ and $Q^{\prime}=$ $-0.05 Q$ (or equivalently $100 \frac{P^{\prime}}{P}=10$ and $100 \frac{Q^{\prime}}{Q}=-5$ ). Substituting in these values:

$$
\begin{aligned}
F^{\prime} & =\frac{P^{\prime} Q-P Q^{\prime}}{Q^{2}} \\
& =\frac{0.1 P Q-P(-0.05 Q)}{Q^{2}} \\
& =\frac{0.15 P Q}{Q^{2}} \\
& =0.15 \frac{P}{Q} \\
& =0.15 F \\
\Longrightarrow \quad F^{\prime} & =0.15 F
\end{aligned}
$$

or $100 \frac{F^{\prime}}{F}=15 \%$. That is, the instantaneous percentage rate of change of $F$ is $15 \%$.

## 3.2 .2 .5 . *. Solution.

- The distance $z(t)$ between the particles at any moment in time is

$$
z^{2}(t)=x(t)^{2}+y(t)^{2}
$$

where $x(t)$ is the position on the $x$-axis of the particle A at time $t$ (measured in seconds) and $y(t)$ is the position on the $y$-axis of the particle B at the same time $t$.

- We differentiate the above equation with respect to $t$ and get

$$
2 z \cdot z^{\prime}=2 x \cdot x^{\prime}+2 y \cdot y^{\prime}
$$

- We are told that $x^{\prime}=-2$ and $y^{\prime}=-3$. (The values are negative because $x$ and $y$ are decreasing.) It will take 3 seconds for particle $A$ to reach $x=4$, and in this time particle $B$ will reach $y=3$.
- At this point $z=\sqrt{x^{2}+y^{2}}=\sqrt{3^{2}+4^{2}}=5$.
- Hence

$$
\begin{aligned}
10 z^{\prime} & =8 \cdot(-2)+6 \cdot(-3)=-34 \\
z^{\prime} & =-\frac{34}{10}=-\frac{17}{5} \text { units per second. }
\end{aligned}
$$

### 3.2.2.6. *. Solution.

- We compute the distance $z(t)$ between the two particles after $t$ seconds as

$$
z^{2}(t)=3^{2}+\left(y_{A}(t)-y_{B}(t)\right)^{2},
$$

where $y_{A}(t)$ and $y_{B}(t)$ are the $y$-coordinates of particles $A$ and $B$ after $t$ seconds, and the horizontal distance between the two particles is always 3 units.

- We are told the distance between the particles is 5 units, this happens when

$$
\begin{aligned}
\left(y_{A}-y_{B}\right)^{2} & =5^{2}-3^{2}=16 \\
y_{A}-y_{B} & =4
\end{aligned}
$$

That is, when the difference in $y$-coordinates is 4 . This happens when $t=4$.

- We differentiate the distance equation (from the first bullet point) with respect to $t$ and get

$$
2 z \cdot z^{\prime}=2\left(y_{A}^{\prime}-y_{B}^{\prime}\right)\left(y_{A}-y_{B}\right),
$$

- We know that $\left(y_{A}-y_{B}\right)=4$, and we are told that $z=5, y_{A}^{\prime}=3$, and $y_{B}^{\prime}=2$. Hence

$$
10 z^{\prime}(4)=2 \times 1 \times 4=8
$$

- Therefore

$$
z^{\prime}(4)=\frac{8}{10}=\frac{4}{5} \text { units per second. }
$$

### 3.2.2.7. *. Solution.



As in the above figure, let $x(t)$ be the distance between H (Hawaii) and ship B , and $y(t)$ be the distance between H and ship A , and $z(t)$ be the distance between ships A and B , all at time $t$. Then

$$
x(t)^{2}+y(t)^{2}=z(t)^{2}
$$

Differentiating with respect to $t$,

$$
\begin{aligned}
2 x(t) x^{\prime}(t)+2 y(t) y^{\prime}(t) & =2 z(t) z^{\prime}(t) \\
x(t) x^{\prime}(t)+y(t) y^{\prime}(t) & =z(t) z^{\prime}(t)
\end{aligned}
$$

At the specified time, $x(t)$ is decreasing, so $x^{\prime}(t)$ is negative, and $y(t)$ is increasing, so $y^{\prime}(t)$ is positive.

$$
\begin{aligned}
(300)(-15)+(400)(20) & =\sqrt{300^{2}+400^{2}} z^{\prime}(t) \\
500 z^{\prime}(t) & =3500 \\
z^{\prime}(t) & =7 \mathrm{mph}
\end{aligned}
$$

### 3.2.2.8. *. Solution.

- We compute the distance $d(t)$ between the two snails after $t$ minutes as

$$
d^{2}(t)=30^{2}+\left(y_{1}(t)-y_{2}(t)\right)^{2}
$$

where $y_{1}(t)$ is the altitude of the first snail, and $y_{2}(t)$ the altitude of the second snail after $t$ minutes.

- We differentiate the above equation with respect to $t$ and get

$$
\begin{aligned}
2 d \cdot d^{\prime} & =2\left(y_{1}^{\prime}-y_{2}{ }^{\prime}\right)\left(y_{1}-y_{2}\right) \\
d \cdot d^{\prime} & =\left(y_{1}^{\prime}-y_{2}^{\prime}\right)\left(y_{1}-y_{2}\right)
\end{aligned}
$$

- We are told that $y_{1}{ }^{\prime}=25$ and $y_{2}{ }^{\prime}=15$. It will take 4 minutes for the first snail to reach $y_{1}=100$, and in this time the second snail will reach $y_{2}=60$.
- At this point $d^{2}=30^{2}+(100-60)^{2}=900+1600=2500$, hence $d=50$.
- Therefore

$$
\begin{aligned}
50 d^{\prime} & =(25-15) \times(100-60) \\
d^{\prime} & =\frac{400}{50}=8 \mathrm{~cm} \text { per minute } .
\end{aligned}
$$

### 3.2.2.9. *. Solution.

- If we write $z(t)$ for the length of the ladder at time $t$ and $y(t)$ for the height of the top end of the ladder at time $t$ we have

$$
z(t)^{2}=5^{2}+y(t)^{2}
$$

- We differentiate the above equation with respect to $t$ and get

$$
2 z \cdot z^{\prime}=2 y \cdot y^{\prime}
$$

- We are told that $z^{\prime}(t)=-2$, so $z(3.5)=20-3.5 \cdot 2=13$.
- At this point $y=\sqrt{z^{2}-5^{2}}=\sqrt{169-25}=\sqrt{144}=12$.
- Hence

$$
\begin{aligned}
2 \cdot 13 \cdot(-2) & =2 \cdot 12 y^{\prime} \\
y^{\prime} & =-\frac{2 \cdot 13}{12}=-\frac{13}{6} \text { meters per second. }
\end{aligned}
$$

3.2.2.10. Solution. What we're given is $\frac{\mathrm{d} V}{\mathrm{~d} t}$ (where $V$ is volume of water in the trough, and $t$ is time), and what we are asked for is $\frac{\mathrm{d} h}{\mathrm{~d} t}$ (where $h$ is the height of the water). So, we need an equation relating $V$ and $h$. First, let's get everything in the same units: centimetres.


We can calculate the volume of water in the trough by multiplying the area of its trapezoidal cross section by 200 cm . A trapezoid with height $h$ and bases $b_{1}$ and $b_{2}$ has area $h\left(\frac{b_{1}+b_{2}}{2}\right)$. (To see why this is so, draw the trapezoid as a rectangle flanked by two triangles.) So, using $w$ as the width of the top of the water (as in the diagram above), the area of the cross section of the water in the trough is

$$
A=h\left(\frac{60+w}{2}\right)
$$

and therefore the volume of water in the trough is

$$
V=100 h(60+w) \mathrm{cm}^{3} .
$$

We need a formula for $w$ in terms of $h$. If we draw lines straight up from the bottom corners of the trapezoid, we break it into rectangles and triangles.


Using similar triangles, $\frac{a}{h}=\frac{20}{50}$, so $a=\frac{2}{5} h$. Then

$$
\begin{aligned}
w & =60+2 a \\
& =60+2\left(\frac{2}{5} h\right)=60+\frac{4}{5} h
\end{aligned}
$$

so

$$
\begin{aligned}
V & =100 h(60+w) \\
& =100 h\left(120+\frac{4}{5} h\right) \\
& =80 h^{2}+12000 h
\end{aligned}
$$

This is the equation we need, relating $V$ and $h$. Differentiating implicitly with respect to $t$ :

$$
\begin{aligned}
\frac{\mathrm{d} V}{\mathrm{~d} t} & =2 \cdot 80 h \cdot \frac{\mathrm{~d} h}{\mathrm{~d} t}+12000 \frac{\mathrm{~d} h}{\mathrm{~d} t} \\
& =(160 h+12000) \frac{\mathrm{d} h}{\mathrm{~d} t}
\end{aligned}
$$

We are given that $h=25$ and $\frac{\mathrm{d} V}{\mathrm{~d} t}=3$ litres per minute. Converting to cubic centimetres, $\frac{\mathrm{d} V}{\mathrm{~d} t}=-3000$ cubic centimetres per minute. So:

$$
\begin{aligned}
-3000 & =(160 \cdot 25+12000) \frac{\mathrm{d} h}{\mathrm{~d} t} \\
\frac{\mathrm{~d} h}{\mathrm{~d} t} & =-\frac{3}{16}=-.1875 \frac{\mathrm{~cm}}{\mathrm{~min}}
\end{aligned}
$$

So, the water level is dropping at $\frac{3}{16}$ centimetres per minute.
3.2.2.11. Solution. If $V$ is the volume of the water in the tank, and $t$ is time, then we are given $\frac{\mathrm{d} V}{\mathrm{~d} t}$. What we want to know is $\frac{\mathrm{d} h}{\mathrm{~d} t}$, where $h$ is the height of the water in the tank. A reasonable plan is to find an equation relating $V$ and $h$, and differentiate it implicitly with respect to $t$.
Let's be a little careful about units. The volume of water in the tank is

$$
\text { (area of cross section of water }) \times(\text { length of tank })
$$

If we measure these values in metres (area in square metres, length in metres), then the volume is going to be in cubic metres. So, when we differentiate with respect to time, our units will be cubic metres per second. The water is flowing in at one litre per second, or 1000 cubic centimetres per second. So, we either have to measure our areas and distances in centimetres, or convert litres to cubic metres. We'll do the latter, but both are fine.
If we imagine one cubic metre as a cube, with each side of length 1 metre, then it's
easy to see the volume inside is $(100)^{3}=10^{6}$ cubic centimetres: it's the volume of a cube with each side of length 100 cm . Since a litre is $10^{3}$ cubic centimetres, and a cubic metre is $10^{6}$ cubic centimetres, one litre is $10^{-3}$ cubic metres. So, $\frac{\mathrm{d} V}{\mathrm{~d} t}=\frac{1}{10^{3}}$ cubic metres per second.
Let $h$ be the height of the water (in metres). We can figure out the area of the cross section by breaking it into three pieces: a triangle on the left, a rectangle in the middle, and a trapezoid on the right.


- The triangle on the left has height $h$ metres. Let its base be $a$ metres. It forms a similar triangle with the triangle whose height is 1.25 metres and width is 1 metre, so:

$$
\begin{aligned}
\frac{a}{h} & =\frac{1}{1.25} \\
a & =\frac{4}{5} h
\end{aligned}
$$

So, the area of the triangle on the left is

$$
\frac{1}{2} a h=\frac{2}{5} h^{2}
$$

- The rectangle in the middle has length 3 metres and height $h$ metres, so its area is $3 h$ square metres.
- The trapezoid on the right is a portion of a triangle with base 3 metres and height 1.25 metres. So, its area is

$$
\underbrace{\left(\frac{1}{2}(3)(1.25)\right)}_{\text {area of big triangle }}-\underbrace{\left(\frac{1}{2}(b)(1.25-h)\right)}_{\text {area of little triangle }}
$$

The little triangle (of base $b$ and height $1.25-h$ ) is formed by the air on the right side of the tank. It is a similar triangle to the triangle of base 3 and height 1.25 , so

$$
\begin{aligned}
\frac{b}{1.25-h} & =\frac{3}{1.25} \\
b & =\frac{3}{1.25}(1.25-h)
\end{aligned}
$$

So, the area of the trapezoid on the right is

$$
\begin{aligned}
& \frac{1}{2}(3)(1.25)-\frac{1}{2}\left(\frac{3}{1.25}\right)(1.25-h)(1.25-h) \\
& =3 h-\frac{6}{5} h^{2}
\end{aligned}
$$

So, the area $A$ of the cross section of the water is

$$
\begin{aligned}
A & =\underbrace{\frac{2}{5} h^{2}}_{\text {triangle }}+\underbrace{3 h}_{\text {rectangle }}+\underbrace{3 h-\frac{6}{5} h^{2}}_{\text {trapezoid }} \\
& =6 h-\frac{4}{5} h^{2}
\end{aligned}
$$

So, the volume of water is

$$
V=5\left(6 h-\frac{4}{5} h^{2}\right)=30 h-4 h^{2}
$$

Differentiating with respect to time, $t$ :

$$
\frac{\mathrm{d} V}{\mathrm{~d} t}=30 \frac{\mathrm{~d} h}{\mathrm{~d} t}-8 h \frac{\mathrm{~d} h}{\mathrm{~d} t}
$$

When $h=\frac{1}{10}$ metre, and $\frac{\mathrm{d} V}{\mathrm{~d} t}=\frac{1}{10^{3}}$ cubic metres per second,

$$
\begin{aligned}
\frac{1}{10^{3}} & =30 \frac{\mathrm{~d} h}{\mathrm{~d} t}-8\left(\frac{1}{10}\right) \frac{\mathrm{d} h}{\mathrm{~d} t} \\
\frac{\mathrm{~d} h}{\mathrm{~d} t} & =\frac{1}{29200} \text { metres per second }
\end{aligned}
$$

This is about 1 centimetre every five minutes. You might want a bigger hose.
3.2.2.12. Solution. Let $\theta$ be the angle of your head, where $\theta=0$ means you are looking straight ahead, and $\theta=\frac{\pi}{2}$ means you are looking straight up. We are interested in $\frac{\mathrm{d} \theta}{\mathrm{d} t}$, but we only have information about $h$. So, a reasonable plan is to find an equation relating $h$ and $\theta$, and differentiate with respect to time.


The right triangle formed by you, the rocket, and the rocket's original position has adjacent side (to $\theta$ ) length 2 km , and opposite side (to $\theta$ ) length $h(t)$ kilometres, so

$$
\tan \theta=\frac{h}{2}
$$

Differentiating with respect to $t$ :

$$
\begin{aligned}
\sec ^{2} \theta \cdot \frac{\mathrm{~d} \theta}{\mathrm{~d} t} & =\frac{1}{2} \frac{\mathrm{~d} h}{\mathrm{~d} t} \\
\frac{\mathrm{~d} \theta}{\mathrm{~d} t} & =\frac{1}{2} \cos ^{2} \theta \cdot \frac{\mathrm{~d} h}{\mathrm{~d} t}
\end{aligned}
$$

We know $\tan \theta=\frac{h}{2}$. We draw a right triangle with angle $\theta$ (filling in the sides using SOH CAH TOA and the Pythagorean theorem) to figure out $\cos \theta$ :


Using the triangle, $\cos \theta=\frac{2}{\sqrt{h^{2}+4}}$, so

$$
\begin{aligned}
\frac{\mathrm{d} \theta}{\mathrm{~d} t} & =\frac{1}{2}\left(\frac{2}{\sqrt{h^{2}+4}}\right)^{2} \cdot \frac{\mathrm{~d} h}{\mathrm{~d} t} \\
& =\left(\frac{2}{h^{2}+4}\right) \frac{\mathrm{d} h}{\mathrm{~d} t}
\end{aligned}
$$

So, the quantities we need to know one minute after liftoff (that is, when $t=\frac{1}{60}$ ) are $h\left(\frac{1}{60}\right)$ and $\frac{\mathrm{d} h}{\mathrm{~d} t}\left(\frac{1}{60}\right)$. Recall $h(t)=61750 t^{2}$.

$$
h\left(\frac{1}{60}\right)=\frac{61750}{3600}=\frac{1235}{72}
$$

$$
\begin{aligned}
\frac{\mathrm{d} h}{\mathrm{~d} t} & =2(61750) t \\
\frac{\mathrm{~d} h}{\mathrm{~d} t}\left(\frac{1}{60}\right) & =\frac{2(61750)}{60}=\frac{6175}{3}
\end{aligned}
$$

Returning to the equation $\frac{\mathrm{d} \theta}{\mathrm{d} t}=\left(\frac{2}{h^{2}+4}\right) \frac{\mathrm{d} h}{\mathrm{~d} t}$ :

$$
\frac{\mathrm{d} \theta}{\mathrm{~d} t}\left(\frac{1}{60}\right)=\left(\frac{2}{\left(\frac{1235}{72}\right)^{2}+4}\right)\left(\frac{6175}{3}\right) \approx 13.8 \frac{\mathrm{rad}}{\text { hour }} \approx 0.0038 \frac{\mathrm{rad}}{\mathrm{sec}}
$$

3.2.2.13. *. Solution. 3.2.2.13.a Let $x(t)$ be the distance of the train along the track at time $t$, measured from the point on the track nearest the camera. Let $z(t)$ be the distance from the camera to the train at time $t$.


Then $x^{\prime}(t)=2$ and at the time in question, $z(t)=1.3 \mathrm{~km}$ and $x(t)=\sqrt{1.3^{2}-0.5^{2}}=$ 1.2 km . So

$$
\begin{aligned}
z(t)^{2} & =x(t)^{2}+0.5^{2} \\
2 z(t) z^{\prime}(t) & =2 x(t) x^{\prime}(t) \\
2 \times 1.3 z^{\prime}(t) & =2 \times 1.2 \times 2 \\
z^{\prime}(t) & =\frac{2 \times 1.2}{1.3} \approx 1.85 \mathrm{~km} / \mathrm{min}
\end{aligned}
$$

3.2.2.13.b Let $\theta(t)$ be the angle shown at time $t$. Then

$$
\sin (\theta(t))=\frac{x(t)}{z(t)}
$$

Differentiating with respect to $t$ :

$$
\begin{aligned}
\theta^{\prime}(t) \cos (\theta(t)) & =\frac{x^{\prime}(t) z(t)-x(t) z^{\prime}(t)}{z(t)^{2}} \\
\theta^{\prime}(t) & =\frac{x^{\prime}(t) z(t)-x(t) z^{\prime}(t)}{z(t)^{2} \cos (\theta(t))}
\end{aligned}
$$

From our diagram, we see $\cos (\theta(t))=\frac{0.5}{z(t)}$, so:

$$
=2 \frac{x^{\prime}(t) z(t)-x(t) z^{\prime}(t)}{z(t)}
$$

Substituting in $x^{\prime}(t)=2, z(t)=1.3, x(t)=1.2$, and $z^{\prime}(t)=\frac{2 \times 1.2}{1.3}$ :

$$
\theta^{\prime}(t)=2 \frac{2 \times 1.3-1.2 \times \frac{2 \times 1.2}{1.3}}{1.3} \approx .592 \text { radians } / \mathrm{min}
$$

3.2.2.14. Solution. Let $\theta$ be the angle between the two hands.


The Law of Cosines (Appendix B.4.1) tells us that

$$
\begin{aligned}
& D^{2}=5^{2}+10^{2}-2 \cdot 5 \cdot 10 \cdot \cos \theta \\
& D^{2}=125-100 \cos \theta
\end{aligned}
$$

Differentiating with respect to time $t$,

$$
2 D \frac{\mathrm{~d} D}{\mathrm{~d} t}=100 \sin \theta \cdot \frac{\mathrm{~d} \theta}{\mathrm{~d} t}
$$

Our tasks now are to find $D, \theta$ and $\frac{\mathrm{d} \theta}{\mathrm{d} t}$ when the time is 4:00. At 4:00, the minute hand is straight up, and the hour hand is $\frac{4}{12}=\frac{1}{3}$ of the way around the clock, so $\theta=\frac{1}{3}(2 \pi)=\frac{2 \pi}{3}$ at 4:00. Then $D^{2}=125-100 \cos \left(\frac{2 \pi}{3}\right)=125-100\left(-\frac{1}{2}\right)=175$, so $D=\sqrt{175}=5 \sqrt{7}$ at 4:00.
To calculate $\frac{\mathrm{d} \theta}{\mathrm{d} t}$, remember that both hands are moving. The hour hand makes a full rotation every 12 hours, so its rotational speed is $\frac{2 \pi}{12}=\frac{\pi}{6}$ radians per hour. The hour hand is being chased by the minute hand. The minute hand makes a full rotation every hour, so its rotational speed is $\frac{2 \pi}{1}=2 \pi$ radians per hour. Therefore, the angle $\theta$ between the two hands is changing at a rate of

$$
\frac{\mathrm{d} \theta}{\mathrm{~d} t}=-\left(2 \pi-\frac{\pi}{6}\right)=\frac{-11 \pi}{6} \frac{\mathrm{rad}}{\mathrm{hr}}
$$

Now, we plug in $D, \theta$, and $\frac{\mathrm{d} \theta}{\mathrm{d} t}$ to find $\frac{\mathrm{d} D}{\mathrm{~d} t}$ :

$$
\begin{aligned}
2 D \frac{\mathrm{~d} D}{\mathrm{~d} t} & =100 \sin \theta \cdot \frac{\mathrm{~d} \theta}{\mathrm{~d} t} \\
2(5 \sqrt{7}) \frac{\mathrm{d} D}{\mathrm{~d} t} & =100 \sin \left(\frac{2 \pi}{3}\right)\left(\frac{-11 \pi}{6}\right) \\
10 \sqrt{7} \frac{\mathrm{~d} D}{\mathrm{~d} t} & =100\left(\frac{\sqrt{3}}{2}\right)\left(\frac{-11 \pi}{6}\right)=-\frac{275 \pi}{\sqrt{3}}
\end{aligned}
$$

$$
\frac{\mathrm{d} D}{\mathrm{~d} t}=\frac{-55 \sqrt{21} \pi}{42} \frac{\mathrm{~cm}}{\mathrm{hr}}
$$

So $D$ is decreasing at $\frac{55 \sqrt{21} \pi}{42} \approx 19$ centimetres per hour.
3.2.2.15. *. Solution. The area at time $t$ is the area of the outer circle minus the area of the inner circle:

$$
\begin{aligned}
A(t) & =\pi\left(R(t)^{2}-r(t)^{2}\right) \\
\text { So, } A^{\prime}(t) & =2 \pi\left(R(t) R^{\prime}(t)-r(t) r^{\prime}(t)\right)
\end{aligned}
$$

Plugging in the given data,

$$
A^{\prime}=2 \pi(3 \cdot 2-1 \cdot 7)=-2 \pi
$$

So the area is shrinking at a rate of $2 \pi \frac{\mathrm{~cm}^{2}}{\mathrm{~s}}$.
3.2.2.16. Solution. The volume between the spheres, while the little one is inside the big one, is

$$
V=\frac{4}{3} \pi R^{3}-\frac{4}{3} \pi r^{3}
$$

Differentiating implicitly with respect to $t$ :

$$
\frac{\mathrm{d} V}{\mathrm{~d} t}=4 \pi R^{2} \frac{\mathrm{~d} R}{\mathrm{~d} t}-4 \pi r^{2} \frac{\mathrm{~d} r}{\mathrm{~d} t}
$$

We differentiate $R=10+2 t$ and $r=6 t$ to find $\frac{\mathrm{d} R}{\mathrm{~d} t}=2$ and $\frac{\mathrm{d} r}{\mathrm{~d} t}=6$. When $R=2 r$, $10+2 t=2(6 t)$, so $t=1$. When $t=1, R=12$ and $r=6$. So:

$$
\frac{\mathrm{d} V}{\mathrm{~d} t}=4 \pi\left(12^{2}\right)(2)-4 \pi\left(6^{2}\right)(6)=288 \pi
$$

So the volume between the two spheres is increasing at $288 \pi$ cubic units per unit time.
Remark: when the radius of the inner sphere increases, we are "subtracting" more area. Since the radius of the inner sphere grows faster than the radius of the outer sphere, we might expect the area between the spheres to be decreasing. Although the radius of the outer sphere grows more slowly, a small increase in the radius of the outer sphere results in a larger change in volume than the same increase in the radius of the inner sphere. So, a result showing that the volume between the spheres is increasing is not unreasonable.
3.2.2.17. Solution. We know something about the rate of change of the height $h$ of the triangle, and we want to know something about the rate of change of its area, $A$. A reasonable plan is to find an equation relating $A$ and $h$, and differentiate
implicitly with respect to $t$. The area of a triangle with height $h$ and base $b$ is

$$
A=\frac{1}{2} b h
$$

Note, $b$ will change with time as well as $h$. So, differentiating with respect to time, $t$ :

$$
\frac{\mathrm{d} A}{\mathrm{~d} t}=\frac{1}{2}\left(\frac{\mathrm{~d} b}{\mathrm{~d} t} \cdot h+b \cdot \frac{\mathrm{~d} h}{\mathrm{~d} t}\right)
$$

We are given $\frac{\mathrm{d} h}{\mathrm{~d} t}$ and $h$, but those $b$ 's are a mystery. We need to relate them to $h$. We can do this by breaking our triangle into two right triangles and using the Pythagorean Theorem:


So, the base of the triangle is

$$
b=\sqrt{150^{2}-h^{2}}+\sqrt{200^{2}-h^{2}}
$$

Differentiating with respect to $t$ :

$$
\begin{aligned}
\frac{\mathrm{d} b}{\mathrm{~d} t} & =\frac{-2 h \frac{\mathrm{~d} h}{\mathrm{~d} t}}{2 \sqrt{150^{2}-h^{2}}}+\frac{-2 h \frac{\mathrm{~d} h}{\mathrm{~d} t}}{2 \sqrt{200^{2}-h^{2}}} \\
& =\frac{-h \frac{\mathrm{~d} h}{\mathrm{~d} t}}{\sqrt{150^{2}-h^{2}}}+\frac{-h \frac{\mathrm{~d} h}{\mathrm{~d} t}}{\sqrt{200^{2}-h^{2}}}
\end{aligned}
$$

Using $\frac{\mathrm{d} h}{\mathrm{~d} t}=-3$ centimetres per minute:

$$
\frac{\mathrm{d} b}{\mathrm{~d} t}=\frac{3 h}{\sqrt{150^{2}-h^{2}}}+\frac{3 h}{\sqrt{200^{2}-h^{2}}}
$$

When $h=120, \sqrt{150^{2}-h^{2}}=90$ and $\sqrt{200^{2}-h^{2}}=160$. So, at this moment in time:

$$
\begin{aligned}
b & =90+160=250 \\
\frac{\mathrm{~d} b}{\mathrm{~d} t} & =\frac{3(120)}{90}+\frac{3(120)}{160}=4+\frac{9}{4}=\frac{25}{4}
\end{aligned}
$$

We return to our equation relating the derivatives of $A, b$, and $h$.

$$
\frac{\mathrm{d} A}{\mathrm{~d} t}=\frac{1}{2}\left(\frac{\mathrm{~d} b}{\mathrm{~d} t} \cdot h+b \cdot \frac{\mathrm{~d} h}{\mathrm{~d} t}\right)
$$

When $h=120 \mathrm{~cm}, b=250, \frac{\mathrm{~d} h}{\mathrm{~d} t}=-3$, and $\frac{\mathrm{d} b}{\mathrm{~d} t}=\frac{25}{4}$ :

$$
\begin{aligned}
\frac{\mathrm{d} A}{\mathrm{~d} t} & =\frac{1}{2}\left(\frac{25}{4}(120)+250(-3)\right) \\
& =0
\end{aligned}
$$

Remark: What does it mean that $\left.\frac{\mathrm{d} A}{\mathrm{~d} t}\right|_{h=120}=0$ ? Certainly, as the height changes, the area changes as well. As the height sinks to 120 cm , the area is increasing, but after it sinks past 120 cm , the area is decreasing. So, at the instant when the height is exactly 120 cm , the area is neither increasing nor decreasing: it is at a local maximum. You'll learn more about this kind of problem in Section 3.5.
3.2.2.18. Solution. Let $S$ be the flow of salt (in cubic centimetres per second). We want to know $\frac{\mathrm{d} S}{\mathrm{~d} t}$ : how fast the flow is changing at time $t$. We are given an equation for $S$ :

$$
S=\frac{1}{5} A
$$

where $A$ is the uncovered area of the cut-out. So,

$$
\frac{\mathrm{d} S}{\mathrm{~d} t}=\frac{1}{5} \frac{\mathrm{~d} A}{\mathrm{~d} t}
$$

If we can find $\frac{\mathrm{d} A}{\mathrm{~d} t}$, then we can find $\frac{\mathrm{d} S}{\mathrm{~d} t}$. We are given information about how quickly the door is rotating. If we let $\theta$ be the angle made by the leading edge of the door and the far edge of the cut-out (shown below), then $\frac{\mathrm{d} \theta}{\mathrm{d} t}=-\frac{\pi}{6}$ radians per second. (Since the door is covering more and more of the cut-out, $\theta$ is getting smaller, so $\frac{\mathrm{d} \theta}{\mathrm{d} t}$ is negative.)


Since we know $\frac{\mathrm{d} \theta}{\mathrm{d} t}$, and we want to know $\frac{\mathrm{d} A}{\mathrm{~d} t}$ (in order to get $\frac{\mathrm{d} S}{\mathrm{~d} t}$ ), it is reasonable to look for an equation relating $A$ and $\theta$, and differentiate it implicitly with respect to $t$ to get an equation relating $\frac{\mathrm{d} A}{\mathrm{~d} t}$ and $\frac{\mathrm{d} \theta}{\mathrm{d} t}$.
The area of an annulus with outer radius 6 cm and inner radius 1 cm is $\pi \cdot 6^{2}-\pi$. $1^{2}=35 \pi$ square centimetres. A sector of that same annulus with angle $\theta$ has area $\left(\frac{\theta}{2 \pi}\right)(35 \pi)$, since $\frac{\theta}{2 \pi}$ is the ratio of the sector to the entire annulus. (For example, if $\theta=\pi$, then the sector is half of the entire annulus, so its area is $(1 / 2) 35 \pi$.)
So, when $0 \leq \theta \leq \frac{\pi}{2}$, the area of the cutout that is open is

$$
A=\frac{\theta}{2 \pi}(35 \pi)=\frac{35}{2} \theta
$$

This is the formula we wanted, relating $A$ and $\theta$. Differentiating with respect to $t$,

$$
\frac{\mathrm{d} A}{\mathrm{~d} t}=\frac{35}{2} \frac{\mathrm{~d} \theta}{\mathrm{~d} t}=\frac{35}{2}\left(-\frac{\pi}{6}\right)=-\frac{35 \pi}{12}
$$

Since $\frac{\mathrm{d} S}{\mathrm{~d} t}=\frac{1}{5} \frac{\mathrm{~d} A}{\mathrm{~d} t}$,

$$
\frac{\mathrm{d} S}{\mathrm{~d} t}=-\frac{1}{5} \frac{35 \pi}{12}=-\frac{7 \pi}{12} \approx-1.8 \frac{\mathrm{~cm}^{3}}{\mathrm{sec}^{2}}
$$

Remark: the change in flow of salt is constant while the door covers more and more of the cut-out, so we never used the fact that precisely half of the cut-out was open. We also never used the radius of the lid, which is immaterial to the flow of salt.
3.2.2.19. Solution. Let $F$ be the flow of water through the pipe, so $F=\frac{1}{5} A$. We want to know $\frac{\mathrm{d} F}{\mathrm{~d} t}$, so differentiating implicitly with respect to $t$, we find

$$
\frac{\mathrm{d} F}{\mathrm{~d} t}=\frac{1}{5} \frac{\mathrm{~d} A}{\mathrm{~d} t}
$$

If we can find $\frac{\mathrm{d} A}{\mathrm{~d} t}$, then we can find $\frac{\mathrm{d} F}{\mathrm{~d} t}$. We know something about the shape of the uncovered area of the pipe; a reasonable plan is to find an equation relating the height of the door with the uncovered area of the pipe. Let $h$ be the distance from the top of the pipe to the bottom of the door, measured in metres.


Since the radius of the pipe is 1 metre, the orange line has length $1-h$ metres, and the blue line has length 1 metre. Using the Pythagorean Theorem, the green line has length $\sqrt{1^{2}-(1-h)^{2}}=\sqrt{2 h-h^{2}}$ metres.
The uncovered area of the pipe can be broken up into a triangle (of height $1-h$ and base $2 \sqrt{2 h-h^{2}}$ ) and a sector of a circle (with angle $2 \pi-2 \theta$ ). The area of the triangle is

$$
\underbrace{(1-h)}_{\text {height }} \underbrace{\sqrt{2 h-h^{2}}}_{\frac{1}{2} \text { base }}
$$

The area of the sector is

$$
\underbrace{\left(\frac{2 \pi-2 \theta}{2 \pi}\right)}_{\begin{array}{c}
\text { fraction } \\
\text { of circle }
\end{array}} \underbrace{\left(\pi \cdot 1^{2}\right)}_{\begin{array}{c}
\text { of circa } \\
\text { arcle }
\end{array}}=\pi-\theta
$$

Remember: what we want is to find $\frac{\mathrm{d} A}{\mathrm{~d} t}$, and what we know is $\frac{\mathrm{d} h}{\mathrm{~d} t}=0.01$ metres per second. If we find $\theta$ in terms of $h$, we find $A$ in terms of $h$, and then differentiate with respect to $t$.
Since $\theta$ is an angle in a right triangle with hypotenuse 1 and adjacent side length $1-h, \cos \theta=\frac{1-h}{1}=1-h$. We want to conclude that $\theta=\arccos (1-h)$, but let's be a little careful: remember that the range of the arccosine function is angles in $[0, \pi]$. We must be confident that $0 \leq \theta \leq \pi$ in order to conclude $\theta=\arccos (1-h)$-but clearly, $\theta$ is in this range. (Remark: we could also have said $\sin \theta=\frac{\sqrt{2 h-h^{2}}}{1}$, and so $\theta=\arcsin \left(\sqrt{2 h-h^{2}}\right)$. This would require $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, which is true when
$h<1$, but false for $h>1$. Since our problem asks about $h=0.25$, we could also use arcsine.)
Now, we know the area of the open pipe in terms of $h$.

$$
\begin{aligned}
A & =(\text { area of triangle })+(\text { area of sector }) \\
& =(1-h) \sqrt{2 h-h^{2}}+(\pi-\theta) \\
& =(1-h) \sqrt{2 h-h^{2}}+\pi-\arccos (1-h)
\end{aligned}
$$

We want to differentiate with respect to $t$. Using the chain rule:

$$
\begin{aligned}
\frac{\mathrm{d} A}{\mathrm{~d} t} & =\frac{\mathrm{d} A}{\mathrm{~d} h} \cdot \frac{\mathrm{~d} h}{\mathrm{~d} t} \\
\frac{\mathrm{~d} A}{\mathrm{~d} t} & =\left((1-h) \frac{2-2 h}{2 \sqrt{2 h-h^{2}}}+(-1) \sqrt{2 h-h^{2}}+\frac{-1}{\sqrt{1-(1-h)^{2}}}\right) \frac{\mathrm{d} h}{\mathrm{~d} t} \\
& =\left(\frac{(1-h)^{2}}{\sqrt{2 h-h^{2}}}-\sqrt{2 h-h^{2}}-\frac{1}{\sqrt{2 h-h^{2}}}\right) \frac{\mathrm{d} h}{\mathrm{~d} t} \\
& =\left(\frac{(1-h)^{2}-1}{\sqrt{2 h-h^{2}}}-\sqrt{2 h-h^{2}}\right) \frac{\mathrm{d} h}{\mathrm{~d} t} \\
& =\left(\frac{-\left(2 h-h^{2}\right)}{\sqrt{2 h-h^{2}}}-\sqrt{2 h-h^{2}}\right) \frac{\mathrm{d} h}{\mathrm{~d} t} \\
& =\left(-\sqrt{2 h-h^{2}}-\sqrt{2 h-h^{2}}\right) \frac{\mathrm{d} h}{\mathrm{~d} t} \\
& =-2 \sqrt{2 h-h^{2}} \frac{\mathrm{~d} h}{\mathrm{~d} t}
\end{aligned}
$$

We note here that the negative sign makes sense: as the door lowers, $h$ increases and $A$ decreases, so $\frac{\mathrm{d} h}{\mathrm{~d} t}$ and $\frac{\mathrm{d} A}{\mathrm{~d} t}$ should have opposite signs.

When $h=\frac{1}{4}$ metres, and $\frac{\mathrm{d} h}{\mathrm{~d} t}=\frac{1}{100}$ metres per second:

$$
\frac{\mathrm{d} A}{\mathrm{~d} t}=-2 \sqrt{\frac{2}{4}-\frac{1}{4^{2}}}\left(\frac{1}{100}\right)=-\frac{\sqrt{7}}{200} \frac{\mathrm{~cm}^{2}}{\mathrm{~s}}
$$

Since $\frac{\mathrm{d} F}{\mathrm{~d} t}=\frac{1}{5} \frac{\mathrm{~d} A}{\mathrm{~d} t}$ :

$$
\frac{\mathrm{d} F}{\mathrm{~d} t}=-\frac{\sqrt{7}}{1000} \frac{\mathrm{~m}^{3}}{\mathrm{sec}^{2}}
$$

That is, the flow is decreasing at a rate of $\frac{\sqrt{7}}{1000} \frac{\mathrm{~m}^{3}}{\mathrm{sec}^{2}}$.
3.2.2.20. Solution. We are given the rate of change of the volume of liquid, and are asked for the rate of change of the height of the liquid. So, we need an equation relating volume and height.
The volume $V$ of a cone with height $h$ and radius $r$ is $\frac{1}{3} \pi r^{2} h$. Since we know $\frac{\mathrm{d} V}{\mathrm{~d} t}$, and want to know $\frac{\mathrm{d} h}{\mathrm{~d} t}$, we need to find a way to deal with the unwanted variable $r$. We can find $r$ in terms of $h$ by using similar triangles. Viewed from the side, the conical glass is an equilateral triangle, as is the water in it. Using the Pythagorean Theorem, the cone has height $5 \sqrt{3}$.


Using similar triangles, $\frac{r}{h}=\frac{5}{5 \sqrt{3}}$, so $r=\frac{h}{\sqrt{3}}$. (Remark: we could also use the fact that the water forms a cone that looks like an equilateral triangle when viewed from the side to conclude $r=\frac{h}{\sqrt{3}}$.)
Now, we can write the volume of water in the cone in terms of $h$, and no other variables.

$$
\begin{aligned}
V & =\frac{1}{3} \pi r^{2} h \\
& =\frac{1}{3} \pi\left(\frac{h}{\sqrt{3}}\right)^{2} h \\
& =\frac{\pi}{9} h^{3}
\end{aligned}
$$

Differentiating with respect to $t$ :

$$
\frac{\mathrm{d} V}{\mathrm{~d} t}=\frac{\pi}{3} h^{2} \frac{\mathrm{~d} h}{\mathrm{~d} t}
$$

When $h=7 \mathrm{~cm}$ and $\frac{\mathrm{d} V}{\mathrm{~d} t}=-5 \mathrm{~mL}$ per minute,

$$
\begin{aligned}
-5 & =\frac{\pi}{3}(49) \frac{\mathrm{d} h}{\mathrm{~d} t} \\
\frac{\mathrm{~d} h}{\mathrm{~d} t} & =\frac{-15}{49 \pi} \approx-0.097 \mathrm{~cm} \text { per minute }
\end{aligned}
$$

## Exercises - Stage 3

3.2.2.21. Solution. As is so often the case, we use a right triangle in this problem to relate the quantities.


$$
\begin{aligned}
\sin \theta & =\frac{D}{2} \\
D & =2 \sin \theta
\end{aligned}
$$

Using the chain rule, we differentiate both sides with respect to time, $t$.

$$
\frac{\mathrm{d} D}{\mathrm{~d} t}=2 \cos \theta \cdot \frac{\mathrm{~d} \theta}{\mathrm{~d} t}
$$

So, if $\frac{\mathrm{d} \theta}{\mathrm{d} t}=0.25$ radians per hour and $\theta=\frac{\pi}{4}$ radians, then
(a) $\frac{\mathrm{d} D}{\mathrm{~d} t}=2 \cos \left(\frac{\pi}{4}\right) \cdot 0.25=2\left(\frac{1}{\sqrt{2}}\right) \frac{1}{4}=\frac{1}{2 \sqrt{2}}$ metres per hour.

Setting aside part (b) for a moment, let's think about (c). If $\frac{\mathrm{d} \theta}{\mathrm{d} t}$ and $\frac{\mathrm{d} D}{\mathrm{~d} t}$ have different signs, then because $\frac{\mathrm{d} D}{\mathrm{~d} t}=2 \cos \theta \cdot \frac{\mathrm{~d} \theta}{\mathrm{~d} t}$, that means $\cos \theta<0$. We have to have a nonnegative depth, so $D>0$ and $D=2 \sin \theta$ implies $\sin \theta>0$. If $\sin \theta \geq 0$ and $\cos \theta<0$, then $\theta \in(\pi / 2, \pi]$. On the diagram, that looks like this:


That is: the water has reversed direction. This happens, for instance, when a river empties into the ocean and the tide is high. Skookumchuck Narrows provincial park, in the Sunshine Coast, has reversing rapids.
Now, let's return to (b). If the rope is only 2 metres long, and the river rises higher than 2 metres, then our equation $D=2 \sin \theta$ doesn't work any more: the buoy might be stationary underwater while the water rises or falls (but stays at or above 2 metres deep).
3.2.2.22. Solution. (a) When the point is at $(0,-2)$, its $y$-coordinate is not changing, because it is moving along a horizontal line. So, the rate at which the
particle moves is simply $\frac{\mathrm{d} x}{\mathrm{~d} t}$. Let $\theta$ be the angle an observer would be looking at, in order to watch the point. Since we know $\frac{\mathrm{d} \theta}{\mathrm{d} t}$, a reasonable plan is to find an equation relating $\theta$ and $x$, and then differentiate implicitly with respect to $t$. To do this, let's return to our diagram.


When the point is a little to the right of $(0,-2)$, then we can make a triangle with the origin, as shown. If we let $\theta$ be the indicated angle, then $\frac{\mathrm{d} \theta}{\mathrm{d} t}=1$ radian per second. (It is given that the observer is turning one radian per second, so this is how fast $\theta$ is increasing.) From the right triangle in the diagram, we see

$$
\tan \theta=\frac{x}{2}
$$

Now, we have to take care of a subtle point. The diagram we drew only makes sense for the point when it is at a position a little to the right of $(0,-2)$. So, right now, we've only made a set-up that will find the derivative from the right. But, with a little more thought, we see that even when $x$ is negative (that is, when the point is a little to the left of $(0,-2)$ ), our equation holds if we are careful about how we define $\theta$. Let $\theta$ be the angle between the line connecting the point and the origin, and the $y$-axis, where $\theta$ is negative when the point is to the left of the $y$-axis.


Since $x$ and $\theta$ are both negative when the point is to the left of the $y$-axis,

$$
\begin{aligned}
\tan |\theta| & =\frac{2}{|x|} \\
\tan (-\theta) & =\frac{-x}{2}
\end{aligned}
$$

So, since $\tan (-\theta)=-\tan (\theta)$ :

$$
\tan \theta=\frac{x}{2}
$$

So, we've shown that the relationship $\tan \theta=\frac{x}{2}$ holds when our point is at $(x,-2)$, regardless of the sign of $x$.
Moving on, since we are given $\frac{\mathrm{d} \theta}{\mathrm{d} t}$ and asked for $\frac{\mathrm{d} x}{\mathrm{~d} t}$, we differentiate with respect to $t$ :

$$
\sec ^{2} \theta \cdot \frac{\mathrm{~d} \theta}{\mathrm{~d} t}=\frac{1}{2} \cdot \frac{\mathrm{~d} x}{\mathrm{~d} t}
$$

When the point is at $(0,-2)$, since the observer is turning at one radian per second, also $\frac{\mathrm{d} \theta}{\mathrm{d} t}=1$. Also, looking at the diagram, $\theta=0$. Plugging in these values:

$$
\begin{aligned}
\sec ^{2}(0) \cdot(1) & =\frac{1}{2} \cdot \frac{\mathrm{~d} x}{\mathrm{~d} t} \\
1 & =\frac{1}{2} \cdot \frac{\mathrm{~d} x}{\mathrm{~d} t} \\
\frac{\mathrm{~d} x}{\mathrm{~d} t} & =2
\end{aligned}
$$

So, the particle is moving at 2 units per second.
(b) When the point is at $(0,2)$, it is moving along a line with slope $-\frac{1}{2}$ and $y$ intercept 2 . So, it is on the line

$$
y=2-\frac{1}{2} x
$$

That is, at time $t$, if the point is at $(x(t), y(t))$, then $x(t)$ and $y(t)$ satisfy $y(t)=$ $2-\frac{1}{2} x(t)$. Implicitly differentiating with respect to $t$ :

$$
\frac{\mathrm{d} y}{\mathrm{~d} t}=-\frac{1}{2} \cdot \frac{\mathrm{~d} x}{\mathrm{~d} t}
$$

So, when $\frac{\mathrm{d} x}{\mathrm{~d} t}=1, \frac{\mathrm{~d} y}{\mathrm{~d} t}=-\frac{1}{2}$. That is, its $y$-coordinate is decreasing at $\frac{1}{2}$ unit per second.
For the question "How fast is the point moving?", remember that the velocity of an object can be found by differentiating (with respect to time) the equation that gives the position of the object. The complicating factors in this case are that (1) the position of our object is not given as a function of time, and (2) the position of our object is given in two dimensions (an $x$ coordinate and a $y$ coordinate), not one.
Remark: the solution below is actually pretty complicated. It is within your abilities to figure it out, but later on in your mathematical career you will learn an easier way, using vectors. For now, take this as a relatively tough exercise, and a motivation to keep learning: your intuition that there must be an easier way is well founded!
The point is moving along a straight line. So, to take care of complication (2), we can give its position as a point on the line. We can take the line as a sort of axis. We'll need to choose a point on the axis to be the "origin": $(2,1)$ is a convenient point. Let $D$ be the point's (signed) distance along the "axis" from $(2,1)$. When the point is a distance of one unit to the left of $(2,1)$, we'll have $D=-1$, and when the point is a distance of one unit to the right of $(2,1)$, we'll have $D=1$. Then $D$ changes with respect to time, and $\frac{\mathrm{d} D}{\mathrm{~d} t}$ is the velocity of the point. Since we know $\frac{\mathrm{d} x}{\mathrm{~d} t}$ and $\frac{\mathrm{d} y}{\mathrm{~d} t}$, a reasonable plan is to find an equation relating $x, y$, and $D$, and differentiate implicitly with respect to $t$. (This implicit differentiation takes care of complication (1).) Using the Pythagorean Theorem:

$$
D^{2}=(x-2)^{2}+(y-1)^{2}
$$

Differentiating with respect to $t$ :

$$
2 D \cdot \frac{\mathrm{~d} D}{\mathrm{~d} t}=2(x-2) \cdot \frac{\mathrm{d} x}{\mathrm{~d} t}+2(y-1) \cdot \frac{\mathrm{d} y}{\mathrm{~d} t}
$$

We plug in $x=0, y=2, \frac{\mathrm{~d} x}{\mathrm{~d} t}=1, \frac{\mathrm{~d} y}{\mathrm{~d} t}=-\frac{1}{2}$, and $D=-\sqrt{(0-2)^{2}+(2-1)^{2}}=-\sqrt{5}$ (negative because the point is to the left of $(2,1)$ ):

$$
-2 \sqrt{5} \cdot \frac{\mathrm{~d} D}{\mathrm{~d} t}=2(-2)(1)+2(1)\left(-\frac{1}{2}\right)
$$

$$
\frac{\mathrm{d} D}{\mathrm{~d} t}=\frac{\sqrt{5}}{2} \text { units per second }
$$

3.2.2.23. Solution. (a) Since the perimeter of the bottle is unchanged (you aren't stretching the plastic), it is always the same as the perimeter before it was smooshed, which is the circumference of a circle of radius 5 , or $2 \pi(5)=10 \pi$. So, using our approximation for the perimeter of an ellipse,

$$
\begin{aligned}
10 \pi & =\pi[3(a+b)-\sqrt{(a+3 b)(3 a+b)}] \\
10 & =3(a+b)-\sqrt{(a+3 b)(3 a+b)}
\end{aligned}
$$

(b) The area of the base of the bottle is $\pi a b$ (see Section A.10), and its height is 20 cm , so the volume of the bottle is

$$
V=20 \pi a b
$$

(c) As you smoosh the bottle, its volume decreases, so the water spills out. (If it turns out that the volume is increasing, then no water is spilling out-but life experience suggests, and our calculations verify, that this is not the case.) The water will spill out at a rate of $-\frac{\mathrm{d} V}{\mathrm{~d} t}$ cubic centimetres per second, where $V$ is the volume inside the bottle. We know something about $a$ and $\frac{\mathrm{d} a}{\mathrm{~d} t}$, so a reasonable plan is to differentiate the equation from (b) (relating $V$ and $a$ ) with respect to $t$.
Using the product rule, we differentiate the equation in (b) implicitly with respect to $t$ and get

$$
\frac{\mathrm{d} V}{\mathrm{~d} t}=20 \pi\left(\frac{\mathrm{~d} a}{\mathrm{~d} t} b+a \frac{\mathrm{~d} b}{\mathrm{~d} t}\right)
$$

So, we need to find the values of $a, b, \frac{\mathrm{~d} a}{\mathrm{~d} t}$, and $\frac{\mathrm{d} b}{\mathrm{~d} t}$ at the moment when $a=2 b$.
The equation from (a) tells us $10=3(a+b)-\sqrt{(a+3 b)(3 a+b)}$. So, when $a=2 b$,

$$
\begin{aligned}
10 & =3(2 b+b)-\sqrt{(2 b+3 b)(6 b+b)} \\
10 & =9 b-\sqrt{(5 b)(7 b)}=b(9-\sqrt{35}) \\
b & =\frac{10}{9-\sqrt{35}}
\end{aligned}
$$

where we use the fact that $b$ is a positive number, so $\sqrt{b^{2}}=|b|=b$.

Since $a=2 b$,

$$
a=\frac{20}{9-\sqrt{35}}
$$

Now we know $a$ and $b$ at the moment when $a=2 b$. We still need to know $\frac{\mathrm{d} a}{\mathrm{~d} t}$ and $\frac{\mathrm{d} b}{\mathrm{~d} t}$ at that moment. Since $a=5+t$, always $\frac{\mathrm{d} a}{\mathrm{~d} t}=1$. The equation from (a) relates $a$ and $b$, so differentiating both sides with respect to $t$ will give us an equation relating $\frac{\mathrm{d} a}{\mathrm{~d} t}$ and $\frac{\mathrm{d} b}{\mathrm{~d} t}$. When differentiating the portion with a square root, be careful not to forget the chain rule.

$$
0=3\left(\frac{\mathrm{~d} a}{\mathrm{~d} t}+\frac{\mathrm{d} b}{\mathrm{~d} t}\right)-\frac{\left(\frac{\mathrm{d} a}{\mathrm{~d} t}+3 \frac{\mathrm{~d} b}{\mathrm{~d} t}\right)(3 a+b)+(a+3 b)\left(3 \frac{\mathrm{~d} a}{\mathrm{~d} t}+\frac{\mathrm{d} b}{\mathrm{~d} t}\right)}{2 \sqrt{(a+3 b)(3 a+b)}}
$$

Since $\frac{\mathrm{d} a}{\mathrm{~d} t}=1$ :

$$
0=3\left(1+\frac{\mathrm{d} b}{\mathrm{~d} t}\right)-\frac{\left(1+3 \frac{\mathrm{~d} b}{\mathrm{~d} t}\right)(3 a+b)+(a+3 b)\left(3+\frac{\mathrm{d} b}{\mathrm{~d} t}\right)}{2 \sqrt{(a+3 b)(3 a+b)}}
$$

At this point, we could plug in the values we know for $a$ and $b$ at the moment when $a=2 b$. However, the algebra goes a little smoother if we start by plugging in $a=2 b$ :

$$
\begin{aligned}
0 & =3\left(1+\frac{\mathrm{d} b}{\mathrm{~d} t}\right)-\frac{\left(1+3 \frac{\mathrm{~d} b}{\mathrm{~d} t}\right)(7 b)+(5 b)\left(3+\frac{\mathrm{d} b}{\mathrm{~d} t}\right)}{2 \sqrt{(5 b)(7 b)}} \\
0 & =3\left(1+\frac{\mathrm{d} b}{\mathrm{~d} t}\right)-\frac{b\left(7+21 \frac{\mathrm{~d} b}{\mathrm{~d} t}+15+5 \frac{\mathrm{~d} b}{\mathrm{~d} t}\right)}{2 b \sqrt{35}} \\
0 & =3\left(1+\frac{\mathrm{d} b}{\mathrm{~d} t}\right)-\frac{22+26 \frac{\mathrm{~d} b}{\mathrm{~d} t}}{2 \sqrt{35}} \\
0 & =3+3 \frac{\mathrm{~d} b}{\mathrm{~d} t}-\frac{11}{\sqrt{35}}-\frac{13}{\sqrt{35}} \frac{\mathrm{~d} b}{\mathrm{~d} t} \\
-3+\frac{11}{\sqrt{35}} & =\left(3-\frac{13}{\sqrt{35}}\right) \frac{\mathrm{d} b}{\mathrm{~d} t} \\
\frac{\mathrm{~d} b}{\mathrm{~d} t} & =\frac{-3+\frac{11}{\sqrt{35}}}{3-\frac{13}{\sqrt{35}}}=\frac{-3 \sqrt{35}+11}{3 \sqrt{35}-13}
\end{aligned}
$$

Now, we can calculate $\frac{\mathrm{d} V}{\mathrm{~d} t}$ at the moment when $a=2 b$. We already found

$$
\frac{\mathrm{d} V}{\mathrm{~d} t}=20 \pi\left(\frac{\mathrm{~d} a}{\mathrm{~d} t} b+a \frac{\mathrm{~d} b}{\mathrm{~d} t}\right)
$$

So, plugging in the values of $a, b, \frac{\mathrm{~d} a}{\mathrm{~d} t}$, and $\frac{\mathrm{d} b}{\mathrm{~d} t}$ at the moment when $a=2 b$ :

$$
\frac{\mathrm{d} V}{\mathrm{~d} t}=20 \pi\left((1)\left(\frac{10}{9-\sqrt{35}}\right)+\left(\frac{20}{9-\sqrt{35}}\right)\left(\frac{-3 \sqrt{35}+11}{3 \sqrt{35}-13}\right)\right)
$$

$$
\begin{aligned}
& =\frac{200 \pi}{9-\sqrt{35}}\left(1-2\left(\frac{3 \sqrt{35}-11}{3 \sqrt{35}-13}\right)\right) \\
& \approx-375.4 \frac{\mathrm{~cm}^{3}}{\mathrm{sec}}
\end{aligned}
$$

So the water is spilling out of the cup at about 375.4 cubic centimetres per second. Remark: the algebra in this problem got a little nasty, but the method behind its solution is no more difficult than most of the problems in this section. One of the reasons why calculus is so widely taught in universities is to give you lots of practice with problem-solving: taking a big problem, breaking it into pieces you can manage, solving the pieces, and getting a solution.
A problem like this can sometimes derail people. Breaking it up into pieces isn't so hard, but when you actually do those pieces, you can get confused and forget why you are doing the calculations you're doing. If you find yourself in this situation, look back a few steps to remind yourself why you started the calculation you just did. It can also be helpful to write notes, like "We are trying to find $\frac{\mathrm{d} V}{\mathrm{~d} t}$. We already know that $\frac{\mathrm{d} V}{\mathrm{~d} t}=\ldots$. We still need to find $a, b, \frac{\mathrm{~d} a}{\mathrm{~d} t}$ and $\frac{\mathrm{d} b}{\mathrm{~d} t}$."
3.2.2.24. Solution. Since $A=0$, the equation relating the variables tells us:

$$
\begin{aligned}
& 0=\log \left(C^{2}+D^{2}+1\right) \\
& 1=C^{2}+D^{2}+1 \\
& 0=C^{2}+D^{2} \\
& 0=C=D
\end{aligned}
$$

This will probably be useful information. Since we're also given the value of a derivative, let's differentiate the equation relating the variables implicitly with respect to $t$. For ease of notation, we will write $\frac{\mathrm{d} A}{\mathrm{~d} t}=A^{\prime}$, etc.

$$
A^{\prime} B+A B^{\prime}=\frac{2 C C^{\prime}+2 D D^{\prime}}{C^{2}+D^{2}+1}
$$

At $t=10, A=C=D=0$ :

$$
\begin{aligned}
A^{\prime} B+0 & =\frac{0+0}{0+0+1} \\
A^{\prime} B & =0
\end{aligned}
$$

at $t=10, A^{\prime}=2$ units per second:

$$
\begin{aligned}
2 B & =0 \\
B & =0 .
\end{aligned}
$$

## 3.3 - Exponential Growth and Decay - a First Look at Differential Equations

### 3.3.4 • Exercises

## - Exercises for § 3.3.1

## Exercises - Stage 1

3.3.4.1. Solution. In the beginning of this section, the text says "A differential equation is an equation for an unknown function that involves the derivative of the unknown function." Our unknown function is $y$, so a differential equation is an equation that relates $y$ and $\frac{\mathrm{d} y}{\mathrm{~d} x}$. This applies to (a) and (b), but not (c), (d), or (e). Note that $\frac{\mathrm{d} x}{\mathrm{~d} x}=1$ : this is the derivative of $x$ with respect to $x$.
3.3.4.2. Solution. Theorem 3.3.2 tells us that a function is a solution to the differential equation $\frac{\mathrm{d} Q}{\mathrm{~d} t}=k Q(t)$ if and only if the function has the form $Q(t)=$ $C e^{k t}$ for some constant $C$. In our case, we want $Q(t)=5 \frac{\mathrm{~d} Q}{\mathrm{~d} t}$, so $\frac{\mathrm{d} Q}{\mathrm{~d} t}=\frac{1}{5} Q(t)$. So, the theorem tells us that the solutions are the functions of the form $Q(t)=C e^{t / 5}$. This applies to (a) (with $C=0$ ) and (d) (with $C=1$ ), but none of the other functions.
We don't actually need a theorem to answer this question, though: we can just test every option.
a $\frac{\mathrm{d} Q}{\mathrm{~d} t}=0$, so $Q(t)=0=5 \cdot 0=5 \frac{\mathrm{~d} Q}{\mathrm{~d} t}$, so (a) is a solution.
b $\frac{\mathrm{d} Q}{\mathrm{~d} t}=5 e^{t}=Q(t)$, so $Q(t)=\frac{\mathrm{d} Q}{\mathrm{~d} t} \neq 5 \frac{\mathrm{~d} Q}{\mathrm{~d} t}$, so $(\mathrm{b})$ is not a solution.
c $\frac{\mathrm{d} Q}{\mathrm{~d} t}=5 e^{5 t}=5 Q(t)$, so $Q(t)=\frac{1}{5} \frac{\mathrm{~d} Q}{\mathrm{~d} t} \neq 5 \frac{\mathrm{~d} Q}{\mathrm{~d} t}$, so (c) is not a solution.
d $\frac{\mathrm{d} Q}{\mathrm{~d} t}=\frac{1}{5} e^{t / 5}=\frac{1}{5} Q(t)$, so $Q(t)=5 \frac{\mathrm{~d} Q}{\mathrm{~d} t}$, so (d) is a solution.
$\mathrm{e} \frac{\mathrm{d} Q}{\mathrm{~d} t}=\frac{1}{5} e^{t / 5}=\frac{1}{5}(Q(t)-1)$, so $Q(t)=5 \frac{\mathrm{~d} Q}{\mathrm{~d} t}+1$, so $(\mathrm{e})$ is not a solution.
3.3.4.3. Solution. What we're asked to find is when

$$
Q(t)=0
$$

That is,

$$
C e^{-k t}=0
$$

If $C=0$, then this is the case for all $t$. There was no isotope to begin with, and there will continue not being any undecayed isotope forever.
If $C>0$, then since $e^{-k t}>0$, also $Q(t)>0$ : so $Q(t)$ is never 0 for any value of $t$. (But as $t$ gets bigger and bigger, $Q(t)$ gets closer and closer to 0 .)
Remark: The last result is somewhat disturbing: surely at some point the last atom has decayed. The differential equation we use is a model that assumes $Q$ runs continuously. This is a good approximation only when there is a very large number of atoms. In practice, that is almost always the case.

## Exercises - Stage 2

3.3.4.4. *. Solution. The two pieces of information give us

$$
f(0)=A=5 \quad f(7)=A e^{7 k}=\pi
$$

Thus we know that $A=5$ and so $\pi=f(7)=5 e^{7 k}$. Hence

$$
\begin{aligned}
e^{7 k} & =\frac{\pi}{5} \\
7 k & =\log (\pi / 5) \\
k & =\frac{1}{7} \cdot \log (\pi / 5)
\end{aligned}
$$

where we use $\log$ to mean natural logarithm, $\log _{e}$.
3.3.4.5. *. Solution. In Theorem 3.3.2, we saw that if $y$ is a function of $t$, and $\frac{\mathrm{d} y}{\mathrm{~d} t}=-k y$, then $y=C e^{-k t}$ for some constant $C$.
Our equation $y$ satisfies $\frac{\mathrm{d} y}{\mathrm{~d} t}=-3 y$, so the theorem tells us $y=C e^{-3 t}$ for some constant $C$.
We are also told that $y(1)=2$. So, $2=C e^{-3 \times 1}$ tells us $C=2 e^{3}$. Then:

$$
y=2 e^{3} \cdot e^{-3 t}=2 e^{-3(t-1)}
$$

3.3.4.6. Solution. The amount of Carbon-14 in the sample $t$ years after the animal died will be

$$
Q(t)=5 e^{-k t}
$$

for some constant $k$ (where 5 is the amount of Carbon-14 in the sample at time $t=0$ ). So, the answer we're looking for is $Q(10000)$. We need to replace $k$ with an actual number to evaluate $Q(10000)$, and the key to doing this is the half-life. The text tells us that the half-life of Carbon-14 is 5730 years, so we know:

$$
\begin{aligned}
Q(5730) & =\frac{5}{2} \\
5 e^{-k \cdot 5730} & =\frac{5}{2} \\
\left(e^{-k}\right)^{5730} & =\frac{1}{2} \\
e^{-k} & =\sqrt[5730]{\frac{1}{2}}=2^{-\frac{1}{5730}}
\end{aligned}
$$

So:

$$
\begin{aligned}
Q(t) & =5\left(e^{-k}\right)^{t} \\
& =5 \cdot 2^{-\frac{t}{5730}}
\end{aligned}
$$

Now, we can evaluate:

$$
Q(10000)=5 \cdot 2^{-\frac{10000}{5730}} \approx 1.5 \mu g
$$

Remark: after $2(5730)=11,460$ years, the sample will have been sitting for two half-lives, so its remaining Carbon- 14 will be a quarter of its original amount, or $1.25 \mu \mathrm{~g}$. It makes sense that at 10,000 years, the sample will contain slightly more Carbon-14 than at 11,460 years. Indeed, 1.5 is slightly larger than 1.25 , so our answer seems plausible.
It's a good habit to look for ways to quickly check whether your answer seems plausible, since a small algebra error can easily turn into a big error in your solution.
3.3.4.7. Solution. Let 100 years ago be the time $t=0$. Then if $Q(t)$ is the amount of Radium-226 in the sample, $Q(0)=1$, and

$$
Q(t)=e^{-k t}
$$

for some positive constant $k$. When $t=100$, the amount of Radium-226 left is 0.9576 grams, so

$$
\begin{aligned}
0.9576=Q(100) & =e^{-k \cdot 100}=\left(e^{-k}\right)^{100} \\
e^{-k} & =0.9576 \frac{1}{100}
\end{aligned}
$$

This tells us

$$
Q(t)=0.9576^{\frac{t}{100}}
$$

So, if half the original amount of Radium-226 is left,

$$
\begin{aligned}
\frac{1}{2} & =0.9576^{\frac{t}{100}} \\
\log \left(\frac{1}{2}\right) & =\log \left(0.9576 \frac{t}{100}\right) \\
-\log 2 & =\frac{t}{100} \log (0.9576) \\
t & =-100 \frac{\log 2}{\log 0.9576} \approx 1600
\end{aligned}
$$

So, the half life of Radium-226 is about 1600 years.
3.3.4.8. *. Solution. Let $Q(t)$ denote the mass at time $t$. Then $\frac{\mathrm{d} Q}{\mathrm{~d} t}$ is the rate at which the mass is changing. Since the rate the mass is decreasing is proportional to the mass remaining, we know $\frac{\mathrm{d} Q}{\mathrm{~d} t}=-k Q(t)$, where $k$ is a positive constant. (Remark: since $Q$ is decreasing, $\frac{\mathrm{d} Q}{\mathrm{~d} t}$ is negative. Since we cannot have a negative mass, if we choose $k$ to be positive, then $k$ and $Q$ are both positive-this is why we added the negative sign.)
The information given in the question is:

$$
Q(0)=6 \quad \frac{\mathrm{~d} Q}{\mathrm{~d} t}=-k Q(t) \quad Q(1)=1
$$

for some constant $k>0$. By Theorem 3.3.2, we know

$$
Q(t)=C e^{-k t}
$$

for some constant $C$. Since $Q(0)=C e^{0}=C$, the given information tells us $6=C$. (This is the initial mass of our sample.) So, $Q(t)=6 e^{-k t}$. To get the full picture of the behaviour of $Q$, we should find $k$. We do this using the given information $Q(1)=1$ :

$$
\begin{aligned}
1 & =Q(1)=6 e^{-k(1)} \\
6^{-1}=\frac{1}{6} & =e^{-k}
\end{aligned}
$$

So, all together,

$$
Q(t)=6\left(e^{-k}\right)^{t}=6 \cdot\left(6^{-1}\right)^{t}=6^{1-t}
$$

The question asks us to determine the time $t_{h}$ which obeys $Q\left(t_{h}\right)=\frac{6}{2}=3$. Now that we know the equation for $Q(t)$, we simply solve:

$$
\begin{aligned}
Q(t) & =6^{1-t} \\
3=Q\left(t_{h}\right) & =6^{1-t_{h}} \\
\log 3 & =\log \left(6^{1-t_{h}}\right)=\left(1-t_{h}\right) \log 6 \\
\frac{\log 3}{\log 6} & =1-t_{h} \\
t_{h} & =1-\frac{\log 3}{\log 6}=\frac{\log 6-\log 3}{\log 6}=\frac{\log 2}{\log 6}
\end{aligned}
$$

The half-life of Polonium-210 is $\frac{\log 2}{\log 6}$ years, or about 141 days.
Remark: The actual half-life of Polonium-210 is closer to 138 days. The numbers in the question are made to work out nicely, at the expense of some accuracy.
3.3.4.9. Solution. The amount of Radium-221 in a sample will be

$$
Q(t)=C e^{-k t}
$$

where $C$ is the amount in the sample at time $t=0$, and $k$ is some positive constant. We know the half-life of the isotope, so we can find $e^{-k}$ :

$$
\begin{aligned}
\frac{C}{2}=Q(30) & =C e^{-k \cdot 30} \\
\frac{1}{2} & =\left(e^{-k}\right)^{30} \\
2^{-\frac{1}{30}} & =e^{-k}
\end{aligned}
$$

So,

$$
Q(t)=C\left(e^{-k}\right)^{t}=C \cdot 2^{-\frac{t}{30}}
$$

When only $0.01 \%$ of the original sample is left, $Q(t)=0.0001 C$ :

$$
\begin{aligned}
0.0001 C=Q(t) & =C \cdot 2^{-\frac{t}{30}} \\
0.0001 & =2^{-\frac{t}{30}} \\
\log (0.0001) & =\log \left(2^{-\frac{t}{30}}\right) \\
\log \left(10^{-4}\right) & =-\frac{t}{30} \log 2 \\
-4 \log 10 & =-\frac{t}{30} \log 2 \\
t & =120 \cdot \frac{\log 10}{\log 2} \approx 398.6
\end{aligned}
$$

It takes about 398.6 seconds (that is, roughly 6 and a half minutes) for all but $0.01 \%$ of the sample to decay.
Remark: we can do another reality check here. The half-life is 30 seconds. 6 and a half minutes represents 13 half-lives. So, the sample is halved 13 times: $\left(\frac{1}{2}\right)^{13} \approx 0.00012=0.012 \%$. So these 13 half-lives should reduce the sample to about $0.01 \%$ of its original amount, as desired.

## Exercises - Stage 3

3.3.4.10. Solution. We know that the amount of Polonium-210 in a sample after $t$ days is given by

$$
Q(t)=C e^{-k t}
$$

where $C$ is the original amount of the sample, and $k$ is some positive constant.
The question asks us what percentage of the sample decays in a day. Since $t$ is measured in days, the amount that decays in a day is $Q(t)-Q(t+1)$. The percentage of $Q(t)$ that this represents is $100 \frac{Q(t)-Q(t+1)}{Q(t)}$. (For example, if there were two grams at time $t$, and one gram at time $t+1$, then $100 \frac{2-1}{1}=50: 50 \%$ of the sample decayed in a day.)
In order to simplify, we should figure out a better expression for $Q(t)$. As usual, we make use of the half-life.

$$
\begin{aligned}
Q(138) & =\frac{C}{2} \\
C e^{-k \cdot 138} & =\frac{C}{2} \\
\left(e^{-k}\right)^{138} & =\frac{1}{2}=2^{-1} \\
e^{-k} & =2^{-\frac{1}{138}}
\end{aligned}
$$

Now, we have a better formula for $Q(t)$ :

$$
Q(t)=C\left(e^{-k}\right)^{t}
$$

$$
Q(t)=C \cdot 2^{-\frac{t}{138}}
$$

Finally, we can evaluate what percentage of the sample decays in a day.

$$
\begin{aligned}
100 \frac{Q(t)-Q(t+1)}{Q(t)} & =100 \frac{C \cdot 2^{-\frac{t}{138}}-C \cdot 2^{-\frac{t+1}{138}}}{C \cdot 2^{-\frac{t}{138}}}\left(\frac{\frac{1}{C}}{\frac{1}{C}}\right) \\
& =100 \frac{2^{-\frac{t}{138}}-2^{-\frac{t+1}{138}}}{2^{-\frac{t}{138}}} \\
& =100\left(2^{-\frac{t}{138}}-2^{-\frac{t+1}{138}}\right) 2^{\frac{t}{138}} \\
& =100\left(1-2^{-\frac{1}{138}}\right) \approx 0.5
\end{aligned}
$$

About $0.5 \%$ of the sample decays in a day.
Remark: when we say that half a percent of the sample decays in a day, we don't mean half a percent of the original sample. If a day starts out with, say, 1 mi crogram, then what decays in the next 24 hours is about half a percent of that 1 microgram, regardless of what the "original" sample (at some time $t=0$ ) held.
In particular, since the sample is getting smaller and smaller, that half of a percent that decays every day represents fewer and fewer actual atoms decaying. That's why we can't say that half of the sample ( $50 \%$ ) will decay after about 100 days, even though $0.5 \%$ decays every day and $100 \times 0.5=50$.
3.3.4.11. Solution. The amount of Uranium-232 in the sample of ore at time $t$ will be

$$
Q(t)=Q(0) e^{-k t}
$$

where $6.9 \leq Q(0) \leq 7.5$. We don't exactly know $Q(0)$, and we don't exactly know the half-life, so we also won't exactly know $Q(10)$ : we can only say that is it between two numbers. Our strategy is to find the highest and lowest possible values of $Q(10)$, given the information in the problem. In order for the most possible Uranium-232 to be in the sample after 10 years, we should start with the most and have the longest half-life (since this represents the slowest decay). So, we take $Q(0)=7.5$ and $Q(70)=\frac{1}{2}(7.5)$.

$$
\begin{aligned}
Q(t) & =7.5 e^{-k t} \\
\frac{1}{2}(7.5)=Q(70) & =7.5 e^{-k(70)} \\
\frac{1}{2} & =\left(e^{-k}\right)^{70} \\
2^{-\frac{1}{70}} & =e^{-k}
\end{aligned}
$$

So, in this secenario,

$$
Q(t)=7.5 \cdot 2^{-\frac{t}{70}}
$$

After ten years,

$$
Q(10)=7.5 \cdot 2^{-\frac{10}{70}} \approx 6.79
$$

So after ten years, the sample contains at most $6.8 \mu \mathrm{~g}$. Now, let's think about the least possible amount of Uranium-232 that could be left after 10 years. We should start with as little as possible, so take $Q(0)=6.9$, and the sample should decay quickly, so take the half-life to be 68.8 years.

$$
\begin{aligned}
Q(t) & =6.9 e^{-k t} \\
\frac{1}{2} 6.9=Q(68.8) & =6.9 e^{-k(68.8)} \\
\frac{1}{2} & =\left(e^{-k}\right)^{68.8} \\
2^{-\frac{1}{68.8}} & =e^{-k}
\end{aligned}
$$

In this scenario,

$$
Q(t)=6.9 \cdot 2^{-\frac{t}{68.8}}
$$

After ten years,

$$
Q(10)=6.9 \cdot 2^{-\frac{10}{68.8}} \approx 6.24
$$

So after ten years, the sample contains at least $6.2 \mu \mathrm{~g}$.
After ten years, the sample contains between 6.2 and $6.8 \mu \mathrm{~g}$ of Uranium-232.

## - Exercises for § 3.3.2

## Exercises - Stage 1

3.3.4.1. Solution. Using Corollary 3.3.8 (with $A=20$ and $K=5$ ), solutions to the differential equation all have the form

$$
T(t)=[T(0)-20] e^{5 t}+20
$$

for some constant $T(0)$. This fits (a) (with $T(0)=20$ ), (c) (with $T(0)=21$ ), and (d) (with $T(0)=40$ ), but not (b) (since the constant has the wrong sign).

Instead of using the corollary, we can also just check each function for ourselves.
a $\frac{\mathrm{d} T}{\mathrm{~d} t}=0=5 \cdot 0=5[T(t)-20]$, so (a) gives a solution to the differential equation.
b $\frac{\mathrm{d} T}{\mathrm{~d} t}=5\left[20 e^{5 t}\right]=5[T+20] \neq 5[T-20]$, so (b) does not give a solution to the differential equation.
c $\frac{\mathrm{d} T}{\mathrm{~d} t}=5\left[e^{5 t}\right]=5[T-20]$, so (c) gives a solution to the differential equation.
$\mathrm{d} \frac{\mathrm{d} T}{\mathrm{~d} t}=5\left[20 e^{5 t}\right]=5[T-20]$, so (d) gives a solution to the differential equation.
3.3.4.2. Solution. From Newton's Law of Cooling and Corollary 3.3.8, the temperature of the object will be

$$
T(t)=[T(0)-A] e^{K t}+A
$$

where $A$ is the ambient temperature (the temperature of the room), $T(0)$ is the initial temperature of the copper, and $K$ is some constant. So, the ambient temperature-the temperature of the room- is -10 degrees. Since the coefficient of the exponential part of the function is positive, the temperature of the object is higher than the temperature of the room.
3.3.4.3. Solution. As $t$ grows very large, $T(t)$ approaches $A$. That is:

$$
\begin{aligned}
\lim _{t \rightarrow \infty} T(t) & =A \\
\lim _{t \rightarrow \infty}[T(0)-A] e^{K t}+A & =A \\
\lim _{t \rightarrow \infty}[T(0)-A] e^{K t} & =0
\end{aligned}
$$

Since the object is warmer than the room, $T(0)-A$ is a nonzero constant. So,

$$
\lim _{t \rightarrow \infty} e^{K t}=0
$$

This tells us that $K$ is a negative number. So, $K$ must be negative - not zero, and not positive.
Remark: in our work, we used the fact that the object and the room have different temperatures (but it didn't matter which one was hotter). If not, then $T(0)=A$, and $T(t)=A$ : that is, the temperature of the object is constant. In this case, our usual form for the temperature of the object looks like this:

$$
T(t)=0 e^{K t}+A
$$

Keeping the exponential piece in there is overkill: the temperature isn't changing, the function is simply constant. If the object and the room have the same temperature, $K$ could be any real number since we're multiplying $e^{K t}$ by zero. Remark: contrast this to Question 3.3.4.9.
3.3.4.4. Solution. We want to know when

$$
[T(0)-A] e^{k t}+A=A
$$

That is, when

$$
[T(0)-A] e^{k t}=0
$$

Since $e^{k t}>0$ for all values of $k$ and $t$, this happens exactly when

$$
T(0)-A=0
$$

So: if the initial temperature of the object is not the same as the ambient temperature, then according to this model, it never will be! (However, as $t$ gets larger and larger, $T(t)$ gets closer and closer to $A$-it just never exactly reaches there.)
If the initial temperature of the object starts out the same as the ambient temperature, then $T(t)=A$ for all values of $t$.

## Exercises - Stage 2

3.3.4.5. Solution. From Newton's Law of Cooling and Corollary 3.3.8, we know the temperature of the copper will be

$$
T(t)=[T(0)-A] e^{K t}+A
$$

where $A$ is the ambient temperature $\left(100^{\circ}\right), T(0)$ is the temperature of the copper at time 0 (let's make this the instant it was dumped in the water, so $T(0)=25^{\circ}$ ), and $K$ is some constant. That is:

$$
\begin{aligned}
T(t) & =[25-100] e^{K t}+100 \\
& =-75 e^{K t}+100
\end{aligned}
$$

The information given tells us that $T(10)=90$, so

$$
\begin{aligned}
90 & =-75 e^{10 K}+100 \\
75\left(e^{K}\right)^{10} & =10 \\
\left(e^{K}\right)^{10} & =\frac{2}{15} \\
e^{K} & =\left(\frac{2}{15}\right)^{\frac{1}{10}}
\end{aligned}
$$

This lets us describe $T(t)$ without any unknown constants.

$$
\begin{aligned}
T(t) & =-75\left(e^{K}\right)^{t}+100 \\
& =-75\left(\frac{2}{15}\right)^{\frac{t}{10}}+100
\end{aligned}
$$

The question asks what value of $t$ gives $T(t)=99.9$.

$$
\begin{aligned}
99.9 & =-75\left(\frac{2}{15}\right)^{\frac{t}{10}}+100 \\
75\left(\frac{2}{15}\right)^{\frac{t}{10}} & =0.1 \\
\left(\frac{2}{15}\right)^{\frac{t}{10}} & =\frac{1}{750} \\
\log \left(\left(\frac{2}{15}\right)^{\frac{t}{10}}\right) & =\log \left(\frac{1}{750}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{t}{10} \log \left(\frac{2}{15}\right) & =-\log (750) \\
t & =\frac{-10 \log (750)}{\log \left(\frac{2}{15}\right)} \approx 32.9
\end{aligned}
$$

It takes about 32.9 seconds.
3.3.4.6. Solution. The temperature of the stone $t$ minutes after taking it from the bonfire is

$$
\begin{aligned}
T(t) & =[T(0)-A] e^{K t}+A \\
& =[500-0] e^{K t}+0 \\
& =500 e^{K t}
\end{aligned}
$$

for some constant $K$. We are given that $T(10)=100$.

$$
\begin{aligned}
100=T(10) & =500 e^{10 K} \\
e^{10 K} & =\frac{1}{5} \\
e^{K} & =5^{-\frac{1}{10}}
\end{aligned}
$$

This gives us the more complete picture for the temperature of the stone.

$$
T(t)=500\left(e^{K}\right)^{t}=500 \cdot 5^{-\frac{t}{10}}
$$

If $T(t)=50:$

$$
\begin{aligned}
50=T(t) & =500 \cdot 5^{-\frac{t}{10}} \\
\frac{1}{10}=10^{-1} & =5^{-\frac{t}{10}} \\
10 & =5^{\frac{t}{10}} \\
\log (10) & =\frac{t}{10} \log (5) \\
t & =10 \frac{\log (10)}{\log (5)} \approx 14.3
\end{aligned}
$$

So the stone has been out of the fire for about 14.3 minutes.

## Exercises - Stage 3

3.3.4.7. *. Solution.

- First scenario: At time 0, Newton mixes 9 parts coffee at temperature $95^{\circ} \mathrm{C}$ with 1 part cream at temperature $5^{\circ} \mathrm{C}$. The resulting mixture has temperature

$$
\frac{9 \times 95+1 \times 5}{9+1}=86^{\circ}
$$

The mixture cools according to Newton's Law of Cooling, with initial temperature $86^{\circ}$ and ambient temperature $22^{\circ}$ :

$$
\begin{aligned}
& T(t)=[86-22] e^{-k t}+22 \\
& T(t)=64 e^{-k t}+22
\end{aligned}
$$

After 10 minutes,

$$
\begin{aligned}
54=T(10) & =22+64 e^{-10 k} \\
e^{-10 k} & =\frac{54-22}{64}=\frac{1}{2}
\end{aligned}
$$

We could compute $k$ from this, but we don't need it.

- Second scenario: At time 0, Newton gets hot coffee at temperature $95^{\circ} \mathrm{C}$. It cools according to Newton's Law of Cooling

$$
T(t)=[T(0)-22] e^{-k t}+22
$$

In this second scenario, $T(0)=95$, so

$$
T(t)=[95-22] e^{-k t}+22=73 e^{-k t}+22
$$

The value of $k$ is the same as in the first scenario, so after 10 minutes

$$
T(10)=22+73 e^{-10 k}=22+73 \frac{1}{2}=58.5
$$

This cooled coffee is mixed with cold cream to yield a mixture of temperature

$$
\frac{9 \times 58.5+1 \times 5}{9+1}=53.15
$$

Under the second (add cream just before drinking) scenario, the coffee ends up cooler by $0.85^{\circ} \mathrm{C}$.

### 3.3.4.8. *. Solution.

a By Newton's law of cooling, the rate of change of temperature is proportional to the difference between $T(t)$ and the ambient temperature, which in this case is $30^{\circ}$. Thus

$$
\frac{\mathrm{d} T}{\mathrm{~d} t}=k[T(t)-30]
$$

for some constant of proportionality $k$. To answer part (a), all we have to do is find $k$.
Under Newton's Law of Cooling, the temperature at time $t$ will be given by

$$
\begin{aligned}
T(t) & =[T(0)-A] e^{k t}+A \\
& =[5-30] e^{k t}+30
\end{aligned}
$$

$$
=-25 e^{k t}+30
$$

We are told $T(5)=10$ :

$$
\begin{aligned}
10 & =-25 e^{5 k}+30 \\
25 e^{5 k} & =20 \\
e^{5 k} & =\frac{4}{5} \\
5 k & =\log (4 / 5) \\
k & =\frac{1}{5} \log (4 / 5)
\end{aligned}
$$

So, the differential equation is

$$
\frac{\mathrm{d} T}{\mathrm{~d} t}(t)=\frac{1}{5} \log (4 / 5)[T(t)-30]
$$

b Since $T(t)=30-25 e^{k t}$, the temperature of the tea is $20^{\circ}$ when

$$
\begin{aligned}
30-25 e^{k t} & =20 \\
e^{k t} & =\frac{10}{25} \\
k t & =\log \left(\frac{10}{25}\right) \\
t & =\frac{1}{k} \log \frac{2}{5} \\
& =\frac{5 \log (2 / 5)}{\log (4 / 5)} \\
& \approx 20.53 \mathrm{~min}
\end{aligned}
$$

3.3.4.9. Solution. As time goes on, temperatures that follow Newton's Law of Cooling get closer and closer to the ambient temperature. So, $\lim _{t \rightarrow \infty} T(t)$ exists. In particular, $\lim _{t \rightarrow \infty} 0.8^{k t}$ exists.

- If $k<0$, then $\lim _{t \rightarrow \infty} 0.8^{k t}=\infty$, since $0.8<1$. So, $k \geq 0$.
- If $k=0$, then $T(t)=16$ for all values of $t$. But, in the statement of the question, the object is changing temperature. So, $k>0$.

Therefore, $k$ is positive.
Remark: contrast this to Question 3.3.4.3. The reason we get a different answer is that our base in this question (0.8) is less than one, while the base in Question 3.3.4.3 $(e)$ is greater than one.
Although the given equation $T(t)$ does not exactly look like the Newton's Law equations we're used to, it is equivalent. Remembering $e^{\log (0.8)}=0.8$ :

$$
T(t)=0.8^{k t}+15
$$

$$
\begin{aligned}
& =\left(e^{\log 0.8}\right)^{k t}+15 \\
& =e^{(k \log 0.8) t}+15 \\
& =[16-15] e^{(k \log 0.8) t}+15 \\
& =[16-15] e^{K t}+15
\end{aligned}
$$

with $K=k \log 0.8$. This is now the more familiar form for Newton's Law of Cooling (with $A=15$ and $T(0)=16$ ).
Since $0.8<1, \log (0.8)$ is negative, so $k$ and $K$ have opposite signs.

## - Exercises for § 3.3.3

## Exercises - Stage 1

3.3.4.1. Solution. Since $b$ is a positive constant, $\lim _{t \rightarrow \infty} e^{b t}=\infty$. Therefore:

$$
\lim _{t \rightarrow \infty} P(t)=\lim _{t \rightarrow \infty} P(0) e^{b t}=\left\{\begin{array}{cl}
0 & \text { if } P(0)=0 \\
\infty & \text { if } P(0)>0
\end{array}\right.
$$

If $P(0)=0$, then the model simply says "a population that starts with no individuals continues to have no individuals indefinitely," which certainly makes sense. If $P(0) \neq 0$, then (since we can't have a negative population) $P(0)>0$, and the model says "a population that starts out with some individuals will end up with any gigantically huge number you can think of, given enough time." This one doesn't make so much sense. Populations only grow to a certain finite amount, due to scarcity of resources and such. In the derivation of the Malthusian model, we assume a constant net birth rate-that the birth and death rates (per individual) don't depend on the population, which is not a reasonable assumption long-term.

## Exercises - Stage 2

3.3.4.2. Solution. The assumption that the animals grow according to the Malthusian model tells us that their population $t$ years after 2015 is given by $P(t)=121 e^{b t}$ for some constant $b$, since $121=P(0)$, the population 0 years after 2015. The information about 2016 tells us

$$
\begin{aligned}
136=P(1) & =121 e^{b} \\
\frac{136}{121} & =e^{b}
\end{aligned}
$$

This gives us a better idea of $P(t)$ :

$$
P(t)=121 e^{b t}=121\left(\frac{136}{121}\right)^{t}
$$

2020 is 5 years after 2015, so in 2020 (assuming the population keeps growing according to the Malthusian model) the population will be

$$
P(5)=121\left(\frac{136}{121}\right)^{5} \approx 217
$$

In 2020, the Malthusian model predicts the herd will number 217 individuals.
3.3.4.3. Solution. Since the initial population of bacteria is 1000 individuals, the Malthusian model says that the population of bacteria $t$ hours after being placed in the dish will be $P(t)=1000 e^{b t}$ for some constant $b$. Since $P(1)=2000$,

$$
\begin{aligned}
2000=P(1) & =1000 e^{b} \\
2 & =e^{b}
\end{aligned}
$$

So, the population at time $t$ is

$$
P(t)=1000 \cdot 2^{t}
$$

We want to know at what time the population triples, to 3,000 individuals.

$$
\begin{aligned}
3000 & =1000 \cdot 2^{t} \\
3 & =2^{t} \\
\log (3) & =\log \left(2^{t}\right)=t \log (2) \\
t & =\frac{\log (3)}{\log (2)} \approx 1.6
\end{aligned}
$$

The population triples in about 1.6 hours. Note that the tripling time depends only on the constant $b$. In particular, it does not depend on the initial condition $P(0)$.
3.3.4.4. Solution. According to the Malthusian Model, if the ship wrecked at year $t=0$ and 2 rats washed up on the island, then $t$ years after the wreck, the population of rats will be

$$
P(t)=2 e^{b t}
$$

for some constant $b$. We want to get rid of this extraneous variable $b$, so we use the given information. If 1928 is $a$ years after the wreck:

$$
\begin{aligned}
1000=P(a) & =2 e^{b a} \\
1500=P(a+1) & =2 e^{b(a+1)}=2 e^{b a} e^{b}
\end{aligned}
$$

So,

$$
(1000)\left(e^{b}\right)=\left(2 e^{b a}\right)\left(e^{b}\right)=1500
$$

Which tells us

$$
e^{b}=\frac{1500}{1000}=1.5
$$

Now, our model is complete:

$$
P(t)=2\left(e^{b}\right)^{t}=2 \cdot 1.5^{t}
$$

Since $P(a)=1000$, we can find $a$ :

$$
1000=P(a)=2 \cdot 1.5^{a}
$$

$$
\begin{aligned}
500 & =1.5^{a} \\
\log (500) & =\log \left(1.5^{a}\right)=a \log (1.5) \\
a & =\frac{\log (500)}{\log (1.5)} \approx 15.3
\end{aligned}
$$

So, the year 1928 was about 15.3 years after the shipwreck. Since we aren't given a month when the rats reached exactly 1000 in number, that puts the shipwreck at the year 1912 or 1913.
3.3.4.5. Solution. The Malthusian model suggests that, if we start with $P(0)$ cochineals, their population after 3 months will be

$$
P(t)=P(0) e^{b t}
$$

for some constant $b$. The constant $b$ is the net birthrate per population member per unit time. Assuming that the net birthrate for a larger population will be the same as the test population, we can use the data from the test to find $e^{b}$. Let $Q(t)$ be the number of individuals in the test population at time $t$.

$$
\begin{aligned}
Q(t) & =Q(0) e^{b t}=200 e^{b t} \\
1000 & =Q(3)=200 e^{3 b} \\
5 & =e^{3 b} \\
e^{b} & =5^{1 / 3}
\end{aligned}
$$

Now that we have an idea of the birthrate, we predict

$$
P(t)=P(0)\left(e^{b}\right)^{t}=P(0) \cdot 5^{\frac{t}{3}}
$$

We want $P(12)=10^{6}+P(0)$.

$$
\begin{aligned}
10^{6}+P(0)=P(12) & =P(0) \cdot 5^{\frac{12}{3}}=P(0) \cdot 5^{4} \\
10^{6} & =P(0) \cdot 5^{4}-P(0)=P(0)\left[5^{4}-1\right] \\
P(0) & =\frac{10^{6}}{5^{4}-1} \approx 1603
\end{aligned}
$$

The farmer should use an initial population of (at least) about 1603 individuals. Remark: if we hadn't specified that we need to save $P(0)$ individuals to start next year's population, the number of individual cochineals we would want to start with to get a million in a year would be 1600 -almost the same!

## Exercises - Stage 3

### 3.3.4.6. Solution.

- [(a)] Since $f(t)$ gives the amount of the radioactive isotope in the sample at time $t$, the amount of the radioactive isotope in the sample when $t=0$ is $f(0)=100 e^{0}=100$ units. Since the sample is decaying, $f(t)$ is decreasing,
so $f^{\prime}(t)$ is negative. Differentiating, $f^{\prime}(t)=k\left(100 e^{k t}\right)$. Since $100 e^{k t}$ is positive and $f^{\prime}(t)$ is negative, $k$ is negative.
- [(b)] Since $f(t)$ gives the size of the population at time $t$, the number of individuals in the population when $t=0$ is $f(0)=100 e^{0}=100$. Since the population is growing, $f(t)$ is increasing, so $f^{\prime}(t)$ is positive. Differentiating, $f^{\prime}(t)=k\left(100 e^{k t}\right)$. Since $100 e^{k t}$ is positive and $f^{\prime}(t)$ is positive, $k$ is also positive.
- [(c)] Newton's Law of Cooling gives the temperature of an object at time $t$ as $f(t)=[f(0)-A] e^{k t}+A$, where $A$ is the ambient temperature surrounding the object. In our case, the ambient temperature is 0 degrees. In an object whose temperature is being modelled by Newton's Law of Cooling, it doesn't matter whether the object is heating or cooling, $k$ is negative. We saw this in Question 3.3.4.3 of Section 3.3.2, but it bears repeating. Since $f(t)$ approaches the ambient temperature (in this case, 0 ) as $t$ goes to infinty:

$$
\lim _{t \rightarrow \infty} 100 e^{k t}=0
$$

so $k$ is negative.

## - Further problems for § 3.3

3.3.4.1. *. Solution. The first piece of information given tells us $\frac{\mathrm{d} f}{\mathrm{~d} x}=\pi f(x)$. Then by Theorem 3.3.2,

$$
f(x)=C e^{\pi x}
$$

for some constant $C$. The second piece of given information tells us $f(0)=2$. Using this, we can solve for $C$ :

$$
2=f(0)=C e^{0}=C
$$

Now, we know $f(x)$ entirely:

$$
f(x)=2 e^{\pi t}
$$

So, we can evaluate $f(2)$

$$
f(2)=2 e^{2 \pi}
$$

3.3.4.2. Solution. To use Corollary 3.3.8, we re-write the differential equation as

$$
\frac{\mathrm{d} T}{\mathrm{~d} t}=7\left[T-\left(-\frac{9}{7}\right)\right]
$$

Now, $A=-\frac{9}{7}$ and $K=7$, so we see that the solutions to the differential equation have
the form

$$
T(t)=\left[T(0)+\frac{9}{7}\right] e^{7 t}-\frac{9}{7}
$$

for some constant $T(0)$.

We can check that this is reasonable: if

$$
T(t)=\left[T(0)+\frac{9}{7}\right] e^{7 t}-\frac{9}{7}
$$

then

$$
\begin{aligned}
\frac{\mathrm{d} T}{\mathrm{~d} t} & =7\left[T(0)+\frac{9}{7}\right] e^{7 t} \\
& =7\left[T+\frac{9}{7}\right] \\
& =7 T+9
\end{aligned}
$$

3.3.4.3. *. Solution. Let $Q(t)$ denote the amount of radioactive material after $t$ days. Then $Q(t)=Q(0) e^{k t}$. We are told

$$
Q(8)=0.8 Q(0)
$$

So,

$$
\begin{aligned}
Q(0) e^{8 k} & =0.8 Q(0) \\
e^{8 k} & =0.8 \\
e^{k} & =0.8^{\frac{1}{8}}
\end{aligned}
$$

If $Q(0)=100$, the time $t$ at which $Q(t)=40$ is determined by

$$
40=Q(t)=Q(0) e^{k t}=100 e^{k t}=100\left(0.8^{\frac{1}{8}}\right)^{t}=100 \cdot 0.8^{\frac{t}{8}}
$$

Solving for $t$ :

$$
\begin{aligned}
\frac{40}{100} & =0.8^{\frac{t}{8}} \\
\log (0.4) & =\log \left(0.8^{\frac{t}{8}}\right)=\frac{t}{8} \log (0.8) \\
t & =\frac{8 \log (0.4)}{\log (0.8)} \approx 32.85 \text { days }
\end{aligned}
$$

100 grams will decay to 40 grams in about 32.85 days.
3.3.4.4. Solution. Let $t=0$ be the time the boiling water is left in the room, and let $T(t)$ be the temperature of the water $t$ minutes later, so $T(0)=100$. Using Newton's Law of Cooling, we model the temperature of the water at time $t$ as

$$
T(t)=[100-A] e^{K t}+A
$$

where $A$ is the room temperature, and $K$ is some constant. We are told that $T(15)=85$ and $T(30)=73$, so:

$$
\begin{aligned}
85 & =T(15)=[100-A] e^{15 K}+A \\
73 & =T(30)=[100-A] e^{30 K}+A
\end{aligned}
$$

Rearranging both equations:

$$
\begin{aligned}
& \frac{85-A}{100-A}=e^{15 K} \\
& \frac{73-A}{100-A}=e^{30 K}=\left(e^{15 K}\right)^{2}
\end{aligned}
$$

Using these equations:

$$
\begin{aligned}
\left(\frac{85-A}{100-A}\right)^{2}=\left(e^{15 K}\right)^{2} & =e^{30 K}=\frac{73-A}{100-A} \\
\frac{(85-A)^{2}}{100-A} & =73-A \\
(85-A)^{2} & =(73-A)(100-A) \\
85^{2}-170 A+A^{2} & =7300-173 A+A^{2} \\
173 A-170 A & =7300-85^{2} \\
3 A & =75 \\
A & =25
\end{aligned}
$$

The room temperature is $25^{\circ} \mathrm{C}$.

### 3.3.4.5. *. Solution.

a The amount of money at time $t$ obeys

$$
\frac{\mathrm{d} A}{\mathrm{~d} t}=0.05 A+2,000=0.05[A-(-40,000)]
$$

Using Corollary 3.3.8

$$
\begin{aligned}
A(t) & =[A(0)+40,000] e^{0.05 t}-40,000 \\
& =90,000 \cdot e^{0.05 t}-40,000
\end{aligned}
$$

where $t=0$ corresponds to the year when the graduate is 25 .

When the graduate is 65 years old, $t=40$, so

$$
A(40)=90,000 e^{0.05 \times 40}-40,000 \approx 625,015.05
$$

b When the graduate stops depositing money and instead starts withdrawing money at a rate $W$, the equation for $A$ becomes

$$
\frac{\mathrm{d} A}{\mathrm{~d} t}=0.05 A-W=0.05[A(t)-20 W]
$$

Using Corollary 3.3.8 and assuming that the interest rate remains 5\%

$$
\begin{aligned}
A(t) & =[A(0)-20 W] e^{0.05 t}+20 W \\
& =[625,015.05-20 W] e^{0.05 t}+20 W
\end{aligned}
$$

Note that, for part (b), we only care about what happens when the graduate starts withdrawing money. We take $t=0$ to correspond to the year when the graduate is 65 -so we're using a different $t$ from part (a). Then from part (a), $A(0)=625,025.05$.

We want the account to be depleted when the graduate is 85 . So, we want

$$
\begin{aligned}
0 & =A(20) \\
0 & =20 W+e^{0.05 \times 20}(625,015.05-20 W) \\
0 & =20 W+e(625,015.05-20 W) \\
20(e-1) W & =625,015.05 e \\
W & =\frac{625,015.05 e}{20(e-1)} \approx \$ 49,437.96
\end{aligned}
$$

3.3.4.6. *. Solution. 3.3.4.6.a The amount of money at time $t$ obeys

$$
\frac{\mathrm{d} A}{\mathrm{~d} t}=0.06 A-9,000=0.06[A-150,000]
$$

Using Corollary 3.3.8

$$
\begin{aligned}
A(t) & =[A(0)-150,000] e^{0.06 t}+150,000 \\
& =[120,000-150,000] e^{0.06 t}+150,000 \\
& =-30,000 e^{0.06 t}+150,000
\end{aligned}
$$

3.3.4.6.b The money runs out when $A(t)=0$.

$$
\begin{aligned}
A(t) & =0 \\
150,000-30,000 e^{0.06 t} & =0 \\
30,000 e^{0.06 t} & =150,000 \\
e^{0.06 t} & =5 \\
0.06 t & =\log 5 \\
t & =\frac{\log 5}{0.06} \approx 26.8 \mathrm{yrs}
\end{aligned}
$$

The money runs out in about 26.8 years.
Remark: without earning any interest, the money would have run out in about 13.3 years.
3.3.4.7. *. Solution. Let $Q(t)$ denote the number of bacteria at time $t$. We are told that $Q^{\prime}(t)=k Q(t)$ for some constant of proportionality $k$. Consequently, $Q(t)=$ $Q(0) e^{k t}$ (Corollary 3.3.2). We are also told

$$
\begin{aligned}
Q(9) & =3 Q(0) \\
\text { So, } \quad Q(0) e^{9 k} & =3 Q(0) \\
e^{9 k} & =3 \\
e^{k} & =3^{\frac{1}{9}}
\end{aligned}
$$

The doubling time $t$ obeys:

$$
\begin{aligned}
Q(t) & =2 Q(0) \\
\text { So, } \quad Q(0) e^{k t} & =2 Q(0) \\
e^{k t} & =2 \\
3^{\frac{t}{9}} & =2 \\
\frac{t}{9} \log 3 & =\log 2 \\
t & =9 \frac{\log 2}{\log 3} \approx 5.68 \mathrm{hr}
\end{aligned}
$$

3.3.4.8. *. Solution. (a) We want our differential equation to have the format of the equation in Corollary 3.3.8:

$$
\begin{aligned}
\frac{d v}{d t}(t) & =-g-k v(t) \\
& =-k\left(v(t)+\frac{g}{k}\right) \\
& =-k\left(v(t)-\left(-\frac{g}{k}\right)\right)
\end{aligned}
$$

So, we can use the corollary, with $K=-k, T=v$, and $A=-\frac{g}{k}$.

$$
\begin{aligned}
v(t) & =\left(v(0)-\left(-\frac{g}{k}\right)\right) e^{-k t}-\frac{g}{k} \\
& =\left(v_{0}+\frac{g}{k}\right) e^{-k t}-\frac{g}{k}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\lim _{t \rightarrow \infty} v(t) & =\lim _{t \rightarrow \infty}\left[\left(v_{0}+\frac{g}{k}\right) e^{-k t}-\frac{g}{k}\right] \\
& =\left(v_{0}+\frac{g}{k}\right)\left(\lim _{t \rightarrow \infty} e^{-k t}\right)-\frac{g}{k}
\end{aligned}
$$

Since $k$ is positive:

$$
\begin{aligned}
& =\left(v_{0}+\frac{g}{k}\right)(0)-\frac{g}{k} \\
& =-\frac{g}{k}
\end{aligned}
$$

Remark: This means, as the object falls, instead of accelerating without bound, it approaches some maximum speed. The velocity is negative because the object is moving in the negative direction-downwards.

### 3.4 Approximating Functions Near a Specified Point - Taylor Polynomials

### 3.4.11 • Exercises

- Exercises for § 3.4.1


## Exercises - Stage 1

3.4.11.1. Solution. Since $f(0)$ is closer to $g(0)$ than it is to $h(0)$, you would probably want to estimate $f(0) \approx g(0)=1+2 \sin (1)$ if you had the means to efficiently figure out what $\sin (1)$ is, and if you were concerned with accuracy. If you had a calculator, you could use this estimation. Also, later in this chapter we will learn methods of approximating $\sin (1)$ that do not require a calculator, but they do require time.
Without a calculator, or without a lot of time, using $f(0) \approx h(0)=0.7$ probably makes the most sense. It isn't as accurate as $f(0) \approx g(0)$, but you get an estimate very quickly, without worrying about figuring out what $\sin (1)$ is.
Remark: when you're approximating something in real life, there probably won't be an obvious "correct" way to do it. There's usually a trade-off between accuracy and ease.

## Exercises - Stage 2

3.4.11.2. Solution. 0.93 is pretty close to 1 , and we know $\log (1)=0$, so we estimate $\log (0.93) \approx \log (1)=0$.

3.4.11.3. Solution. We don't know $\arcsin (0.1)$, but 0.1 is reasonably close to 0 , and $\arcsin (0)=0$. So, we estimate $\arcsin (0.1) \approx 0$.
3.4.11.4. Solution. We don't know $\tan (1)$, but we do know $\tan \left(\frac{\pi}{3}\right)=\sqrt{3}$. Since $\frac{\pi}{3} \approx 1.047$ is pretty close to 1 , we estimate $\sqrt{3} \tan (1) \approx \sqrt{3} \tan \left(\frac{\pi}{3}\right)=(\sqrt{3})^{2}=3$.

## Exercises - Stage 3

3.4.11.5. Solution. Since 10.1 is pretty close to 10 , we estimate $10.1^{3} \approx 10^{3}=$ 1000.

Remark: these kinds of approximations are very useful when you are doing computations. It's easy to make a mistake in your work, and having in mind that $10.1^{3}$ should be about a thousand is a good way to check that whatever answer you have makes sense.

## - Exercises for § 3.4.2

## Exercises - Stage 1

3.4.11.1. Solution. The linear approximation is $L(x)=3 x-9$. Since we're approximating at $x=5, f(5)=L(5)$, and $f^{\prime}(5)=L^{\prime}(5)$. However, there is no guarantee that $f(x)$ and $L(x)$ have the same value when $x \neq 5$. So:
(a) $f(5)=L(5)=6$
(b) $f^{\prime}(5)=L^{\prime}(5)=3$
(c) there is not enough information to find $f(0)$.
3.4.11.2. Solution. The linear approximation is a line, passing through $(2, f(2))$, with slope $f^{\prime}(2)$. That is, the linear approximation to $f(x)$ about $x=2$ is the tangent line to $f(x)$ at $x=2$. It is shown below in red.

3.4.11.3. Solution. For any constant $a, f(a)=(2 a+5)$, and $f^{\prime}(a)=2$, so our approximation gives us

$$
f(x) \approx(2 a+5)+2(x-a)=2 x+5
$$

Since $f(x)$ itself is a linear function, the linear approximation is actually just $f(x)$ itself. As a consequence, the linear approximation is perfectly accurate for all values of $x$.

## Exercises - Stage 2

3.4.11.4. Solution. We have no idea what $f(0.93)$ is, but 0.93 is pretty close to 1 , and we definitely know $f(1)$. The linear approximation of $f(x)$ about $x=1$ is given by

$$
f(x) \approx f(1)+f^{\prime}(1)(x-1)
$$

So, we calculate:

$$
\begin{aligned}
f(1) & =\log (1)=0 \\
f^{\prime}(x) & =\frac{1}{x} \\
f^{\prime}(1) & =\frac{1}{1}=1
\end{aligned}
$$

Therefore,

$$
f(x) \approx 0+1(x-1)=x-1
$$

When $x=0.93$ :

$$
f(0.93) \approx 0.93-1=-0.07
$$


3.4.11.5. Solution. We approximate the function $f(x)=\sqrt{x}$ about $x=4$, since 4 is a perfect square and it is close to 5 .

$$
\begin{aligned}
f(4) & =\sqrt{4}=2 \\
f^{\prime}(x) & =\frac{1}{2 \sqrt{x}} \quad \Rightarrow \quad f^{\prime}(4)=\frac{1}{2 \sqrt{4}}=\frac{1}{4} \\
f(x) & \approx f(4)+f^{\prime}(4)(x-4)=2+\frac{1}{4}(x-4) \\
f(5) & \approx 2+\frac{1}{4}(5-4)=\frac{9}{4}
\end{aligned}
$$

We estimate $\sqrt{5} \approx \frac{9}{4}$.
Remark: $\left(\frac{9}{4}\right)^{2}=\frac{81}{16}$, which is pretty close to $\frac{80}{16}=5$. Our approximation seems pretty good.
3.4.11.6. Solution. We approximate the function $f(x)=\sqrt[5]{x}$. We need to centre the approximation about some value $x=a$ such that we know $f(a)$ and $f^{\prime}(a)$, and $a$ is not too far from 30 .

$$
\begin{aligned}
f(x) & =\sqrt[5]{x}=x^{\frac{1}{5}} \\
f^{\prime}(x) & =\frac{1}{5} x^{-\frac{4}{5}}=\frac{1}{5 \sqrt[5]{x}}
\end{aligned}
$$

$a$ needs to be a number whose fifth root we know. Since $\sqrt[5]{32}=2$, and 32 is reasonably close to $30, a=32$ is a great choice.

$$
\begin{aligned}
f(32) & =\sqrt[5]{32}=2 \\
f^{\prime}(32) & =\frac{1}{5 \cdot 2^{4}}=\frac{1}{80}
\end{aligned}
$$

The linear approximation of $f(x)$ about $x=32$ is

$$
f(x) \approx 2+\frac{1}{80}(x-32)
$$

When $x=30$ :

$$
f(30) \approx 2+\frac{1}{80}(30-32)=2-\frac{1}{40}=\frac{79}{40}
$$

We estimate $\sqrt[5]{30} \approx \frac{79}{40}$.
Remark: $\frac{79}{40}=1.975$, while $\sqrt[5]{30} \approx 1.97435$. This is a decent estimation.

## Exercises - Stage 3

3.4.11.7. Solution. If $f(x)=x^{3}$, then $f(10.1)=10.1^{3}$, which is the value we want to estimate. Let's take the linear approximation of $f(x)$ about $x=10$ :

$$
\begin{aligned}
f(10) & =10^{3}=1000 \\
f^{\prime}(x) & =3 x^{2} \\
f^{\prime}(10) & =3\left(10^{2}\right)=300 \\
f(a) & \approx f(10)+f^{\prime}(10)(x-10) \\
& =1000+300(x-10) \\
f(10.1) & \approx 1000+300(10.1-10)=1030
\end{aligned}
$$

We estimate $10.1^{3} \approx 1030$. If we calculate $10.1^{3}$ exactly (which is certainly possible to do by hand), we get 1030.301.
Remark: in the previous subsection, we used a constant approximation to estimate $10.1^{3} \approx 1000$. That approximation was easy to do in your head, in a matter of seconds. The linear approximation is more accurate, but not much faster than simply calculating $10.1^{3}$.
3.4.11.8. Solution. There are many possible answers. One is:

$$
f(x)=\sin x \quad a=0 \quad b=\pi
$$

We know that $f(\pi)=0$ and $f(0)=0$. Using a constant approximation of $f(x)$ about $x=0$, our estimation is $f(\pi) \approx f(0)=0$, which is exactly the correct value. However, is we make a linear approximation of $f(x)$ about $x=0$, we get

$$
f(\pi) \approx f(0)+f^{\prime}(0)(\pi-0)=\sin (0)+\cos (0) \pi=\pi
$$

which is not exactly the correct value.


Remark: in reality, we wouldn't estimate $\sin (\pi)$, because we know its value exactly. The purpose of this problem is to demonstrate that fancier approximations are not always more accurate. At the of this section, we'll talk about how to bound the error of your estimations, to make sure that you are finding something sufficiently accurate.
3.4.11.9. Solution. The linear approximation $L(x)$ of $f(x)$ about $x=a$ is chosen so that $L(a)=f(a)$ and $L^{\prime}(a)=f^{\prime}(a)$. So,

$$
\begin{aligned}
L^{\prime}(a) & =f^{\prime}(a)=\frac{1}{1+a^{2}} \\
\frac{1}{4} & =\frac{1}{1+a^{2}} \\
a & = \pm \sqrt{3}
\end{aligned}
$$

We've narrowed down $a$ to $\sqrt{3}$ or $-\sqrt{3}$. Recall the linear approximation of $f(x)$ about $x=a$ is $f(a)+f^{\prime}(a)(x-a)$, so its constant term is $f(a)-a f^{\prime}(a)=\arctan (a)-$
$\frac{a}{1+a^{2}}$. We compute this for $a=\sqrt{3}$ and $a=-\sqrt{3}$.

$$
\begin{aligned}
a=\sqrt{3}: \quad \arctan (a)-\frac{a}{1+a^{2}} & =\arctan (\sqrt{3})-\frac{\sqrt{3}}{1+(\sqrt{3})^{2}} \\
& =\frac{\pi}{3}-\frac{\sqrt{3}}{4}=\frac{4 \pi-\sqrt{27}}{12} \\
a=-\sqrt{3}: \quad \arctan (a)-\frac{a}{1+a^{2}} & =\arctan (-\sqrt{3})-\frac{-\sqrt{3}}{1+(-\sqrt{3})^{2}} \\
& =-\frac{\pi}{3}+\frac{\sqrt{3}}{4}=\frac{-4 \pi+\sqrt{27}}{12}
\end{aligned}
$$

So, when $a=\sqrt{3}$,

$$
L(x)=\frac{1}{4} x+\frac{4 \pi-\sqrt{27}}{12}
$$

and this does not hold when $a=-\sqrt{3}$. We conclude $a=\sqrt{3}$.

## - Exercises for § 3.4.3

## Exercises - Stage 1

3.4.11.1. Solution. If $Q(x)$ is the quadratic approximation of $f$ about 3, then $Q(3)=f(3), Q^{\prime}(3)=f^{\prime}(3)$, and $Q^{\prime \prime}(3)=f^{\prime \prime}(3)$. There is no guarantee that $f(x)$ and $Q(x)$ share the same third derivative, though, so we do not have enough information to know $f^{\prime \prime \prime}(3)$.

$$
\begin{aligned}
f(3) & =-3^{2}+6(3)=9 \\
f^{\prime}(3) & =\left.\frac{\mathrm{d}}{\mathrm{~d} x}\left\{-x^{2}+6 x\right\}\right|_{x=3}=-2 x+\left.6\right|_{x=3}=0 \\
f^{\prime \prime}(3) & =\left.\frac{\mathrm{d}^{2}}{\mathrm{~d} x^{2}}\left\{-x^{2}+6 x\right\}\right|_{x=3}=\left.\frac{\mathrm{d}}{\mathrm{~d} x}\{-2 x+6\}\right|_{x=3}=-2
\end{aligned}
$$

3.4.11.2. Solution. The quadratic approximation of $f(x)$ about $x=a$ is

$$
f(x) \approx f(a)+f^{\prime}(a)(x-a)+\frac{1}{2} f^{\prime \prime}(a)(x-a)^{2}
$$

We subsitute $f(a)=2 a+5, f^{\prime}(a)=2$, and $f^{\prime \prime}(a)=0$ :

$$
f(x) \approx(2 a+5)+2(x-a)=2 x+5
$$

So, our approximation is $f(x) \approx 2 x+5$.
Remark: Our approximation is exact for every value of $x$. This will always happen with a quadratic approximation of a function that is quadratic, linear, or constant.

## Exercises - Stage 2

3.4.11.3. Solution. We approximate the function $f(x)=\log x$ about the point $x=1$. We choose 1 because it is close to 0.93 , and we can evaluate $f(x)$ and its first two derivatives at $x=1$.

$$
\begin{aligned}
f(1) & =0 \\
f^{\prime}(x) & =\frac{1}{x} \quad \Rightarrow \quad f^{\prime}(1)=1 \\
f^{\prime \prime}(x) & =\frac{-1}{x^{2}} \quad \Rightarrow \quad f^{\prime \prime}(1)=-1
\end{aligned}
$$

So,

$$
\begin{aligned}
f(x) & \approx f(1)+f^{\prime}(1)(x-1)+\frac{1}{2} f^{\prime \prime}(1)(x-1)^{2} \\
& =0+(x-1)-\frac{1}{2}(x-1)^{2}
\end{aligned}
$$

When $x=0.93$ :

$$
\begin{aligned}
f(0.93) & \approx(0.93-1)-\frac{1}{2}(0.93-1)^{2}=-0.07-\frac{1}{2}(0.0049) \\
& =-0.07245
\end{aligned}
$$

We estimate $\log (0.93) \approx-0.07245$.
Remark: a calculator approximates $\log (0.93) \approx-0.07257$. We're pretty close.
3.4.11.4. Solution. We approximate the function $f(x)=\cos x$. We can easily evaluate $\cos x$ and $\sin x(\sin x$ will appear in the first derivative) at $x=0$, and 0 is quite close to $\frac{1}{15}$, so we centre our approximation about $x=0$.

$$
\begin{aligned}
f(0) & =1 \\
f^{\prime}(x) & =-\sin x \\
f^{\prime}(0) & =-\sin (0)=0 \\
f^{\prime \prime}(x) & =-\cos x \\
f^{\prime \prime}(0) & =-\cos (0)=-1
\end{aligned}
$$

Using the quadratic approximation $f(x) \approx f(0)+f^{\prime}(0)(x-0)+\frac{1}{2} f^{\prime \prime}(0)(x-0)^{2}$ :

$$
\begin{aligned}
f(x) & \approx 1-\frac{1}{2} x^{2} \\
f\left(\frac{1}{15}\right) & \approx 1-\frac{1}{2 \cdot 15^{2}}=\frac{449}{450}
\end{aligned}
$$

We approximate $\cos \left(\frac{1}{15}\right) \approx \frac{449}{450}$.
Remark: $\frac{449}{450}=0.99 \overline{77}$, while a calculator gives $\cos \left(\frac{1}{15}\right) \approx 0.9977786$. Our approximation has an error of about 0.000001 .
3.4.11.5. Solution. The quadratic approximation of a function $f(x)$ about $x=a$ is

$$
f(x) \approx f(a)+f^{\prime}(a)(x-a)+\frac{1}{2} f^{\prime \prime}(a)(x-a)^{2}
$$

We compute derivatives.

$$
\begin{aligned}
f(0) & =e^{0}=1 \\
f^{\prime}(x) & =2 e^{2 x} \\
f^{\prime}(0) & =2 e^{0}=2 \\
f^{\prime \prime}(x) & =4 e^{2 x} \\
f^{\prime \prime}(0) & =4 e^{0}=4
\end{aligned}
$$

Substituting:

$$
\begin{aligned}
& f(x) \approx 1+2(x-0)+\frac{4}{2}(x-0)^{2} \\
& f(x) \approx 1+2 x+2 x^{2}
\end{aligned}
$$

3.4.11.6. Solution. There are a few functions we could choose to approximate. For example:

- $f(x)=x^{4 / 3}$. In this case, we would probably choose to approximate $f(x)$ about $x=8$ (since 8 is a cube, $8^{4 / 3}=2^{4}=16$ is something we can evaluate) or $x=1$.
- $f(x)=5^{x}$. We can easily figure out $f(x)$ when $x$ is a whole number, so we would want to centre our approximation around some whole number $x=a$, but then $f^{\prime}(a)=5^{a} \log (5)$ gives us a problem: without a calculator, it's hard to know what $\log (5)$ is.
- Since $5^{4 / 3}=5 \sqrt[3]{5}$, we can use $f(x)=5 \sqrt[3]{x}$. As in the first bullet, we would centre about $x=8$, or $x=1$.

There isn't much difference between the first and third bullets. We'll go with $f(x)=$ $5 \sqrt[3]{x}$, centred about $x=8$.

$$
\begin{aligned}
f(x)=5 x^{\frac{1}{3}} & \Rightarrow f(8)=5 \cdot 2=10 \\
f^{\prime}(x)=\frac{5}{3} x^{-\frac{2}{3}} & \Rightarrow f^{\prime}(8)=\frac{5}{3}\left(2^{-2}\right)=\frac{5}{12} \\
f^{\prime \prime}(x)=\frac{5}{3}\left(-\frac{2}{3}\right) x^{-\frac{5}{3}}=-\frac{10}{9} x^{-\frac{5}{3}} & \Rightarrow f^{\prime \prime}(8)=-\frac{10}{9}\left(2^{-5}\right)=-\frac{5}{144}
\end{aligned}
$$

Using the quadratic approximation $f(x) \approx f(a)+f^{\prime}(a)(x-a)+\frac{1}{2} f^{\prime \prime}(a)(x-a)^{2}$ :

$$
\begin{aligned}
& f(x) \approx 10+\frac{5}{12}(x-8)-\frac{5}{288}(x-8)^{2} \\
& f(5) \approx 10+\frac{5}{12}(-3)-\frac{5}{288}(9)=\frac{275}{32}
\end{aligned}
$$

We estimate $5^{4 / 3} \approx \frac{275}{32}$
Remark: $\frac{275}{32}=8.59375$, and a calculator gives $5^{4 / 3} \approx 8.5499$. Although 5 and 8 are somewhat far apart, our estimate is only off by about 0.04 .

### 3.4.11.7. Solution.

- [3.4.11.7.a] For every value of $n$, the term being added is simply the constant

1. So, $\sum_{n=5}^{30} 1=1+1+\cdots+1$. The trick is figuring out how many 1 s are added.

Our index $n$ takes on all integers from 5 to 30, including 5 and 30, which is 26 numbers. So, $\sum_{n=5}^{30}=26$.
If you're having a hard time seeing why the sum is 26 , instead of 25 , think of it this way: there are thirty numbers in the collection $\{1,2,3,4,5,6, \ldots, 29,30\}$. If we remove the first four, we get $30-4=26$ numbers in the collection $\{5,6, \ldots, 30\}$.

- [3.4.11.7.b]

$$
\begin{aligned}
& \sum_{n=1}^{3}\left[2(n+3)-n^{2}\right] \\
& \quad=\underbrace{2(1+3)-1^{2}}_{n=1}+\underbrace{2(2+3)-2^{2}}_{n=2}+\underbrace{2(3+3)-3^{2}}_{n=3} \\
& \quad=8-1+10-4+12-9=16
\end{aligned}
$$

- [3.4.11.7.c]

$$
\begin{aligned}
\sum_{n=1}^{10}\left[\frac{1}{n}-\frac{1}{n+1}\right] & =\underbrace{\frac{1}{1}-\frac{1}{1+1}}_{n=1}+\underbrace{\frac{1}{2}-\frac{1}{2+1}}_{n=2}+\underbrace{\frac{1}{3}-\frac{1}{3+1}}_{n=3} \\
& +\underbrace{\frac{1}{4}-\frac{1}{4+1}}_{n=4}+\underbrace{\frac{1}{5}-\frac{1}{5+1}}_{n=5}+\underbrace{\frac{1}{6}-\frac{1}{6+1}}_{n=6} \\
& +\underbrace{\frac{1}{7}-\frac{1}{7+1}}_{n=7}+\underbrace{\frac{1}{8}-\frac{1}{8+1}}_{n=8}+\underbrace{\frac{1}{9}-\frac{1}{9+1}}_{n=9} \\
& +\underbrace{\frac{1}{10}-\frac{1}{10+1}}_{n=10}
\end{aligned}
$$

Most of these cancel!

$$
=\frac{1}{1} \underbrace{-\frac{1}{2}+\frac{1}{2}}_{0} \underbrace{-\frac{1}{3}+\frac{1}{3}}_{0} \underbrace{-\frac{1}{4}+\frac{1}{4}}_{0} \underbrace{-\frac{1}{5}+\frac{1}{5}}_{0} \underbrace{-\frac{1}{6}+\frac{1}{6}}_{0}
$$

$$
\begin{aligned}
& \underbrace{-\frac{1}{7}+\frac{1}{7}}_{0} \underbrace{-\frac{1}{8}+\frac{1}{8}}_{0} \underbrace{-\frac{1}{9}+\frac{1}{9}}_{0} \underbrace{-\frac{1}{10}+\frac{1}{10}}_{0}-\frac{1}{11} \\
& =1-\frac{1}{11}=\frac{10}{11}
\end{aligned}
$$

- [3.4.11.7.d]

$$
\begin{aligned}
\sum_{n=1}^{4} \frac{5 \cdot 2^{n}}{4^{n+1}} & =5 \sum_{n=1}^{4} \frac{2^{n}}{4 \cdot 4^{n}}=\frac{5}{4} \sum_{n=1}^{4} \frac{2^{n}}{4^{n}}=\frac{5}{4} \sum_{n=1}^{4} \frac{1}{2^{n}} \\
& =\frac{5}{4}(\underbrace{\frac{1}{2}}_{n=1}+\underbrace{\frac{1}{4}}_{n=2}+\underbrace{\frac{1}{8}}_{n=3}+\underbrace{\frac{1}{16}}_{n=4})=\frac{75}{64}
\end{aligned}
$$

3.4.11.8. Solution. For each of these, there are many solutions. We provide some below.
a $1+2+3+4+5=\sum_{n=1}^{5} n$
b $2+4+6+8=\sum_{n=1}^{4} 2 n$
c $3+5+7+9+11=\sum_{n=1}^{5}(2 n+1)$
d $9+16+25+36+49=\sum_{n=3}^{7} n^{2}$
e $9+4+16+5+25+6+36+7+49+8=\sum_{n=3}^{7}\left(n^{2}+n+1\right)$
f $8+15+24+35+48=\sum_{n=3}^{7}\left(n^{2}-1\right)$
g $3-6+9-12+15-18=\sum_{n=1}^{6}(-1)^{n+1} 3 n$
Remark: if we had written $(-1)^{n}$ instead of $(-1)^{n+1}$, with everything else the same, the signs would have been reversed.

## Exercises - Stage 3

3.4.11.9. Solution. Let's start by taking the first two derivative of $f(x)$.

$$
\begin{array}{rlrl}
f(x) & =2 \arcsin x & & \\
f^{\prime}(x) & =\frac{2}{\sqrt{1-x^{2}}} & & f(0)=2(0)=0 \\
f^{\prime \prime}(x) & =\frac{\mathrm{d}}{\mathrm{~d} x}\left\{2\left(1-x^{2}\right)^{-\frac{1}{2}}\right\} & & f^{\prime}(0)=\frac{2}{1}=2 \\
& =2\left(-\frac{1}{2}\right)\left(1-x^{2}\right)^{-\frac{3}{2}}(-2 x) & & \\
& =\frac{2 x}{\left(\sqrt{1-x^{2}}\right)^{3}} & & \\
\text { (chain rule) } \\
& & & f^{\prime \prime}(0)=0
\end{array}
$$

Now, we can find the quadratic approximation about $x=0$.

$$
\begin{aligned}
f(x) & \approx f(0)+f^{\prime}(0) x+\frac{1}{2} f^{\prime \prime}(0) x^{2} \\
& =2 x \\
f(1) & \approx 2
\end{aligned}
$$

Our quadratic approximation gives $2 \arcsin (1) \approx 2$. However, $2 \arcsin (1)$ is exactly equal to $2\left(\frac{\pi}{2}\right)=\pi$. We've just made the rather unfortunate approximation $\pi \approx 2$.
3.4.11.10. Solution. From the text, the quadratic approximation of $e^{x}$ about $x=0$ is

$$
e^{x} \approx 1+x+\frac{1}{2} x^{2}
$$

So,

$$
e=e^{1} \approx 1+1+\frac{1}{2}=2.5
$$

We estimate $e \approx 2.5$.
Remark: actually, $e \approx 2.718$.

### 3.4.11.11. Solution.

- First, we'll show that 3.4.11.11.a,3.4.11.11.d, 3.4.11.11.e are equivalent:

$$
\begin{aligned}
3.4 .11 .11 . \mathrm{d} & =2 \sum_{n=1}^{10} n=2(1+2+\cdots+10) \\
& =2(1)+2(2)+\cdots+2(10)=\sum_{n=1}^{10} 2 n \\
& =3 \cdot 4 \cdot 11.11 \cdot \mathrm{a}
\end{aligned}
$$

So 3.4.11.11.a and 3.4.11.11.d are equivalent.

$$
\text { 3.4.11.11.e }=2 \sum_{n=2}^{11}(n-1)=2(1+2+\cdots+10)=3.4 .11 .11 . \mathrm{d}
$$

So 3.4.11.11.e and 3.4.11.11.d are equivalent.

- Second, we'll show that 3.4.11.11.b and 3.4.11.11.g are equivalent.

$$
\begin{aligned}
\text { 3.4.11.11.g } & =\frac{1}{4} \sum_{n=1}^{10}\left(\frac{4^{n+1}}{2^{n}}\right)=\frac{1}{4} \sum_{n=1}^{10}\left(\frac{4 \cdot 4^{n}}{2^{n}}\right) \\
& =\frac{4}{4} \sum_{n=1}^{10}\left(\frac{4^{n}}{2^{n}}\right)=\sum_{n=1}^{10}\left(\frac{4}{2}\right)^{n} \\
& =\sum_{n=1}^{10} 2^{n}=3.4 .11 .11 . \mathrm{b}
\end{aligned}
$$

- Third, we'll show that 3.4.11.11.c and 3.4.11.11.f are equivalent.

$$
\begin{aligned}
\text { 3.4.11.11.f } & =\sum_{n=5}^{14}(n-4)^{2}=1^{2}+2^{2}+\cdots+10^{2}=\sum_{n=1}^{10} n^{2} \\
& =3 \cdot 4.11 .11 . \mathrm{c}
\end{aligned}
$$

- Now, we have three groups, where each group consists of equivalent expressions. To be quite thorough, we should show that no two of these groups contain expressions that are secretly equivalent. They would be hard to evaluate, but we can bound them and show that no two expressions in two separate groups could possibly be equivalent. Notice that

$$
\begin{aligned}
& \sum_{n=1}^{10} 2^{n}=2^{1}+2^{2}+\cdots+2^{10}>2^{10}=1024 \\
& \sum_{n=1}^{10} n^{2}<\sum_{n=1}^{10} 10^{2}=10(100)=1000 \\
& \sum_{n=1}^{10} n^{2}=1^{2}+2^{2}+\cdots 8^{2}+9^{2}+10^{2}>8^{2}+9^{2}+10^{2}=245 \\
& \sum_{n=1}^{10} 2 n<\sum_{n=1}^{10} 20=200
\end{aligned}
$$

The expressions in the blue group add to less than 200, but the expressions in the green group add to more than 245 , and the expressions in the red group add to more than 1024, so the blue groups expressions can't possibly simplify to the same number as the red and green group expressions.
The expressions in the green group add to less than 1000. Since the expressions in the red group add to more than 1024, the expressions in the green and red groups can't possibly simplify to the same numbers.
We group our expressions in to collections of equivalent expressions as follows:

- [3.4.11.11. $\mathrm{a}=3 \cdot 4 \cdot 11.11 . \mathrm{d}=3 \cdot 4.11 .11 . \mathrm{e}]$ and


## Exercises for ${ }^{[3.11 .11} \mathrm{b}=3.31 .41 .2$, and <br> - [3.4.11.11. $\mathrm{c}=3.4 .11 .11 . \mathrm{f}]$.

## Exercises - Stage 1

3.4.11.1. Solution. Since $T_{3}(x)$ is the third-degree Taylor polynomial for $f(x)$ about $x=1$ :

- $T_{3}(1)=f(1)$
- $T_{3}^{\prime}(1)=f^{\prime}(1)$
- $T_{3}^{\prime \prime}(1)=f^{\prime \prime}(1)$
- $T_{3}^{\prime \prime \prime}(1)=f^{\prime \prime \prime}(1)$

In particular, $f^{\prime \prime}(1)=T_{3}^{\prime \prime}(1)$.

$$
\begin{aligned}
T_{3}^{\prime}(x) & =3 x^{2}-10 x+9 \\
T_{3}^{\prime \prime}(x) & =6 x-10 \\
T_{3}^{\prime \prime}(1) & =6-10=-4
\end{aligned}
$$

So, $f^{\prime \prime}(1)=-4$.
3.4.11.2. Solution. In Question 3.4.11.1, we differentiated the Taylor polynomial to find its derivative. We don't really want to differentiate this ten times, though, so let's look for another way. Unlike Question 3.4.11.1, our Taylor polynomial is given to us in a form very similar to its definition. The $n$th degree Taylor polynomial for $f(x)$ about $x=5$ is

$$
T_{n}(x)=\sum_{k=0}^{n} \frac{f^{(k)}(5)}{k!}(x-5)^{k}
$$

So,

$$
\sum_{k=0}^{n} \frac{f^{(k)}(5)}{k!}(x-5)^{k}=\sum_{k=0}^{n} \frac{2 k+1}{3 k-9}(x-5)^{k}
$$

For any $k$ from 0 to $n$,

$$
\frac{f^{(k)}(5)}{k!}=\frac{2 k+1}{3 k-9}
$$

In particular, when $k=10$,

$$
\begin{aligned}
\frac{f^{(10)}(5)}{10!} & =\frac{20+1}{30-9}=1 \\
f^{(10)}(5) & =10!
\end{aligned}
$$

## Exercises - Stage 3

3.4.11.3. Solution. The fourth-degree Maclaurin polynomial for $f(x)$ is

$$
T_{4}(x)=f(0)+f^{\prime}(0) x+\frac{1}{2} f^{\prime \prime}(0) x^{2}+\frac{1}{3!} f^{\prime \prime \prime}(0) x^{3}+\frac{1}{4!} f^{(4)}(0) x^{4}
$$

while the third-degree Maclaurin polynomial for $f(x)$ is

$$
T_{3}(x)=f(0)+f^{\prime}(0) x+\frac{1}{2} f^{\prime \prime}(0) x^{2}+\frac{1}{3!} f^{\prime \prime \prime}(0) x^{3}
$$

So, we simply "chop off" the part of $T_{4}(x)$ that includes $x^{4}$ :

$$
T_{3}(x)=-x^{3}+x^{2}-x+1
$$

3.4.11.4. Solution. We saw this kind of problem in Question 3.4.11.3. The fourth-degree Taylor polynomial for $f(x)$ about $x=1$ is

$$
\begin{gathered}
T_{4}(x)=f(1)+f^{\prime}(1)(x-1)+\frac{1}{2} f^{\prime \prime}(1)(x-1)^{2}+\frac{1}{3!} f^{\prime \prime \prime}(1)(x-1)^{3} \\
+\frac{1}{4!} f^{(4)}(1)(x-1)^{4}
\end{gathered}
$$

while the third-degree Taylor polynomial for $f(x)$ about $x=1$ is

$$
T_{3}(x)=f(1)+f^{\prime}(1)(x-1)+\frac{1}{2} f^{\prime \prime}(1)(x-1)^{2}+\frac{1}{3!} f^{\prime \prime \prime}(1)(x-1)^{3}
$$

In in Question 3.4.11.3 we "chopped off" the term of degree 4 to get $T_{3}(x)$. However, our polynomial is not in this form. It's not clear, right away, what the term $f^{(4)}(x-$ $1)^{4}$ is in our given $T_{4}(x)$. So, we will use a different method from Question 3.4.11.3. One option is to do some fancy algebra to get $T_{4}(x)$ into the standard form of a Taylor polynomial. Another option (which we will use) is to recover $f(1), f^{\prime}(1)$, $f^{\prime \prime}(1)$, and $f^{\prime \prime \prime}(1)$ from $T_{4}(x)$.

Recall that $T_{4}(x)$ and $f(x)$ have the same values at $x=1$ (although maybe not anywhere else!), and they also have the same first, second, third, and fourth derivatives at $x=1$ (but again, maybe not anywhere else, and maybe their fifth derivatives don't agree). This tells us the following:

$$
\begin{aligned}
T_{4}(x) & =x^{4}+x^{3}-9 & \Rightarrow & f(1)=T_{4}(1)=-7 \\
T_{4}^{\prime}(x) & =4 x^{3}+3 x^{2} & \Rightarrow & f^{\prime}(1)=T_{4}^{\prime}(1)=7 \\
T_{4}^{\prime \prime}(x) & =12 x^{2}+6 x & \Rightarrow & f^{\prime \prime}(1)=T_{4}^{\prime \prime}(1)=18 \\
T_{4}^{\prime \prime \prime}(x) & =24 x+6 & \Rightarrow & f^{\prime \prime \prime}(1)=T_{4}^{\prime \prime \prime}(1)=30
\end{aligned}
$$

Now, we can write the third-degree Taylor polynomial for $f(x)$ about $x=1$ :

$$
T_{3}(x)=-7+7(x-1)+\frac{1}{2}(18)(x-1)^{2}+\frac{1}{3!}(30)(x-1)^{3}
$$

$$
=-7+7(x-1)+9(x-1)^{2}+5(x-1)^{3}
$$

Remark: expanding the expression above, we get the equivalent polynomial $T_{3}(x)=$ $5 x^{3}-6 x^{2}+4 x-10$. From this, it is clear that we can't just "chop off" the term with $x^{4}$ to change $T_{4}(x)$ into $T_{3}(x)$ when the Taylor polynomial is not centred about $x=0$.
3.4.11.5. Solution. The $n$th degree Taylor polynomial for $f(x)$ about $x=5$ is

$$
T_{n}(x)=\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(5)(x-5)^{k}
$$

We expand this somewhat:

$$
\begin{aligned}
& T_{n}(x)=f(5)+f^{\prime}(x-5)+\cdots+ \frac{1}{10!} f^{(10)}(5)(x-5)^{10} \\
&+\cdots \\
&+\frac{1}{n!} f^{(n)}(5)(x-1)^{n}
\end{aligned}
$$

So, the coefficient of $(x-5)^{10}$ is $\frac{1}{10!} f^{(10)}(5)$. Expanding the given form of the Taylor polynomial:

$$
\begin{aligned}
T_{n}(x)= & \sum_{k=0}^{n / 2} \frac{2 k+1}{3 k-9}(x-5)^{2 k} \\
= & \underbrace{\frac{1}{-9}}_{k=0}+\underbrace{\frac{3}{-6}(x-5)^{2}}_{k=1}+\cdots+\underbrace{\frac{11}{6}(x-5)^{10}}_{k=5}+\cdots \\
& +\underbrace{\frac{n+1}{(3 / 2) n-9}(x-5)^{n}}_{k=n / 2}
\end{aligned}
$$

Equating the coefficients of $(x-5)^{10}$ in the two expressions:

$$
\begin{aligned}
\frac{1}{10!} f^{(10)}(5) & =\frac{11}{6} \\
f^{(10)}(5) & =\frac{11 \cdot 10!}{6}
\end{aligned}
$$

3.4.11.6. Solution. Since $T_{3}(x)$ is the third-degree Taylor polynomial for $f(x)$ about $x=a$, we know the following things to be true:

- $f(a)=T_{3}(a)$
- $f^{\prime}(a)=T_{3}^{\prime}(a)$
- $f^{\prime \prime}(a)=T_{3}^{\prime \prime}(a)$
- $f^{\prime \prime \prime}(a)=T_{3}^{\prime \prime \prime}(a)$

But, some of these don't look super useful. For instance, if we try to use the first bullet, we get this equation:

$$
a^{3}\left[2 \log a-\frac{11}{3}\right]=-\frac{2}{3} \sqrt{e^{3}}+3 e a-6 \sqrt{e} a^{2}+a^{3}
$$

Solving this would be terrible. Instead, let's think about how the equations look when we move further down the list. Since $T_{3}(x)$ is a cubic equation, $T_{3}^{\prime \prime \prime}(x)$ is a constant (and so $T_{3}^{\prime \prime \prime}(a)$ does not depend on $a$ ). That sounds like it's probably the simplest option. Let's start differentiating. We'll need to know both $f^{\prime \prime \prime}(a)$ and $T_{3}^{\prime \prime \prime}(a)$.

$$
\begin{aligned}
& f(x)=x^{3}\left[2 \log x-\frac{11}{3}\right] \\
& f^{\prime}(x)=x^{3}\left[\frac{2}{x}\right]+3 x^{2}\left[2 \log x-\frac{11}{3}\right]=6 x^{2} \log x-9 x^{2} \\
& f^{\prime \prime}(x)=6 x^{2} \cdot \frac{1}{x}+12 x \log x-18 x=12 x \log x-12 x \\
& f^{\prime \prime \prime}(x)=12 x \cdot \frac{1}{x}+12 \log x-12=12 \log x \\
& f^{\prime \prime \prime}(a)=12 \log a
\end{aligned}
$$

Now, let's move to the Taylor polynomial. Remember that $e$ is a constant.

$$
\begin{aligned}
T_{3}(x) & =-\frac{2}{3} \sqrt{e^{3}}+3 e x-6 \sqrt{e} x^{2}+x^{3} \\
T_{3}^{\prime}(x) & =3 e-12 \sqrt{e} x+3 x^{2} \\
T_{3}^{\prime \prime}(x) & =-12 \sqrt{e}+6 x \\
T_{3}^{\prime \prime \prime}(x) & =6 \\
T_{3}^{\prime \prime \prime}(a) & =6
\end{aligned}
$$

The final bullet point gives us the equation:

$$
\begin{aligned}
f^{\prime \prime \prime}(a) & =T_{3}^{\prime \prime \prime}(a) \\
12 \log a & =6 \\
\log a & =\frac{1}{2} \\
a & =e^{\frac{1}{2}}
\end{aligned}
$$

So, $a=\sqrt{e}$.

## - Exercises for § 3.4.5

## Exercises - Stage 1

3.4.11.1. Solution. If we were to find the 16 th degree Maclaurin polynomial for a generic function, we might expect to have to differentiate 16 times (ugh). But, we know that the derivatives of sines and cosines repeat themselves. So, it's enough to figure out the pattern:

$$
\begin{array}{rlrl}
f(x) & =\sin x+\cos x & f(0) & =1 \\
f^{\prime}(x) & =\cos x-\sin x & f^{\prime}(0) & =1 \\
f^{\prime \prime}(x) & =-\sin x-\cos x & f^{\prime \prime}(0) & =-1 \\
f^{\prime \prime \prime}(x) & =-\cos x+\sin x & f^{\prime \prime \prime}(0) & =-1 \\
f^{(4)} & =\sin x+\cos x & f^{(4)}(0) & =1
\end{array}
$$

Since $f^{(4)}(x)=f(x)$, our derivatives repeat from here. They follow the pattern: $+1,+1,-1,-1$.

$$
\begin{aligned}
T_{16}(x)=1 & +x-\frac{1}{2} x^{2}-\frac{1}{3!} x^{3}+\frac{1}{4!} x^{4}+\frac{1}{5!} x^{5}-\frac{1}{6!} x^{6}-\frac{1}{7!} x^{7}+\frac{1}{8!} x^{8}+\frac{1}{9!} x^{9} \\
& -\frac{1}{10!} x^{10}-\frac{1}{11!} x^{11}+\frac{1}{12!} x^{12}+\frac{1}{13!} x^{13}-\frac{1}{14!} x^{14}-\frac{1}{15!} x^{15} \\
& +\frac{1}{16!} x^{16}
\end{aligned}
$$

3.4.11.2. Solution. A Taylor polynomial gives a polynomial approximation for a function $s(t)$. Since $s(t)$ is itself a polynomial, any $n$ th-degree Taylor polynomial, with $n$ greater than or equal to the degree of $s(t)$, will simply give $s(t)$. So, in our case, $T_{100}(t)=s(t)=4.9 t^{2}-t+10$.
If that isn't satisfying, you can go through the normal method of calculating $T_{100}(t)$.

$$
\begin{array}{rlrl}
s(t) & =4.9 t^{2}-t+10 & s(5) & =4.9(25)-5+10=127.5 \\
s^{\prime}(t) & =9.8 t-1 & s^{\prime}(5) & =9.8(5)-1=48 \\
s^{\prime \prime}(t) & =9.8 & s^{\prime \prime}(5) & =9.8
\end{array}
$$

The rest of the derivatives of $s(t)$ are identically zero, so they are (in particular) zero when $t=5$. Therefore,

$$
\begin{aligned}
T_{100}(t) & =127.5+48(t-5)+\frac{1}{2} 9.8(t-5)^{2} \\
& =127.5+48(t-5)+4.9(t-5)^{2}
\end{aligned}
$$

We can now check that $T_{100}(t)$ really is the same as $s(t)$.

$$
\begin{aligned}
T_{100}(t) & =127.5+48(t-5)+4.9(t-5)^{2} \\
& =127.5+48(t-5)+4.9\left(t^{2}-10 t+25\right) \\
& =[127.5+48(-5)+4.9(25)]+[48-4.9(10)] t+4.9 t^{2} \\
& =10-t+4.9 t^{2}=s(t)
\end{aligned}
$$

3.4.11.3. Solution. Let's start by differentiating $f(x)$ and looking for a pattern.

Remember that $\log 2=\log _{e} 2$ is a constant number.

$$
\begin{aligned}
f(x) & =2^{x} \\
f^{\prime}(x) & =2^{x} \log 2 \\
f^{\prime \prime}(x) & =2^{x}(\log 2)^{2} \\
f^{(3)}(x) & =2^{x}(\log 2)^{3} \\
f^{(4)}(x) & =2^{x}(\log 2)^{4} \\
f^{(5)}(x) & =2^{x}(\log 2)^{5}
\end{aligned}
$$

So, in general,

$$
f^{(k)}(x)=2^{x}(\log 2)^{k}
$$

We notice that this formula works even when $k=0$ and $k=1$. When $x=1$,

$$
f^{(k)}(1)=2(\log 2)^{k}
$$

The $n$th degree Taylor polynomial of $f(x)$ about $x=1$ is

$$
\begin{aligned}
T_{n}(x) & =\sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^{k} \\
& =\sum_{k=0}^{n} \frac{2(\log 2)^{k}}{k!}(x-1)^{k}
\end{aligned}
$$

3.4.11.4. Solution. We need to know the first six derivatives of $f(x)$ at $x=1$. Let's get started.

$$
\begin{array}{rlrl}
f(x) & =x^{2} \log x+2 x^{2}+5 & & f(1)=7 \\
f^{\prime}(x) & =\left(x^{2}\right) \frac{1}{x}+(2 x) \log x+4 x & & \\
& =2 x \log x+5 x & & f^{\prime}(1)=5 \\
f^{\prime \prime}(x) & =(2 x) \frac{1}{x}+(2) \log x+5 & & f^{\prime \prime}(1)=7 \\
& =2 \log x+7 & f^{\prime \prime \prime}(1)=2 \\
f^{\prime \prime \prime}(x) & =2 x^{-1} & f^{(4)}(1)=-2 \\
f^{(4)} & =-2 x^{-2} & & f^{(5)}(1)=4 \\
f^{(5)} & =4 x^{-3} & f^{(6)}(1)=-12
\end{array}
$$

Now, we can plug in.

$$
\begin{aligned}
T_{6}(x)= & f(1)+f^{\prime}(1)(x-1)+\frac{1}{2} f^{\prime \prime}(1)(x-1)^{2}+\frac{1}{3!} f^{\prime \prime \prime}(1)(x-1)^{3} \\
& +\frac{1}{4!} f^{(4)}(1)(x-1)^{4}+\frac{1}{5!} f^{(5)}(1)(x-1)^{5}+\frac{1}{6!} f^{(6)}(1)(x-1)^{6}
\end{aligned}
$$

$$
\begin{aligned}
= & 7+5(x-1)+\frac{1}{2}(7)(x-1)^{2}+\frac{1}{3!}(2)(x-1)^{3} \\
& +\frac{1}{4!}(-2)(x-1)^{4}+\frac{1}{5!}(4)(x-1)^{5}+\frac{1}{6!}(-12)(x-1)^{6} \\
= & 7+5(x-1)+\frac{7}{2}(x-1)^{2}+\frac{1}{3}(x-1)^{3}-\frac{1}{12}(x-1)^{4} \\
& +\frac{1}{30}(x-1)^{5}-\frac{1}{60}(x-1)^{6}
\end{aligned}
$$

### 3.4.11.5. Solution. We'll start by differentiating and looking for a pattern.

$$
f(x)=\frac{1}{1-x}=(1-x)^{-1}
$$

Using the chain rule,

$$
\begin{aligned}
f^{\prime}(x) & =(-1)(1-x)^{-2}(-1)=(1-x)^{-2} \\
f^{\prime \prime}(x) & =(-2)(1-x)^{-3}(-1)=2(1-x)^{-3} \\
f^{(3)}(x) & =(-3)(2)(1-x)^{-4}(-1)=2(3)(1-x)^{-4} \\
f^{(4)}(x) & =(-4)(2)(3)(1-x)^{-5}(-1)=2(3)(4)(1-x)^{-5} \\
f^{(5)}(x) & =(-5)(2)(3)(4)(1-x)^{-6}(-1)=2(3)(4)(5)(1-x)^{-6}
\end{aligned}
$$

Recognizing the pattern,

$$
\begin{aligned}
& f^{(k)}(x)=k!(1-x)^{-(k+1)} \\
& f^{(k)}(0)=k!(1)^{-(k+1)}=k!
\end{aligned}
$$

The $n$th degree Maclaurin polynomial for $f(x)$ is

$$
\begin{aligned}
T_{n}(x) & =\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^{k} \\
& =\sum_{k=0}^{n} \frac{k!}{k!} x^{k} \\
& =\sum_{k=0}^{n} x^{k}
\end{aligned}
$$

## Exercises - Stage 3

3.4.11.6. Solution. We'll need to know the first three derivatives of $x^{x}$ at $x=1$. This is a good review of logarithmic differentiation, covered in Section 2.10.

$$
\begin{aligned}
f(x) & =x^{x} \\
& \Longrightarrow f(1)=1 \\
\log (f(x)) & =\log \left(x^{x}\right)=x \log x \\
\frac{\mathrm{~d}}{\mathrm{~d} x}\{\log (f(x))\} & =\frac{\mathrm{d}}{\mathrm{~d} x}\{x \log x\}
\end{aligned}
$$

$$
\frac{f^{\prime}(x)}{f(x)}=x \cdot \frac{1}{x}+\log x=1+\log x
$$

So

$$
\begin{aligned}
f^{\prime}(x) & =x^{x}[1+\log x] \\
& \Longrightarrow f^{\prime}(1)=1 \\
f^{\prime \prime}(x) & =\frac{\mathrm{d}}{\mathrm{~d} x}\left\{x^{x}\right\}[1+\log x]+x^{x} \frac{\mathrm{~d}}{\mathrm{~d} x}\{1+\log x\} \\
& =\left(x^{x}[1+\log x]\right)[1+\log x]+x^{x} \cdot \frac{1}{x} \\
& =x^{x}\left((1+\log x)^{2}+\frac{1}{x}\right) \\
& \Longrightarrow f^{\prime \prime}(1)=2 \\
f^{\prime \prime \prime}(x) & =\frac{\mathrm{d}}{\mathrm{~d} x}\left\{x^{x}\right\}\left((1+\log x)^{2}+\frac{1}{x}\right)+x^{x} \frac{\mathrm{~d}}{\mathrm{~d} x}\left\{(1+\log x)^{2}+\frac{1}{x}\right\} \\
& =x^{x}[1+\log x]\left((1+\log x)^{2}+\frac{1}{x}\right)+x^{x}\left[\frac{2}{x}(1+\log x)-\frac{1}{x^{2}}\right] \\
& =x^{x}\left((1+\log x)^{3}+\frac{3}{x}(1+\log x)-\frac{1}{x^{2}}\right) \\
& \Longrightarrow f^{\prime \prime \prime}(1)=3
\end{aligned}
$$

Now, we can plug in:

$$
\begin{aligned}
T_{3}(x) & =f(1)+f^{\prime}(1)(x-1)+\frac{1}{2} f^{\prime \prime}(1)(x-1)^{2}+\frac{1}{3!} f^{\prime \prime \prime}(1)(x-1)^{3} \\
& =1+1(x-1)+\frac{1}{2}(2)(x-1)^{2}+\frac{1}{6}(3)(x-1)^{3} \\
& =1+(x-1)+(x-1)^{2}+\frac{1}{2}(x-1)^{3}
\end{aligned}
$$

3.4.11.7. Solution. We note that $6 \arctan \left(\frac{1}{\sqrt{3}}\right)=6\left(\frac{\pi}{6}\right)=\pi$. We will find the 5th-degree Maclaurin polynomial $T_{5}(x)$ for $f(x)=6 \arctan x$. Then $\pi=f\left(\frac{1}{\sqrt{3}}\right) \approx$ $T_{5}\left(\frac{1}{\sqrt{3}}\right)$. Let's begin by finding the first five derivatives of $f(x)=6 \arctan x$.

$$
\begin{aligned}
f(x) & =6 \arctan x \\
& \Longrightarrow f(0)=0 \\
f^{\prime}(x) & =6\left(\frac{1}{1+x^{2}}\right) \\
& \Longrightarrow f^{\prime}(0)=6 \\
f^{\prime \prime}(x) & =6\left(\frac{0-2 x}{\left(1+x^{2}\right)^{2}}\right)=-12\left(\frac{x}{\left(1+x^{2}\right)^{2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \Longrightarrow f^{\prime \prime}(0)=0 \\
f^{\prime \prime \prime}(x) & =-12\left(\frac{\left(1+x^{2}\right)^{2}-x \cdot 2\left(1+x^{2}\right)(2 x)}{\left(1+x^{2}\right)^{4}}\right) \\
& =-12\left(\frac{\left(1+x^{2}\right)-4 x^{2}}{\left(1+x^{2}\right)^{3}}\right) \\
& =-12\left(\frac{1-3 x^{2}}{\left(1+x^{2}\right)^{3}}\right) \\
& \Longrightarrow f^{\prime \prime \prime}(0)=-12 \\
f^{(4)}(x) & =-12\left(\frac{\left(1+x^{2}\right)^{3}(-6 x)-\left(1-3 x^{2}\right) \cdot 3\left(1+x^{2}\right)^{2}(2 x)}{\left(1+x^{2}\right)^{6}}\right) \\
& =-12\left(\frac{-6 x\left(1+x^{2}\right)-6 x\left(1-3 x^{2}\right)}{\left(1+x^{2}\right)^{4}}\right) \\
& =144\left(\frac{x-x^{3}}{\left(1+x^{2}\right)^{4}}\right) \\
& \Longrightarrow f^{(4)}(0)=0 \\
f^{(5)}(x) & =144\left(\frac{\left(1+x^{2}\right)^{4}\left(1-3 x^{2}\right)-\left(x-x^{3}\right) \cdot 4\left(1+x^{2}\right)^{3}(2 x)}{\left(1+x^{2}\right)^{8}}\right) \\
& =144\left(\frac{\left(1+x^{2}\right)\left(1-3 x^{2}\right)-8 x\left(x-x^{3}\right)}{\left(1+x^{2}\right)^{5}}\right) \\
& =144 \frac{5 x^{4}-10 x^{2}+1}{\left(1+x^{2}\right)^{5}} \\
& \Longrightarrow f^{(5)}(0)=144
\end{aligned}
$$

We now use these values to compute the 5th-degree Maclaurin polynomial for $f(x)$.

$$
\begin{aligned}
T_{5}(x) & =f(0)+f^{\prime}(0) x+\frac{1}{2} f^{\prime \prime}(0) x^{2} \\
& +\frac{1}{3!} f^{\prime \prime \prime}(0) x^{3}+\frac{1}{4!} f^{(4)}(0) x^{4} \\
& +\frac{1}{5!} f^{(5)}(0) x^{5} \\
& =6 x-\frac{12}{6} x^{3}+\frac{144}{120} x^{5} \\
& =6 x-2 x^{3}+\frac{6}{5} x^{5}
\end{aligned}
$$

Now, if we want to approximate $f\left(\frac{1}{\sqrt{3}}\right)=6 \arctan \left(\frac{1}{\sqrt{3}}\right)=\pi$ :

$$
\begin{aligned}
\pi & =f\left(\frac{1}{\sqrt{3}}\right) \approx T_{5}\left(\frac{1}{\sqrt{3}}\right)=\frac{6}{\sqrt{3}}-\frac{2}{\sqrt{3}^{3}}+\frac{6}{5 \sqrt{3}^{5}} \\
& =2 \sqrt{3}\left(1-\frac{1}{3 \cdot 3}+\frac{1}{5 \cdot 9}\right) \approx 3.156
\end{aligned}
$$

Remark: There are numerous methods for computing $\pi$ to any desired degree of accuracy. Many of them use the Maclaurin expansion of $\arctan x$. In 1706 John Machin computed $\pi$ to 100 decimal digits by using the Maclaurin expansion together with $\pi=16 \arctan \frac{1}{5}-4 \arctan \frac{1}{239}$.
3.4.11.8. Solution. Let's start by differentiating, and looking for a pattern.

$$
\begin{array}{rlrl}
f(x) & =x(\log x-1) & f(1) & =-1 \\
f^{\prime}(x) & =x\left(\frac{1}{x}\right)+\log x-1=\log x & f^{\prime}(1) & =0 \\
f^{\prime \prime}(x) & =\frac{1}{x}=x^{-1} & & f^{\prime \prime}(1)=1 \\
f^{(3)}(x) & =(-1) x^{-2} & f^{(3)}(1)=-1 \\
f^{(4)}(x) & =(-2)(-1) x^{-3}=2!x^{-3} & f^{(4)}(1)=2! \\
f^{(5)}(x) & =(-3)(-2)(-1) x^{-4}=-3!x^{-4} & f^{(4)}(1)=-3! \\
f^{(6)}(x) & =(-4)(-3)(-2)(-1) x^{-5}=4!x^{-5} & f^{(4)}(1)=4! \\
f^{(7)}(x) & =(-5)(-4)(-3)(-2)(-1) x^{-6}=-5!x^{-6} & & f^{(7)}(1)=-5! \\
f^{(8)}(x) & =(-6)(-5)(-4)(-3)(-2)(-1) x^{-7}=6!x^{-7} & & f^{(8)}(1)=6!
\end{array}
$$

When $k \geq 2$, making use of the fact that $0!=1$ and $(-1)^{k-2}=(-1)^{k}$ :

$$
f^{(k)}(x)=(-1)^{k-2}(k-2)!x^{-(k-1)} \quad f^{(k)}(1)=(-1)^{k}(k-2)!
$$

Now we use the standard form of a Taylor polynomial. Since the first two terms don't fit the pattern, we add those outside of the sigma.

$$
\begin{aligned}
T_{100}(x) & =\sum_{k=0}^{100} \frac{f^{(k)}(1)}{k!}(x-1)^{k} \\
& =f(1)+f^{\prime}(1)(x-1)+\sum_{k=2}^{100} \frac{f^{(k)}(1)}{k!}(x-1)^{k} \\
& =-1+0(x-1)+\sum_{k=2}^{100} \frac{(-1)^{k}(k-2)!}{k!}(x-1)^{k} \\
& =-1+\sum_{k=2}^{100} \frac{(-1)^{k}}{k(k-1)}(x-1)^{k}
\end{aligned}
$$

### 3.4.11.9. Solution. Recall that

$$
T_{2 n}(x)=\sum_{k=0}^{2 n} \frac{f^{(k)}\left(\frac{\pi}{4}\right)}{k!}\left(x-\frac{\pi}{4}\right)^{k}
$$

Let's start by taking some derivatives. Of course, since we're differentiating sine, the derivatives will repeat every four iterations.

$$
\begin{array}{ll}
f(x)=\sin x & f\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}} \\
f^{\prime}(x)=\cos x & f^{\prime}\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}} \\
f^{\prime \prime}(x)=-\sin x & f^{\prime \prime}\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}
\end{array}
$$

$$
f^{\prime \prime \prime}(x)=-\cos x \quad f^{\prime \prime \prime}\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}
$$

So, the pattern of derivatives is $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}$, etc. This is a little tricky to write in sigma notation. We can deal with the "doubles" by separating the even and odd powers. The first few terms of $T_{2 n}$ that contain even powers of $\left(x-\frac{\pi}{4}\right)$ are

$$
\underbrace{\frac{1}{\sqrt{2}}}_{k=0}-\underbrace{\frac{1}{2!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{2}}_{k=2}+\underbrace{\frac{1}{4!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{4}}_{k=4}
$$

Observe that the signs alternate between successive terms. So if we rename $k$ to $2 \ell$ these terms are

$$
\underbrace{\frac{1}{\sqrt{2}}}_{\ell=0}-\underbrace{\frac{1}{2!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{2}}_{\ell=1}+\underbrace{\frac{1}{4!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{4}}_{\ell=2}
$$

and the $\ell^{\text {th }}$ term here is $\frac{(-1)^{\ell}}{(2 \ell)!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{2 \ell}$. To verify that this really is the $\ell^{\text {th }}$ term, evaluate this for $\ell=0,1,2$ explicitly. When $k=2 n, \ell=n$ so that

$$
\sum_{\substack{0 \leq k \leq 2 n \\ k \text { even }}} \frac{f^{(k)}\left(\frac{\pi}{4}\right)}{k!}\left(x-\frac{\pi}{4}\right)^{k}=\sum_{\ell=0}^{n} \frac{(-1)^{\ell}}{(2 \ell)!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{2 \ell}
$$

Now for the odd powers. The first few terms of $T_{2 n}$ that contain odd powers of $\left(x-\frac{\pi}{4}\right)$ are

$$
\underbrace{\frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right)}_{k=1}-\underbrace{\frac{1}{3!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{3}}_{k=3}+\underbrace{\frac{1}{5!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{5}}_{k=5}
$$

Observe that the signs again alternate between successive terms. So if we rename $k$ to $2 \ell+1$ these terms are

$$
\underbrace{\frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right)}_{\ell=0}-\underbrace{\frac{1}{3!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{3}}_{\ell=1}+\underbrace{\frac{1}{5!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{5}}_{\ell=2}
$$

and the $\ell^{\text {th }}$ term here is $\frac{(-1)^{\ell}}{(2 \ell+1)!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{2 \ell+1}$. To verify that this really is the $\ell^{\text {th }}$ term, evaluate this for $\ell=0,1,2$ explicitly. The largest odd integer that is smaller than $2 n$ is $2 n-1$ and when $k=2 n-1=2 \ell+1, \ell=n-1$ so that

$$
\sum_{\substack{0 \leq k \leq 2 n \\ \text { ondd }}} \frac{f^{(k)}\left(\frac{\pi}{4}\right)}{k!}\left(x-\frac{\pi}{4}\right)^{k}=\sum_{\ell=0}^{n-1} \frac{(-1)^{\ell}}{(2 \ell+1)!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{2 \ell+1}
$$

Putting the even and odd powers together

$$
T_{2 n}(x)=\sum_{\ell=0}^{n} \frac{(-1)^{\ell}}{(2 \ell)!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{2 \ell}+\sum_{\ell=0}^{n-1} \frac{(-1)^{\ell}}{(2 \ell+1)!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{2 \ell+1}
$$

3.4.11.10. Solution. From Example 3.4 .12 in the text, we see that the $n$th Maclaurin polynomial for $f(x)=e^{x}$ is

$$
T_{n}(x)=\sum_{k=0}^{n} \frac{1}{k!} x^{k}=1+x+\frac{x^{2}}{2}+\frac{x^{3}}{3!}+\frac{x^{4}}{4!}+\cdots+\frac{x^{n}}{n!}
$$

If $n=157$ and $x=1$,

$$
T_{157}(1)=\sum_{k=0}^{157} \frac{1}{k!}=1+1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{157!}
$$

Although we wouldn't expect $T_{157}(1)$ to be exactly equal to $e^{1}$, it's probably pretty close. So, we estimate

$$
1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{157!} \approx e-1
$$

3.4.11.11. Solution. While you're working with sums, it's easy to mistake a constant for a function. The sum given in this question is some number: $\pi$ is a constant, and $k$ is an index- if you wrote out all 100 terms of this sum, there would be no letter $k$. So, the sum given is indeed a number, but we don't want to have to add 100 terms to get a good idea of what number it is.
From Example 3.4.14
in the text, we see that the $(2 n)$ th-degree Maclaurin polynomial for $f(x)=\cos x$ is

$$
T_{2 n}(x)=\sum_{k=0}^{n} \frac{(-1)^{k}}{(2 k)!} \cdot x^{2 k}
$$

If $n=100$ and $x=\frac{5 \pi}{4}$, this equation becomes

$$
T_{200}\left(\frac{5 \pi}{4}\right)=\sum_{k=0}^{100} \frac{(-1)^{k}}{(2 k)!} \cdot\left(\frac{5 \pi}{4}\right)^{2 k}
$$

So, the sum corresponds to the 200th Maclaurin polynomial for $f(x)=\cos x$ evaluated at $x=\frac{5 \pi}{4}$. We should be careful to understand that $T_{200}(x)$ is not equal to $f(x)$, in general. However, when $x$ is reasonably close to 0 , these two functions are approximations of one another. So, we estimate

$$
\sum_{k=0}^{100} \frac{(-1)^{k}}{2 k!}\left(\frac{5 \pi}{4}\right)^{2 k}=T_{200}\left(\frac{5 \pi}{4}\right) \approx \cos \left(\frac{5 \pi}{4}\right)=-\frac{1}{\sqrt{2}}
$$

## - Exercises for § 3.4.6

## Exercises - Stage 1

### 3.4.11.1. Solution.


3.4.11.2. Solution. Let $f(x)$ be the number of problems finished after $x$ minutes of work. The question tells us that $5 \Delta y \approx \Delta x$. So, if $\Delta x=15, \Delta y \approx 3$. That is, in 15 minutes more, you will finish about 3 more problems.
Remark: the math behind this problem is intended to be easy! Looking at symbols like $\Delta y$ and $f(x+\Delta x)$ can be confusing, but the basic idea is pretty simple.

## Exercises - Stage 2

3.4.11.3. Solution. First, let's find the first and second derivatives of $f$ when $x=5$.

$$
\begin{aligned}
f^{\prime}(x) & =\frac{1}{1+x^{2}} & f^{\prime}(5) & =\frac{1}{26} \\
f^{\prime \prime}(x) & =\frac{-2 x}{\left(1+x^{2}\right)^{2}} & f^{\prime \prime}(5) & =\frac{-10}{26^{2}}
\end{aligned}
$$

The linear approximation for $\Delta y$ when $\Delta x=\frac{1}{10}$ is

$$
\Delta y \approx f^{\prime}(5) \Delta x=\frac{1}{26} \cdot \frac{1}{10}=\frac{1}{260} \approx 0.003846
$$

The quadratic approximation for $\Delta y$ when $\Delta x=\frac{1}{10}$ is

$$
\begin{aligned}
\Delta y & \approx f^{\prime}(5) \Delta x+\frac{1}{2} f^{\prime \prime}(5)(\Delta x)^{2}=\frac{1}{26} \cdot \frac{1}{10}+\frac{1}{2} \cdot \frac{-10}{26^{2}} \cdot \frac{1}{10^{2}} \\
& =\frac{51}{13520} \approx 0.003772
\end{aligned}
$$

Remark: $\Delta y=\arctan (5.1)-\arctan (5) \approx 0.003774$.
3.4.11.4. Solution. (a) When $x$ is near 4 , the linear approximation of $\Delta y$ is

$$
\Delta y \approx s^{\prime}(4) \Delta x
$$

From the problem, $\Delta x=5-4=1$. Differentiating,

$$
\begin{aligned}
& s^{\prime}(t)=\sqrt{19.6} \cdot \frac{1}{2 \sqrt{x}}=\sqrt{\frac{4.9}{x}}, \text { so } \\
& s^{\prime}(4)=\sqrt{\frac{4.9}{4}}
\end{aligned}
$$

The linear approximation gives us

$$
\Delta y \approx \sqrt{\frac{4.9}{4}}(1) \approx 1.1
$$

So moving from 4 metres to 5 metres increases the speed with which you hit the water by about 1.1 metres per second.
(b) Again, we'll use the linear approximation

$$
\begin{aligned}
\Delta y & \approx s^{\prime}(a) \Delta x \\
& =\sqrt{\frac{4.9}{a}} \cdot \Delta x
\end{aligned}
$$

The difference in height between the first two jumps and between the last two jumps is the same, $\Delta x$. The initial height of the first jump is smaller than the initial height of the second jump. So, the value corresponding to $a$ is smaller for the first jump than for the second. Therefore, $\Delta y$ is larger between the first two jumps than it is between the last two jumps. So, the increase in speed from the first jump to the second is larger than the increase in speed from the second jump to the third.
To put that more symbolically, the change in terminal speed between the first two jumps is

$$
\Delta y \approx \sqrt{\frac{4.9}{x}} \cdot \Delta x
$$

while the change in terminal speed between the next two jumps is

$$
\Delta y \approx \sqrt{\frac{4.9}{x+\Delta x}} \cdot \Delta x
$$

Since $\sqrt{\frac{4.9}{x}} \cdot \Delta x>\sqrt{\frac{4.9}{x+\Delta x}} \cdot \Delta x$ (when $x$ and $\Delta x$ are positive), the change in terminal speed is greater between the first two jumps than between the next two jumps.

## - Exercises for § 3.4.7

## Exercises - Stage 1

3.4.11.1. Solution. False. The linear approximation is an approximation. It tells us

$$
\Delta y \approx f^{\prime}\left(x_{0}\right) \Delta x=f^{\prime}(2)(1)=25
$$

However, from our definition of $\Delta y$,

$$
\Delta y=f\left(x_{0}+\Delta x\right)-f\left(x_{0}\right)=f(2+1)-f(2)=58-26=32
$$

Remark: this is to emphasize that the calculations in this subsection are estimations of error bounds, rather than actual error bounds. All we can say is that we estimate the error will be no more than some number-we don't guarantee it.
In the next subsection, we will introduce an error bound that is guaranteed to be accurate. It is usually harder to calculate than the estimations in this section.
3.4.11.2. Solution. When an exact value $Q_{0}$ is measured as $Q_{0}+\Delta Q$, Definition 3.4.25 gives us the absolute error as $|\Delta Q|$, and the percentage error as $100 \frac{|\Delta Q|}{Q_{0}}$.
In our situation, $Q_{0}=5.83$ and $Q_{0}+\Delta Q=6$, so $\Delta Q=0.17$. So, the absolute error is 0.17 , and the percentage error is

$$
100 \frac{0.17}{5.83} \approx 2.92 \%
$$

3.4.11.3. Solution. Since $f^{\prime}(x)=6 x$, when $x=10, f^{\prime}(10)=60$. If $\Delta y=$ $f(11)-f(10)$, and $\Delta x=11-10$, then the linear approximation tells us

$$
\Delta y \approx 60 \Delta x=60
$$

So, the linear approximates estimates the error in $f(x)$ to be about 60 .
Since $f^{\prime \prime}(x)=6$, the quadratic approximation (using $f^{\prime}(10)=60, f^{\prime \prime}(10)=6$, and $\Delta x=1$ ) tells us

$$
\Delta y \approx f^{\prime}(10) \Delta x+\frac{1}{2} f^{\prime \prime}(1)(\Delta x)^{2}=60 \cdot 1+\frac{1}{2}(6)(1)^{2}=63
$$

So, the quadratic approximates estimates the error in $f(x)$ to be about 63. (Indeed, the exact value of $f(11)-f(10)$ is 63 . It is not a fluke that our estimated error, using a quadratic approximation, is exactly the same as our actual error. It is a consequence of the fact that $f(x)$ is a quadratic function.)

## Exercises - Stage 2

3.4.11.4. Solution. Let $A$ be the area of a pen of radius $r$. Then

$$
A(r)=\pi r^{2}
$$

Differentiating with respect to $r$,

$$
A^{\prime}(r)=2 \pi r
$$

The exact area desired is $A_{0}$. Let the corresponding exact radius desired be $r_{0}$.

Using the linear approximation formula, where $\Delta A$ is the change in $A$ corresponding to a change in $r$ of $\Delta r$,

$$
\begin{aligned}
\Delta A & \approx A^{\prime}\left(r_{0}\right) \Delta r=2 \pi r_{0} \Delta r \\
\Delta r & \approx \frac{\Delta A}{2 \pi r_{0}}
\end{aligned}
$$

What we're interested in is the percent error $r$ can have. The percent error is:

$$
\begin{aligned}
100 \frac{\Delta r}{r_{0}} & \approx 100 \frac{\Delta A}{2 \pi r_{0} \cdot r_{0}} \\
& =100 \frac{\Delta A}{2\left(\pi r_{0}^{2}\right)} \\
& =100 \frac{\Delta A}{2 \cdot A_{0}} \\
& =\left(100 \frac{\Delta A}{A_{0}}\right) \frac{1}{2} \\
& \leq(2) \frac{1}{2}=1
\end{aligned}
$$

(To get the last line, we used the given information that the percent error in the area, $100 \frac{\Delta A}{A_{0}}$, must be less than $2 \%$.)
We conclude the error in $r$ cannot be more than $1 \%$.
3.4.11.5. Solution. (a) The area removed represents a proportion of $\frac{\theta}{2 \pi}$ of the entire circle, whose area is $\pi\left(3^{2}\right)=9 \pi$. So, the area of the sector removed is

$$
\frac{\theta}{2 \pi} \cdot 9 \pi=\frac{9}{2} \theta
$$

(b) To find $\theta$ from $d$, we cut our triangle (with angle $\theta$ opposite side of length $d$ ) into two equivalent right triangles, as shown below.


Using the information that the radius of the circle (also the hypotenuse of the right
triangle) is 3 ,

$$
\sin \left(\frac{\theta}{2}\right)=\frac{\frac{d}{2}}{3}=\frac{d}{6}
$$

Since the question tells us the sector is no more than a quarter of the circle, we know $0 \leq \theta \leq \frac{\pi}{2}$, so $0 \leq \frac{\theta}{2} \leq \frac{\pi}{4}$. This puts $\frac{\theta}{2}$ well within the range of arcsine.

$$
\theta=2 \arcsin \left(\frac{d}{6}\right)
$$

(c) First, let's get an expression for the area of the sector in terms of $d$.

$$
\begin{aligned}
A & =\frac{9}{2} \theta=\frac{9}{2}\left(2 \arcsin \left(\frac{d}{6}\right)\right) \\
& =9 \arcsin \left(\frac{d}{6}\right)
\end{aligned}
$$

Differentiating,

$$
\begin{aligned}
A^{\prime}(d) & =\frac{9}{\sqrt{1-\left(\frac{d}{6}\right)^{2}}} \cdot \frac{1}{6} \\
& =\frac{9}{\sqrt{36-d^{2}}}
\end{aligned}
$$

Let $\Delta A$ is the error in $A$ corresponding to an error of $\Delta d$ in $d$. Since we measured $d$ to be 0.7 instead of 0.68 , in the linear approximation we take $\Delta d=0.02$ and $d_{0}=0.68$.

$$
\begin{aligned}
\Delta A & \approx A^{\prime}\left(d_{0}\right) \cdot \Delta d \\
& =A^{\prime}(0.68) \cdot 0.02 \\
& =\frac{9}{\sqrt{36-0.68^{2}}} \cdot 0.02 \\
& \approx 0.03
\end{aligned}
$$

So, the error in $A$ is about 0.03 .
3.4.11.6. Solution. Suppose we have a function $V(h)$ that gives the volume of water in the tank as a function of its height.
Let $h_{0}=0.5$ metres, $\Delta h=-0.05$, and $\Delta V=V\left(h_{0}+\Delta h\right)-V\left(h_{0}\right)=V(0.45)-$ $V(0.5)$. Then, by the linear approximation,

$$
\Delta V \approx V^{\prime}(0.5) \cdot \Delta h=-0.05 V^{\prime}(0.5)
$$

In order to solve the problem, we will find a function $V(h)$ giving the volume of water in terms of the height, then find $V^{\prime}(0.5)$, and finally approximate that the
change in the volume of the water is $\Delta V \approx-0.05 V^{\prime}(0.5)$.
The water in the tank forms a cone. The volume of a cone of height $h$ and radius $r$ is

$$
V=\frac{1}{3} \pi r^{2} h
$$

We need to get rid of the variable $r$. We can do this using similar triangles. The diagram below shows the side view of the tank and the water.


The side view of the tank forms a triangle that is similar to the triangle formed by the side view of the water, so

$$
\begin{aligned}
\frac{1}{2} & =\frac{2 r}{h} \\
r & =\frac{h}{4}
\end{aligned}
$$

Using this, we find our equation for the volume of the water in terms of $h$.

$$
V(h)=\frac{1}{3} \pi r^{2} h=\frac{\pi}{3}\left(\frac{h}{4}\right)^{2} h=\frac{\pi h^{3}}{48}
$$

Differentiating,

$$
\begin{aligned}
V^{\prime}(h) & =\frac{\pi h^{2}}{16} \\
V^{\prime}(0.5) & =\frac{0.25 \pi}{16}=\frac{\pi}{64}
\end{aligned}
$$

Finally, using the approximation $\Delta V \approx-0.05 V^{\prime}(0.5)$,

$$
\Delta V \approx \frac{-0.05 \pi}{64}=-\frac{\pi}{1280} \approx-0.00245 \mathrm{~m}^{3}
$$

We estimate that the volume decreased by about 0.00245 cubic metres, or about 2450 cubic centimetres.

## Exercises - Stage 3

3.4.11.7. Solution. Let $q$ be the measured amount of the isotope remaining after 3 years. Let $h(q)$ be the half-life of the isotope that we calculate using $q$. We measured $q=0.9$, but we want to know what the change in $h$ is if $q$ moves by 0.05 . So, let $\Delta q= \pm 0.05$, and let $\Delta h$ be the corresponding change in $h$. The linear approximation tells us

$$
\Delta h \approx h^{\prime}(0.9) \Delta q
$$

So,

$$
|\Delta h| \approx\left|h^{\prime}(0.9)\right||\Delta q|=h^{\prime}(0.9) \cdot 0.05
$$

This suggests a plan for solving the problem. We will find the equation $h(q)$ giving the calculated half-life of the isotope based on the measurement $q$. Then, we will find $h^{\prime}(0.9)$. Finally, the equation $|\Delta h|=h^{\prime}(0.9) \cdot 0.05$ will tell us the change in $h$ that corresponds with a change of 0.05 in our measurement.
Let us find the half-life of the isotope, if after three years $q \mu \mathrm{~g}$ is remaining. The amount of the isotope present after $t$ years is given by

$$
Q(t)=Q(0) e^{-k t}
$$

for some constant $k$. Let's take $t=0$ to be the time when precisely one $\mu \mathrm{g}$ was present. Then

$$
Q(t)=e^{-k t}
$$

After three years, $q$ is the amount of the isotope remaining, so

$$
\begin{aligned}
q & =e^{-k \cdot 3} \\
q^{\frac{1}{3}} & =e^{-k} \\
Q(t) & =\left(e^{-k}\right)^{t}=q^{\frac{t}{3}}
\end{aligned}
$$

The half-life is the value of $t$ for which $Q(t)=\frac{1}{2} Q(0)=\frac{1}{2}$.

$$
\begin{aligned}
\frac{1}{2} & =Q(t)=q^{\frac{t}{3}} \\
\log \left(\frac{1}{2}\right) & =\log \left(q^{\frac{t}{3}}\right) \\
-\log 2 & =\frac{t}{3} \log q \\
t & =\frac{-3 \log 2}{\log q}
\end{aligned}
$$

So, we calculate the half-life to be $\frac{-3 \log 2}{\log q}$. This gives us our first goal: a function $h(q)$ that tells us the calculated half-life of the element.

$$
h(q)=\frac{-3 \log 2}{\log q}
$$

Following our plan, we find $h^{\prime}(0.9)$.

$$
\begin{aligned}
h^{\prime}(q) & =\frac{\mathrm{d}}{\mathrm{~d} q}\left\{\frac{-3 \log 2}{\log q}\right\} \\
& =-3 \log 2 \cdot \frac{\mathrm{~d}}{\mathrm{~d} q}\left\{(\log q)^{-1}\right\} \\
& =-3 \log 2 \cdot(-1)(\log q)^{-2} \cdot \frac{1}{q} \\
& =\frac{3 \log 2}{q \log ^{2} q} \\
h^{\prime}(0.9) & =\frac{3 \log 2}{0.9 \log ^{2}(0.9)} \approx 208
\end{aligned}
$$

Finally, as outlined in our plan,

$$
\begin{aligned}
|\Delta h| & =h^{\prime}(0.9) \cdot 0.05 \\
& =\frac{3 \log 2}{18 \log ^{2}(0.9)} \approx 10.4
\end{aligned}
$$

If our measurement changes by $\pm 0.05 \mu \mathrm{~g}$, then we estimate our calculated half-life changes by about $\pm 10.4$ years. Since our measurement is accurate to within 0.05 $\mu \mathrm{g}$, that means we estimate our calculated half-life to be accurate to within about 10.4 years.

Remark: since $h(0.9)=\frac{-3 \log 2}{\log 0.9} \approx 19.7$, an absolute error of 10.4 years corresponds to a percentage error of $100 \frac{10.4}{19.7} \approx 53 \%$. The question did not specify absolute or percentage error. Since both make sense, you can use either one.

## - Exercises for § 3.4.8

## Exercises - Stage 1

3.4.11.1. Solution. From the given information,

$$
|R(10)|=|f(10)-F(10)|=|-3-5|=|-8|=8
$$

So, (a) is false (since 8 is not less than or equal to 7 ), while (b), (c), and (d) are true.
Remark: $R(x)$ is the error in our approximation. As mentioned in the text, we almost never know $R$ exactly, but we can give a bound. We don't need the tightest bound-just a reasonable one that is easy to calculate. If we were dealing with real functions and approximations, we might not know that $|R(10)|=8$, but if we knew it was at most 9 , that would be a pretty decent approximation.
Often in this section, we will make simplifying assumptions to get a bound that is easy to calculate. But, don't go overboard! It is a true statement to say that our absolute error is at most 100, but this statement would probably not be very helpful as a bound.
3.4.11.2. Solution. Equation 3.4.33 tells us that, when $T_{n}(x)$ is the $n$th degree Taylor polynomial for a function $f(x)$ about $x=a$, then

$$
\left|f(x)-T_{n}(x)\right|=\left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right|
$$

for some $c$ strictly between $x$ and $a$. In our case, $n=3, a=0, x=2$, and $f^{(4)}(c)=e^{c}$, so

$$
\begin{aligned}
\left|f(2)-T_{3}(2)\right| & =\left|\frac{f^{(4)}(c)}{4!}(2-0)^{4}\right| \\
& =\frac{2^{4}}{4!} e^{c}=\frac{2}{3} e^{c}
\end{aligned}
$$

Since $c$ is strictly between 0 and $2, e^{c}<e^{2}$ :

$$
\leq \frac{2}{3} e^{2}
$$

but this isn't a number we really know. Indeed: $e^{2}$ is the very number we're trying to approximate. So, we use the estimation $e<3$ :

$$
<\frac{2}{3} \cdot 3^{2}=6
$$

We conclude that the error $\left|f(2)-T_{3}(2)\right|$ is less than 6.
Now we'll get a more exact idea of the error using a calculator. (Calculators will also only give approximations of numbers like $e$, but they are generally very good approximations.)

$$
\begin{aligned}
\left|f(2)-T_{3}(2)\right| & =\left|e^{2}-\left(1+2+\frac{1}{2} \cdot 2^{2}+\frac{1}{3!} \cdot 2^{3}\right)\right| \\
& =\left|e^{2}-\left(1+2+2+\frac{4}{3}\right)\right| \\
& =\left|e^{2}-\frac{19}{3}\right| \approx 1.056
\end{aligned}
$$

So, our actual answer was only off by about 1 .
Remark: $1<6$, so this does not in any way contradict our bound $\left|f(2)-T_{3}(2)\right|<6$.
3.4.11.3. Solution. Whenever you approximate a polynomial with a Taylor polynomial of greater or equal degree, your Taylor polynomial is exactly the same as the function you are approximating. So, the error is zero.
3.4.11.4. Solution. The constant approximation gives

$$
\sin (33) \approx \sin (0)=0
$$

while the linear approximation gives

$$
f(x) \approx f(0)+f^{\prime}(0) x
$$

$$
\begin{aligned}
\sin (x) & \approx \sin (0)+\cos (0) x \\
& =x \\
\sin (33) & \approx 33
\end{aligned}
$$

Since $-1 \leq \sin (33) \leq 1$, the constant approximation is better. (But both are a little silly.)


## Exercises - Stage 2

3.4.11.5. Solution. Equation 3.4.33 tells us that, when $T_{n}(x)$ is the $n$th degree Taylor polynomial for a function $f(x)$ about $x=a$, then

$$
\left|f(x)-T_{n}(x)\right|=\left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right|
$$

for some $c$ strictly between $x$ and $a$. In our case, $n=5, a=11, x=11.5$, and $f^{(6)}(c)=\frac{6!(2 c-5)}{c+3}$.

$$
\begin{aligned}
\left|f(11.5)-T_{5}(11.5)\right| & =\left|\frac{1}{6!}\left(\frac{6!(2 c-5)}{c+3}\right)(11.5-11)^{6}\right| \\
& =\left|\frac{2 c-5}{c+3}\right| \cdot \frac{1}{2^{6}}
\end{aligned}
$$

for some $c$ in $(11,11.5)$. We don't know exactly which $c$ this is true for, but since we know that $c$ lies in $(11,11.5)$, we can provide bounds.

- $2 c-5<2(11.5)-5=18$
- $c+3>11+3=14$
- Therefore, $\left|\frac{2 c-5}{c+3}\right|=\frac{2 c-5}{c+3}<\frac{18}{14}=\frac{9}{7}$ when $c \in(11,11.5)$.

With this bound, we see

$$
\begin{aligned}
\left|f(11.5)-T_{5}(11.5)\right| & =\left|\frac{2 c-5}{c+3}\right| \cdot \frac{1}{2^{6}} \\
& <\left(\frac{9}{7}\right)\left(\frac{1}{2^{6}}\right) \approx 0.0201
\end{aligned}
$$

Our error is less than 0.02.
3.4.11.6. Solution. Equation 3.4 .33 tells us that, when $T_{n}(x)$ is the $n$th degree Taylor polynomial for a function $f(x)$ about $x=a$, then

$$
\left|f(x)-T_{n}(x)\right|=\left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right|
$$

for some $c$ strictly between $x$ and $a$. In our case, $n=2, a=0$, and $x=0.1$, so

$$
\begin{aligned}
\left|f(0.1)-T_{2}(0.1)\right| & =\left|\frac{f^{(3)}(c)}{3!}(0.1-0)^{3}\right| \\
& =\frac{\left|f^{\prime \prime \prime}(c)\right|}{6000}
\end{aligned}
$$

for some $c$ in $(0,0.1)$.
We will find $f^{\prime \prime \prime}(x)$, and use it to give an upper bound for

$$
\left|f(0.1)-T_{2}(0.1)\right|=\frac{\left|f^{\prime \prime \prime}(c)\right|}{6000}
$$

when $c$ is in $(0,0.1)$.

$$
\begin{aligned}
f(x) & =\tan x \\
f^{\prime}(x) & =\sec ^{2} x \\
f^{\prime \prime}(x) & =2 \sec x \cdot \sec x \tan x \\
& =2 \sec ^{2} x \tan x \\
f^{\prime \prime \prime}(x) & =\left(2 \sec ^{2} x\right) \sec ^{2} x+(4 \sec x \cdot \sec x \tan x) \tan x \\
& =2 \sec ^{4} x+4 \sec ^{2} x \tan ^{2} x
\end{aligned}
$$

When $0<c<\frac{1}{10}$, also $0<c<\frac{\pi}{6}$, so:

- $\tan c<\tan \left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}$
- $\cos c>\cos \left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$
- $\sec c<\frac{2}{\sqrt{3}}$

With these bounds in mind for secant and tangent, we return to the expression we found for our error.

$$
\begin{aligned}
\left|f(0.1)-T_{2}(0.1)\right| & =\frac{\left|f^{\prime \prime \prime}(c)\right|}{6000}=\frac{\left|2 \sec ^{4} x+4 \sec ^{2} x \tan ^{2} x\right|}{6000} \\
& <\frac{2\left(\frac{2}{\sqrt{3}}\right)^{4}+4\left(\frac{2}{\sqrt{3}}\right)^{2}\left(\frac{1}{\sqrt{3}}\right)^{2}}{6000} \\
& =\frac{1}{1125}
\end{aligned}
$$

The error is less than $\frac{1}{1125}$.
3.4.11.7. Solution. Equation 3.4.33 tells us that, when $T_{n}(x)$ is the $n$th degree Taylor polynomial for a function $f(x)$ about $x=a$, then

$$
\left|f(x)-T_{n}(x)\right|=\left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right|
$$

for some $c$ strictly between $x$ and $a$. In our case, $n=5, a=0$, and $x=-\frac{1}{4}$, so

$$
\begin{aligned}
\left|f\left(-\frac{1}{4}\right)-T_{5}\left(-\frac{1}{4}\right)\right| & =\left|\frac{f^{(6)}(c)}{6!}\left(-\frac{1}{4}-0\right)^{6}\right| \\
& =\frac{\left|f^{(6)}(c)\right|}{6!\cdot 4^{6}}
\end{aligned}
$$

for some $c$ in $\left(-\frac{1}{4}, 0\right)$. We'll need to know the sixth derivative of $f(x)$.

$$
\begin{aligned}
f(x) & =\log (1-x) \\
f^{\prime}(x) & =-(1-x)^{-1} \\
f^{\prime \prime}(x) & =-(1-x)^{-2} \\
f^{\prime \prime \prime}(x) & =-2(1-x)^{-3} \\
f^{(4)}(x) & =-3!(1-x)^{-4} \\
f^{(5)}(x) & =-4!(1-x)^{-5} \\
f^{(6)}(x) & =-5!(1-x)^{-6}
\end{aligned}
$$

Plugging in $\left|f^{(6)}(c)\right|=\frac{5!}{(1-c)^{6}}$ :

$$
\left|f\left(-\frac{1}{4}\right)-T_{5}\left(-\frac{1}{4}\right)\right|=\frac{5!}{6!\cdot 4^{6} \cdot(1-c)^{6}}=\frac{1}{6 \cdot 4^{6} \cdot(1-c)^{6}}
$$

for some $c$ in $\left(-\frac{1}{4}, 0\right)$.
We're interested in an upper bound for the error: we want to know the worst case scenario, so we can say that the error is no worse than that. We need to know
what the biggest possible value of $\frac{1}{6 \cdot 4^{6} \cdot(1-c)^{6}}$ is, given $-\frac{1}{4}<c<0$. That means we want to know the biggest possible value of $\frac{1}{(1-c)^{6}}$. This corresponds to the smallest possible value of $(1-c)^{6}$, which in turn corresponds to the smallest absolute value of $1-c$.

- Since $-\frac{1}{4} \leq c \leq 0$, the smallest absolute value of $1-c$ occurs when $c=0$. In other words, $|1-c| \leq 1$.
- That means the smallest possible value of $(1-c)^{6}$ is $1^{6}=1$.
- Then the largest possible value of $\frac{1}{(1-c)^{6}}$ is 1 .
- Then the largest possible value of $\frac{1}{6 \cdot 4^{6}} \cdot \frac{1}{(1-c)^{6}}$ is $\frac{1}{6 \cdot 4^{6}} \approx 0.0000407$.

Finally, we conclude

$$
\left|f\left(-\frac{1}{4}\right)-T_{5}\left(-\frac{1}{4}\right)\right|=\frac{1}{6 \cdot 4^{6} \cdot(1-c)^{6}}<\frac{1}{6 \cdot 4^{6}}<0.00004
$$

3.4.11.8. Solution. Equation 3.4.33 tells us that, when $T_{n}(x)$ is the $n$th degree Taylor polynomial for a function $f(x)$ about $x=a$, then

$$
\left|f(x)-T_{n}(x)\right|=\left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right|
$$

for some $c$ strictly between $x$ and $a$. In our case, $n=3, a=30$, and $x=32$, so

$$
\begin{aligned}
\left|f(30)-T_{3}(30)\right| & =\left|\frac{f^{(4)}(c)}{4!}(30-32)^{4}\right| \\
& =\frac{2}{3}\left|f^{(4)}(c)\right|
\end{aligned}
$$

for some $c$ in $(30,32)$.
We will now find $f^{(4)}(x)$. Then we can give an upper bound on $\left|f(30)-T_{3}(30)\right|=$ $\frac{2}{3}\left|f^{(4)}(c)\right|$ when $c \in(30,32)$.

$$
\begin{aligned}
f(x) & =x^{\frac{1}{5}} \\
f^{\prime}(x) & =\frac{1}{5} x^{-\frac{4}{5}} \\
f^{\prime \prime}(x) & =-\frac{4}{5^{2}} x^{-\frac{9}{5}} \\
f^{\prime \prime \prime}(x) & =\frac{4 \cdot 9}{5^{3}} x^{-\frac{14}{5}} \\
f^{(4)}(x) & =-\frac{4 \cdot 9 \cdot 14}{5^{4}} x^{-\frac{19}{5}}
\end{aligned}
$$

Using this,

$$
\begin{aligned}
\left|f(30)-T_{3}(30)\right| & =\frac{2}{3}\left|f^{(4)}(c)\right| \\
& =\frac{2}{3}\left|-\frac{4 \cdot 9 \cdot 14}{5^{4}} c^{-\frac{19}{5}}\right| \\
& =\frac{336}{5^{4} \cdot c^{\frac{19}{5}}}
\end{aligned}
$$

Since $30<c<32$,

$$
\begin{aligned}
& <\frac{336}{5^{4} \cdot 30^{\frac{19}{5}}}=\frac{336}{5^{4} \cdot 30^{3} \cdot 30^{\frac{4}{5}}} \\
& =\frac{14}{5^{7} \cdot 9 \cdot 30^{\frac{4}{5}}}
\end{aligned}
$$

This isn't a number we know. We're trying to find the error in our estimation of $\sqrt[5]{30}$, but $\sqrt[5]{30}$ shows up in our error. From here, we have to be a little creative to get a bound that actually makes sense to us. There are different ways to go about it. You could simply use $30^{\frac{4}{5}}>1$. We will be a little more careful, and use the following estimation:

$$
\begin{aligned}
\frac{14}{5^{7} \cdot 9 \cdot 30^{\frac{4}{5}}} & =\frac{14 \cdot 30^{\frac{1}{5}}}{5^{7} \cdot 9 \cdot 30} \\
& <\frac{14 \cdot 32^{\frac{1}{5}}}{5^{7} \cdot 9 \cdot 30} \\
& <\frac{14 \cdot 2}{5^{7} \cdot 9 \cdot 30} \\
& <\frac{14}{5^{7} \cdot 9 \cdot 15} \\
& <0.000002
\end{aligned}
$$

We conclude $\left|f(30)-T_{3}(30)\right|<0.000002$.
3.4.11.9. Solution. Equation 3.4.33 tells us that, when $T_{n}(x)$ is the $n$th degree Taylor polynomial for a function $f(x)$ about $x=a$, then

$$
\left|f(x)-T_{n}(x)\right|=\left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right|
$$

for some $c$ strictly between $x$ and $a$. In our case, $n=1, a=\frac{1}{\pi}$, and $x=0.01$, so

$$
\begin{aligned}
\left|f(0.01)-T_{1}(0.01)\right| & =\left|\frac{f^{\prime \prime}(c)}{2}\left(0.01-\frac{1}{\pi}\right)^{2}\right| \\
& =\frac{1}{2}\left(\frac{100-\pi}{100 \pi}\right)^{2} \cdot\left|f^{\prime \prime}(c)\right|
\end{aligned}
$$

for some $c$ in $\left(\frac{1}{100}, \frac{1}{\pi}\right)$.
Let's find $f^{\prime \prime}(x)$.

$$
\begin{aligned}
f(x) & =\sin \left(\frac{1}{x}\right) \\
f^{\prime}(x) & =\cos \left(\frac{1}{x}\right) \cdot \frac{-1}{x^{2}}=\frac{-\cos \left(\frac{1}{x}\right)}{x^{2}} \\
f^{\prime \prime}(x) & =\frac{x^{2} \sin \left(\frac{1}{x}\right)\left(-x^{-2}\right)+\cos \left(\frac{1}{x}\right)(2 x)}{x^{4}} \\
& =\frac{2 x \cos \left(\frac{1}{x}\right)-\sin \left(\frac{1}{x}\right)}{x^{4}}
\end{aligned}
$$

Now we can plug in a better expression for $f^{\prime \prime}(c)$ :

$$
\begin{aligned}
\left|f(0.01)-T_{1}(0.01)\right| & =\frac{1}{2}\left(\frac{100-\pi}{200 \pi}\right)^{2} \cdot\left|f^{\prime \prime}(c)\right| \\
& =\frac{1}{2}\left(\frac{100-\pi}{100 \pi}\right)^{2} \cdot \frac{\left|2 c \cos \left(\frac{1}{c}\right)-\sin \left(\frac{1}{c}\right)\right|}{c^{4}}
\end{aligned}
$$

for some $c$ in $\left(\frac{1}{100}, \frac{1}{\pi}\right)$.
What we want to do now is find an upper bound on this expression containing $c$, $\frac{1}{2}\left(\frac{100-\pi}{100 \pi}\right)^{2} \cdot \frac{\left|2 c \cos \left(\frac{1}{c}\right)-\sin \left(\frac{1}{c}\right)\right|}{c^{4}}$.

- Since $c \geq \frac{1}{100}$, it follows that $c^{4} \geq \frac{1}{100^{4}}$, so $\frac{1}{c^{4}} \leq 100^{4}$.
- For any value of $x,|\cos x|$ and $|\sin x|$ are at most 1 . Since $|c|<1$, also $\left|c \cos \left(\frac{1}{c}\right)\right|<\left|\cos \left(\frac{1}{c}\right)\right| \leq 1$. So, $\left|2 c \cos \left(\frac{1}{c}\right)-\sin \left(\frac{1}{c}\right)\right|<3$
- Therefore,

$$
\begin{aligned}
\mid f(0.01) & -T_{n}(0.01) \mid \\
& =\frac{1}{2}\left(\frac{100-\pi}{100 \pi}\right)^{2} \cdot \frac{1}{c^{4}} \cdot\left|2 c \cos \left(\frac{1}{c}\right)-\sin \left(\frac{1}{c}\right)\right| \\
& <\frac{1}{2}\left(\frac{100-\pi}{100 \pi}\right)^{2} \cdot 100^{4} \cdot 3 \\
& =\frac{3 \cdot 100^{2}}{2}\left(\frac{100}{\pi}-1\right)^{2}
\end{aligned}
$$

Equation 3.4.33 gives the bound $\left|f(0.01)-T_{1}(0.01)\right| \leq \frac{3 \cdot 100^{2}}{2}\left(\frac{100}{\pi}-1\right)^{2}$.
The bound above works out to approximately fourteen million. One way to understand why the bound is so high is that $\sin \left(\frac{1}{x}\right)$ moves about crazily when $x$ is
near zero-it moves up and down incredibly fast, so a straight line isn't going to approximate it very well at all.
That being said, because $\sin \left(\frac{1}{x}\right)$ is still "sine of something", we know $-1 \leq$ $f(0.01) \leq 1$. To get a better bound on the error, let's find $T_{1}(x)$.

$$
\begin{aligned}
& f(x)=\sin \left(\frac{1}{x}\right) \\
& \begin{aligned}
f^{\prime}(x)=\frac{-\cos \left(\frac{1}{x}\right)}{x^{2}} & f\left(\frac{1}{\pi}\right)=\sin (\pi)=0 \\
T_{1}(x) & =f\left(\frac{1}{\pi}\right)+f^{\prime}\left(\frac{1}{\pi}\right)\left(x-\frac{1}{\pi}\right) \\
& =0+\pi^{2}\left(x-\frac{1}{\pi}\right)=-\pi^{2} \cos (\pi)=\pi^{2} \\
& =\pi^{2} x-\pi \\
T_{1}(0.01) & =\frac{\pi^{2}}{100}-\pi
\end{aligned}
\end{aligned}
$$

Now that we know $T_{1}(0.01)$, and we know $-1 \leq f(0.01) \leq 1$, we can give the bound

$$
\begin{aligned}
\left|f(0.01)-T_{1}(0.01)\right| & \leq|f(0.01)|+\left|T_{1}(0.01)\right| \\
& \leq 1+\left|\frac{\pi^{2}}{100}-\pi\right| \\
& =1+\pi\left|1-\frac{\pi}{100}\right| \\
& <1+\pi \\
& <1+4=5
\end{aligned}
$$

A more reasonable bound on the error is that it is less than 5 .
Still more reasonably, we would not use $T_{1}(x)$ to evaluate $\sin (100)$ approximately. We would write $\sin (100)=\sin (100-32 \pi)$ and approximate the right hand side, which is roughly $\sin (-\pi / 6)$.
3.4.11.10. Solution. Equation 3.4 .33 tells us that, when $T_{n}(x)$ is the $n$th degree Taylor polynomial for a function $f(x)$ about $x=a$, then

$$
\left|f(x)-T_{n}(x)\right|=\left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right|
$$

for some $c$ strictly between $x$ and $a$. In our case, $n=2, a=0$, and $x=\frac{1}{2}$, so

$$
\begin{aligned}
\left|f\left(\frac{1}{2}\right)-T_{2}\left(\frac{1}{2}\right)\right| & =\left|\frac{f^{(3)}(c)}{3!}\left(\frac{1}{2}-0\right)^{3}\right| \\
& =\frac{\left|f^{(3)}(c)\right|}{3!\cdot 2^{3}}
\end{aligned}
$$

for some $c$ in $\left(0, \frac{1}{2}\right)$. The next task that suggests itself is finding $f^{(3)}(x)$.

$$
\begin{aligned}
f(x) & =\arcsin x \\
f^{\prime}(x) & =\frac{1}{\sqrt{1-x^{2}}}=\left(1-x^{2}\right)^{-\frac{1}{2}} \\
f^{\prime \prime}(x) & =-\frac{1}{2}\left(1-x^{2}\right)^{-\frac{3}{2}}(-2 x) \\
& =x\left(1-x^{2}\right)^{-\frac{3}{2}} \\
f^{\prime \prime \prime}(x) & =x\left(-\frac{3}{2}\right)\left(1-x^{2}\right)^{-\frac{5}{2}}(-2 x)+\left(1-x^{2}\right)^{-\frac{3}{2}} \\
& =3 x^{2}\left(1-x^{2}\right)^{-\frac{5}{2}}+\left(1-x^{2}\right)^{-\frac{5}{2}+1} \\
& =\left(1-x^{2}\right)^{-\frac{5}{2}}\left(3 x^{2}+\left(1-x^{2}\right)\right) \\
& =\left(1-x^{2}\right)^{-\frac{5}{2}}\left(2 x^{2}+1\right)
\end{aligned}
$$

Since $\left|f\left(\frac{1}{2}\right)-T_{2}\left(\frac{1}{2}\right)\right|=\frac{\left|f^{(3)}(c)\right|}{3!\cdot 2^{3}}$ for some $c$ in $\left(0, \frac{1}{2}\right)$,

$$
\left|f\left(\frac{1}{2}\right)-T_{2}\left(\frac{1}{2}\right)\right|=\frac{\left|\frac{1+2 c^{2}}{\left(\sqrt{1-c^{2}}\right)^{5}}\right|}{3!\cdot 2^{3}}=\frac{1+2 c^{2}}{48\left(\sqrt{1-c^{2}}\right)^{5}}
$$

for some $c$ in $\left(0, \frac{1}{2}\right)$.
We want to know what is the worst case scenario-what's the biggest this expression can be. So, now we find an upper bound on $\frac{1+2 c^{2}}{48\left(\sqrt{1-c^{2}}\right)^{5}}$ when $0 \leq c \leq \frac{1}{2}$. Remember that our bound doesn't have to be exact, but it should be relatively easy to calculate.

- When $0 \leq c \leq \frac{1}{2}$, the biggest $1+2 c^{2}$ can be is $1+2\left(\frac{1}{2}\right)^{2}=\frac{3}{2}$.

So, the numerator of $\frac{1+2 c^{2}}{48\left(\sqrt{1-c^{2}}\right)^{5}}$ is at most $\frac{3}{2}$.

- The smallest $1-c^{2}$ can be is $1-\left(\frac{1}{2}\right)^{2}=\frac{3}{4}$.
- So, the smallest $\left(\sqrt{1-c^{2}}\right)^{5}$ can be is $\left(\sqrt{\frac{3}{4}}\right)^{5}=\left(\frac{\sqrt{3}}{2}\right)^{5}$.
- Then smallest possible value for the denominator of $\frac{1+2 c^{2}}{48\left(\sqrt{1-c^{2}}\right)^{5}}$ is $48\left(\frac{\sqrt{3}}{2}\right)^{5}$
- Then

$$
\begin{aligned}
\frac{1+2 c^{2}}{48\left(\sqrt{1-c^{2}}\right)^{5}} & \leq \frac{\frac{3}{2}}{48\left(\frac{\sqrt{3}}{2}\right)^{5}} \\
& =\frac{1}{\sqrt{3}^{5}}=\frac{1}{9 \sqrt{3}} \\
& <\frac{1}{10}
\end{aligned}
$$

Let's put together these pieces. We found that

$$
\left|f\left(\frac{1}{2}\right)-T_{2}\left(\frac{1}{2}\right)\right|=\frac{1+2 c^{2}}{48\left(\sqrt{1-c^{2}}\right)^{5}}
$$

for some $c$ in $\left(0, \frac{1}{2}\right)$. We also found that

$$
\frac{1+2 c^{2}}{48\left(\sqrt{1-c^{2}}\right)^{5}}<\frac{1}{10}
$$

when $c$ is in $\left(0, \frac{1}{2}\right)$. We conclude

$$
\left|f\left(\frac{1}{2}\right)-T_{2}\left(\frac{1}{2}\right)\right|<\frac{1}{10} .
$$

For the second part of the question, we need to find $f\left(\frac{1}{2}\right)$ and $T_{2}\left(\frac{1}{2}\right)$. Finding $f\left(\frac{1}{2}\right)$ is not difficult.

$$
\begin{aligned}
f(x) & =\arcsin x \\
f\left(\frac{1}{2}\right) & =\arcsin \left(\frac{1}{2}\right)=\frac{\pi}{6}
\end{aligned}
$$

In order to find $T_{2}\left(\frac{1}{2}\right)$, we need to find $T_{2}(x)$.

$$
T_{2}(x)=f(0)+f^{\prime}(0) x+\frac{1}{2} f^{\prime \prime}(0) x^{2}
$$

Conveniently, we've already found the first few derivatives of $f(x)$.

$$
\begin{aligned}
T_{2}(x) & =\arcsin (0)+\left(\frac{1}{\sqrt{1-0^{2}}}\right) x+\frac{1}{2}\left(\frac{0}{\left(\sqrt{1-0^{2}}\right)^{3}}\right) x^{2} \\
& =0+x+0 \\
& =x \\
T_{2}\left(\frac{1}{2}\right) & =\frac{1}{2}
\end{aligned}
$$

## Exercises actistageo3 is

 constant $c$, so let's find an equation for $f^{(n)}(x)$. This has been done before in A calculator tells us that this is about 0.02 .
the text, but we'll do it again here: we'll take several derivatives, then notice the pattern.

$$
\begin{aligned}
f(x) & =\log x \\
f^{\prime}(x) & =x^{-1} \\
f^{\prime \prime}(x) & =-x^{-2} \\
f^{\prime \prime \prime}(x) & =2!x^{-3} \\
f^{(4)}(x) & =-3!x^{-4} \\
f^{(5)}(x) & =4!x^{-5}
\end{aligned}
$$

So, when $n \geq 1$,

$$
f^{(n)}(x)=(-1)^{n-1}(n-1)!\cdot x^{-n}
$$

Now that we know the derivative of $f(x)$, we have a better idea what the error in our approximation looks like.

$$
\begin{aligned}
\left|f(1.1)-T_{n}(1.1)\right| & =\left|\frac{f^{(n+1)}(c)}{(n+1)!}(1.1-1)^{n+1}\right| \\
& =\left|f^{(n+1)}(c)\right| \frac{0.1^{n+1}}{(n+1)!} \\
& =\left|\frac{n!}{c^{n+1}}\right| \frac{1}{10^{n+1}(n+1)!} \\
& =\frac{1}{|c|^{n+1} \cdot 10^{n+1} \cdot(n+1)}
\end{aligned}
$$

for some $c$ in $(1,1.1)$

$$
\begin{aligned}
& <\frac{1}{(n+1) 10^{n+1} \cdot 1^{n+1}} \\
& =\frac{1}{(n+1) 10^{n+1}}
\end{aligned}
$$

What we've shown so far is

$$
\left|f(1.1)-T_{n}(1.1)\right|<\frac{1}{(n+1) 10^{n+1}}
$$

If we can show that $\frac{1}{(n+1) 10^{n+1}} \leq 10^{-4}$, then we'll be able to conclude

$$
\left|f(1.1)-T_{n}(1.1)\right|<\frac{1}{(n+1) 10^{n+1}} \leq 10^{-4}
$$

That is, our error is less than $10^{-4}$.
So, our goal for the problem is to find a value of $n$ that makes $\frac{1}{(n+1) 10^{n+1}} \leq 10^{-4}$. Certainly, $n=3$ is such a number. Therefore, any $n$ greater than or equal to 3 is an acceptable value.
3.4.11.12. Solution. We will approximate $f(x)=x^{\frac{1}{7}}$ using a Taylor polynomial. Since $3^{7}=2187$, we will use $x=2187$ as our centre.
We need to figure out which degree Taylor polynomial will result in a small-enough error.
If we use the $n$th Taylor polynomial, our error will be

$$
\begin{aligned}
\left|f(2200)-T_{n}(2200)\right| & =\left|\frac{f^{(n+1)}(c)}{(n+1)!}(2200-2187)^{n+1}\right| \\
& =\left|f^{(n+1)}(c)\right| \cdot \frac{13^{n+1}}{(n+1)!}
\end{aligned}
$$

for some $c$ in $(2187,2200)$. In order for this to be less than 0.001 , we need

$$
\begin{aligned}
\left|f^{(n+1)}(c)\right| \cdot \frac{13^{n+1}}{(n+1)!} & <0.001 \\
\left|f^{(n+1)}(c)\right| & <\frac{(n+1)!}{1000 \cdot 13^{n+1}}
\end{aligned}
$$

It's a tricky thing to figure out which $n$ makes this true. Let's make a table. We won't show all the work of filling it in, but the work is standard.

| $n$ | $\frac{(n+1)!}{1000 \cdot 13^{n+1}}$ | $\left\|f^{(n+1)}(c)\right\|$ | $\frac{(n+1)!}{1000 \cdot 13^{n+1}}$ |
| :---: | :---: | :---: | :---: |
| 0 | $\frac{1}{1000 \cdot 13}$ | $\left\|f^{\prime}(c)\right\|=\frac{1}{7 c^{6 / 7}}<\frac{1}{7 \cdot 3^{6}}$ | $\frac{1}{1000 \cdot 13^{1}}$ |
| 1 | $\frac{2}{1000 \cdot 13^{2}}$ | $\left\|f^{\prime \prime}(c)\right\|=\frac{6}{7^{2} \cdot c^{\frac{13}{7}}}<\frac{6}{7^{2} \cdot 3^{13}}$ | $\frac{(2)!}{1000 \cdot 13^{2}}$ |

Since

$$
\frac{7^{2} \cdot 3^{13}}{3}=26,040,609>169,000=1000 \cdot 13^{2}
$$

we have $\left|f^{(n+1)}(c)\right|<\frac{(n+1)!}{1000 \cdot 13^{n+1}}$ when $n=1$.
That is: if we use the first-degree Taylor polynomial, then for some $c$ between 2187 and 2200,

$$
\begin{aligned}
\left|f(2200)-T_{1}(2200)\right| & =\left|f^{\prime \prime}(c)\right| \cdot \frac{13^{2}}{2!} \\
& =\frac{6}{7^{2} \cdot c^{\frac{13}{7}}} \cdot \frac{13^{2}}{2} \\
& <\frac{6}{7^{2} \cdot 3^{13}} \cdot \frac{13^{2}}{2} \\
& =\frac{3 \cdot 13^{2}}{7^{2} \cdot 3^{13}} \approx 0.0000065
\end{aligned}
$$

So, actually, the linear Taylor polynomial (or any higher-degree Taylor polynomial) will result in an approximation that is much more accurate than required. (We don't know, however, that the constant approximation will be accurate enough-so we'd better stick with $n \geq 1$.)
Now that we know we can take the first-degree Taylor polynomial, let's compute $T_{1}(x)$. Recall we are taking the Taylor polynomial for $f(x)=x^{\frac{1}{7}}$ about $x=2187$.

$$
\begin{aligned}
f(2187) & =2187^{\frac{1}{7}}=3 \\
f^{\prime}(x) & =\frac{1}{7} x^{-\frac{6}{7}} \\
f^{\prime}(2187) & =\frac{1}{7 \sqrt[7]{2187}^{6}}=\frac{1}{7 \cdot 3^{6}} \\
T_{1}(x) & =f(2187)+f^{\prime}(2187)(x-2187) \\
& =3+\frac{x-2187}{7 \cdot 3^{6}} \\
T_{1}(2200) & =3+\frac{2200-2187}{7 \cdot 3^{6}} \\
& =3+\frac{13}{7 \cdot 3^{6}} \\
& \approx 3.00255
\end{aligned}
$$

We conclude $\sqrt[7]{2200} \approx 3.00255$.
3.4.11.13. Solution. If we're going to use Equation 3.4.33, then we'll probably be taking a Taylor polynomial. Using Example 3.4.16, the 6th-degree Maclaurin polynomial for $\sin x$ is

$$
T_{6}(x)=T_{5}(x)=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}
$$

so let's play with this a bit. Equation 3.4.33 tells us that the error will depend on the seventh derivative of $f(x)$, which is $-\cos x$ :

$$
\begin{aligned}
f(1)-T_{6}(1) & =f^{(7)}(c) \frac{1^{7}}{7!} \\
\sin (1)-\left(1-\frac{1}{3!}+\frac{1}{5!}\right) & =\frac{-\cos c}{7!} \\
\sin (1)-\frac{101}{5!} & =\frac{-\cos c}{7!} \\
\sin (1) & =\frac{4242-\cos c}{7!}
\end{aligned}
$$

for some $c$ between 0 and 1 . Since $-1 \leq \cos c \leq 1$,

$$
\begin{aligned}
& \frac{4242-1}{7!} \leq \sin (1) \leq \frac{4242+1}{7!} \\
& \frac{4241}{7!} \leq \sin (1) \leq \frac{4243}{7!} \\
& \frac{4241}{5040} \leq \sin (1) \leq \frac{4243}{5040}
\end{aligned}
$$

Remark: there are lots of ways to play with this idea to get better estimates. One way is to take a higher-degree Maclaurin polynomial. Another is to note that, since $0<c<1<\frac{\pi}{3}$, then $\frac{1}{2}<\cos c<1$, so

$$
\begin{aligned}
& \frac{4242-1}{7!}<\sin (1)<\frac{4242-\frac{1}{2}}{7!} \\
& \frac{4241}{5040}<\sin (1)<\frac{8483}{10080}<\frac{4243}{5040}
\end{aligned}
$$

If you got tighter bounds than asked for in the problem, congratulations!
3.4.11.14. Solution. 3.4.11.14.a For every whole number $n$, the $n$th derivative of $e^{x}$ is $e^{x}$. So:

$$
T_{4}(x)=\sum_{n=0}^{4} \frac{e^{0}}{n!} x^{n}=\sum_{n=0}^{4} \frac{x^{n}}{n!}
$$

3.4.11.14.b

$$
\begin{aligned}
T_{4}(1) & =\sum_{n=0}^{4} \frac{1^{n}}{n!}=\sum_{n=0}^{4} \frac{1}{n!} \\
& =\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!} \\
& =\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{6}+\frac{1}{24} \\
& =\frac{65}{24}
\end{aligned}
$$

3.4.11.14.c Using Equation 3.4.33,

$$
\begin{aligned}
e^{1}-T_{4}(1) & =\frac{1}{5!} e^{c} \quad \text { for some strictly between } 0 \text { and } 1 . \text { So, } \\
e-\frac{65}{24} & =\frac{e^{c}}{120} \\
e & =\frac{65}{24}+\frac{e^{c}}{120}
\end{aligned}
$$

Since $e^{x}$ is a strictly increasing function, and $0<c<1$, we conclude $e^{0}<e^{c}<e^{1}$ :

$$
\frac{65}{24}+\frac{1}{120}<e<\frac{65}{24}+\frac{e}{120}
$$

Simplifying the left inequality, we see

$$
\frac{326}{120}<e
$$

From the right inequality, we see

$$
e<\frac{65}{24}+\frac{e}{120}
$$

$$
\begin{aligned}
e-\frac{e}{120} & <\frac{65}{24} \\
e \cdot \frac{119}{120} & <\frac{65}{24} \\
e & <\frac{65}{24} \cdot \frac{120}{119}=\frac{325}{119}
\end{aligned}
$$

So, we conclude

$$
\frac{326}{120}<e<\frac{325}{119}
$$

as desired.
Remark: $\frac{\dot{3} 26}{120} \approx 2.717$, and $\frac{325}{119} \approx 2.731$.

## - Further problems for § 3.4

## Exercises - Stage 1

3.4.11.1. *. Solution. The third Maclaurin polynomial for $f(x)$ is

$$
f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2} \cdot x^{2}+\frac{f^{\prime \prime \prime}(0)}{6} \cdot x^{3}=4+3 x^{2}+\frac{1}{2} x^{3} .
$$

The coefficient of $x$ is $f^{\prime}(0)$ on one side, and 0 on the other, so $f^{\prime}(0)=0$.
The coefficient of $x^{2}$ is $\frac{1}{2} f^{\prime \prime}(0)$ on one side, and 3 on the other, so $f^{\prime \prime}(0)=6$.
3.4.11.2. *. Solution. The third Maclaurin polynomial for $h(x)$ is

$$
h(0)+h^{\prime}(0) x+\frac{h^{\prime \prime}(0)}{2} \cdot x^{2}+\frac{h^{(3)}(0)}{3!} \cdot x^{3}=1+4 x-\frac{1}{3} x^{2}+\frac{2}{3} x^{3}
$$

The coefficient of $x^{3}$ is $\frac{1}{3!} h^{(3)}(0)$ on one side, and $\frac{2}{3}$ on the other. Thus $\frac{h^{(3)}(0)}{6}=\frac{2}{3}$, so $h^{(3)}(0)=6 \cdot \frac{2}{3}=4$.
3.4.11.3. *. Solution. The third-degree Taylor polynomial for $h(x)$ about $x=2$ is

$$
h(2)+h^{\prime}(2)(x-2)+\frac{h^{\prime \prime}(2)}{2} \cdot(x-2)^{2}+\frac{h^{\prime \prime \prime}(2)}{6} \cdot(x-2)^{3}
$$

The coefficient of $(x-2)$ is $h^{\prime}(2)$ in the definition, and $\frac{1}{2}$ in the given function, so $h^{\prime}(2)=\frac{1}{2}$.
The coefficient of $(x-2)^{2}$ is $\frac{1}{2} h^{\prime \prime}(2)$ in the definition, and 0 in the given function, so $h^{\prime}(2)=0$.

## Exercises - Stage 2

3.4.11.4. *. Solution.
a For $x$ near 3 ,

$$
f(x) \approx f(3)+f^{\prime}(3)(x-3)=2+4(x-3)
$$

In particular

$$
f(2.98) \approx 2+4(2.98-3)=2-0.08=1.92
$$

b For $x$ near 3 ,

$$
\begin{aligned}
f(x) & \approx f(3)+f^{\prime}(3)(x-3)+\frac{1}{2} f^{\prime \prime}(3)(x-3)^{2} \\
& =2+4(x-3)-\frac{1}{2} 10(x-3)^{2}
\end{aligned}
$$

In particular

$$
\begin{aligned}
f(2.98) & \approx 2+4(2.98-3)-5(2.98-3)^{2} \\
& =2-0.08-0.002=1.918
\end{aligned}
$$

3.4.11.5. *. Solution. Let's name $g(x)=x^{1 / 3}$. Then $g^{\prime}(x)=\frac{1}{3} x^{-2 / 3}$ and $g^{\prime \prime}(x)=-\frac{2}{9} x^{-5 / 3}$. In particular, $g(8)=2, g^{\prime}(8)=\frac{1}{12}$, and $g^{\prime \prime}(x)<0$ for all $x>0$. The tangent line approximation to $10^{1 / 3}$ is

$$
\begin{aligned}
g(10) & \approx g(8)+g^{\prime}(8)(10-8) \\
& =2+\frac{1}{12}(2)=\frac{13}{6}
\end{aligned}
$$

Using the error formula:

$$
g(10)=g(8)+g^{\prime}(8)(10-8)+\frac{1}{2} g^{\prime \prime}(c)(10-8)^{2}
$$

for some $8<c<10$. Since $g^{\prime \prime}(c)=-\frac{2}{9 c^{5 / 3}}, g^{\prime \prime}(c)$ is negative, so $g(10)$ is $\frac{13}{6}$ plus some negative quantity. So, the tangent line approximation is too big.
3.4.11.6. *. Solution. We use the function $f(x)=\sqrt{x}$ and point $a=1$ as the centre of our approximation, since we can easily calculate

$$
f(a)=f(1)=\sqrt{1}=1
$$

We compute $f^{\prime}(x)=\frac{1}{2 \sqrt{x}}$, so

$$
f^{\prime}(1)=\frac{1}{2 \sqrt{1}}=\frac{1}{2} .
$$

So, a linear approximation of $\sqrt{2}=f(2)$ is

$$
\begin{aligned}
\sqrt{2} & \approx T_{1}(2)=f(1)+f^{\prime}(1) \cdot(2-1) \\
& =1+\frac{1}{2}=\frac{3}{2}
\end{aligned}
$$

3.4.11.7. *. Solution. We use the function $f(x)=x^{1 / 3}$ and point $a=27$ as the centre of our approximation since we can easily compute

$$
f(27)=3
$$

We compute $f^{\prime}(x)=\frac{1}{3} x^{-2 / 3}$, so

$$
f^{\prime}(27)=\frac{1}{3} \cdot(27)^{-2 / 3}=\frac{1}{27}
$$

So, the linear approximation of $26^{1 / 3}=f(26)$ is

$$
\begin{aligned}
26^{1 / 3} & \approx T_{1}(26)=f(27)+f^{\prime}(27) \cdot(26-27) \\
& =3-\frac{1}{27}=\frac{80}{27}
\end{aligned}
$$

3.4.11.8. *. Solution. We use the function $f(x)=x^{5}$ and point $a=10$ as the centre of our approximation since we know that $f(a)=f(10)=10^{5}$.
Since $f^{\prime}(x)=5 x^{4}$ we have $f^{\prime}(10)=50,000$.
So, a linear approximation of $10.1^{5}$ is

$$
\begin{aligned}
T_{1}(10.1) & =f(10)+f^{\prime}(10) \cdot(10.1-10) \\
& =100,000+50,000 \cdot 0.1=105,000
\end{aligned}
$$

3.4.11.9. *. Solution. We use the function $f(x)=\sin (x)$ and point $a=\pi$ as the centre of our approximation since we know that

$$
\sin (a)=f(\pi)=\sin \pi=0
$$

and $\pi$ is reasonably close to $\frac{101 \pi}{100}$. We compute $f^{\prime}(x)=\cos (x)$, so

$$
f^{\prime}(\pi)=\cos (\pi)=-1
$$

So, the linear approximation of $\sin \left(\frac{101 \pi}{100}\right)$ is

$$
\begin{aligned}
f\left(\frac{101 \pi}{100}\right) & \approx T_{1}\left(\frac{101 \pi}{100}\right) \\
& =f(\pi)+f^{\prime}(\pi) \cdot\left(\frac{101 \pi}{100}-\pi\right) \\
& =0+(-1) \cdot \frac{\pi}{100}=-\frac{\pi}{100}
\end{aligned}
$$

3.4.11.10. *. Solution. Set $f(x)=\arctan (x)$. Then $f^{\prime}(x)=\frac{1}{1+x^{2}}$, so $f^{\prime}(1)=$ $\frac{1}{2}$ and

$$
f(1.1) \approx f(1)+f^{\prime}(1)(1.1-1)=\frac{\pi}{4}+\frac{1}{20}
$$

3.4.11.11. *. Solution. Set $f(x)=(2+x)^{3}$, so we are approximating $f(0.001)$. The obvious choice of $a$ is $a=0$.
Then $f^{\prime}(x)=3(2+x)^{2}$, so

$$
(2.001)^{3}=f(0.001) \approx f(0)+f^{\prime}(0)(0.001-0)=8+\frac{12}{1000}=\frac{8012}{1000}
$$

Remark: if we had chosen $f(x)=x^{3}$ and $a=2$, the result would have been exactly the same.
3.4.11.12. *. Solution. We set $f(x)=(8+x)^{2 / 3}$, and choose $a=0$ as our centre. Then $f^{\prime}(x)=\frac{2}{3}(8+x)^{-1 / 3}$, so that

$$
\begin{aligned}
(8.06)^{2 / 3}=f(0.06) & \approx f(0)+f^{\prime}(0) \cdot 0.06 \\
& =8^{2 / 3}+\frac{2}{3} 8^{-1 / 3} \cdot 0.06 \\
& =\sqrt[3]{8}^{2}+\frac{2}{3 \sqrt[3]{8}} \cdot 0.06 \\
& =2^{2}+\frac{2}{3 \cdot 2} \cdot 0.06 \\
& =4+\frac{1}{3} \cdot 0.06 \\
& =4.02=\frac{402}{100}
\end{aligned}
$$

3.4.11.13. *. Solution. We begin by finding the derivatives of $f$ at $x=0$.

$$
\begin{array}{rlrl}
f(x) & =(1-3 x)^{-1 / 3} & f(0) & =1 \\
f^{\prime}(x) & =(-3) \frac{-1}{3}(1-3 x)^{-4 / 3}=(1-3 x)^{-4 / 3} & f^{\prime}(0) & =1 \\
f^{\prime \prime}(x) & =(-3) \frac{-4}{3}(1-3 x)^{-7 / 3}=4(1-3 x)^{-7 / 3} & f^{\prime \prime}(0) & =4 \\
f^{(3)}(x) & =(-3)(4) \frac{-7}{3}(1-3 x)^{-10 / 3}=28(1-3 x)^{-10 / 3} & f^{(3)}(0) & =28
\end{array}
$$

Plugging these into the definition of a Taylor Polynomial, we find that the thirdorder Taylor polynomial for $f$ around $x=0$ is

$$
\begin{aligned}
T_{3}(x) & =1+x+\frac{4}{2!} x^{2}+\frac{28}{3!} x^{3} \\
& =1+x+2 x^{2}+\frac{14}{3} x^{3}
\end{aligned}
$$

### 3.4.11.14. *. Solution.

- By Equation 3.4.33, the absolute value of the error is

$$
\left|\frac{f^{\prime \prime \prime}(c)}{3!} \cdot(2-1)^{3}\right|=\left|\frac{c}{6\left(22-c^{2}\right)}\right|
$$

for some $c \in(1,2)$.

- When $1 \leq c \leq 2$, we know that $18 \leq 22-c^{2} \leq 21$, and that numerator and denominator are non-negative, so

$$
\begin{aligned}
\left|\frac{c}{6\left(22-c^{2}\right)}\right| & =\frac{c}{6\left(22-c^{2}\right)} \leq \frac{2}{6\left(22-c^{2}\right)} \leq \frac{2}{6 \cdot 18} \\
& =\frac{1}{54} \leq \frac{1}{50}
\end{aligned}
$$

as required.

- Alternatively, notice that $c$ is an increasing function of $c$, while $22-c^{2}$ is a decreasing function of $c$. Hence the fraction is an increasing function of $c$ and takes its largest value at $c=2$. Hence

$$
\left|\frac{c}{6\left(22-c^{2}\right)}\right| \leq \frac{2}{6 \times 18}=\frac{1}{54} \leq \frac{1}{50} .
$$

### 3.4.11.15. *. Solution.

- By Equation 3.4.33, there is $c \in(0,0.5)$ such that the error is

$$
\begin{aligned}
R_{4} & =\frac{f^{(4)}(c)}{4!}(0.5-0)^{4} \\
& =\frac{1}{24 \cdot 16} \cdot \frac{\cos \left(c^{2}\right)}{3-c}
\end{aligned}
$$

- For any $c$ we have $\left|\cos \left(c^{2}\right)\right| \leq 1$, and for $c<0.5$ we have $3-c>2.5$, so that

$$
\left|\frac{\cos \left(c^{2}\right)}{3-c}\right| \leq \frac{1}{2.5}
$$

- We conclude that

$$
\left|R_{4}\right| \leq \frac{1}{2.5 \cdot 24 \cdot 16}=\frac{1}{60 \cdot 16}<\frac{1}{60 \cdot 10}=\frac{1}{600}<\frac{1}{500}
$$

### 3.4.11.16. *. Solution.

- By Equation 3.4.33, there is $c \in(0,1)$ such that the error is

$$
\left|\frac{f^{\prime \prime \prime}(c)}{3!} \cdot(1-0)^{3}\right|=\left|\frac{e^{-c}}{6\left(8+c^{2}\right)}\right|
$$

- When $0<c<1$, we know that $1>e^{-c}>e^{-1}$ and $8 \leq 8+c^{2}<9$, so

$$
\begin{aligned}
\left|\frac{e^{-c}}{6\left(8+c^{2}\right)}\right| & =\frac{e^{-c}}{6\left(8+c^{2}\right)} \\
& <\frac{1}{6\left|8+c^{2}\right|} \\
& <\frac{1}{6 \times 8}=\frac{1}{48}<\frac{1}{40}
\end{aligned}
$$

as required.

- Alternatively, notice that $e^{-c}$ is a decreasing function of $c$, while for $0<c 8+c^{2}$ is an increasing function of $c$. Hence the fraction is a decreasing function of $c$ and takes its largest value at $c=0$. Hence

$$
\left|\frac{e^{c}}{6\left(8+c^{2}\right)}\right| \leq \frac{1}{6 \times 8}=\frac{1}{48}<\frac{1}{40}
$$

3.4.11.17. *. Solution. 3.4.11.17.a, 3.4.11.17.b:

Let $f(x)=x^{1 / 3}$ and $x_{0}=27$. Then

$$
\begin{aligned}
f(x) & =x^{1 / 3} & f^{\prime}(x) & =\frac{1}{3} x^{-2 / 3} \\
f(27) & =27^{1 / 3}=3 & f^{\prime}(27) & =\frac{1}{3} \cdot \frac{1}{3^{2}}=\frac{1}{27}
\end{aligned} \begin{aligned}
f^{\prime \prime}(x) & =-\frac{2}{9} x^{-5 / 3} \\
f^{\prime \prime}(27) & =-\frac{2}{9} \cdot \frac{1}{3^{5}} \\
& =-\frac{2}{2187}
\end{aligned}
$$

so that, with $x=25$,

$$
\begin{aligned}
5^{2 / 3} & =f(25) \approx f(27)+f^{\prime}(27)(25-27)=3-\frac{2}{27} \\
& \approx 2.9259 \text { (linear approx) } \\
5^{2 / 3} & =f(25) \approx f(27)+f^{\prime}(27)(25-27)+\frac{1}{2} f^{\prime \prime}(27)(25-27)^{2} \\
& =3-\frac{2}{27}-\frac{1}{2} \cdot \frac{2 \cdot 4}{2187} \approx 2.9241 \text { (quadratic app) }
\end{aligned}
$$

3.4.11.17.c To obtain an error estimate for the linear approximation, we use that

$$
5^{2 / 3}=f(25)=f(27)+f^{\prime}(27)(25-27)+\frac{1}{2} f^{\prime \prime}(z)(25-27)^{2}
$$

for some $z$ between 25 and 27 . The error is exactly

$$
\left|\frac{1}{2} f^{\prime \prime}(z)(25-27)^{2}\right|=\left|\frac{1}{2}\left(-\frac{2}{9} x^{-5 / 3}\right)(-2)^{2}\right|=\frac{4}{9} z^{-5 / 3}
$$

For $z$ between 25 and $27, z^{-5 / 3}$ is between $25^{-5 / 3}$ and $27^{-5 / 3}$. The biggest this can be is $25^{-5 / 3}$, so the maximum possible error is $\left\{\frac{4}{9} 25^{-5 / 3}\right\}$.
To get a better idea of what this number is, we note $2.9^{3}<25$, so $\frac{4}{9} 25^{-5 / 3}<$ ${ }_{9}^{4} 2.9^{-5}=0.0022$.

## Exercises - Stage 3

3.4.11.18. Solution. The fourth-degree Maclaurin polynomial for $f(x)$ is

$$
T_{4}(x)=f(0)+f^{\prime}(0) x+\frac{1}{2} f^{\prime \prime}(0) x^{2}+\frac{1}{3!} f^{\prime \prime \prime}(0) x^{3}+\frac{1}{4!} f^{(4)}(0) x^{4}
$$

while the third-degree Maclaurin polynomial for $f(x)$ is

$$
T_{3}(x)=f(0)+f^{\prime}(0) x+\frac{1}{2} f^{\prime \prime}(0) x^{2}+\frac{1}{3!} f^{\prime \prime \prime}(0) x^{3}
$$

So, we simply "chop off" the part of $T_{4}(x)$ that includes $x^{4}$. Since that's already 0 , in this case $T_{3}(x)=T_{4}(x)$.

$$
T_{3}(x)=5 x^{2}-9
$$

3.4.11.19. *. Solution. $y$ is a function of $x$ that obeys

$$
y(x)^{4}+x y(x)=x^{2}-1
$$

By implicit differentiation (and then subbing in $x=2, y(2)=1$ )

$$
\begin{aligned}
4 y(x)^{3} y^{\prime}(x)+y(x)+x y^{\prime}(x) & =2 x \\
4 y^{\prime}(2)+1+2 y^{\prime}(2) & =4 \\
y^{\prime}(2) & =\frac{1}{2}
\end{aligned}
$$

Differentiating with respect to $x$ a second time and then subbing in $x=2, y(2)=1$, $y^{\prime}(2)=\frac{1}{2}$ :

$$
\begin{aligned}
12 y(x)^{2} y^{\prime}(x)^{2}+4 y(x)^{3} y^{\prime \prime}(x)+y^{\prime}(x)+y^{\prime}(x)+x y^{\prime \prime}(x) & =2 \\
12 \times 1 \times \frac{1}{4}+4 y^{\prime \prime}(2)+\frac{1}{2}+\frac{1}{2}+2 y^{\prime \prime}(2) & =2 \\
6 y^{\prime \prime}(2) & =-2 \\
y^{\prime \prime}(2) & =-\frac{1}{3}
\end{aligned}
$$

The tangent line approximation to $y(x)$ at $x=2$ is

$$
y(x) \approx y(2)+y^{\prime}(2)(x-2)=1+\frac{1}{2}(x-2)
$$

In particular,

$$
y(2.1) \approx y(2)+y^{\prime}(2)(2.1-2)=1+\frac{1}{2}(.1)=1.05
$$

The quadratic approximation to $y(x)$ at $x=2$ is

$$
y(x) \approx y(2)+y^{\prime}(2)(x-2)+\frac{1}{2} y^{\prime \prime}(2)(x-2)^{2}
$$

$$
=1+\frac{1}{2}(x-2)-\frac{1}{6}(x-2)^{2}
$$

In particular,

$$
\begin{aligned}
y(2.1) & \approx y(2)+y^{\prime}(2)(2.1-2)+\frac{1}{2} y^{\prime \prime}(2)(2.1-2)^{2} \\
& =1+\frac{1}{2}(.1)-\frac{1}{6}(.1)^{2}=1.0483
\end{aligned}
$$

At $x=2, y=1$ and $y^{\prime}=\frac{1}{2}$. So the tangent line passes through $(2,1)$ and has slope $\frac{1}{2}$. At $x=2, y^{\prime \prime}=-\frac{1}{3}$, so the graph $y=f(x)$ (locally!) looks like a parabola pointing down near $x=2$. This gives the graph fragment below.
Alternatively, we could observe that, near $x=2, y(x)$ will be quite close to its quadratic approximation, $1+\frac{1}{2}(x-2)-\frac{1}{6}(x-2)^{2}$.

3.4.11.20. *. Solution. 3.4.11.20.a $y$ is a function of $x$ that obeys

$$
1=x^{4}+y(x)+x y(x)^{4}
$$

By implicit differentiation (and then subbing in $x=-1, y(-1)=1$ )

$$
\begin{aligned}
0 & =4 x^{3}+y^{\prime}(x)+y(x)^{4}+4 x y(x)^{3} y^{\prime}(x) \\
0 & =-4+y^{\prime}(-1)+1-4 y^{\prime}(-1) \\
-1 & =y^{\prime}(-1)
\end{aligned}
$$

Differentiating with respect to $x$ a second time and then subbing in $x=-1, y(-1)=$ 1 , and $y^{\prime}(-1)=-1$ :

$$
\begin{aligned}
& 0=12 x^{2}+y^{\prime \prime}(x)+4 y(x)^{3} y^{\prime}(x)+4 y(x)^{3} y^{\prime}(x)+12 x y(x)^{2} y^{\prime}(x)^{2} \\
&+4 x y(x)^{3} y^{\prime \prime}(x) \\
& 0=12+y^{\prime \prime}(-1)-4-4-12-4 y^{\prime \prime}(-1) \\
&-8=3 y^{\prime \prime}(-1) \\
& y^{\prime \prime}(-1)=-\frac{8}{3}
\end{aligned}
$$

The tangent line approximation to $y(x)$ at $x=-1$ is

$$
y(x) \approx y(-1)+y^{\prime}(-1)(x+1)=1-(x+1)=-x
$$

In particular,

$$
y(-0.9) \approx 0.9
$$

3.4.11.20.b The quadratic approximation to $y(x)$ at $x=-1$ is

$$
\begin{aligned}
y(x) & \approx y(-1)+y^{\prime}(-1)(x+1)+\frac{1}{2} y^{\prime \prime}(-1)(x+1)^{2} \\
& =1-(x+1)-\frac{4}{3}(x+1)^{2}
\end{aligned}
$$

In particular,

$$
y(-0.9) \approx 1-(.1)-\frac{4}{3}(.1)^{2} \approx 0.8867
$$

3.4.11.20.c At $x=-1$, the slope of the curve is $y^{\prime}(-1)=-1$. Its tangent line is falling at $45^{\circ}$. At $x=-1, y^{\prime \prime}(-1)=-\frac{8}{3}$, so the slope of the curve is decreasing as $x$ passes through -1 . Zoomed in very close, the curve looks like a parabola opening downwards. This gives the figure

$$
y=y(x) \approx-1-(x+1)-\frac{4}{3}(x+1)^{2} \text {-1 }
$$

3.4.11.21. *. Solution. Let $f(x)=\log x$ and $x_{0}=10$. Then

$$
\left.\begin{array}{rlrl}
f(x) & =\log x & f^{\prime}(x) & =\frac{1}{x}
\end{array} r f^{\prime \prime}(x)=-\frac{1}{x^{2}}\right)
$$

so that, with $x=10.3$,

$$
\begin{aligned}
\log 10.3 & =f(10.3) \approx f(10)+f^{\prime}(10)(10.3-10)=2.30259+\frac{0.3}{10} \\
& =2.33259
\end{aligned}
$$

The error in this approximation (excluding the error in the given data $\log 10 \approx$ 2.30259) is $\frac{1}{2} f^{\prime \prime}(z)(10.3-10)^{2}$ for some $z$ between 10 and 10.3. Because $f^{\prime \prime}(z)=$ $-\frac{1}{z^{2}}$ increases as $z$ increases, it must be between $-\frac{1}{10^{2}}$ and $-\frac{1}{10.3^{2}}$. This forces $\frac{1}{2} f^{\prime \prime}(z)(10.3-10)^{2}$ to be between $-\frac{1}{2} \cdot \frac{1}{10^{2}}(0.3)^{2}=-0.00045$ and $-\frac{1}{2} \cdot \frac{1}{10.3^{2}}(0.3)^{2}<$ -0.00042 .
3.4.11.22. *. Solution. We begin by finding the values of the derivatives of $f$ at $x=0$. We can use the chain rule, or the formula we found in Question 2.14.2.19, Section 2.14.

$$
\begin{array}{rlrl}
f(x) & =e^{e^{x}} & f(0) & =e \\
f^{\prime}(x) & =e^{x} e^{e^{x}} & f^{\prime}(0) & =e \\
f^{\prime \prime}(x) & =\left(e^{x}+e^{2 x}\right) e^{e^{x}} & f^{\prime \prime}(0) & =2 e \\
f^{\prime \prime \prime}(x) & =\left(e^{x}+3 e^{2 x}+e^{3 x}\right) e^{e^{x}} &
\end{array}
$$

(a) $L(x)=f(0)+f^{\prime}(0)(x-0)=e+e x$
(b) $Q(x)=f(0)+f^{\prime}(0)(x-0)+\frac{1}{2} f^{\prime \prime}(0)(x-0)^{2}=e+e x+e x^{2}$
(c) Since $e x^{2}>0$ for all $x>0, L(x)<Q(x)$ for all $x>0$.

From the error formula, we know that

$$
\begin{aligned}
f(x) & =f(0)+f^{\prime}(0) x+\frac{1}{2} f^{\prime \prime}(0) x^{2}+\frac{1}{3!} f^{\prime \prime \prime}(c) x^{3} \\
& =Q(x)+\frac{1}{6}\left(e^{c}+3 e^{2 c}+e^{3 c}\right) e^{e^{c}} x^{3}
\end{aligned}
$$

for some $c$ between 0 and $x$. Since $\frac{1}{6}\left(e^{c}+3 e^{2 c}+e^{3 c}\right) e^{e^{c}}$ is positive for any $c$, for all $x>0, \frac{1}{6}\left(e^{c}+3 e^{2 c}+e^{3 c}\right) e^{e^{c}} x^{3}>0$, so $Q(x)<f(x)$.
(d) Write $g(x)=e^{x}=1+x+\frac{1}{2!} e^{c} x^{2}$, for some $c$ between 0 and $x$. For $x=0.1$ we have $0<c<0.1$ and $1<e^{c}<e^{0.1}<e<3$. So

$$
\begin{aligned}
& e^{0.1}=f(0.1)=1+0.1+\frac{1}{2} e^{c}(0.1)^{2}>1+0.1+\frac{1}{2}(1)(0.1)^{2}=1.105 \\
& e^{0.1}=f(0.1)=1+0.1+\frac{1}{2} e^{c}(0.1)^{2}<1+0.1+\frac{1}{2}(3)(0.1)^{2}=1.115
\end{aligned}
$$

That is, $1.105<e^{0.1}<1.115$.

## 3.5 - Optimisation

### 3.5.4 • Exercises

- Exercises for § 3.5.1


## Exercises - Stage 1

### 3.5.4.1. Solution.



When $x=0$, the curve $y=f(x)$ appears to have a flat tangent line, so the $x=0$ is a critical point. However, it is not a local extremum: it is not true that $f(0) \geq f(x)$ for all $x$ near 0 , and it is not true that $f(0) \leq f(x)$ for all $x$ near 0 .
To the right of the $x$-axis, there is a spike where the derivative of $f(x)$ does not exist. The $x$-value corresponding to this spike (call it $a$ ) is a singular point, and $f(x)$ has a local maximum at $x=a$.

### 3.5.4.2 Solution.



The $x$-coordinate corresponding to the blue dot (let's call it $a$ ) is a critical point, because the tangent line to $f(x)$ at $x=a$ is horizontal. There is no lower point nearby, and actually no lower point on the whole interval shown, so $f(x)$ has both a local minimum and a global minimum at $x=a$.
If a function is not continuous at a point, then it is not differentiable at that point. So, the $x$-coordinate corresponding to the discontinuity (let's call it $b$ ) is a singular
point. Values of $f(x)$ immediately to the right of $b$ are lower, and values immediately to the left of $b$ are higher, so $f(x)$ has no local (or global) extremum at $x=b$.
3.5.4.3. Solution. One possible answer is shown below.


For every $x$ in the red interval shown below, $f(2) \geq f(x)$, so $f(2)$ is a local maximum. However, the point marked with a blue dot shows that $f(x)>f(2)$ for some $x$, so $f(2)$ is not a global maximum.


## Exercises - Stage 2

3.5.4.4. Solution. Critical points are those values of $x$ for which $f^{\prime}(x)=0$, and singular points are those values of $x$ for which $f(x)$ is not differentiable. So, we ought to find $f^{\prime}(x)$. Using the quotient rule,

$$
f^{\prime}(x)=\frac{(1)\left(x^{2}+3\right)-(x-1)(2 x)}{\left(x^{2}+3\right)^{2}}
$$

$$
\begin{aligned}
& =\frac{-x^{2}+2 x+3}{\left(x^{2}+3\right)^{2}} \\
& =-\frac{(x-3)(x+1)}{\left(x^{2}+3\right)^{2}}
\end{aligned}
$$

(a) The derivative $f^{\prime}(x)$ is zero when $x=3$ and when $x=-1$, so those are the critical points.
(b) The denominator of $f^{\prime}(x)$ is never zero, so the derivative $f^{\prime}(x)$ exists for all $x$ and $f(x)$ has no singular points.
(c) Theorem 3.5.4 tells us that local extrema of $f(x)$ can only occur at critical points and singular points. So, the possible points where extrema of $f(x)$ may exist are $x=3$ and $x=-1$.

## Exercises - Stage 3

### 3.5.4.5. Solution.



For the first curve, the function's value at $x=2$ (that is, the $y$-value of the solid dot) is higher than anything around it. So, it's a local maximum.
For the second curve, the function's value at $x=2$ (that is, the $y$-value of the solid dot) is higher than everything to the left, but lower than values immediately to the right. (On the graph reproduced below, $f(x)$ is higher than everything in the red section, and lower than everything in the blue section.) So, it is neither a local max nor a local min.


Similarly, for the third curve, $f(2)$ is lower than the values to the right of it, and higher than values to the left of it, so it is neither a local minimum nor a local maximum.
In the final curve, $f(2)$ (remember-this is the $y$-value of the solid dot) is higher
than everything immediately to the left or right of it (for instance, over the interval marked in red below), so it is a local maximum.

3.5.4.6. Solution. The question specifies that $x=2$ must not be an endpoint. By Theorem 3.5.4, if $x=2$ not a critical point, then it must be a singular point. That is, $f(x)$ is not differentiable at $x=2$. Two possibilities are shown below, but there are infinitely many possible answers.

3.5.4.7. Solution. Critical points are those values of $x$ for which $f^{\prime}(x)=0$, and singular points are those values of $x$ for which $f(x)$ is not differentiable. So, we ought to find $f^{\prime}(x)$. Since $f(x)$ has an absolute value sign, let's re-write it in a version that is friendlier to differentiation. Remember that $|X|=X$ when $X \geq 0$, and $|X|=-X$ when $X<0$.

$$
\begin{aligned}
f(x) & =\sqrt{|(x-5)(x+7)|} \\
& =\left\{\begin{array}{cl}
\sqrt{(x-5)(x+7)} & \text { if }(x-5)(x+7) \geq 0 \\
\sqrt{-(x-5)(x+7)} & \text { if }(x-5)(x+7)<0
\end{array}\right.
\end{aligned}
$$

The product $(x-5)(x+7)$ is positive when $(x-5)$ and $(x+7)$ have the same sign, and negative when they have opposite signs, so

$$
f(x)= \begin{cases}\sqrt{(x-5)(x+7)} & \text { if } x \in(-\infty,-7] \cup[5, \infty) \\ \sqrt{-(x-5)(x+7)} & \text { if } x \in(-7,5)\end{cases}
$$

Now, when $x \neq-7,5$, we can differentiate, using the chain rule.

$$
\begin{aligned}
f^{\prime}(x) & = \begin{cases}\frac{\frac{d}{d x}\{(x-5)(x+7)\}}{2 \sqrt{(x-5)(x+7)}} & \text { if } x \in(-\infty,-7) \cup(5, \infty) \\
\frac{d}{d x}\{-(x-5)(x+7)\} \\
2 \sqrt{-(x-5)(x+7)} & \text { if } x \in(-7,5) \\
? ? & \text { if } x=-7, x=5\end{cases} \\
& = \begin{cases}\frac{2 x+2}{2 \sqrt{(x-5)(x+7)}} & \text { if } x \in(-\infty,-7) \cup(5, \infty) \\
\frac{-2 x-2}{2 \sqrt{-(x-5)(x+7)}} & \text { if } x \in(-7,5) \\
? ? & \text { if } x=-7, x=5\end{cases}
\end{aligned}
$$

We are tempted to say that the derivative doesn't exist when $x=-7$ and $x=5$, but be careful- we don't actually know that yet. The formulas we have for the $f^{\prime}(x)$ are only good when $x$ is not -7 or 5 .
The middle formula $\frac{-2 x-2}{2 \sqrt{-(x-5)(x+7)}}$ tells us $x=-1$ is a critical point: when $x=-1, f^{\prime}(x)$ is given by the middle line, and it is 0 . Note that $x=-1$ also makes the top formula 0 , but $f^{\prime}(-1)$ is not given by the top formula, so that doesn't matter.
What we've concluded so far is that $x=-1$ is a critical point of $f(x)$, and $f(x)$ has no other critical points or singular points when $x \neq-7,5$. It remains to figure out what's going on at -7 and 5 . One way to do this is to use the definition of the derivative to figure out what $f^{\prime}(-7)$ and $f^{\prime}(5)$ are, if they exist. This is somewhat laborious. Let's look for a better way.

- First, let's notice that $f(x)$ is defined for all values of $x$, thanks to that handy absolute value sign.
- Next, notice $f(x) \geq 0$ for all $x$, since square roots never give a negative value.
- Then if there is some value of $x$ that gives $f(x)=0$, that $x$ gives a global minimum, and therefore a local minimum.
- $f(x)=0$ exactly when $(x-5)(x+7)=0$, which occurs at $x=-7$ and $x=5$
- Therefore, $f(x)$ has global and local minima at $x=-7$ and $x=5$
- So, $x=-7$ and $x=5$ are critical points or singular points by Theorem 3.5.4.

So, all together:
$x=-1$ is a critical point, and $x=-7$ and $x=5$ are critical points or singular points (but we don't know which).
Remark: if you would like a review of how to use the definition of the derivative, below we show that $f(x)$ is not differentiable at $x=-7$. (In fact, $x=-7$ and $x=5$ are both singular points.)

$$
f^{\prime}(-7)=\lim _{h \rightarrow 0} \frac{f(-7+h)-f(-7)}{h}
$$

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{\sqrt{|(-13+h)(h)|}-\sqrt{|0|}}{h} \\
& =\lim _{h \rightarrow 0} \frac{\sqrt{|(-13+h)(h)|}}{h}
\end{aligned}
$$

Let's first consider the case $h>0$.

$$
\begin{aligned}
\lim _{h \rightarrow 0^{+}} \frac{\sqrt{|(-13+h)(h)|}}{h} & =\lim _{h \rightarrow 0^{+}} \frac{\sqrt{(13-h)(h)}}{h} \\
& =\lim _{h \rightarrow 0^{+}} \frac{\sqrt{13 h-h^{2}}}{\sqrt{h^{2}}} \\
& =\lim _{h \rightarrow 0^{+}} \sqrt{\frac{13 h-h^{2}}{h^{2}}} \\
& =\lim _{h \rightarrow 0^{+}} \sqrt{\frac{13}{h}-1} \\
& =\infty
\end{aligned}
$$

Since one side of the limit doesn't exist,

$$
\lim _{h \rightarrow 0} \frac{f(-7+h)-(-7)}{h}=D N E
$$

so $f^{\prime}(x)$ is not differentiable at $x=-7$. Therefore, $x=-7$ is a singular point.
3.5.4.8. Solution. For any real number $c, c$ is in the domain of $f(x)$ and $f^{\prime}(c)$ exists and is equal to zero. So, following Definition 3.5.6, every real number is a critical point of $f(x)$, and $f(x)$ has no singular points.
For every number $c$, let $a=c-1$ and $b=c+1$, so $a<c<b$. Then $f(x)$ is defined for every $x$ in the interval $[a, b]$, and $f(x)=f(c)$ for every $a \leq x \leq b$. That means $f(x) \leq f(c)$ and $f(x) \geq f(c)$. So, comparing with Definition 3.5.3, we see that $f(x)$ has a global and local maximum AND minimum at every real number $x=c$.

## - Exercises for § 3.5.2

## Exercises - Stage 1

3.5.4.1. Solution. Two examples are given below, but many are possible.


If $f(x)=-x^{2}$ or $f(x)=-\sqrt{|x|}$, then $f(x)$ has a global maximum at $x=0$. Since $f(x)$ keeps getting more and more strongly negative as $x$ gets farther and farther from $0, f(x)$ has no global minimum.
3.5.4.2. Solution. Two examples are given below, but many are possible.


If $f(x)=e^{x}$, then $f(x)>0$ for all $x$. As we move left along the $x$-axis, $f(x)$ gets smaller and smaller, approaching 0 but never reaching it. Since $f(x)$ gets smaller and smaller as we move left, there is no global minimum. Likewise, $f(x)$ increases more and more as we move right, so there is no maximum.


If $f(x)=\arctan (x)+2$, then $f(x)>\left(-\frac{\pi}{2}\right)+2>0$ for all $x$.
As we move left along the $x$-axis, $f(x)$ gets smaller and smaller, approaching $\left(-\frac{\pi}{2}+2\right)$ but never reaching it. Since $f(x)$ gets smaller and smaller as we move left, there is no global minimum.
Likewise, as we move right along the $x$-axis, $f(x)$ gets bigger and bigger, approaching $\left(\frac{\pi}{2}+2\right)$ but never reaching it. Since $f(x)$ gets bigger and bigger as we move right, there is no global maximum.
3.5.4.3. Solution. Since $f(5)$ is a global minimum, $f(5) \leq f(x)$ for all $x$, and so in particular $f(5) \leq f(-5)$.
Similarly, $f(-5) \leq f(x)$ for all $x$, so in particular $f(-5) \leq f(5)$.
Since $f(-5) \leq f(5)$ AND $f(5) \leq f(-5)$, it must be true that $f(-5)=f(5)$.
A sketch of one such graph is below.


## Exercises - Stage 2

3.5.4.4. Solution. Global extrema will occur at critical or singular points in the interval $(-5,5)$ or at the endpoints $x=5, x=-5$.
$f^{\prime}(x)=2 x+6$. Since this is defined for all real numbers, there are no singular points. The only time $f^{\prime}(x)=0$ is when $x=-3$. This is inside the interval $[-5,5]$.

So, our points to check are $x=-3, x=-5$, and $x=5$.

| $c$ | -3 | -5 | 5 |
| :---: | :---: | :---: | :---: |
| type | critical point | endpoint | endpoint |
| $f(c)$ | -19 | -15 | 45 |

The global maximum is 45 at $x=5$ and the global minimum is -19 at $x=-3$.
3.5.4.5. Solution. Global extrema will occur at the endpoints of the interval, $x=-4$ and $x=0$, or at singular or critical points inside the interval. Since $f(x)$ is a polynomial, it is differentiable everywhere, so there are no singular points. To find the critical points, we set the derivative equal to zero.

$$
\begin{aligned}
f^{\prime}(x) & =2 x^{2}-4 x-30 \\
0 & =2 x^{2}-4 x-30 \\
x & =5,-3
\end{aligned} \quad=(2 x-10)(x+3)
$$

The only critical point inside the interval is $x=-3$.

| $c$ | -3 | -4 | 0 |
| :---: | :---: | :---: | :---: |
| type | critical point | endpoint | endpoint |
| $f(c)$ | 61 | $\frac{157}{3}=52+\frac{1}{3}$ | 7 |

The global maximum over the interval is 61 at $x=-3$, and the global minimum is 7 at $x=0$.

## - Exercises for § 3.5.3

## Exercises - Stage 1

3.5.4.1. *. Solution. We compute $f^{\prime}(x)=5 x^{4}-5$, which means that $f(x)$ has no singular points (i.e., it is differentiable for all values of $x$ ), but it has two critical points:

$$
\begin{aligned}
& 0=5 x^{4}-5 \\
& 0=x^{4}-1=\left(x^{2}+1\right)\left(x^{2}-1\right) \\
& 0=x^{2}-1 \\
& x= \pm 1
\end{aligned}
$$

Note, however, that 1 is not in the interval $[-2,0]$.
The global maximum and the global minimum for $f(x)$ on the interval $[-2,0]$ will occur at $x=-2, x=0$, or $x=-1$.

| $c$ | -2 | 0 | -1 |
| :---: | :---: | :---: | :---: |
| type | endpoint | endpoint | critical point |
| $f(c)$ | -20 | 2 | 6 |

So, the global maximum is $f(-1)=6$ while the global minimum is $f(-2)=-20$.
3.5.4.2. *. Solution. We compute $f^{\prime}(x)=5 x^{4}-5$, which means that $f(x)$ has no singular points (i.e., it is differentiable for all values of $x$ ), but it has two critical points:

$$
\begin{aligned}
& 0=5 x^{4}-5 \\
& 0=x^{4}-1=\left(x^{2}+1\right)\left(x^{2}-1\right) \\
& 0=x^{2}-1 \\
& x= \pm 1
\end{aligned}
$$

Note, however, that -1 is not in the interval [ 0,2 ].
The global maximum and the global minimum for $f(x)$ on the interval $[0,2]$ will occur at $x=2, x=0$, or $x=1$.

| $c$ | 2 | 0 | 1 |
| :---: | :---: | :---: | :---: |
| type | endpoint | endpoint | critical point |
| $f(c)$ | 12 | -10 | -14 |

So, the global maximum is $f(2)=12$ while the global minimum is $f(1)=-14$.
3.5.4.3. *. Solution. We compute $f^{\prime}(x)=6 x^{2}-12 x=6 x(x-2)$, which means that $f(x)$ has no singular points (i.e., it is differentiable for all values of $x$ ), but it has the two critical points: $x=0$ and $x=2$. Note, however, 0 is not in the interval [1, 4].

| $c$ | 1 | 4 | 2 |
| :---: | :---: | :---: | :---: |
| type | endpoint | endpoint | critical point |
| $f(c)$ | -6 | 30 | -10 |

So, the global maximum is $f(4)=30$ while the global minimum is $f(2)=-10$.
3.5.4.4. *. Solution. Since $h(x)$ is a polynomial, it has no singular points. We compute its critical points:

$$
\begin{aligned}
h^{\prime}(x) & =3 x^{2}-12 \\
0 & =3 x^{2}-12 \\
x & = \pm 2
\end{aligned}
$$

Notice as $x \rightarrow \infty, h(x) \rightarrow \infty$, and as $x \rightarrow-\infty h(x) \rightarrow-\infty$. So Theorem 3.5.17 doesn't exactly apply. Instead, let's consider the signs of $h^{\prime}(x)$.

| $x$ | $(-\infty,-2)$ | $(-2,2)$ | $(2, \infty)$ |
| :---: | :---: | :---: | :---: |
| $h^{\prime}(x)$ | $>0$ | $<0$ | $>0$ |
| $h(x)$ | increasing | decreasing | increasing |

So, $h(x)$ increases until $x=-2$, then decreases. That means $h(x)$ has a local maximum at $x=-2$. The function decreases from -2 until 2 , after which is
increases, so $h(x)$ has a local minimum at $x=2$. We compute $f(-2)=20$ and $f(2)=-12$.
3.5.4.5. *. Solution. Since $h(x)$ is a polynomial, it has no singular points. We compute its critical points:

$$
\begin{aligned}
h^{\prime}(x) & =6 x^{2}-24 \\
0 & =6 x^{2}-24 \\
x & = \pm 2
\end{aligned}
$$

Notice as $x \rightarrow \infty, h(x) \rightarrow \infty$, and as $x \rightarrow-\infty h(x) \rightarrow-\infty$. So Theorem 3.5.17 doesn't exactly apply. Instead, let's consider the signs of $h^{\prime}(x)$.

| $x$ | $(-\infty,-2)$ | $(-2,2)$ | $(2, \infty)$ |
| :---: | :---: | :---: | :---: |
| $h^{\prime}(x)$ | $>0$ | $<0$ | $>0$ |
| $h(x)$ | increasing | decreasing | increasing |

So, $h(x)$ increases until $x=-2$, then decreases. That means $h(x)$ has a local maximum at $x=-2$. The function decreases from -2 until 2 , after which is increases, so $h(x)$ has a local minimum at $x=2$.
We compute $f(-2)=33$ and $f(2)=-31$.
3.5.4.6. *. Solution. Suppose that $Q$ is a distance of $x$ from $A$. Then it is a distance of $18-x$ from $B$.


Using the Pythagorean Theorem, the distance from $P$ to $Q$ is $\sqrt{12^{2}+x^{2}}$ kilometres, and the buggy travels 15 kph over this off-road stretch. The travel time from $P$ to $Q$ is $\frac{\sqrt{12^{2}+x^{2}}}{15}$ hours.
The distance from $Q$ to $B$ is $18-x$ kilometres, and the dune buggy travels 30 kph along this road. The travel time from $Q$ to $B$ is $\frac{18-x}{30}$ hours. So, the total travel time is

$$
f(x)=\frac{\sqrt{12^{2}+x^{2}}}{15}+\frac{18-x}{30} .
$$

We wish to minimize this for $0 \leq x \leq 18$. We will test all singular points, critical points, and endpoints to find which yields the smallest value of $f(x)$. Since there
are no singular points, we begin by locating the critical points.

$$
\begin{aligned}
0=f^{\prime}(x) & =\frac{1}{15} \cdot \frac{1}{2}\left(144+x^{2}\right)^{-1 / 2}(2 x)-\frac{1}{30} \\
\frac{1}{15} \cdot \frac{x}{\sqrt{144+x^{2}}} & =\frac{1}{30} \\
\frac{x}{\sqrt{144+x^{2}}} & =\frac{1}{2} \\
\frac{x^{2}}{144+x^{2}} & =\frac{1}{4} \\
4 x^{2} & =144+x^{2} \\
x & =\frac{12}{\sqrt{3}}=4 \sqrt{3}
\end{aligned}
$$

So the minimum travel times must be one of $f(0), f(18)$, and $f(4 \sqrt{3})$.

$$
\begin{aligned}
f(0) & =\frac{12}{15}+\frac{18}{30}=1.4 \\
f(18) & =\frac{\sqrt{12^{2}+18^{2}}}{15} \approx 1.44 \\
f(4 \sqrt{3}) & =\frac{\sqrt{144+144 / 3}}{15}+\frac{18-12 / \sqrt{3}}{30} \approx 1.29
\end{aligned}
$$

So $Q$ should be $4 \sqrt{3} \mathrm{~km}$ from $A$.
3.5.4.7. *. Solution. Let $\ell, w$ and $h$ denote the length, width and height of the box respectively. We are told that $\ell w h=4500$ and that $\ell=3 w$. Hence $h=\frac{4500}{\ell w}=\frac{4500}{3 w^{2}}=\frac{1500}{w^{2}}$. The surface area of the box is

$$
\begin{aligned}
A & =2 \ell w+2 \ell h+2 w h=2\left(3 w^{2}+3 w \frac{1500}{w^{2}}+w \frac{1500}{w^{2}}\right) \\
& =2\left(3 w^{2}+\frac{6000}{w}\right)=6\left(w^{2}+\frac{2000}{w}\right)
\end{aligned}
$$



As $w$ tends to zero or to infinity, the surface area approaches infinity. By Theo-
rem 3.5.17 the minimum surface area must occur at a critical point of $w^{2}+\frac{2000}{w}$.

$$
\begin{aligned}
0 & =\frac{\mathrm{d}}{\mathrm{~d} w}\left\{w^{2}+\frac{2000}{w}\right\} \\
& =2 w-\frac{2000}{w^{2}} \\
2 w & =\frac{2000}{w^{2}} \\
w^{3} & =1000 \\
w & =10
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\ell & =3 w=30 \\
h & =\frac{1500}{w^{2}}=15
\end{aligned}
$$

The dimensions of the box with minimum surface area are $10 \times 30 \times 15$.
3.5.4.8. *. Solution. Let the length of the sides of the square base be $b$ metres and let the height be $h$ metres. The area of the base is $b^{2}$, the area of the top is $b^{2}$ and the area of each of the remaining four sides is $b h$ so the total cost is

$$
\underbrace{5\left(b^{2}\right)}_{\text {cost of base }}+\underbrace{1\left(b^{2}+4 b h\right)}_{\text {cost of } 5 \text { sides }}=6 b^{2}+4 b h=72
$$

Solving for $h$

$$
\begin{aligned}
h & =\frac{72-6 b^{2}}{4 b} \\
& =\frac{6}{4}\left(\frac{12-b^{2}}{b}\right) \\
& =\frac{3}{2}\left(\frac{12-b^{2}}{b}\right)
\end{aligned}
$$

The volume is

$$
\begin{aligned}
V=b^{2} h & =b^{2} \cdot \frac{3}{2}\left(\frac{12-b^{2}}{b}\right) \\
& =18 b-\frac{3}{2} b^{3} .
\end{aligned}
$$

This is the function we want to maximize. Since volume is never negative, the endpoints of the functions are the values of $b$ that make the volume 0 . So, the maximum volume will not occur at an endpoint, it will occur at a critical point. The only critical point is $b=2$ :

$$
0=\frac{\mathrm{d}}{\mathrm{~d} b}\left\{18 b-\frac{3}{2} b^{3}\right\}
$$

$$
\begin{aligned}
& =18-\frac{9}{2} b^{2} \\
b^{2} & =4 \\
b & =2, h=\frac{3}{2}\left(\frac{12-4}{2}\right)=6
\end{aligned}
$$

The desired dimensions are $2 \times 2 \times 6$.
3.5.4.9. *. Solution. It suffices to consider $X$ and $Y$ such that the line $X Y$ is tangent to the circle. Otherwise we could reduce the area of the triangle by, for example, holding $X$ fixed and reducing $Y$. So let $X$ and $Y$ be the $x-$ and $y$ intercepts of the line tangent to the circle at $(\cos \theta, \sin \theta)$. Then $\frac{1}{X}=\cos \theta$ and $\frac{1}{Y}=\cos \left(\frac{\pi}{2}-\theta\right)=\sin \theta$. The area of the triangle is

$$
\frac{1}{2} X Y=\frac{1}{2 \cos \theta \sin \theta}=\frac{1}{\sin (2 \theta)}
$$

(

This is a minimum when $\sin (2 \theta)$ is a maximum. That is when $2 \theta=\frac{\pi}{2}$. Hence $X=\frac{1}{\cos (\pi / 4)}$ and $Y=\frac{1}{\sin (\pi / 4)}$. That is, $X=Y=\sqrt{2}$.
3.5.4.10. *. Solution. For ease of notation, we place the semicircle on a Cartesian plane with diameter along the $x$-axis and centre at the origin.


If $x$ is the point where the rectangle touches the diameter to the right of the $y$-axis, then $2 x$ is the width of the rectangle. The origin and the two right corners of the rectangle form a right triangle with hypotenuse $R$, so by the Pythagorean Theorem,
the upper right hand corner of the rectangle is at $\left(x, \sqrt{R^{2}-x^{2}}\right)$. The perimeter of the rectangle is given by the function:

$$
P(x)=4 x+2 \sqrt{R^{2}-x^{2}}
$$

So, this is what we optimize. The endpoints of the domain for this function are $x=0$ and $x=R$. To find the critical points, we differentiate:

$$
\begin{aligned}
P^{\prime}(x) & =4-\frac{2 x}{\sqrt{R^{2}-x^{2}}} \\
P^{\prime}(x)=0 \Longleftrightarrow 4 & =\frac{2 x}{\sqrt{R^{2}-x^{2}}} \\
x & =2 \sqrt{R^{2}-x^{2}} \\
x^{2} & =4\left(R^{2}-x^{2}\right) \\
5 x^{2} & =4 R^{2} \\
x & =\frac{2}{\sqrt{5}} R
\end{aligned}
$$

Note that since our perimeter formula was defined to work only for $x$ in $[0, R]$, we neglect the negative square root, $-\frac{2}{\sqrt{5}} R$.
Now, we find the size of the perimeter at the critical point and the endpoints:

| $c$ | 0 | $R$ | $\frac{2}{\sqrt{5}} R$ |
| :---: | :---: | :---: | :---: |
| type | endpoint | endpoint | critical point |
| $P(c)$ | $2 R$ | $4 R$ | $2 \sqrt{5} R$ |

So, the largest possible perimeter is $2 \sqrt{5} R$ and the smallest possible perimeter is $2 R$.
Remark: as a check on the correctness of our formula for $P(x)$, when $x=0$ the rectangle degenerates to the line segment from $(0,0)$ to $(0, R)$. The perimeter of this "width zero rectangle" is $2 R$, agreeing with $P(0)$. Similarly, when $x=R$ the rectangle degenerates to the line segment from $(R, 0)$ to $(-R, 0)$. The perimeter of this "width zero rectangle" is $4 R$, agreeing with $P(R)$.
3.5.4.11. *. Solution. Let the cylinder have radius $r$ and height $h$. If we imagine popping off the ends, they are two circular disks, each with surface area $\pi r^{2}$. Then we imagine unrolling the remaining tube. It has height $h$, and its other dimension is given by the circumference of the disks, which is $2 \pi r$. Then the area of the "unrolled tube" is $2 \pi r h$.


So, the surface area is $2 \pi r^{2}+2 \pi r h$. Since the area is given as $A$, we can solve for $h$ :

$$
\begin{aligned}
A & =2 \pi r^{2}+2 \pi r h \\
2 \pi r h & =A-2 \pi r^{2} \\
h & =\frac{A-2 \pi r^{2}}{2 \pi r} .
\end{aligned}
$$

Then we can write the volume as a function of the variable $r$ and the constant $A$ :

$$
\begin{aligned}
V(r) & =\pi r^{2} h \\
& =\pi r^{2}\left(\frac{A-2 \pi r^{2}}{2 \pi r}\right) \\
& =\frac{1}{2}\left(A r-2 \pi r^{3}\right)
\end{aligned}
$$

This is the function we want to maximize. Let's find its critical points.

$$
\begin{aligned}
V^{\prime}(r)= & \frac{1}{2}\left(A-6 \pi r^{2}\right) \\
V^{\prime}(r)=0 & \Longleftrightarrow A=6 \pi r^{2} \Longleftrightarrow r=\sqrt{\frac{A}{6 \pi}}
\end{aligned}
$$

since negative values of $r$ don't make sense. At this critical point,

$$
\begin{aligned}
V\left(\sqrt{\frac{A}{6 \pi}}\right) & =\frac{1}{2}\left[A\left(\sqrt{\frac{A}{6 \pi}}\right)-2 \pi\left(\sqrt{\frac{A}{6 \pi}}\right)^{3}\right] \\
& =\frac{1}{2}\left[\frac{A^{3 / 2}}{\sqrt{6 \pi}}-\frac{2 \pi A^{3 / 2}}{6 \pi \sqrt{6 \pi}}\right] \\
& =\frac{1}{2}\left[\frac{A^{3 / 2}}{\sqrt{6 \pi}}-\frac{A^{3 / 2}}{3 \sqrt{6 \pi}}\right]
\end{aligned}
$$

$$
=\frac{A^{3 / 2}}{3 \sqrt{6 \pi}}
$$

We should also check the volume of the cylinder at the endpoints of the function. Since $r \geq 0$, one endpoint is $r=0$. Since $h \geq 0$, and $r$ grows as $h$ shrinks, the other endpoint is whatever value of $r$ causes $h$ to be 0 . We could find this value of $r$, but it's not strictly necessary: when $r=0$, the volume of the cylinder is zero, and when $h=0$, the volume of the cylinder is still zero. So, the maximum volume does not occur at the endpoints.
Therefore, the maximum volume is achieved at the critical point, where

$$
V_{\max }=\frac{A^{3 / 2}}{3 \sqrt{6 \pi}}
$$

Remark: as a check, $A$ has units $m^{2}$ and, because of the $A^{3 / 2}$, our answer has units $m^{3}$, which are the correct units for a volume.
3.5.4.12. *. Solution. Denote by $r$ the radius of the semicircle, and let $h$ be the height of the recangle.


Since the perimeter is required to be $P$, the height, $h$, of the rectangle must obey

$$
\begin{aligned}
P & =\pi r+2 r+2 h \\
h & =\frac{1}{2}(P-\pi r-2 r)
\end{aligned}
$$

So the area is

$$
\begin{aligned}
A(r) & =\frac{1}{2} \pi r^{2}+2 r h \\
& =\frac{1}{2} \pi r^{2}+r(P-\pi r-2 r)
\end{aligned}
$$

$$
=r P-\frac{1}{2}(\pi+4) r^{2}
$$

Finding all critical points:

$$
\begin{aligned}
0=A^{\prime}(r) & =P-(\pi+4) r \\
r & =\frac{P}{\pi+4}
\end{aligned}
$$

Now we want to know what radius yields the maximum area. We notice that $A^{\prime}(r)>0$ for $r<\frac{P}{\pi+4}$ and $A^{\prime}(r)<0$ for $r>\frac{P}{\pi+4}$. So, $A(r)$ is increasing until the critical point, then decreasing after it. That means the global maximum occurs at the critical point, $r=\frac{P}{\pi+4}$. The maximum area is

$$
\begin{aligned}
r P-\frac{1}{2}(\pi+4) r^{2} & =\frac{P^{2}}{\pi+4}-\frac{1}{2}(\pi+4) \frac{P^{2}}{(\pi+4)^{2}} \\
& =\frac{P^{2}}{2(\pi+4)}
\end{aligned}
$$

Remark: another way to see that the global maximum occurs at the critical point is to compare the area at the critical point to the areas at the endpoints of the function. The smallest value of $r$ is 0 , while the biggest is $\frac{P}{\pi+2}$ (when the shape is simply a half-circle). Comparing $A(0), A\left(\frac{P}{\pi+2}\right)$, and $A\left(\frac{P}{\pi+4}\right)$ is somewhat laborious, but certainly possible.

### 3.5.4.13. *. Solution.


a The surface area of the pan is

$$
\begin{aligned}
x y+2 x z+2 y z & =p x^{2}+2 x z+2 p x z \\
& =p x^{2}+2(1+p) x z
\end{aligned}
$$

and the volume of the pan is $x y z=p x^{2} z$. Assuming that all $A \mathrm{~cm}^{2}$ is used, we have the constraint

$$
p x^{2}+2(1+p) x z=A \quad \text { or } \quad z=\frac{A-p x^{2}}{2(1+p) x}
$$

So

$$
\begin{aligned}
V(x) & =x y z=x(p x)\left(\frac{A-p x^{2}}{2(1+p) x}\right) \\
& =\frac{p}{2(1+p)} x\left(A-p x^{2}\right)
\end{aligned}
$$

Using the product rule,

$$
\begin{aligned}
V^{\prime}(x) & =\frac{p}{2(1+p)}\left[x(-2 p x)+\left(A-p x^{2}\right)\right] \\
& =\frac{p}{2(1+p)}\left[A-3 p x^{2}\right]
\end{aligned}
$$

The derivative $V^{\prime}(x)$ is 0 when $x=\sqrt{\frac{A}{3 p}}$. The derivative is positive (i.e. $V(x)$ is increasing) for $x<\sqrt{\frac{A}{3 p}}$ and is negative (i.e. $V(x)$ is decreasing) for $x>\sqrt{\frac{A}{3 p}}$. So the pan of maximum volume has dimensions $x=\sqrt{\frac{A}{3 p}}$, $y=p \sqrt{\frac{A}{3 p}}=\sqrt{\frac{A p}{3}}$ and $z=\frac{2 A / 3}{2(1+p) \sqrt{A /(3 p)}}=\frac{\sqrt{A p}}{\sqrt{3}(1+p)}$.
b The volume of the pan from part (a) is

$$
V(p)=\left(\sqrt{\frac{A}{3 p}}\right)\left(p \sqrt{\frac{A}{3 p}}\right) \frac{\sqrt{A p}}{\sqrt{3}(1+p)}=\left(\frac{A}{3}\right)^{3 / 2} \frac{\sqrt{p}}{1+p}
$$

Since

$$
\frac{\mathrm{d}}{\mathrm{~d} p}\left\{\frac{\sqrt{p}}{1+p}\right\}=\frac{\frac{1}{2}(1+p) / \sqrt{p}-\sqrt{p}}{(1+p)^{2}}=\frac{\sqrt{p}\left(\frac{1}{p}-1\right)}{2(1+p)^{2}}
$$

the volume is increasing with $p$ for $p<1$ and decreasing with $p$ for $p>1$. So the maximum volume is achieved for $p=1$ (a square base).

## Exercises - Stage 3

3.5.4.14. *. Solution. 3.5.4.14.a We use logarithmic differentiation.

$$
\begin{aligned}
f(x) & =x^{x} \\
\log f(x) & =\log \left(x^{x}\right)=x \log x \\
\frac{\mathrm{~d}}{\mathrm{~d} x}\{\log f(x)\} & =\frac{\mathrm{d}}{\mathrm{~d} x}\{x \log x\} \\
\frac{f^{\prime}(x)}{f(x)} & =x\left(\frac{1}{x}\right)+\log x=1+\log x \\
f^{\prime}(x) & =f(x)(1+\log x)=x^{x}(1+\log x)
\end{aligned}
$$

3.5.4.14.b Since $x>0, x^{x}>0$. Therefore,

$$
f^{\prime}(x)=0 \Longleftrightarrow 1+\log x=0 \Longleftrightarrow \log x=-1 \Longleftrightarrow x=\frac{1}{e}
$$

3.5.4.14.c Since $x>0, x^{x}>0$. So, the sign of $f^{\prime}(x)$ is the same as the sign of $1+\log x$.
For $x<\frac{1}{e}, \log x<-1$ and $f^{\prime}(x)<0$. That is, $f(x)$ decreases as $x$ increases, when $x<\frac{1}{e}$. For $x>\frac{1}{e}, \log x>-1$ and $f^{\prime}(x)>0$. That is, $f(x)$ increases as $x$ increases, when $x>\frac{1}{e}$. Hence $f(x)$ is a \{local minimum $\}$ at $x=\frac{1}{e}$.
3.5.4.15. *. Solution. Call the length of the wire $L$ units and suppose that it is cut $\ell$ units from one end. Make the square from the piece of length $\ell$, and make the circle from the remaining piece of length $L-\ell$.
The square has perimeter $\ell$, so its side length is $\ell / 4$ and its area is $\left(\frac{\ell}{4}\right)^{2}$. The circle has circumference $L-\ell$, so its radius is $\frac{L-\ell}{2 \pi}$ and its area is $\pi\left(\frac{L-\ell}{2 \pi}\right)^{2}=\frac{(L-\ell)^{2}}{4 \pi}$. The area enclosed by the shapes, when the square is made from a length of size $\ell$, is

$$
A(\ell)=\frac{\ell^{2}}{16}+\frac{(L-\ell)^{2}}{4 \pi}
$$

We want to find the global max and min for this function, given the constraint $0 \leq \ell \leq L$, so we find its derivative:

$$
A^{\prime}(\ell)=\frac{\ell}{8}-\frac{L-\ell}{2 \pi}=\frac{\pi+4}{8 \pi} \ell-\frac{L}{2 \pi}
$$

Now, we find the critical point.

$$
\begin{aligned}
A^{\prime}(\ell) & =0 \\
\frac{\pi+4}{8 \pi} \ell & =\frac{L}{2 \pi} \\
\ell & =\frac{4 L}{\pi+4}
\end{aligned}
$$

| $\ell$ | 0 | $L$ | $\frac{4 L}{\pi+4}$ |
| :---: | :---: | :---: | :---: |
| type | endpoint | endpoint | critical point |
| $A(\ell)$ | $\frac{L^{2}}{4 \pi}$ | $\frac{L^{2}}{16}$ | $A\left(\frac{4 L}{\pi+4}\right)$ |

It seems obnoxious to evaluate $A\left(\frac{4 L}{\pi+4}\right)$, and the problem doesn't ask for it-but we still have to figure out whether it is a global max or min.
When $\ell<\frac{4 L}{\pi+4}, A^{\prime}(\ell)<0$, and when $\ell>\frac{4 L}{\pi+4}, A^{\prime}(\ell)>0$. So, $A(\ell)$ is decreasing
until $\ell=\frac{4 L}{\pi+4}$, then increasing. That means our critical point $\ell=\frac{4 L}{\pi+4}$ is a local minimum.
So, the minimum occurs at the only critical point, which is $\ell=\left(\frac{4}{4+\pi}\right) L$. This corresponds to $\frac{\ell}{L}=\frac{4}{4+\pi}$ : the proportion of the wire that is cut is $\frac{4}{4+\pi}$.
The maximum has to be either at $\ell=0$ or at $\ell=L$. As $A(0)=\frac{L^{2}}{4 \pi}>A(L)=\frac{L^{2}}{16}$, the maximum has $\ell=0$ (that is, no square).

## 3.6 • Sketching Graphs

### 3.6.7 • Exercises

- Exercises for § 3.6.1


## Exercises - Stage 1

3.6.7.1. Solution. In general, this is false. For example, the function $f(x)=$ $\frac{x^{2}-9}{x^{2}-9}$ has no vertical asymptotes, because it is equal to 1 in every point in its domain (and is undefined when $x= \pm 3$ ).
However, it is certainly possible that $f(x)$ has a vertical asymptote at $x=-3$. For example, $f(x)=\frac{1}{x^{2}-9}$ has a vertical asymptote at $x=-3$. More generally, if $g(x)$ is continuous and $g(-3) \neq 0$, then $f(x)$ has a vertical asympotote at $x=-3$.

## Exercises - Stage 2

3.6.7.2. Solution. Since $x^{2}+1$ and $x^{2}+4$ are always positive, $f(x)$ and $h(x)$ are defined over all real numbers. So, $f(x)$ and $h(x)$ correspond to $A(x)$ and $B(x)$. Which is which? $A(0)=1=f(0)$ while $B(0)=2=h(0)$, so $A(x)=f(x)$ and $B(x)=h(x)$.
That leaves $g(x)$ and $k(x)$ matching to $C(x)$ and $D(x)$. The domain of $g(x)$ is all $x$ such that $x^{2}-1 \geq 0$. That is, $|x| \geq 1$, like $C(x)$. The domain of $k(x)$ is all $x$ such that $x^{2}-4 \geq 0$. That is, $|x| \geq 2$, like $D(x)$. So, $C(x)=g(x)$ and $D(x)=k(x)$.
3.6.7.3. Solution. (a) Since $f(0)=2$, we solve

$$
\begin{aligned}
2 & =\sqrt{\log ^{2}(0+p)} \\
& =\sqrt{\log ^{2} p} \\
& =|\log p| \\
\log p & = \pm 2 \\
p & =e^{ \pm 2} \\
p & =e^{2} \text { or } p=\frac{1}{e^{2}}
\end{aligned}
$$

We know that $p$ is $e^{2}$ or $\frac{1}{e^{2}}$, but we have to decide between the two. In both cases,
$f(0)=2$. Let's consider the domain of $f(x)$. Since $\log ^{2}(x+p)$ is never negative, the square root does not restrict our domain. However, we can only take the logarithm of positive numbers. Therefore, the domain is

$$
\begin{aligned}
& x \text { such that } x+p>0 \\
& x \text { such that } x>-p
\end{aligned}
$$

If $p=\frac{1}{e^{2}}$, then the domain of $f(x)$ is $\left(-\frac{1}{e^{2}}, \infty\right)$. In particular, since $-\frac{1}{e^{2}}>-1$, the domain of $f(x)$ does not include $x=-1$. However, it is clear from the graph that $f(-1)$ exists. So, $p=e^{2}$.
(b) Now, we need to figure out what $b$ is. Notice that $b$ is the end of the domain of $f(x)$, which we already found to be $(-p, \infty)$. So, $b=-p=-e^{2}$.
(As a quick check, if we take $e \approx 2.7$, then $-e^{2}=-7.29$, and this looks about right on the graph.)
(c) The $x$-intercept is the value of $x$ for which $f(x)=0$ :

$$
\begin{aligned}
& 0=\sqrt{\log ^{2}(x+p)} \\
& 0=\log (x+p) \\
& 1=x+p \\
& x=1-p=1-e^{2}
\end{aligned}
$$

The $x$-intercept is $1-e^{2}$.
(As another quick check, the $x$-intercept we found is a distance of 1 from the vertical asymptote, and this looks about right on the graph.)
3.6.7.4. Solution. Vertical asymptotes occur where the function blows up. In rational functions, this can only happen when the denominator goes to 0 . In our case, the denominator is 0 when $x=3$, and in this case the numerator is 147 . That means that as $x$ gets closer and closer to 3 , the numerator gets closer and closer to 147 while the denominator gets closer and closer to 0 , so $|f(x)|$ grows without bound. That is, there is a vertical asymptote at $x=3$.
The horizontal asymptotes are found by taking the limits as $x$ goes to infinity and negative infinity. In our case, they are the same, so we condense our work.

$$
\lim _{x \rightarrow \pm \infty} \frac{x(2 x+1)(x-7)}{3 x^{3}-81}=\lim _{x \rightarrow \pm \infty} \frac{2 x^{3}+a x^{2}+b x+c}{3 x^{3}-81}
$$

where $a, b$, ad $c$ are some constants. Remember, for rational functions, you can figure out the end behaviour by looking only at the terms with the highest degreethe others won't matter, so we don't bother finding them. From here, we divide the numerator and denominator by the highest power of $x$ in the denominator, $x^{3}$.

$$
\begin{aligned}
& =\lim _{x \rightarrow \pm \infty} \frac{2 x^{3}+a x^{2}+b x+c}{3 x^{3}-81}\left(\frac{\frac{1}{x^{3}}}{\frac{1}{x^{3}}}\right) \\
& =\lim _{x \rightarrow \pm \infty} \frac{2+\frac{a}{x}+\frac{b}{x^{2}}+\frac{c}{x^{3}}}{3-\frac{81}{x^{3}}}
\end{aligned}
$$

$$
=\frac{2+0+0+0}{3-0}=\frac{2}{3}
$$

So there is a horizontal asymptote of $y=\frac{2}{3}$ both as $x \rightarrow \infty$ and as $x \rightarrow-\infty$.
3.6.7.5. Solution. Since $f(x)$ is continuous over all real numbers, it has no vertical asymptote.
To find the horizontal asymptotes, we evaluate $\lim _{x \rightarrow \pm \infty} f(x)$.

$$
\lim _{x \rightarrow \infty} 10^{3 x-7}=\underbrace{\lim _{X \rightarrow \infty} 10^{X}}_{\text {let } X=3 x-7}=\infty
$$

So, there's no horizontal asymptote as $x \rightarrow \infty$.

$$
\begin{aligned}
\lim _{x \rightarrow-\infty} 10^{3 x-7} & =\underbrace{\lim _{X \rightarrow-\infty} 10^{X}}_{\text {let } X=3 x-7} \\
& =\underbrace{\lim _{X^{\prime} \rightarrow \infty} 10^{-X^{\prime}}}_{\text {let } X^{\prime}=-X} \\
& =\lim _{X^{\prime} \rightarrow \infty} \frac{1}{10^{X^{\prime}}} \\
& =0
\end{aligned}
$$

That is, $y=0$ is a horizontal asymptote as $x \rightarrow-\infty$.

## - Exercises for § 3.6.2

## Exercises - Stage 1

3.6.7.1. Solution. Functions $A(x)$ and $B(x)$ share something in common that sets them apart from the others: they have a horizontal tangent line only once. In particular, $A^{\prime}(-2) \neq 0$ and $B^{\prime}(2) \neq 0$. The only listed functions that do not have two distinct roots are $l(x)$ and $p(x)$. Since $l(-2) \neq 0$ and $p(2) \neq 0$, we conclude

$$
A^{\prime}(x)=l(x) \quad B^{\prime}(x)=p(x)
$$

Function $C(x)$ is never decreasing. Its tangent line is horizontal when $x= \pm 2$, but the curve never decreases, so $C^{\prime}(x) \geq 0$ for all $x$ and $C^{\prime}(2)=C^{\prime}(-2)=0$. The only function that matches this is $n(x)=(x-2)^{2}(x+2)^{2}$. Since its linear terms have even powers, it is never negative, and its roots are precisely $x= \pm 2$.

$$
C^{\prime}(x)=n(x)
$$

For the functions $D(x)$ and $E(x)$ we consider their behaviour near $x=0 . D(x)$ is decreasing near $x=0$, so $D^{\prime}(0)<0$, which matches with $o(0)<0$. Contrastingly, $E(x)$ is increasing near zero, so $E^{\prime}(0)>0$, which matches with $m(0)>0$.

$$
D^{\prime}(x)=o(x) \quad E^{\prime}(x)=m(x)
$$

## Exercises - Stage 2

3.6.7.2. *. Solution. The domain of $f(x)$ is all real numbers except -3 (because when $x=-3$ the denominator is zero). For $x \neq-3$, we differentiate using the quotient rule:

$$
f^{\prime}(x)=\frac{e^{x}(x+3)-e^{x}(1)}{(x+3)^{2}}=\frac{e^{x}}{(x+3)^{2}}(x+2)
$$

Since $e^{x}$ and $(x+3)^{2}$ are positive for every $x$ in the domain of $f(x)$, the sign of $f^{\prime}(x)$ is the same as the sign of $x+2$. We conclude that $f(x)$ is increasing for every $x$ in its domain with $x+2>0$. That is, over the open interval $(-2, \infty)$.
3.6.7.3. *. Solution. Since we can't take the square root of a negative number, $f(x)$ is only defined when $x \geq 1$. Furthermore, since we can't have zero as a denominator, $x=-2$ is not in the domain - but as long as $x \geq 1$, we also have $x \neq-2$. So, the domain of the function is $[1, \infty)$.
In order to find where $f(x)$ is increasing, we find where $f^{\prime}(x)$ is positive.

$$
f^{\prime}(x)=\frac{\frac{2 x+4}{2 \sqrt{x-1}}-2 \sqrt{x-1}}{(2 x+4)^{2}}=\frac{(x+2)-2(x-1)}{\sqrt{x-1}(2 x+4)^{2}}=\frac{-x+4}{\sqrt{x-1}(2 x+4)^{2}}
$$

The denominator is never negative, so $f(x)$ is increasing when the numerator of $f^{\prime}(x)$ is positive, i.e. when $4-x>0$, or $x<4$. Recalling that the domain of definition for $f(x)$ is $[1,+\infty)$, we conclude that $f(x)$ is increasing on the open interval ( 1,4 ).
3.6.7.4. *. Solution. The domain of arctangent is all real numbers. The domain of the logarithm function is all positive numbers, and $1+x^{2}$ is positive for all $x$. So, the domain of $f(x)$ is all real numbers.
In order to find where $f(x)$ is increasing, we find where $f^{\prime}(x)$ is positive.

$$
f^{\prime}(x)=\frac{2}{1+x^{2}}-\frac{2 x}{1+x^{2}}=\frac{2-2 x}{1+x^{2}}
$$

Since the denominator is always positive, $f(x)$ is increasing when when $2-2 x>0$. We conclude that $f(x)$ is increasing on the open interval $(-\infty, 1)$.

## - Exercises for § 3.6.3

## Exercises - Stage 1

### 3.6.7.1. Solution.



In the graph above, the concave-up sections are marked in red. These are where the graph has an increasing derivative; equivalently, where the graph lies above its tangent lines; more descriptively, where it curves like a smiley face.
Concave-down sections are marked in blue. These are where the graph has a decreasing derivative; equivalently, where the graph lies below its tangent lines; more descriptively, where it curves like a frowney face.
3.6.7.2. Solution. The most basic shape of the graph is given by the last two bullet points:


The curve is concave down over the interval $(-5,5)$, so let's give it a frowney-face curvature there.


Finally, when $x>5$ or $x<-5$, our curve should be concave up, so let's give it smiley-face curvature there, without changing its basic increasing/decreasing shape.


This finishes our sketch.
3.6.7.3. Solution. An inflection point is where the concavity of a function changes. It is possible that $x=3$ is an inflection point, but it is also possible that is not. So, the statement is false, in general.
For example, let $f(x)=(x-3)^{4}$. Since $f(x)$ is a polynomial, all its derivatives exist and are continuous. $f^{\prime \prime}(x)=12(x-3)^{2}$, so $f^{\prime \prime}(3)=0$. However, since $f^{\prime \prime}(x)$ is something squared, it is never negative, so $f(x)$ is never concave down. Since $f(x)$ is never concave down, it never changes concavity, so it has no inflection points.
Remark: finding inflection points is somewhat reminiscent of finding local extrema. To find local extrema, we first find all critical and singular points, since local extrema can only occur there or at endpoints. Then, we have to figure out which critical and singular points are actually local extrema. Similarly, if you want to find inflection points, start by finding where $f^{\prime \prime}(x)$ is zero or non-existant, because inflection points can only occur there (see Question 3.6.7.7). Then, you still have to check whether those points are actually inflection points.

## Exercises - Stage 2

3.6.7.4. *. Solution. Inflection points occur where $f^{\prime \prime}(x)$ changes sign. Since $f(x)$ is a polynomial, its first and second derivatives exist everywhere, and are themselves polynomials. In particular,

$$
\begin{aligned}
f(x) & =3 x^{5}-5 x^{4}+13 x \\
f^{\prime}(x) & =15 x^{4}-20 x^{3}+13 \\
f^{\prime \prime}(x) & =60 x^{3}-60 x^{2}=60 x^{2}(x-1)
\end{aligned}
$$

The second derivative is negative for $x<1$ and positive for $x>1$. Thus the concavity changes between concave up and concave down at $\{x=1, y=11\}$.
This is the only inflection point. It is true that $f^{\prime \prime}(0)=0$, but for values of $x$ both a little larger than and a little smaller than $0, f^{\prime \prime}(x)<0$, so the concavity does not change at $x=0$.

## Exercises - Stage 3

3.6.7.5. *. Solution. In order to show that $f(x)$ has exactly one inflection point, we will show that is has at least one, and no more than one.
Let

$$
g(x)=f^{\prime \prime}(x)=x^{3}+5 x-20
$$

Then $g^{\prime}(x)=3 x^{2}+5$, which is always positive. That means $g(x)$ is strictly increasing for all $x$. So, $g(x)$ can change signs once, from negative to positive, but it can never change back to negative. An inflection point of $f(x)$ occurs when $g(x)$ changes signs. So, $f(x)$ has at most one inflection point. (At this point, we don't know that $f(x)$ has any inflection points: maybe $g(x)$ is always positive.)
Since $g(x)$ is continuous, we can apply the Intermediate Value Theorem to it. Notice $g(3)>0$ while $g(0)<0$. By the IVT, $g(x)=0$ for at least one $x \in(0,3)$. Since $g(x)$ is strictly increasing, at the point where $g(x)=0, g(x)$ changes from negative to positive. So, the concavity of $f(x)$ changes. Therefore, $f(x)$ has at least one inflection point.
Now that we've shown that $f(x)$ has at most one inflection point, and at least one inflection point, we conclude it has exactly one inflection point.
3.6.7.6. *. Solution. 3.6.7.6.a Let

$$
g(x)=f^{\prime}(x)
$$

Then $f^{\prime \prime}(x)$ is the derivative of $g(x)$. Since $f^{\prime \prime}(x)>0$ for all $x, g(x)=f^{\prime}(x)$ is strictly increasing for all $x$. In other words, if $y>x$ then $g(y)>g(x)$.
Suppose $g(x)=0$. Then for every $y$ that is larger than $x, g(y)>g(x)$, so $g(y) \neq 0$. Similarly, for every $y$ that is smaller than $x, g(y)<g(x)$, so $g(y) \neq 0$. Therefore, $g(x)$ can only be zero for at most one value of $x$. Since $g(x)=f^{\prime}(x)$, that means $f(x)$ can have at most one critical point.
Suppose $f^{\prime}(c)=0$. Since $f^{\prime}(x)$ is a strictly increasing function, $f^{\prime}(x)<0$ for all $x<c$ and $f^{\prime}(x)>0$ for all $x>c$.


Then $f(x)$ is decreasing for $x<c$ and increasing for $x>c$. So $f(x)>f(c)$ for all $x<c$ and $f(x)>f(c)$ for all $x>c$.


We have concluded that $f(x)>f(c)$ for all $x \neq c$, so $c$ is an absolute minimum for $f(x)$.
3.6.7.6.b We know that the maximum over an interval occurs at an endpoint, at a critical point, or at a singular point.

- Since $f^{\prime}(x)$ exists everywhere, there are no singular points.
- If the maximum were achieved at a critical point, that critical point would have to provide both the absolute maximum and the absolute minimum (by part a). So, the function would have to be a constant and consequently could not have a nonzero second derivative. So the maximum is not at a critical point.

That leaves only the endpoints of the interval.
3.6.7.7. Solution. If $x=3$ is an inflection point, then the concavity of $f(x)$ changes at $x=3$. That is, there is some interval strictly containing 3 , with endpoints $a$ and $b$, such that

- $f^{\prime \prime}(a)<0$ and $f^{\prime \prime}(x)<0$ for every $x$ between $a$ and 3 , and
- $f^{\prime \prime}(b)>0$ and $f^{\prime \prime}(x)>0$ for every $x$ between $b$ and 3 .

Remark: we are leaving unknown whether $a<3<b$ or $b<3<a$. Since we don't know whether $f(x)$ changes from concave up to concave down, or from concave down to concave up, by remaining vague we cover both cases.


Since $f^{\prime \prime}(a)<0$ and $f^{\prime \prime}(b)>0$, and since $f^{\prime \prime}(x)$ is continuous, the Intermediate Value Theorem tells us that there exists some $x$ strictly between $a$ and $b$ with $f^{\prime \prime}(x)=0$. So, we know $f^{\prime \prime}(x)=0$ somewhere between $a$ and $b$. The question is, where exactly could that be?

- $f^{\prime \prime}(x)<0$ (and hence $f^{\prime \prime}(x) \neq 0$ ) for all $x$ between $a$ and 3
- $f^{\prime \prime}(x)>0$ (and hence $f^{\prime \prime}(x) \neq 0$ ) for all $x$ between $b$ and 3
- So, any number between $a$ and $b$ that is not 3 has $f^{\prime \prime}(x) \neq 0$.

So, $x=3$ is the only possible place between $a$ and $b$ where $f^{\prime \prime}(x)$ could be zero. Therefore, $f^{\prime \prime}(3)=0$.
Remark: this is why, in general, we set $f^{\prime \prime}(x)=0$ to find inflection points. (They can also occur where $f^{\prime \prime}(x)$ does not exist.)

## - Exercises for § 3.6.4

## Exercises - Stage 1

3.6.7.1. Solution. This function is symmetric across the $y$-axis, so it is even.
3.6.7.2. Solution. The function is not even, because it is not mirrored across the $y$-axis.
Assuming it continues as shown, the function is periodic, because the unit shown below is repeated:


Additionally, $f(x)$ is odd. In a function with odd symmetry, if we mirror the righthand portion of the curve (the portion to the right of the $y$-axis) across both the $y$-axis and the $x$-axis, it lines up with the left-hand portion of the curve.

$\square$


Since reflecting the right-hand portion of the graph across the $y$-axis, then the $x$-axis, gives us $f(x)$, we conclude $f(x)$ is odd.
3.6.7.3. Solution. Since the function is even, we simply reflect the portion shown across the $y$-axis to complete the sketch.

3.6.7.4. Solution. Since the function is odd, to complete the sketch, we reflect the portion shown across the $y$-axis (shown dashed), then the $x$-axis (shown in red).


## Exercises - Stage 2

3.6.7.5. Solution. A function is even if $f(-x)=f(x)$.

$$
\begin{aligned}
f(-x) & =\frac{(-x)^{4}-(-x)^{6}}{e^{(-x)^{2}}} \\
& =\frac{x^{4}-x^{6}}{e^{x^{2}}} \\
& =f(x)
\end{aligned}
$$

So, $f(x)$ is even.
3.6.7.6. Solution. For any real number $x$, we will show that $f(x)=f(x+4 \pi)$.

$$
\begin{aligned}
f(x+4 \pi) & =\sin (x+4 \pi)+\cos \left(\frac{x+4 \pi}{2}\right) \\
& =\sin (x+4 \pi)+\cos \left(\frac{x}{2}+2 \pi\right) \\
& =\sin (x)+\cos \left(\frac{x}{2}\right) \\
& =f(x)
\end{aligned}
$$

So, $f(x)$ is periodic.
3.6.7.7. Solution. $f(x)$ is not periodic. (You don't really have to justify this, but if you wanted to, you could say something like this. Notice $f(0)=1$. Whenever $x>10, f(x)>1$. Then the value of $f(0)$ is not repeated indefinitely, so $f(x)$ is not
periodic.)
To decide whether $f(x)$ is even, odd, or neither, simplify $f(-x)$ :

$$
\begin{aligned}
f(-x) & =(-x)^{4}+5(-x)^{2}+\cos \left((-x)^{3}\right) \\
& =x^{4}+5 x^{3}+\cos (-x) \\
& =x^{4}+5 x^{3}+\cos (x) \\
& =f(x)
\end{aligned}
$$

Since $f(-x)=f(x)$, our function is even.
3.6.7.8. Solution. It should be clear that $f(x)$ is not periodic. (If you wanted to justify this, you could note that $f(x)=0$ has exactly two solutions, $x=0,-5$. Since the value of $f(0)$ is repeated only twice, and not indefinitely, $f(x)$ is not periodic.)
To decide whether $f(x)$ is odd, even, or neither, we simplify $f(-x)$.

$$
\begin{aligned}
f(-x) & =(-x)^{5}+5(-x)^{4} \\
& =-x^{5}+5 x^{4}
\end{aligned}
$$

We see that $f(-x)$ is not equal to $f(x)$ or to $-f(x)$. For instance, when $x=1$ :

- $f(-x)=f(-1)=4$,
- $f(x)=f(1)=6$, and
- $-f(x)=-f(1)=-6$.

Since $f(-x)$ is not equal to $f(x)$ or to $-f(x), f(x)$ is neither even nor odd.
3.6.7.9. Solution. Recall the period of $g(X)=\tan X$ is $\pi$.

$$
\tan (X+\pi)=\tan (X) \quad \text { for any } X \text { in the domain of } \tan X
$$

Replacing $X$ with $\pi x$ :

$$
\begin{aligned}
\tan (\pi x+\pi) & =\tan (\pi x) & & \text { for any } x \text { in the domain of } \tan (\pi x) \\
\tan (\pi(x+1)) & =\tan (\pi x) & & \text { for any } x \text { in the domain of } \tan (\pi x) \\
f(x+1) & =f(x) & & \text { for any } x \text { in the domain of } \tan (\pi x)
\end{aligned}
$$

The period of $f(x)$ is 1 .

## Exercises - Stage 3

3.6.7.10. Solution. Let's consider $g(x)=\tan (3 x)$ and $h(x)=\sin (4 x)$ separately. Recall that $\pi$ is the period of tangent.

$$
\tan X=\tan (X+\pi) \quad \text { for every } X \text { in the domain of } \tan X
$$

Replacing $X$ with $3 x$ :

$$
\begin{array}{rlrl}
\tan (3 x) & =\tan (3 x+\pi) & \text { for every } x \text { in the domain of } \tan 3 x \\
\tan (3 x) & =\tan \left(3\left(x+\frac{\pi}{3}\right)\right) & & \text { for every } x \text { in the domain of } \tan 3 x \\
g(x) & =g\left(x+\frac{\pi}{3}\right) & & \text { for every } x \text { in the domain of } \tan 3 x
\end{array}
$$

So, the period of $g(x)=\tan (3 x)$ is $\frac{\pi}{3}$.
Similarly, $2 \pi$ is the period of sine.

$$
\sin (X)=\sin (X+2 \pi) \quad \text { for every } X \text { in the domain of } \sin (X)
$$

Replacing $X$ with $4 x$ :

$$
\begin{aligned}
\sin (4 x) & =\sin (4 x+2 \pi) & & \text { for every } x \text { in the domain of } \sin (4 x) \\
\sin (4 x) & =\sin \left(4\left(x+\frac{\pi}{2}\right)\right) & & \text { for every } x \text { in the domain of } \sin (4 x) \\
h(x) & =h\left(x+\frac{\pi}{2}\right) & & \text { for every } x \text { in the domain of } \sin (4 x)
\end{aligned}
$$

So, the period of $h(x)=\sin (4 x)$ is $\frac{\pi}{2}$.
All together, $f(x)=g(x)+h(x)$ will repeat when both $g(x)$ and $h(x)$ repeat. The least common integer multiple of $\frac{\pi}{3}$ and $\frac{\pi}{2}$ is $\pi$. Since $g(x)$ repeats every $\frac{\pi}{3}$ units, and $h(x)$ repeats every $\frac{\pi}{2}$ units, they will not both repeat until we move $\pi$ units. So, the period of $f(x)$ is $\pi$.

## - Exercises for § 3.6.6

## Exercises - Stage 1

3.6.7.1. *. Solution. 3.6.7.1.a Since we must have $3-x \geq 0$, this tells us $x \leq 3$. So, the domain is $(-\infty, 3]$.
3.6.7.1.b

$$
f^{\prime}(x)=\sqrt{3-x}-\frac{x}{2 \sqrt{3-x}}=3 \frac{2-x}{2 \sqrt{3-x}}
$$

For every $x$ in the domain of $f^{\prime}(x)$, the denominator is positive, so the sign of $f^{\prime}(x)$ depends only on the numerator.

| $x$ | $(-\infty, 2)$ | 2 | $(2,3)$ | 3 |
| :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | positive | 0 | negative | DNE |
| $f(x)$ | increasing | maximum | decreasing | endpoint |

So, $f$ is increasing for $x<2$, has a local (in fact global) maximum at $x=2$, is decreasing for $2<x<3$, and has a local minimum at $x=3$.
Remark: this shows us the basic skeleton of the graph. It consists of a single hump.

3.6.7.1.c When $x<3$,

$$
f^{\prime \prime}(x)=\frac{1}{4}(3 x-12)(3-x)^{-3 / 2}<0
$$

The domain of $f^{\prime \prime}(x)$ is $(-\infty,-3)$, and over its domain it is always negative (the factor $(3 x-12)$ is negative for all $x<4$ and the factor $(3-x)^{-3 / 2}$ is positive for all $x<3$ ). So, $f(x)$ has no inflection points and is concave down everywhere.
3.6.7.1.d We already found

$$
f^{\prime}(x)=3 \frac{2-x}{2 \sqrt{3-x}}
$$

This is undefined at $x=3$. Indeed,

$$
\lim _{x \rightarrow 3^{-}} 3 \frac{2-x}{2 \sqrt{3-x}}=-\infty
$$

so $f(x)$ has a vertical tangent line at $(3,0)$.
3.6.7.1.e To sketch the curve $y=f(x)$, we already know its intervals of increase and decrease, and its concavity. We also note its intercepts are $(0,0)$ and $(3,0)$.

3.6.7.2. *. Solution.

- Asymptotes:

$$
\lim _{x \rightarrow \pm \infty} f(x)=\lim _{x \rightarrow \pm \infty} \frac{1}{x}-\frac{2}{x^{4}}=0
$$

So $y=0$ is a horizontal asymptote both at $x=\infty$ and $x=-\infty$.

$$
\lim _{x \rightarrow 0} f(x)=\lim _{x \rightarrow 0} \frac{x^{3}-2}{x^{4}}=-\infty
$$

So there is a vertical asymptote at $x=0$, where the function plunges downwards from both the right and the left.

- Intervals of increase and decrease:

$$
f^{\prime}(x)=-\frac{1}{x^{2}}+\frac{8}{x^{5}}=\frac{8-x^{3}}{x^{5}}
$$

The only place where $f^{\prime}(x)$ is zero only at $x=2$. So $f(x)$ has a horizontal tangent at $x=2, y=\frac{3}{8}$. This is a critical point.
The derivative is undefined at $x=0$, as is the function.

| $x$ | $(-\infty, 0)$ | 0 | $(0,2)$ | 2 | $(2, \infty)$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | negative | DNE | positive | 0 | negative |
| $f(x)$ | decreasing | vertical asymptote | increasing | local max | decreasing |

Since the function changes from increasing to decreasing at $x=2$, the only local maximum is at $x=2$.
At this point, we get a rough sketch of $f(x)$.


- Concavity:

$$
f^{\prime \prime}(x)=\frac{2}{x^{3}}-\frac{40}{x^{6}}=\frac{2 x^{3}-40}{x^{6}}
$$

The second derivative of $f(x)$ is positive for $x>\sqrt[3]{2} 0$ and negative for $x<$ $\sqrt[3]{2} 0$. So the curve is concave up for $x>\sqrt[3]{2} 0$ and concave down for $x<\sqrt[3]{2} 0$. There is an inflection point at $x=\sqrt[3]{2} 0 \approx 2.7, y=\frac{18}{20^{4 / 3}} \approx 0.3$.

- Intercepts:

Since $f(x)$ is not defined at $x=0$, there is no $y$-intercept. The only $x$-intercept is $x=\sqrt[3]{2} \approx 1.3$.

- Sketch:

We can add concavity to our skeleton sketched above, and label our intercept and inflection point (the open dot).
$\underbrace{\underbrace{y}_{\text {increasing }} \underbrace{2}_{\text {decreasing }}}_{\underbrace{\underbrace{(2)}_{\text {concave down }}}_{\text {decreasing }} x}$

### 3.6.7.3. *. Solution.

- Asymptotes:

When $x=-1$, the denominator $1+x^{3}$ of $f(x)$ is zero while the numerator is 1 , so $x=-1$ is a vertical asymptote. More precisely,

$$
\lim _{x \rightarrow-1^{-}} f(x)=-\infty \quad \lim _{x \rightarrow-1^{+}} f(x)=\infty
$$

There are no horizontal asymptotes, because

$$
\lim _{x \rightarrow \infty} \frac{x^{4}}{1+x^{3}}=\infty \quad \lim _{x \rightarrow-\infty} \frac{x^{4}}{1+x^{3}}=-\infty
$$

- Intervals of increase and decrease:

We note that $f^{\prime}(x)$ is defined for all $x \neq-1$ and is not defined for $x=-1$. Therefore, the only singular point for $f(x)$ is $x=-1$.
To find critical points, we set

$$
\begin{array}{rll}
f^{\prime}(x)=0 & \\
4 x^{3}+x^{6}=0 & \\
x^{3}\left(4+x^{3}\right)=0 & \\
x^{3}=0 \quad & \text { or } & 4+x^{3}=0 \\
x=0 \quad & \text { or } & x=-\sqrt[3]{4} \approx-1.6
\end{array}
$$

At these critical points, $f(0)=0$ and $f(-\sqrt[3]{4})=\frac{4 \sqrt[3]{4}}{-3}<0$. The denominator of $f^{\prime}(x)$ is never negative, so the sign of $f^{\prime}(x)$ is the same as the sign of its numerator, $x^{3}\left(4+x^{3}\right)$.

| $x$ | $(-\infty,-\sqrt[3]{4})$ | $-\sqrt[3]{4}$ | $(-\sqrt[3]{4},-1)$ | -1 | $(-1,0)$ | 0 | $(0, \infty)$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | positive | 0 | negative | DNE | negative | 0 | positive |
| $f(x)$ | increasing | l. max | decreasing | VA | decreasing | l. min | increasing |

Now, we have enough information to make a skeleton of our graph.


- Concavity:

The second derivative is undefined when $x=-1$. It is zero when $12 x^{2}-6 x^{5}=$ $6 x^{2}\left(2-x^{3}\right)=0$. That is, at $x=\sqrt[3]{2} \approx 1.3$ and $x=0$. Notice that the sign of $f^{\prime \prime}(x)$ does not change at $x=0$, so $x=0$ is not an inflection point.

| $x$ | $(-\infty,-1)$ | -1 | $(-1,0)$ | 0 | $(0, \sqrt[3]{2})$ | $\sqrt[3]{2}$ | $(\sqrt[3]{2}, \infty)$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime \prime}(x)$ | negative | DNE | positive | 0 | positive | 0 | negative |
| $f(x)$ | concave down | VA | concave up |  | concave up | IP | concave down |

Now we can refine our skeleton by adding concavity.


### 3.6.7.4. *. Solution.

- Asymptotes:

$$
\lim _{x \rightarrow-\infty} \frac{x^{3}}{1-x^{2}}=\infty \quad \lim _{x \rightarrow \infty} \frac{x^{3}}{1-x^{2}}=-\infty
$$

So, $f(x)$ has no horizontal asymptotes.
On the other hand $f(x)$ blows up at both $x=1$ and $x=-1$, so there are vertical asymptotes at $x=1$ and $x=-1$. More precisely,

$$
\lim _{x \rightarrow-1^{-}} \frac{x^{3}}{1-x^{2}}=\infty \quad \lim _{x \rightarrow-1^{+}} \frac{x^{3}}{1-x^{2}}=-\infty
$$

$$
\lim _{x \rightarrow 1^{-}} \frac{x^{3}}{1-x^{2}}=\infty \quad \lim _{x \rightarrow 1^{+}} \frac{x^{3}}{1-x^{2}}=-\infty
$$

- Symmetry:
$f(x)$ is an odd function, because

$$
f(-x)=\frac{(-x)^{3}}{1-(-x)^{2}}=\frac{-x^{3}}{1-x^{2}}=-f(x)
$$

- Intercepts:

The only intercept of $f(x)$ is the origin. In particular, that means that out of the three intervals where it is continuous, namely $(-\infty,-1),(-1,1)$ and $(1, \infty)$, in two of them $f(x)$ is always positive or always negative.

- When $x<-1$ : $1-x^{2}<0$ and $x^{3}<0$, so $f(x)>0$.
- When $x>1: 1-x^{2}<0$ and $x^{3}>0$, so $f(x)<0$.
- When $-1<x<0,1-x^{2}>0$ and $x^{3}<0$ so $f(x)<0$.
- When $0<x<1,1-x^{2}>0$ and $x^{3}>0$ so $f(x)>0$.
- Intervals of increase and decrease:

$$
f^{\prime}(x)=\frac{3 x^{2}-x^{4}}{\left(1-x^{2}\right)^{2}}=\frac{x^{2}\left(3-x^{2}\right)}{\left(1-x^{2}\right)^{2}}
$$

The only singular points are $x= \pm 1$, where $f(x)$, and hence $f^{\prime}(x)$, is not defined. The critical points are:

\[

\]

The values of $f$ at its critical points are $f(0)=0, f(\sqrt{3})=-\frac{3 \sqrt{3}}{2} \approx-2.6$ and $f(-\sqrt{3})=\frac{3 \sqrt{3}}{2} \approx 2.6$.
Notice the sign of $f^{\prime}(x)$ is the same as the sign of $3-x^{2}$.

| $x$ | $(-\infty,-\sqrt{3})$ | $-\sqrt{3}$ | $(-\sqrt{3},-1)$ | -1 |
| :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | negative | 0 | positive | DNE |
| $f(x)$ | decreasing | local min | increasing | VA |


| $x$ | $(-1,0)$ | 0 | $(0, \sqrt{3})$ | $\sqrt{3}$ | $(\sqrt{3}, \infty)$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | positive | 0 | positive | 0 | negative |
| $f(x)$ | increasing |  | increasing | local max | decreasing |

Now we have enough information to sketch a skeleton of $f(x)$.


- Concavity:

$$
f^{\prime \prime}(x)=\frac{2 x\left(3+x^{2}\right)}{\left(1-x^{2}\right)^{3}}
$$

The second derivative is zero when $x=0$, and is undefined when $x= \pm 1$.

| $x$ | $(-\infty,-1)$ | $(-1,0)$ | 0 | $(0,1)$ | $(1, \infty)$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime \prime}(x)$ | positive | negative | 0 | positive | negative |
| $f(x)$ | concave up | concave down | inflection point | concave up | concave down |

Now, we can refine our skeleton.

3.6.7.5. *. Solution. 3.6.7.5.a One branch of the function, the exponential function $e^{x}$, is continuous everywhere. So $f(x)$ is continuous for $x<0$. When $x \geq 0, f(x)=\frac{x^{2}+3}{3(x+1)}$, which is continuous whenever $x \neq 1$ (so it's continuous for all $x>0$ ). So, $f(x)$ is continuous for $x>0$. To see that $f(x)$ is continuous at $x=0$, we see:

$$
\begin{aligned}
\lim _{x \rightarrow 0-} f(x)=\lim _{x \rightarrow 0-} e^{x} & =1 \\
\lim _{x \rightarrow 0+} f(x)=\lim _{x \rightarrow 0+} \frac{x^{2}+3}{3(x+1)} & =1 \\
\text { So, } \lim _{x \rightarrow 0} f(x) & =1=f(0)
\end{aligned}
$$

Hence $f(x)$ is continuous at $x=0$, so $f(x)$ is continuous everywhere. 3.6.7.5.b We differentiate the function twice. Notice

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{x^{2}+3}{3(x+1)}\right\} & =\frac{3(x+1)(2 x)-\left(x^{2}+3\right)(3)}{9(x+1)^{2}} \\
& =\frac{x^{2}+2 x-3}{3(x+1)^{2}} \\
& =\frac{(x-1)(x+3)}{3(x+1)^{2}} \quad \text { where } x \neq-1
\end{aligned}
$$

Then

$$
\begin{aligned}
\lim _{x \rightarrow 0^{+}} f^{\prime}(x) & =\frac{(0-1)(0+3)}{3(0+1)^{2}}=-1 \neq 1=e^{0}=\lim _{x \rightarrow 0^{-}} f^{\prime}(x) \\
\text { so } \quad f^{\prime}(x) & = \begin{cases}e^{x} & x<0 \\
D N E & x=0 \\
\frac{(x-1)(x+3)}{3(x+1)^{2}} & x>0\end{cases}
\end{aligned}
$$

Differentiating again,

$$
\begin{aligned}
& \frac{\mathrm{d}^{2}}{\mathrm{~d} x^{2}}\left\{\frac{x^{2}+3}{3(x+1)}\right\}=\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{x^{2}+2 x-3}{3(x+1)^{2}}\right\} \\
&=\frac{3(x+1)^{2}(2 x+2)-\left(x^{2}+2 x-3\right)(6)(x+1)}{9(x+1)^{4}} \\
&=\frac{(x+1)(2 x+2)-2\left(x^{2}+2 x-3\right)}{3(x+1)^{3}} \\
&=\frac{8}{3(x+1)^{3}} \\
& \text { so where } x \neq-1 \\
& \text { so } \quad f^{\prime \prime}(x)= \begin{cases}e^{x} & x<0 \\
D N E & x=0 \\
\frac{8}{3(x+1)^{3}} & x>0\end{cases}
\end{aligned}
$$

i. The only singular point is $x=0$, and the only critical point is $x=1$. (When you're reading off the expression for $f^{\prime}(x)$, remember that the bottom line only applies when $x>0$.)

| $x$ | $(-\infty, 0)$ | 0 | $(0,1)$ | 1 | $(1, \infty)$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | positive | DNE | negative | 0 | positive |
| $f(x)$ | increasing | local max | decreasing | local min | increasing |

The coordinates of the local maximum are $(0,1)$ and the coordinates of the local minimum are $\left(1, \frac{2}{3}\right)$.
ii. When $x \neq 0, f^{\prime \prime}(x)$ is always positive, so $f(x)$ is concave up. iii.

$$
\begin{aligned}
\lim _{x \rightarrow \infty} f(x) & =\lim _{x \rightarrow \infty} \frac{x^{2}+3}{3 x+3} \\
& =\lim _{x \rightarrow \infty} \frac{x+\frac{3}{x}}{1+\frac{3}{x}}=\infty
\end{aligned}
$$

So, there is no horizontal asymptote to the right.

$$
\lim _{x \rightarrow-\infty} f(x)=\lim _{x \rightarrow-\infty} e^{x}=0
$$

So, $y=0$ is a horizontal asymptote to the left.
Since $f(x)$ is continuous everywhere, there are no vertical asymptotes.
3.6.7.5.c


### 3.6.7.6. *. Solution.

- Asymptotes: In the problem statement, we are told:

$$
\lim _{x \rightarrow \pm \infty} \frac{1+2 x}{e^{x^{2}}}=0
$$

So, $y=0$ is a horizontal asymptote both at $x=\infty$ and at $x=-\infty$.
Since $f(x)$ is continuous, it has no vertical asymptotes.

- Intervals of increase and decrease:

The critical points are the zeroes of $1-x-2 x^{2}=(1-2 x)(1+x)$. That is, $x=\frac{1}{2},-1$.

| $x$ | $(-\infty,-1)$ | -1 | $\left(-1, \frac{1}{2}\right)$ | $\frac{1}{2}$ | $\left(\frac{1}{2}, \infty\right)$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | negative | 0 | positive | 0 | negative |
| $f(x)$ | decreasing | local min | increasing | local max | decreasing |

At these critical points, $f\left(\frac{1}{2}\right)=2 e^{-1 / 4}>0$ and $f(-1)=-e^{-1}<0$.
From here, we can sketch a skeleton of the graph.


- Concavity:

We are told that we don't have to actually solve for the inflection points. We just need to know enough to get a basic idea. So, we'll turn the skeleton of the graph into smooth curve.


Inflection points are points where the convexity changes from up to down or vice versa. It looks like our graph is convex down for $x$ from $-\infty$ to about -1.8 , convex up from about $x=-1.8$ to about $x=-0.1$, convex down from about $x=-0.1$ to about $x=1.4$ and convex up from about $x=1.4$ to infinity. So there are three inflection points at roughly $x=-1.8,-0.1,1.4$.
3.6.7.7. *. Solution. 3.6.7.7.a We need to know the first and second derivative of $f(x)$. Using the product and chain rules, $f^{\prime}(x)=e^{-x^{2} / 2}\left(1-x^{2}\right)$. Given to us is $f^{\prime \prime}(x)=\left(x^{3}-3 x\right) e^{-x^{2} / 2}$. (These derivatives are also easy to find using the formula developed in Question 2.14.2.19, Section 2.14.)
Since $e^{-x^{2} / 2}$ is always positive, the sign of $f^{\prime}(x)$ is the same as the sign of $1-x^{2}$. $f(x)$ has no singular points and its only critical points are $x= \pm 1$. At these critical points, $f(-1)=-\frac{1}{\sqrt{e}}$ and $f(1)=\frac{1}{\sqrt{e}}$.

| $x$ | $(-\infty,-1)$ | -1 | $(-1,1)$ | 1 | $(1, \infty)$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | negative | 0 | positive | 0 | negative |
| $f(x)$ | decreasing | local min | increasing | local max | decreasing |

This, together with the observations that $f(x)<0$ for $x<0, f(0)=0$ and $f(x)>0$ for $x>0$ (in fact $f$ is an odd function), is enough to sketch a skeleton of our graph.


We can factor $f^{\prime \prime}(x)=\left(x^{3}-3 x\right) e^{-x^{2} / 2}=x(x+\sqrt{3})(x-\sqrt{3}) e^{-x^{2} / 2}$. Since $e^{-x^{2} / 2}$ is always positive, the sign of $f^{\prime \prime}(x)$ is the same as the sign of $x(x+\sqrt{3})(x-\sqrt{3})$.

| $x$ | $(-\infty,-\sqrt{3})$ | $-\sqrt{3}$ | $(-\sqrt{3}, 0)$ | 0 | $(0, \sqrt{3})$ | $\sqrt{3}$ | $(\sqrt{3}, \infty)$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime \prime}(x)$ | negative | 0 | positive | 0 | negative | 0 | positive |
| $f(x)$ | concave down | IP | concave up | IP | concave down | IP | concave up |

3.6.7.7.b We've already seen that $f(x)$ has a local min at $x=-1$ and a local max at $x=1$.
As $x$ tends to negative infinity, $f(x)$ tends to 0 , and $f(x)$ is decreasing on $(-\infty,-1)$. Then $f(x)$ is between 0 and $f(-1)=\frac{-1}{\sqrt{e}}$ on $(-\infty,-1)$. Then $f(x)$ is increasing on $(-1,1)$ from $f(-1)=\frac{-1}{\sqrt{e}}$ to $f(1)=\frac{1}{\sqrt{e}}$. Finally, for $x>1, f(x)$ is decreasing from $f(1)=\frac{1}{\sqrt{e}}$ and tending to 0 . So when $x>1, f(x)$ is between $\frac{1}{\sqrt{e}}$ and 0 .
So, over its entire domain, $f(x)$ is between $\frac{-1}{\sqrt{e}}$ and $\frac{1}{\sqrt{e}}$, and it only achieves those values at $x=-1$ and $x=1$, respectively. Therefore, the local and global min of $f(x)$ is at $\left(-1, \frac{-1}{\sqrt{e}}\right)$, and the local and global max of $f(x)$ is at $\left(1, \frac{1}{\sqrt{e}}\right)$.
$3.6 .7 .7 . \mathrm{c}$ In the graph below, open dots are inflection points, and solid dots are extrema.


### 3.6.7.8. Solution.

- Symmetry:

$$
f(-x)=-x+2 \sin (-x)=-x-2 \sin x=-f(x)
$$

So, $f(x)$ is an odd function. If we can sketch $y=f(x)$ for nonnegative $x$, we can use symmetry to complete the curve for all $x$.

- Asymptotes:

Since $f(x)$ is continuous, it has no vertical asymptotes. It also has no horizontal asymptotes, since

$$
\lim _{x \rightarrow-\infty} f(x)=-\infty \quad \lim _{x \rightarrow \infty} f(x)=\infty
$$

- Intervals of increase and decrease:

Since $f(x)$ is differentiable everywhere, there are no singular points.

$$
f^{\prime}(x)=1+2 \cos x
$$

So, the critical points of $f(x)$ occur when

$$
\begin{aligned}
\cos x & =-\frac{1}{2} \\
x & =2 \pi n \pm \frac{2 \pi}{3} \text { for any integer } n
\end{aligned}
$$

For instance, $f(x)$ has critical points at $x=\frac{2 \pi}{3}, x=\frac{4 \pi}{3}, x=\frac{8 \pi}{3}$, and $x=\frac{10 \pi}{3}$.
From the unit circle, or the graph of $y=1+2 \cos x$, we see:

| $x$ | $\left(-\frac{2 \pi}{3}, \frac{2 \pi}{3}\right)$ | $\frac{2 \pi}{3}$ | $\left(\frac{2 \pi}{3}, \frac{4 \pi}{3}\right)$ | $\frac{4 \pi}{3}$ | $\left(\frac{4 \pi}{3}, \frac{8 \pi}{3}\right)$ | $\frac{8 \pi}{3}$ | $\left(\frac{8 \pi}{3}, \frac{10 \pi}{3}\right)$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | positive | 0 | negative | 0 | positive | 0 | negative |
| $f(x)$ | increasing | l. max | decreasing | 1. min | increasing | l. max | decreasing |

We have enough information to sketch a skeleton of the curve $y=f(x)$. We use the pattern above for the graph to the right of the $y$-axis, and use odd symmetry for the graph to the left of the $y$-axis.


- Concavity:

$$
f^{\prime \prime}(x)=-2 \sin x
$$

So, $f^{\prime \prime}(x)$ exists everywhere, and is zero for $x=\pi+\pi n$ for every integer $n$.

| $x$ | $(0, \pi)$ | $\pi$ | $(\pi, 2 \pi)$ | $2 \pi$ | $(2 \pi, 3 \pi)$ | $3 \pi$ | $(3 \pi, 4 \pi)$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime \prime}(x)$ | negative | 0 | positive | 0 | negative | 0 | positive |
| $f(x)$ | concave down | IP | concave up | IP | concave down | IP | concave up |

Using these values, and the odd symmetry of $f(x)$, we can refine our skeleton. The closed dots are local extrema, and the open dots are inflection points occurring at every integer multiple of $\pi$.

3.6.7.9. *. Solution. We first compute the derivatives $f^{\prime}(x)$ and $f^{\prime \prime}(x)$.

$$
\begin{aligned}
f^{\prime}(x) & =4 \cos x+4 \sin 2 x=4 \cos x+8 \sin x \cos x \\
& =4 \cos x(1+2 \sin x) \\
f^{\prime \prime}(x) & =-4 \sin x+8 \cos 2 x=-4 \sin x+8-16 \sin ^{2} x \\
& =-4\left(4 \sin ^{2} x+\sin x-2\right)
\end{aligned}
$$

The graph has the following features.

- Symmetry: $f(x)$ is periodic of period $2 \pi$. We'll consider only $-\pi \leq x \leq \pi$. (Any interval of length $2 \pi$ will do.)
- $y$-intercept: $f(0)=-2$
- Intervals of increase and decrease: $f^{\prime}(x)=0$ when $\cos x=0$, i.e. $x= \pm \frac{\pi}{2}$, and when $\sin x=-\frac{1}{2}$, i.e. $x=-\frac{\pi}{6},-\frac{5 \pi}{6}$.

| $x$ | $\left(-\pi,-\frac{5 \pi}{6}\right)$ | $\left(-\frac{5 \pi}{6},-\frac{\pi}{2}\right)$ | $\left(-\frac{\pi}{2},-\frac{\pi}{6}\right)$ | $\left(-\frac{\pi}{6}, \frac{\pi}{2}\right)$ | $\left(\frac{\pi}{2}, \pi\right)$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | negative | positive | negative | positive | negative |
| $f(x)$ | decreasing | increasing | decreasing | increasing | decreasing |

This tells us local maxima occur at $x= \pm \frac{\pi}{2}$ and local minima occur at $x=-\frac{5 \pi}{6}$ and $x=-\frac{\pi}{6}$.
Here is a table giving the value of $f$ at each of its critical points.

| $x$ | $-\frac{5}{6} \pi$ | $-\frac{\pi}{2}$ | $\frac{\pi}{2}$ | $\frac{5}{6} \pi$ |
| :--- | :--- | :--- | :--- | :--- |
| $\sin (x)$ | $-\frac{1}{2}$ | -1 | $-\frac{1}{2}$ | 1 |
| $\cos (2 x)$ | $\frac{1}{2}$ | -1 | $\frac{1}{2}$ | -1 |
| $f(x)$ | -3 | -2 | -3 | 6 |

From here, we can graph a skeleton of of $f(x)$ :


- Concavity: To find the points where $f^{\prime \prime}(x)=0$, set $y=\sin x$, so $f^{\prime \prime}(x)=$
$-4\left(4 y^{2}+y-2\right)$. Then we really need to solve

$$
\begin{aligned}
4 y^{2}+y-2 & =0 & \text { which gives us } \\
y & =\frac{-1 \pm \sqrt{33}}{8} &
\end{aligned}
$$

These two $y$-values map to the following two $x$-values, which we'll name $a$ and $b$ for convenience:

$$
\begin{aligned}
& a=\arcsin \left(\frac{-1+\sqrt{33}}{8}\right) \approx 0.635 \\
& b=\arcsin \left(\frac{-1-\sqrt{33}}{8}\right) \approx-1.003
\end{aligned}
$$

However, these are not the only values of $x$ in $[-\pi, \pi]$ with $\sin x=\frac{-1 \pm \sqrt{33}}{8}$. The analysis above misses the others because the arcsine function only returns numbers in the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The graph below shows that there should be other values of $x$ with $\sin x=\frac{-1 \pm \sqrt{33}}{8}$, and hence $f^{\prime \prime}(x)=0$.


We can recover the other solutions in $[-\pi, \pi]$ by recalling that

$$
\sin (x)=\sin (\pi-x)
$$

(see CLP appendix A7). So, if we choose $x=\arcsin \left(\frac{-1+\sqrt{33}}{8}\right) \approx 0.635$ to make $\sin (x)=\frac{-1+\sqrt{33}}{8}$ so that $f^{\prime \prime}(x)=0$, then setting

$$
x=\pi-a=\pi-\arcsin \left(\frac{-1+\sqrt{33}}{8}\right) \approx 2.507
$$

will also give us $\sin (x)=\frac{-1+\sqrt{33}}{8}$ and $f^{\prime \prime}(x)=0$. Similarly, setting

$$
x=\pi-b=\pi-\arcsin \left(\frac{-1-\sqrt{33}}{8}\right) \approx 4.145
$$

would give us $f^{\prime \prime}(x)=0$. However, this value is outside $[-\pi, \pi]$. To find another solution inside $[-\pi, \pi]$ we use the identity

$$
\sin (x)=\sin (-\pi-x)
$$

(which we can obtain from the identity we used above and the fact that $\sin (\theta)=\sin (\theta \pm 2 \pi)$ for any angle $\theta)$. Using this, we can show that

$$
x=-\pi-b=-\pi-\arcsin \left(\frac{-1-\sqrt{33}}{8}\right) \approx-2.139
$$

also gives $f^{\prime \prime}(x)=0$.
So, all together, $f^{\prime \prime}(x)=0$ when $x=-\pi-b, x=b, x=a$, and $x=\pi-a$.
Now, we should compute the sign of $f^{\prime \prime}(x)$ while $x$ is between $-\pi$ and $\pi$. Recall that, if $y=\sin x$, then $f^{\prime \prime}(x)=-4\left(4 y^{2}+y-2\right)$. So, in terms of $y, f^{\prime \prime}$ is a parabola pointing down, with intercepts $y=\frac{-1 \pm \sqrt{33}}{8}$. Then $f^{\prime \prime}$ is positive when $y$ is in the interval $\left(\frac{-1-\sqrt{33}}{8}, \frac{-1+\sqrt{33}}{8}\right)$, and $f^{\prime \prime}$ is negative otherwise. From the graph of sine, we see that $y$ is between $\frac{-1-\sqrt{33}}{8}$ and $\frac{-1+\sqrt{33}}{8}$ precisely on the intervals $(-\pi,-\pi-b),(b, a)$, and $(\pi-a, \pi)$.
Therefore, $f(x)$ is concave up on the intervals $(-\pi,-\pi-b),(b, a)$, and $(\pi-$ $a, \pi)$, and $f(x)$ is concave down on the intervals $(-\pi-b, b)$ and $(a, \pi-a)$. So, the inflection points of $f$ occur at $x=-\pi-b, x=b, x=a$, and $x=\pi-a$.


To find the maximum and minimum values of $f(x)$ on $[0, \pi]$, we compare the values of $f(x)$ at its critical points in this interval (only $x=\frac{\pi}{2}$ ) with the values of $f(x)$ at its endpoints $x=0, x=\pi$.
Since $f(0)=f(\pi)=-2$, the minimum value of $f$ on $[0, \pi]$ is -2 , achieved at $x=0, \pi$ and the maximum value of $f$ on $[0, \pi]$ is 6 , achieved at $x=\frac{\pi}{2}$.
3.6.7.10. Solution. Let $f(x)=\sqrt[3]{\frac{x+1}{x^{2}}}$.

- Asymptotes: Since $\lim _{x \rightarrow 0} f(x)=\infty, f(x)$ has a vertical asymptote at $x=0$ where the curve reaches steeply upward from both the left and the right.
$\lim _{x \rightarrow \pm \infty} f(x)=0$, so $y=0$ is a horizontal asymptote for $x \rightarrow \pm \infty$.
- Intercepts: $f(-1)=0$.
- Intervals of increase and decrease:

$$
f^{\prime}(x)=\frac{-(x+2)}{3 x^{5 / 3}(x+1)^{2 / 3}}
$$

There is a singular point at $x=-1$ and a critical point at $x=-2$, in addition to a discontinuity at $x=0$. Note that $(x+1)^{2 / 3}=(\sqrt[3]{x+1})^{2}$, which is never negative. Note also that $\lim _{x \rightarrow-1} f^{\prime}(x)=\infty$, so $f(x)$ has a vertical tangent line at $x=-1$.

| $x$ | $(-\infty,-2)$ | -2 | $(-2,-1)$ | -1 | $(-1,0)$ | 0 | $(0, \infty)$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | negative | 0 | positive | DNE | positive | DNE | negative |
| $f(x)$ | decreasing | l. min | increasing | vertical | increasing | VA | decreasing |

This gives us enough information to sketch a skeleton of the curve.


- Concavity:

$$
f^{\prime \prime}(x)=\frac{4 x^{2}+16 x+10}{9 x^{8 / 3}(x+1)^{5 / 3}}
$$

We still have a discontinuity at $x=0$, and $f^{\prime \prime}(x)$ does not exist at $x=-1$. The second derivative is zero when $4 x^{2}+16 x+10=0$. Using the quadratic formula, we find this occurs when $x=-2 \pm \sqrt{1.5} \approx-0.8,-3.2$. Note $x^{8 / 3}=(\sqrt[3]{x})^{8}$ is never negative.

| $x$ | $(-\infty,-2-\sqrt{1.5})$ | $-2-\sqrt{1.5}$ | $(-2-\sqrt{1.5},-1)$ | -1 |
| :--- | :--- | :--- | :--- | :--- |
| $f^{\prime \prime}(x)$ | negative | 0 | positive | DNE |
| $f(x)$ | concave down | IP | concave up | IP |


| $x$ | $(-1,-2+\sqrt{1.5})$ | $-2+\sqrt{1.5}$ | $(-2+\sqrt{1.5}, 0)$ | $(0, \infty)$ |
| :--- | :--- | :--- | :--- | :--- |
| $f^{\prime \prime}(x)$ | negative | 0 | positive | positive |
| $f(x)$ | concave down | IP | concave up | concave up |

Now, we can refine our skeleton. The closed dot is the local minimum, and the open dots are inflection points.


## Exercises - Stage 3

3.6.7.11. *. Solution. The parts of the question are just scaffolding to lead you through sketching the curve. Their answers are given explicitly, in an organized manner, in the "answers" section. In this solution, they are scattered throughout.

- Asymptotes:

Since the function has a derivative at every real number, the function is continuous for every real number, so it has no vertical asymptotes. In the problem statement, you are told $\lim _{x \rightarrow \infty} f(x)=0$, so $y=0$ is a horizontal asymptote as $x$ goes to infinity. It remains to evaluate $\lim _{x \rightarrow-\infty} f(x)$. Let's consider the limit
of $f^{\prime}(x)$ instead. Recall $K$ is a positive constant.

$$
\begin{aligned}
& \lim _{x \rightarrow-\infty} e^{-x}=\lim _{x \rightarrow \infty} e^{x}=\infty \\
& \lim _{x \rightarrow-\infty} K\left(2 x-x^{2}\right)=-\infty
\end{aligned}
$$

So,

$$
\lim _{x \rightarrow-\infty} K\left(2 x-x^{2}\right) e^{-x}=-\infty
$$

That is, as $x$ becomes a hugely negative number, $f^{\prime}(x)$ also becomes a hugely negative number. As we move left along the $x$-axis, $f(x)$ is decreasing with a steeper and steeper slope, as in the sketch below. That means $\lim _{x \rightarrow-\infty} f(x)=\infty$.


- Intervals of increase and decrease:

We are given $f^{\prime}(x)$ (although we don't know $f(x)$ ):

$$
f^{\prime}(x)=K x(2-x) e^{-x}
$$

The critical points of $f(x)$ are $x=0$ and $x=2$, and there are no singular points. Recall $e^{-x}$ is always positive, and $K$ is a positive constant.

| $x$ | $(-\infty, 0)$ | 0 | $(0,2)$ | 2 | $(2, \infty)$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | negative | 0 | positive | 0 | negative |
| $f(x)$ | decreasing | local min | increasing | local max | decreasing |

So, $f(0)=0$ is a local minimum, and $f(2)=2$ is a local maximum.
Looking ahead to part 3.6.7.11.d, we have a skeleton of the curve.


- Concavity:

Since we're given $f^{\prime}(x)$, we can find $f^{\prime \prime}(x)$.

$$
\begin{aligned}
f^{\prime \prime}(x) & =K\left(2-2 x-2 x+x^{2}\right) e^{-x} \\
& =K\left(2-4 x+x^{2}\right) e^{-x} \\
& =K(x-2-\sqrt{2})(x-2+\sqrt{2}) e^{-x}
\end{aligned}
$$

where the last line can be found using the quadratic equation. So, $f^{\prime \prime}(x)=0$ for $x=2 \pm \sqrt{2}$, and $f^{\prime \prime}(x)$ exists everywhere.

| $x$ | $(-\infty, 2-\sqrt{2})$ | $2-\sqrt{2}$ | $(2-\sqrt{2}, 2+\sqrt{2})$ | $2+\sqrt{2}$ | $(2+\sqrt{2}, \infty)$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $f^{\prime}(x)$ | positive | 0 | negative | 0 | positive |
| $f(x)$ | concave up | IP | concave down | IP | concave up |

Now, we can add concavity to our sketch.

3.6.7.12. *. Solution. 3.6.7.12.a You should be familiar with the graph of $y=e^{x}$. You can construct the graph of $y=e^{-x}$ just by reflecting the graph of $y=e^{x}$ across the $y$-axis. To see why this is the case, imagine swapping each value of $x$ with its negative: for example, swapping the point at $x=-1$ with the point at $x=1$, etc. Alternatively, you can graph $y=f(x)=e^{-x}, x \geq 0$, using the methods of this section: at $x=0, y=f(0)=1$; as $x$ increases, $y=f(x)=e^{-x}$ decreases, with no local extrema; and as $x \rightarrow+\infty, y=f(x) \rightarrow 0$.
There are no inflection points or extrema, except the endpoint $(0,1)$.


### 3.6.7.12.b

Recall that, to graph the inverse of a function, we reflect the original function across the line $y=x$. To see why this is true, consider the following. By definition, the inverse function $g$ of $f$ is obtained by solving $y=f(x)$ for $x$ as a function of $y$. So, for any pair of numbers $x$ and $y$, we have

$$
f(x)=y \text { if and only } g(y)=x
$$

That is, $g$ is the function that swaps the input and output of $f$. Now the point $(x, y)$ lies on the graph of $f$ if and only if $y=f(x)$. Similarly, the point $(X, Y)$ lies on the graph of $g$ if and only if $Y=g(X)$. Choosing $Y=x$ and $X=y$, we see that the point $(X, Y)=(y, x)$ lies on the graph of $g$ if and only if $x=g(y)$, which in turn is the case if and only if $y=f(x)$. So
$(y, x)$ is on the graph of $g$ if and only if $(x, y)$ is on the graph of $f$.
To get from the point $(x, y)$ to the point $(y, x)$ we have to exchange $x \leftrightarrow y$, which we can do by reflecting over the line $y=x$. Thus we can construct the graph of $g$ by reflecting the curve $y=f(x)$ over the line $y=x$.

3.6.7.12.c The domain of $g$ is the range of $f$, which is $(0,1]$. The range of $g$ is the domain of $f$, which is $[0, \infty)$.
3.6.7.12.d Since $g$ and $f$ are inverses,

$$
g(f(x))=x
$$

Using the chain rule,

$$
g^{\prime}(f(x)) \cdot f^{\prime}(x)=1
$$

Since $f^{\prime}(x)=-e^{-x}=-f(x)$ :

$$
g^{\prime}(f(x)) \cdot f(x)=-1
$$

We plug in $f(x)=\frac{1}{2}$.

$$
\begin{aligned}
g^{\prime}\left(\frac{1}{2}\right) \cdot \frac{1}{2} & =-1 \\
g^{\prime}\left(\frac{1}{2}\right) & =-2
\end{aligned}
$$

3.6.7.13. *. Solution. (a) First, we differentiate.

$$
f(x)=x^{5}-x \quad f^{\prime}(x)=5 x^{4}-1 \quad f^{\prime \prime}(x)=20 x^{3}
$$

The function and its first derivative tells us the following:

- $\lim _{x \rightarrow \infty} f(x)=\infty, \lim _{x \rightarrow-\infty} f(x)=-\infty$
- $f^{\prime}(x)>0$ (i.e. $f$ is increasing) for $|x|>\frac{1}{\sqrt[4]{5}}$
- $f^{\prime}(x)=0$ (i.e. $f$ has critical points) for $x= \pm \frac{1}{\sqrt[4]{5}} \approx \pm 0.67$
- $f^{\prime}(x)<0$ (i.e. $f$ is decreasing) for $|x|<\frac{1}{\sqrt[4]{5}}$
- $f\left( \pm \frac{1}{\sqrt[4]{5}}\right)=\mp \frac{4}{5 \sqrt[4]{5}} \approx \mp 0.53$

This gives us a first idea of the shape of the graph.


We refine this skeleton using information from the second derivative.

- $f^{\prime \prime}(x)>0$ (i.e. $f$ is concave up) for $x>0$,
- $f^{\prime \prime}(x)=0$ (i.e. $f$ has an inflection point) for $x=0$, and
- $f^{\prime \prime}(x)<0$ (i.e. $f$ is concave down) for $x<0$

Thus

- $f$ has no asymptotes
- $f$ has a local maximum at $x=-\frac{1}{\sqrt[4]{5}}$ and a local minimum at $x=\frac{1}{\sqrt[4]{5}}$
- $f$ has an inflection point at $x=0$
- $f$ is concave down for $x<0$ and concave up for $x>0$

(b) The function $x^{5}-x+k$ has a root at $x=x_{0}$ if and only if $x^{5}-x=-k$ at $x=x_{0}$. So the number of distinct real roots of $x^{5}-x+k$ is the number of times the curve $y=x^{5}-x$ crosses the horizontal line $y=-k$. The local maximum of $x^{5}-x$ (when $x=-\frac{1}{\sqrt[4]{5}}$ ) is $\frac{4}{5 \sqrt[4]{5}}$, and the local minimum of $x^{5}-x\left(\right.$ when $x=\frac{1}{\sqrt[4]{5}}$ ) is $-\frac{4}{5 \sqrt[4]{5}}$. So, looking at the graph of $x^{5}-x$ above, we see that the number of distinct real roots of $x^{5}-x+k$ is
- 1 when $|k|>\frac{4}{5 \sqrt[4]{5}}$
- 2 when $|k|=\frac{4}{5 \sqrt[4]{5}}$
- 3 when $|k|<\frac{4}{5 \sqrt[4]{5}}$
3.6.7.14. *. Solution. (a) You might not be familiar with hyperbolic sine and cosine, but you don't need to be. We can graph them using the same methods as the other curves in this section. The derivatives are given to us:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{\sinh x\} & =\cosh x=\frac{e^{x}+e^{-x}}{2} \\
\frac{\mathrm{~d}}{\mathrm{~d} x}\{\cosh x\} & =\sinh x=\frac{e^{x}-e^{-x}}{2} \\
\left(\frac{\mathrm{~d}}{\mathrm{~d} x}\right)^{2}\{\sinh x\} & =\sinh x=\frac{e^{x}-e^{-x}}{2}
\end{aligned}
$$

$$
\left(\frac{\mathrm{d}}{\mathrm{~d} x}\right)^{2}\{\cosh x\}=\cosh x=\frac{e^{x}+e^{-x}}{2}
$$

Observe that:

- $\sinh (x)$ has a derivative that is always positive, so $\sinh (x)$ is always increasing. The second derivative of $\sinh (x)$ is negative to the left of $x=0$ and positive to the right of $x=0$, so $\sinh (x)$ is concave down to the left of the $y$-axis and concave up to its right, with an inflection point at $x=0$.
- $\cosh (x)$ has a derivative that is positive when $x>0$ and negative when $x<0$. The second derivative of $\cosh (x)$ is always positive, so it is always concave up.
- $\cosh (0)=1$ and $\sinh (0)=0$.
- $\lim _{x \rightarrow \infty} \sinh x=\lim _{x \rightarrow \infty} \cosh x=\lim _{x \rightarrow \infty} \frac{e^{x}}{2}=\infty$, since $\lim _{x \rightarrow \infty} e^{-x}=0$
- 

$$
\lim _{x \rightarrow-\infty} \sinh x=\lim _{x \rightarrow-\infty}\left(\frac{e^{x}}{2}-\frac{e^{-x}}{2}\right)=\lim _{x \rightarrow \infty}\left(\frac{e^{-x}}{2}-\frac{e^{x}}{2}\right)=-\infty
$$

and

$$
\lim _{x \rightarrow-\infty} \cosh x=\lim _{x \rightarrow-\infty}\left(\frac{e^{x}}{2}+\frac{e^{-x}}{2}\right)=\lim _{x \rightarrow \infty}\left(\frac{e^{-x}}{2}+\frac{e^{x}}{2}\right)=\infty
$$

- $\cosh (x)$ is even, since

$$
\cosh (-x)=\frac{e^{-x}+e^{-(-x)}}{2}=\frac{e^{-x}+e^{x}}{2}=\cosh (x)
$$

and $\sinh (x)$ is odd, since

$$
\begin{aligned}
\sinh (-x) & =\frac{e^{-x}-e^{-(-x)}}{2}=\frac{e^{-x}-e^{x}}{2}=\frac{-\left(e^{x}-e^{-x}\right)}{2} \\
& =-\sinh (x)
\end{aligned}
$$


(b)

- As $y$ runs over $(-\infty, \infty)$ the function $\sinh (y)$ takes every real value exactly once. So, for each $x \in(-\infty, \infty)$, define $\sinh ^{-1}(x)$ to be the unique solution of $\sinh (y)=x$.
- As $y$ runs over $[0, \infty)$ the function $\cosh (y)$ takes every real value in $[1, \infty)$ exactly once. In particular, the smallest value of $\cosh (y)$ is $\cosh (0)=1$. So, for each $x \in[1, \infty)$, define $\cosh ^{-1}(x)$ to be the unique $y \in[0, \infty)$ that obeys $\cosh (y)=x$.

To graph the inverse of a (one-to-one) function, we reflect the original function over the line $y=x$. Using this method to graph $y=\sinh ^{-1}(x)$ is straightforward. To graph $y=\cosh (x)$, we need to be careful of the domains: we are restricting $\cosh (x)$ to values of $x$ in $[0, \infty)$. The graphs are

(c) Let $y(x)=\cosh ^{-1}(x)$. Then, using the definition of $\cosh ^{-1}$,

$$
\cosh y(x)=x
$$

We differentiate with respect to $x$ using the chain rule.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{\cosh y(x)\} & =\frac{\mathrm{d}}{\mathrm{~d} x}\{x\} \\
y^{\prime}(x) \sinh y(x) & =1
\end{aligned}
$$

We solve for $y^{\prime}(x)$.

$$
y^{\prime}(x)=\frac{1}{\sinh y(x)}
$$

We want to have our answer in terms of $x$, not $y$. We know that $\cosh y=x$, so if we can convert hyperbolic sine into hyperbolic cosine, we can get rid of $y$. Our tool for this is the identity, given in the question statement, $\cosh ^{2}(x)-\sinh ^{2}(x)=1$. This tells us $\sinh ^{2}(y)=1-\cosh ^{2}(y)$. Now we have to decide whether $\sinh (y)$ is the positive or negative square root of $1-\cosh ^{2}(y)$ in our context. Looking at the graph of $y(x)=\cosh ^{-1}(x)$, we see $y^{\prime}(x)>0$. So we use the positive square root:

$$
y^{\prime}(x)=\frac{1}{\sqrt{\cosh ^{2} y(x)-1}}=\frac{1}{\sqrt{x^{2}-1}}
$$

Remark: $\frac{\mathrm{d}}{\mathrm{d} x}\{\arccos (x)\}=\frac{-1}{\sqrt{1-x^{2}}}$, so again the hyperbolic trigonometric function has properties similar to (but not exactly the same as) its trigonometric counterpart.

## 3.7 • L'Hôpital's Rule, Indeterminate Forms

### 3.7.4 • Exercises

## Exercises - Stage 1

3.7.4.1. Solution. There are many possible answers. Consider these: $f(x)=5 x$, $g(x)=2 x$. Then $\lim _{x \rightarrow \infty} f(x)=\lim _{x \rightarrow \infty} g(x)=\infty$, and $\lim _{x \rightarrow \infty} \frac{f(x)}{g(x)}=\lim _{x \rightarrow \infty} \frac{5 x}{2 x}=\lim _{x \rightarrow \infty} \frac{5}{2}=$ $\frac{5}{2}=2.5$.
3.7.4.2. Solution. There are many possible answers. Consider these: $f(x)=x$, $g(x)=x^{2}$. Then $\lim _{x \rightarrow \infty} f(x)=\lim _{x \rightarrow \infty} g(x)=\infty$, and $\lim _{x \rightarrow \infty} \frac{f(x)}{g(x)}=\lim _{x \rightarrow \infty} \frac{x}{x^{2}}=\lim _{x \rightarrow \infty} \frac{1}{x}=$ 0.
3.7.4.3. Solution. From Example 3.7.20, we know that $\lim _{x \rightarrow 0}(1+x)^{\frac{a}{x}}=e^{a}$, so $\lim _{x \rightarrow 0}(1+x)^{\frac{\log 5}{x}}=e^{\log 5}=5$. However, this is the limit as $x$ goes to 0 , which is not what we were asked. So, we modify the functions by replacing $x$ with $\frac{1}{x}$. If $x \rightarrow 0^{+}$, then $\frac{1}{x} \rightarrow \infty$.
Taking $f(x)=1+\frac{1}{x}$ and $g(x)=x \log 5$, we see:

- 3.7.4.3.i $\lim _{x \rightarrow \infty} f(x)=\lim _{x \rightarrow \infty}\left(1+\frac{1}{x}\right)=1$
- 3.7.4.3.ii $\lim _{x \rightarrow \infty} g(x)=\lim _{x \rightarrow \infty} x \log 5=\infty$
- 3.7.4.3.iii Let us name $\frac{1}{x}=X$. Then as $x \rightarrow \infty, X \rightarrow 0^{+}$, so:

$$
\begin{aligned}
\lim _{x \rightarrow \infty}[f(x)]^{g(x)} & =\lim _{x \rightarrow \infty}\left[1+\frac{1}{x}\right]^{x \log 5}=\lim _{x \rightarrow \infty}\left[1+\frac{1}{x}\right]^{\frac{\log 5}{\frac{1}{x}}} \\
& =\lim _{X \rightarrow 0^{+}}[1+X]^{\frac{\log 5}{x}}=e^{\log 5}=5
\end{aligned}
$$

where in the penultimate step, we used the result of Example 3.7.20.

## Exercises - Stage 2

3.7.4.4. *. Solution. If we plug in $x=1$ to the numerator and the denominator, we find they are both zero. So, we have an indeterminate form appropriate for L'Hôpital's Rule.

$$
\lim _{x \rightarrow 1} \underbrace{\frac{x^{3}-e^{x-1}}{\sin (\pi x)}}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow 0}}=\lim _{x \rightarrow 1} \frac{3 x^{2}-e^{x-1}}{\pi \cos (\pi x)}=-\frac{2}{\pi}
$$

3.7.4.5. *. Solution. Be careful- this is not an indeterminate form!

As $x \rightarrow 0+$, the numerator $\log x \rightarrow-\infty$. That is, the numerator is becoming an increasingly huge, negative number. As $x \rightarrow 0+$, the denominator $x \rightarrow 0+$, which only serves to make the total fraction even larger, and still negative. So, $\lim _{x \rightarrow 0^{+}} \frac{\log x}{x}=-\infty$.
Remark: if we had tried to use l'Hôpital's Rule here, we would have come up with the wrong answer. If we differentiate the numerator and the denominator, the fraction becomes $\frac{\frac{1}{x}}{1}=\frac{1}{x}$, and $\lim _{x \rightarrow 0^{+}} \frac{1}{x}=\infty$. The reason we cannot apply l'Hôpital's Rule is that we do not have an indeterminate form, like both numerator and denominator going to infinity, or both numerator and denominator going to zero.
3.7.4.6. *. Solution. We rearrange the expression to a more natural form:

$$
\lim _{x \rightarrow \infty}(\log x)^{2} e^{-x}=\lim _{x \rightarrow \infty} \underbrace{\frac{(\log x)^{2}}{e^{x}}}_{\substack{\text { num } \rightarrow \infty \\ \text { den } \rightarrow \infty}}
$$

Both the numerator and denominator go to infinity as $x$ goes to infinity. So, we can apply l'Hôpital's Rule. In fact, we end up applying it twice.

$$
\begin{aligned}
\lim _{x \rightarrow \infty}(\log x)^{2} e^{-x} & =\lim _{x \rightarrow \infty} \underbrace{\frac{2 \log x}{x e^{x}}}_{\substack{\text { num } \rightarrow \infty \\
\text { den } \rightarrow \infty}} \\
& =\lim _{x \rightarrow \infty} \frac{2 / x}{x e^{x}+e^{x}}
\end{aligned}
$$

The numerator gets smaller and smaller while the denominator gets larger and larger, so:

$$
\lim _{x \rightarrow \infty}(\log x)^{2} e^{-x}=0
$$

### 3.7.4.7. *. Solution.

$$
\lim _{x \rightarrow \infty} x^{2} e^{-x}=\lim _{x \rightarrow \infty} \underbrace{\frac{x^{2}}{e^{x}}}_{\substack{\text { num } \rightarrow \infty \\ \text { den } \rightarrow \infty}}=\lim _{x \rightarrow \infty} \underbrace{\frac{2 x}{e^{x}}}_{\substack{\text { num } \rightarrow \infty \\ \text { den } \rightarrow \infty}}=\lim _{x \rightarrow \infty} \underbrace{\frac{2}{e^{x}}}_{\substack{\text { num } \rightarrow \infty \\ \text { den } \rightarrow \infty}}=0
$$

### 3.7.4.8. *. Solution.

$$
\begin{aligned}
\lim _{x \rightarrow 0} \underbrace{\frac{x-x \cos x}{x-\sin x}}_{\substack{\text { num } \rightarrow 0 \\
\text { den } \rightarrow 0}} & =\lim _{x \rightarrow 0} \underbrace{\frac{1-\cos x+x \sin x}{1-\cos x}}_{\substack{\text { num } \rightarrow 0 \\
\text { den } \rightarrow 0}} \\
& =\lim _{x \rightarrow 0} \underbrace{\frac{\sin x+\sin x+x \cos x}{\sin x}}_{\substack{\text { num } \rightarrow 0 \\
\text { den } \rightarrow 0}} \\
& =\lim _{x \rightarrow 0} \frac{2 \cos x+\cos x-x \sin x}{\cos x}=3
\end{aligned}
$$

3.7.4.9. Solution. If we plug in $x=0$ to the numerator and denominator, both are zero, so this is a candidate for l'Hôpital's Rule. However, an easier way to evaluate the limit is to factor $x^{2}$ from the numerator and denominator, and cancel.

$$
\begin{aligned}
\lim _{x \rightarrow 0} \frac{\sqrt{x^{6}+4 x^{4}}}{x^{2} \cos x} & =\lim _{x \rightarrow 0} \frac{\sqrt{x^{4}} \sqrt{x^{2}+4}}{x^{2} \cos x} \\
& =\lim _{x \rightarrow 0} \frac{x^{2} \sqrt{x^{2}+4}}{x^{2} \cos x} \\
& =\lim _{x \rightarrow 0} \frac{\sqrt{x^{2}+4}}{\cos x} \\
& =\frac{\sqrt{0^{2}+4}}{\cos (0)}=2
\end{aligned}
$$

### 3.7.4.10. *. Solution.

$$
\lim _{x \rightarrow \infty} \underbrace{\frac{(\log x)^{2}}{x}}_{\substack{\text { num } \rightarrow \infty \\ \text { den } \rightarrow \infty}}=\lim _{x \rightarrow \infty} \frac{2(\log x) \frac{1}{x}}{1}=2 \lim _{x \rightarrow \infty} \underbrace{\frac{\log x}{x}}_{\substack{\text { num } \rightarrow \infty \\ \text { den } \rightarrow \infty}}=2 \lim _{x \rightarrow \infty} \frac{\frac{1}{x}}{1}=0
$$

3.7.4.11. *. Solution.

$$
\lim _{x \rightarrow 0} \underbrace{}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow 0}} \frac{1-\cos x}{\sin ^{2} x}=\lim _{x \rightarrow 0} \frac{\sin x}{2 \sin x \cos x}=\lim _{x \rightarrow 0} \frac{1}{2 \cos x}=\frac{1}{2}
$$

3.7.4.12. Solution. If we plug in $x=0$, the numerator is zero, and the denominator is $\sec 0=\frac{1}{\cos 0}=\frac{1}{1}=1$. So the limit is $\frac{0}{1}=0$.
Be careful: you cannot use l'Hôpital's Rule here, because the fraction does not give an indeterminate form. If you try to differentiate the numerator and the denominator, you get an expression whose limit does not exist:

$$
\lim _{x \rightarrow 0} \frac{1}{\sec x \tan x}=\lim _{x \rightarrow 0} \cos x \cdot \frac{\cos x}{\sin x}=D N E .
$$

3.7.4.13. Solution. If we plug $x=0$ into the denominator, we get 1 . However, the numerator is an indeterminate form: $\tan 0=0$, while $\lim _{x \rightarrow 0^{+}} \csc x=\infty$ and
$\lim _{x \rightarrow 0^{-}} \csc x=-\infty$. If we use $\csc x=\frac{1}{\sin x}$, our expression becomes

$$
\lim _{x \rightarrow 0} \frac{\tan x \cdot\left(x^{2}+5\right)}{\sin x \cdot e^{x}}
$$

Since plugging in $x=0$ makes both the numerator and the denominator equal to zero, this is a candidate for l'Hôspital's Rule. However, a much easier way is to simplify the trig first.

$$
\begin{aligned}
\lim _{x \rightarrow 0} \frac{\tan x \cdot\left(x^{2}+5\right)}{\sin x \cdot e^{x}} & =\lim _{x \rightarrow 0} \frac{\sin x \cdot\left(x^{2}+5\right)}{\cos x \cdot \sin x \cdot e^{x}} \\
& =\lim _{x \rightarrow 0} \frac{x^{2}+5}{\cos x \cdot e^{x}} \\
& =\frac{0^{2}+5}{\cos (0) \cdot e^{0}}=5
\end{aligned}
$$

3.7.4.14. Solution. $\lim _{x \rightarrow \infty} \sqrt{2 x^{2}+1}-\sqrt{x^{2}+x}$ has the indeterminate form $\infty-\infty$. To get a better idea of what's going on, let's rationalize.

$$
\begin{aligned}
& \lim _{x \rightarrow \infty} \sqrt{2 x^{2}+1}-\sqrt{x^{2}+x} \\
&=\lim _{x \rightarrow \infty}\left(\sqrt{2 x^{2}+1}-\sqrt{x^{2}+x}\right)\left(\frac{\sqrt{2 x^{2}+1}+\sqrt{x^{2}+x}}{\sqrt{2 x^{2}+1}+\sqrt{x^{2}+x}}\right) \\
&=\lim _{x \rightarrow \infty} \frac{\left(2 x^{2}+1\right)-\left(x^{2}+x\right)}{\sqrt{2 x^{2}+1}+\sqrt{x^{2}+x}} \\
&=\lim _{x \rightarrow \infty} \frac{x^{2}-x+1}{\sqrt{2 x^{2}+1}+\sqrt{x^{2}+x}}
\end{aligned}
$$

Here, we have the indeterminate form $\frac{\infty}{\infty}$, so l'Hôpital's Rule applies. However, if we try to use it here, we quickly get a huge mess. Instead, remember how we dealt with these kinds of limits in the past: factor out the highest power of the denominator, which is $x$.

$$
\begin{aligned}
& =\lim _{x \rightarrow \infty} \frac{x\left(x-1+\frac{1}{x}\right)}{\sqrt{x^{2}\left(2+\frac{1}{x^{2}}\right)}+\sqrt{x^{2}\left(1+\frac{1}{x}\right)}} \\
& =\lim _{x \rightarrow \infty} \frac{x\left(x-1+\frac{1}{x}\right)}{x\left(\sqrt{2+\frac{1}{x^{2}}}+\sqrt{1+\frac{1}{x}}\right)} \\
& =\lim _{x \rightarrow \infty} \underbrace{\frac{x-1+\frac{1}{x}}{\sqrt{2+\frac{1}{x^{2}}}+\sqrt{1+\frac{1}{x}}}}_{\substack{\text { num } \rightarrow \infty \\
\operatorname{den} \rightarrow \sqrt{2}+1}} \\
& =\infty
\end{aligned}
$$

3.7.4.15. *. Solution. If we plug in $x=0$, both numerator and denominator become zero. So, we have exactly one of the indeterminate forms that l'Hôpital's Rule applies to.

$$
\lim _{x \rightarrow 0} \underbrace{\frac{\sin \left(x^{3}+3 x^{2}\right)}{\sin ^{2} x}}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow 0}}=\lim _{x \rightarrow 0} \frac{\left(3 x^{2}+6 x\right) \cos \left(x^{3}+3 x^{2}\right)}{2 \sin x \cos x}
$$

If we plug in $x=0$, still we find that both the numerator and the denominator go to zero. We could jump in with another iteration of l'Hôpital's Rule. However, the derivatives would be a little messy, so we use limit laws and break up the fraction into the product of two fractions. If both limits exist:

$$
\begin{aligned}
& \lim _{x \rightarrow 0} \frac{\left(3 x^{2}+6 x\right) \cos \left(x^{3}+3 x^{2}\right)}{2 \sin x \cos x} \\
& \quad=\left(\lim _{x \rightarrow 0} \frac{x^{2}+2 x}{\sin x}\right) \cdot\left(\lim _{x \rightarrow 0} \frac{3 \cos \left(x^{3}+3 x^{2}\right)}{2 \cos x}\right)
\end{aligned}
$$

We can evaluate the right-hand limit by simply plugging in $x=0$ :

$$
\begin{aligned}
\lim _{x \rightarrow 0} \frac{\left(3 x^{2}+6 x\right) \cos \left(x^{3}+3 x^{2}\right)}{2 \sin x \cos x} & =\frac{3}{2} \lim _{x \rightarrow 0} \underbrace{\frac{x^{2}+2 x}{\sin x}}_{\substack{\text { num } \rightarrow 0 \\
\text { den } \rightarrow 0}} \\
& =\frac{3}{2} \lim _{x \rightarrow 0} \frac{2 x+2}{\cos x} \\
& =\frac{3}{2}\left(\frac{2}{1}\right)=3
\end{aligned}
$$

### 3.7.4.16. *. Solution.

$$
\lim _{x \rightarrow 1} \frac{\log \left(x^{3}\right)}{x^{2}-1}=\lim _{x \rightarrow 1} \underbrace{\frac{3 \log (x)}{x^{2}-1}}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow 0}}=\lim _{x \rightarrow 1} \frac{3 / x}{2 x}=\frac{3}{2}
$$

### 3.7.4.17. *. Solution.

- Solution 1.

$$
\begin{aligned}
\lim _{x \rightarrow 0} \frac{e^{-1 / x^{2}}}{x^{4}} & =\lim _{x \rightarrow 0} \frac{\frac{1}{x^{4}}}{\underbrace{e^{1 / x^{2}}}_{\substack{\text { num } \rightarrow \infty \\
\text { den } \rightarrow \infty}}}=\lim _{x \rightarrow 0} \frac{\frac{-4}{x^{5}}}{\frac{-2}{x^{3}} e^{1 / x^{2}}}=\lim _{x \rightarrow 0} \frac{\frac{2}{x^{2}}}{\underbrace{e^{1 / x^{2}}}_{\substack{\text { num } \rightarrow \infty \\
\text { den } \rightarrow \infty}}} \\
& =\lim _{x \rightarrow 0} \frac{\frac{-4}{x^{3}}}{\frac{-2}{x^{3}} e^{1 / x^{2}}}=\lim _{x \rightarrow 0} \frac{2}{e^{1 / x^{2}}}=0
\end{aligned}
$$

since, as $x \rightarrow 0$, the exponent $\frac{1}{x^{2}} \rightarrow \infty$ so that $e^{1 / x^{2}} \rightarrow \infty$ and $e^{-1 / x^{2}} \rightarrow 0$.

- Solution 2.

$$
\lim _{x \rightarrow 0} \frac{e^{-1 / x^{2}}}{x^{4}}=\lim _{t=\frac{1}{x^{2}} \rightarrow \infty} \frac{e^{-t}}{t^{-2}}=\lim _{t \rightarrow \infty} \underbrace{\frac{t^{2}}{\overline{e^{t}}}}_{\substack{\text { num } \rightarrow \infty \\ \text { den } \rightarrow \infty}}=\lim _{t \rightarrow \infty} \underbrace{\frac{2 t}{e^{t}}}_{\substack{\text { num } \rightarrow \infty \\ \text { den } \rightarrow \infty}}=\lim _{t \rightarrow \infty} \frac{2}{e^{t}}=0
$$

### 3.7.4.18. *. Solution.

$$
\lim _{x \rightarrow 0} \frac{x e^{x}}{\underbrace{\tan (3 x)}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow 0}}}=\lim _{x \rightarrow 0} \frac{e^{x}+x e^{x}}{3 \sec ^{2}(3 x)}=\frac{1}{3}
$$

3.7.4.19. Solution. $\lim _{x \rightarrow 0} \sin ^{2} x=0$, and $\lim _{x \rightarrow 0} \frac{1}{x^{2}}=\infty$, so we have the form $0^{\infty}$. (Note that $\sin ^{2} x$ is positive, so our root is defined.) This is not an indeterminate form: $\lim _{x \rightarrow 0} \sqrt[x^{2}]{\sin ^{2} x}=0$.
3.7.4.20. Solution. $\lim _{x \rightarrow 0} \cos x=1$ and $\lim _{x \rightarrow 0} \frac{1}{x^{2}}=\infty$, so $\lim _{x \rightarrow 0}(\cos x)^{\frac{1}{x^{2}}}$ has the indeterminate form $1^{\infty}$. We want to use l'Hôpital, but we need to get our function into a fractional indeterminate form. So, we'll use a logarithm.

$$
\begin{aligned}
y: & =(\cos x)^{\frac{1}{x^{2}}} \\
\log y & =\log \left((\cos x)^{\frac{1}{x^{2}}}\right)=\frac{1}{x^{2}} \log (\cos x)=\frac{\log \cos x}{x^{2}} \\
\lim _{x \rightarrow 0} \log y & =\lim _{x \rightarrow 0} \underbrace{\frac{\log \cos x}{x^{2}}}_{\substack{\text { num } \rightarrow 0 \\
\text { den } \rightarrow 0}}=\lim _{x \rightarrow 0} \frac{\frac{-\sin x}{\cos x}}{2 x}=\lim _{x \rightarrow 0} \underbrace{\frac{-\tan x}{2 x}}_{\substack{\text { num } \rightarrow 0 \\
\text { den } \rightarrow 0}} \\
& =\lim _{x \rightarrow 0} \frac{-\sec ^{2} x}{2}=\lim _{x \rightarrow 0} \frac{-1}{2 \cos ^{2} x}=-\frac{1}{2}
\end{aligned}
$$

Therefore, $\lim _{x \rightarrow 0} y=\lim _{x \rightarrow 0} e^{\log y}=e^{-1 / 2}=\frac{1}{\sqrt{e}}$

### 3.7.4.21. Solution.

- Solution 1

$$
\begin{aligned}
y: & =e^{x \log x}=\left(e^{x}\right)^{\log x} \\
\lim _{x \rightarrow 0^{+}} y & =\lim _{x \rightarrow 0^{+}}\left(e^{x}\right)^{\log x}
\end{aligned}
$$

This has the form $1^{-\infty}=\frac{1}{1^{\infty}}$, and $1^{\infty}$ is an indeterminate form. We want to use l'Hôpital, but we need to get a different indeterminate form. So, we'll use logarithms.

$$
\begin{aligned}
\lim _{x \rightarrow 0^{+}} \log y & =\lim _{x \rightarrow 0^{+}} \log \left(\left(e^{x}\right)^{\log x}\right)=\lim _{x \rightarrow 0^{+}} \log x \log \left(e^{x}\right) \\
& =\lim _{x \rightarrow 0^{+}}(\log x) \cdot x
\end{aligned}
$$

This has the indeterminate form $0 \cdot \infty$, so we need one last adjustment before we can use l'Hôpital's Rule.

$$
\lim _{x \rightarrow 0^{+}}(\log x) \cdot x=\lim _{x \rightarrow 0^{+}} \frac{\log x}{\substack{\text { num } \\ \text { den } \rightarrow \infty}}\left|\frac{1}{x}\right| \quad \lim _{x \rightarrow 0^{+}} \frac{\frac{1}{x}}{\frac{1}{x^{2}}}=\lim _{x \rightarrow 0^{+}}-x=0
$$

Now, we can figure out what happens to our original function, $y$ :

$$
\lim _{x \rightarrow 0^{+}} y=\lim _{x \rightarrow 0^{+}} e^{\log y}=e^{0}=1
$$

## - Solution 2

$$
\begin{aligned}
y: & =e^{x \log x}=\left(e^{\log x}\right)^{x}=x^{x} \\
\lim _{x \rightarrow 0^{+}} y & =\lim _{x \rightarrow 0^{+}} x^{x}
\end{aligned}
$$

We have the indeterminate form $0^{0}$. We want to use l'Hôpital, but we need a different indeterminate form. So, we'll use logarithms.

$$
\lim _{x \rightarrow 0^{+}} \log y=\lim _{x \rightarrow 0^{+}} \log \left(x^{x}\right)=\lim _{x \rightarrow 0^{+}} x \log x
$$

Now we have the indeterminate form $0 \cdot \infty$, so we need one last adjustment before we can use l'Hôpital's Rule.

$$
\lim _{x \rightarrow 0^{+}} y=\lim _{x \rightarrow 0^{+}} \underbrace{\frac{\log x}{\frac{1}{x}}}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow-\infty}}=\lim _{x \rightarrow 0^{+}} \frac{\frac{1}{x}}{\frac{-1}{x^{2}}}=\lim _{x \rightarrow 0^{+}}-x=0
$$

Now, we can figure out what happens to our original function, $y$ :

$$
\lim _{x \rightarrow 0^{+}} y=\lim _{x \rightarrow 0^{+}} e^{\log y}=e^{0}=1
$$

3.7.4.22. Solution. First, note that the function exists near $0: x^{2}$ is positive, so $\log \left(x^{2}\right)$ exists; near $0, \log x^{2}$ is negative, so $-\log \left(x^{2}\right)$ is positive, so $\left[-\log \left(x^{2}\right)\right]^{x}$ exists even when $x$ is negative.
Since $\lim _{x \rightarrow 0}-\log \left(x^{2}\right)=\infty$ and $\lim _{x \rightarrow 0} x=0$, we have the indeterminate form $\infty^{0}$. We need l'Hôpital, but we need to manipulate our function into an appropriate form. We do this using logarithms.

$$
\begin{aligned}
& y:=\left[-\log \left(x^{2}\right)\right]^{x} \\
& \log y=\log \left(\left[-\log \left(x^{2}\right)\right]^{x}\right)=\underbrace{x}_{\rightarrow 0} \cdot \underbrace{\log (\underbrace{-\log \left(x^{2}\right)}_{\rightarrow \infty})}_{\rightarrow \infty}
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{\log \left(-\log \left(x^{2}\right)\right)}{\frac{1}{x}} \\
\lim _{x \rightarrow 0} \log y & =\lim _{x \rightarrow 0} \underbrace{\frac{\log \left(-\log \left(x^{2}\right)\right)}{\frac{1}{x}}}_{\substack{\text { num } \rightarrow \infty \\
\text { den } \rightarrow \pm \infty}}=\lim _{x \rightarrow 0} \frac{\frac{-\frac{2}{x}}{-\frac{\log \left(x^{2}\right)}{\frac{1}{x^{2}}}}}{\frac{-1}{x^{2}}}=\lim _{x \rightarrow 0} \underbrace{\log \left(x^{2}\right)}_{\substack{\text { num } \rightarrow 0 \\
\text { den } \rightarrow-\infty}}
\end{aligned}=0
$$

Now, we're ready to figure out our original limit.

$$
\lim _{x \rightarrow 0} y=\lim _{x \rightarrow 0} e^{\log y}=e^{0}=1
$$

3.7.4.23. *. Solution. Both the numerator and denominator converge to 0 as $x \rightarrow 0$. So, by l'Hôpital,

$$
\lim _{x \rightarrow 0} \underbrace{\frac{1+c x-\cos x}{e^{x^{2}}-1}}_{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow 0}}=\lim _{x \rightarrow 0} \frac{c+\sin x}{2 x e^{x^{2}}}
$$

The new denominator still converges to 0 as $x \rightarrow 0$. For the limit to exist, the same must be true for the new numerator. This tells us that if $c \neq 0$, the limit does not exist. We should check whether the limit exists when $c=0$. Using l'Hôpital:

$$
\lim _{x \rightarrow 0} \frac{\sin x}{\substack{\text { num } \rightarrow 0 \\ \text { den } \rightarrow 0}} \frac{2 x e^{x^{2}}}{}=\lim _{x \rightarrow 0} \frac{\cos x}{e^{x^{2}}\left(4 x^{2}+2\right)}=\frac{1}{1(0+2)}=\frac{1}{2}
$$

So, the limit exists when $c=0$.
3.7.4.24. *. Solution. The first thing we notice is, regardless of $k$, when we plug in $x=0$ both numerator and denominator become zero. Let's use this fact, and apply l'Hôpital's Rule.

$$
\begin{aligned}
\lim _{x \rightarrow 0} \underbrace{x^{k}}_{\substack{\text { num } \rightarrow 0 \\
\text { den } \rightarrow 0}} & =\lim _{x \rightarrow 0} \frac{2 k x \cos \left(x^{2}\right) e^{k \sin \left(x^{2}\right)}-4 x}{4 x^{3}} \\
& =\lim _{x \rightarrow 0} \frac{2 k \cos \left(x^{2}\right) e^{k \sin \left(x^{2}\right)}-4}{4 x^{2}}
\end{aligned}
$$

When we plug in $x=0$, the denominator becomes 0 , and the numerator becomes $2 k-4$. So, we'll need some cases, because the behaviour of the limit depends on $k$. For $k=2$ :

$$
\begin{aligned}
& \lim _{x \rightarrow 0} \frac{2 k \cos \left(x^{2}\right) e^{k \sin \left(x^{2}\right)}-4}{4 x^{2}}=\lim _{x \rightarrow 0} \underbrace{\frac{4 \cos ^{2}\left(x^{2}\right) e^{2 \sin \left(x^{2}\right)}-4}{4 x^{2}}}_{\substack{\text { num } \rightarrow 0 \\
\text { den } \rightarrow 0}} \\
& \quad=\lim _{x \rightarrow 0} \frac{-8 x \sin \left(x^{2}\right) e^{2 \sin \left(x^{2}\right)}+16 x \cos ^{2}\left(x^{2}\right) e^{2 \sin \left(x^{2}\right)}}{8 x}
\end{aligned}
$$

$$
\begin{aligned}
& =\lim _{x \rightarrow 0}\left[-\sin \left(x^{2}\right) e^{2 \sin \left(x^{2}\right)}+2 \cos ^{2}\left(x^{2}\right) e^{2 \sin \left(x^{2}\right)}\right] \\
& =2
\end{aligned}
$$

For $k>2$, the numerator goes to $2 k-4$, which is a positive constant, while the denominator goes to 0 from the right, so:

$$
\lim _{x \rightarrow 0} \frac{2 k \cos \left(x^{2}\right) e^{k \sin \left(x^{2}\right)}-4}{4 x^{2}}=\infty
$$

For $k<2$, the numerator goes to $2 k-4$, which is a negative constant, while the denominator goes to 0 from the right, so:

$$
\lim _{x \rightarrow 0} \frac{2 k \cos \left(x^{2}\right) e^{k \sin \left(x^{2}\right)}-4}{4 x^{2}}=-\infty
$$

## Exercises - Stage 3

### 3.7.4.25. Solution.

- We want to find the limit as $n$ goes to infinity of the percentage error, $\lim _{n \rightarrow \infty} 100 \frac{|S(n)-A(n)|}{|S(n)|}$. Since $A(n)$ is a nicer function than $S(n)$, let's simplify: $\lim _{n \rightarrow \infty} 100 \frac{|S(n)-A(n)|}{|S(n)|}=100\left|1-\lim _{n \rightarrow \infty} \frac{A(n)}{S(n)}\right|$.
We figure out this limit the natural way:

$$
\begin{aligned}
100\left|1-\lim _{n \rightarrow \infty} \frac{A(n)}{S(n)}\right| & =100|1-\lim _{n \rightarrow \infty} \underbrace{\frac{5 n^{4}}{5 n^{4}-13 n^{3}-4 n+\log (n)}}_{\substack{\text { num } \rightarrow \infty \\
\text { den } \rightarrow \infty}}| \\
& =100\left|1-\lim _{n \rightarrow \infty} \frac{20 n^{3}}{20 n^{3}-39 n^{2}-4+\frac{1}{n}}\right| \\
& =100\left|1-\lim _{n \rightarrow \infty} \frac{n^{3}}{n^{3}} \cdot \frac{20}{20-\frac{39}{n}-\frac{4}{n^{3}}+\frac{1}{n^{4}}}\right| \\
& =100|1-1|=0
\end{aligned}
$$

So, as $n$ gets larger and larger, the relative error in the approximation gets closer and closer to 0 .

- Now, let's look at the absolute error.

$$
\lim _{n \rightarrow \infty}|S(n)-A(n)|=\lim _{n \rightarrow \infty}\left|-13 n^{3}-4 n+\log n\right|=\infty
$$

So although the error gets small relative to the giant numbers we're talking about, the absolute error grows without bound.

## 4 - Towards Integral Calculus

## 4.1 • Introduction to Antiderivatives

### 4.1.2 • Exercises

## Exercises - Stage 1

4.1.2.1. Solution. An antiderivative of $f^{\prime}(x)$ is a function whose derivative is $f^{\prime}(x)$. Our original function $f(x)$ has this property, so $f(x)$ is an antiderivative of $f^{\prime}(x)$, but it's not the most general. We can add a constant to $f(x)$ without affecting its derivative. The most general antiderivative of $f^{\prime}(x)$ is $f(x)+C$, where $C$ is any constant.
4.1.2.2. Solution. Notice $f(x)$ is nonnegative for an interval covering the left part of the graph, and negative on the right part of the graph. That means its antiderivative is increasing for the left interval, then decreasing for the right interval. This applies to $A(x)$ and $C(x)$, but not $B(x)$.
There are only three points where $A(x)$ has a horizontal tangent line: at its global maximum and the endpoints of the interval shown. By contrast, $C(x)$ has a horizontal tangent line in four places: at its global maximum, at its inflection point, and at the endpoints of the interval shown. Since $f(x)=0$ four times (and these line up with the horizontal portions of $C(x))$ we conclude $C(x)$ is the antiderivative of $f(x)$.

## Exercises - Stage 2

4.1.2.3. Solution. For any constant $n \neq-1$, an antiderivative of $x^{n}$ is $\frac{1}{n+1} x^{n+1}$.

$$
\begin{aligned}
F^{\prime}(x) & =3 x^{2}+5 x^{4}+10 x-9 \\
F(x) & =3\left(\frac{1}{3}\right) x^{3}+5\left(\frac{1}{5}\right) x^{5}+10\left(\frac{1}{2}\right) x^{2}-9\left(\frac{1}{1}\right) x^{1}+C \\
& =x^{3}+x^{5}+5 x^{2}-9 x+C
\end{aligned}
$$

Remark: we can always check by differentiating:

$$
\begin{aligned}
F^{\prime}(x) & =\frac{\mathrm{d}}{\mathrm{~d} x}\left\{x^{3}+x^{5}+5 x^{2}-9 x+C\right\} \\
& =3 x^{2}+5 x^{4}+10 x-9 \\
& =f(x)
\end{aligned}
$$

so $F(x)$ is indeed an antiderivative of $f(x)$.

### 4.1.2.4. Solution.

$$
\begin{aligned}
F^{\prime}(x) & =\frac{3}{5} x^{7}-18 x^{4}+x \\
F(x) & =\left(\frac{3}{5}\right)\left(\frac{1}{8}\right) x^{8}-18\left(\frac{1}{5}\right) x^{5}+\frac{1}{2} x^{2}+C
\end{aligned}
$$

$$
=\frac{3}{40} x^{8}-\frac{18}{5} x^{5}+\frac{1}{2} x^{2}+C
$$

4.1.2.5. Solution. For any constant $n \neq 1$, an antiderivative of $x^{n}$ is $\frac{1}{n+1} x^{n+1}$. The constant $n$ does not have to be an integer.

$$
\begin{aligned}
F^{\prime}(x) & =4 \sqrt[3]{x}-\frac{9}{2 x^{2.7}} \\
& =4 x^{\frac{1}{3}}-\frac{9}{2} x^{-2.7} \\
F(x) & =4\left(\frac{1}{\frac{1}{3}+1}\right) x^{\left(\frac{1}{3}+1\right)}-\left(\frac{9}{2}\right)\left(\frac{1}{-2.7+1}\right) x^{(-2.7+1)}+C \\
& =4\left(\frac{3}{4}\right) x^{\frac{4}{3}}-\left(\frac{9}{2}\right)\left(\frac{10}{-17}\right) x^{-1.7}+C \\
& =3 x^{\frac{4}{3}}+\frac{45}{17 x^{1.7}}+C
\end{aligned}
$$

### 4.1.2.6. Solution.

- Solution 1: We can re-write $f(x)$ to make it a power of $x$.

$$
\begin{aligned}
F^{\prime}(x) & =\frac{1}{7} x^{-\frac{1}{2}} \\
F(x) & =\left(\frac{1}{7}\right)\left(\frac{1}{-\frac{1}{2}+1}\right) x^{\left(-\frac{1}{2}+1\right)}+C \\
& =\left(\frac{1}{7}\right)(2) x^{\frac{1}{2}}+C \\
& =\frac{2}{7} \sqrt{x}+C
\end{aligned}
$$

- Solution 2: We notice that $\frac{1}{7 \sqrt{x}}$ looks a lot like $\frac{1}{2 \sqrt{x}}$, which is the derivative of $\sqrt{x}$. So:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{\sqrt{x}\} & =\frac{1}{2 \sqrt{x}} \\
\frac{\mathrm{~d}}{\mathrm{~d} x}\left\{\frac{2}{7} \sqrt{x}\right\} & =\left(\frac{2}{7}\right) \frac{1}{2 \sqrt{x}}=f(x)
\end{aligned}
$$

So, an antiderivative of $f(x)$ is $\frac{2}{7} \sqrt{x}$. Then the most general antiderivative is $F(x)=\frac{2}{7} \sqrt{x}+C$.
4.1.2.7. Solution. We recall $\frac{\mathrm{d}}{\mathrm{d} x} e^{x}=e^{x}$. That is, $e^{x}$ is its own antiderivative.

So, a first guess for the antiderivative of $f(x)$ might be itself.

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{5 x+11}\right\}=5 e^{5 x+11}
$$

This isn't exactly right, so we modify it by multiplying by a constant.

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{1}{5} e^{5 x+11}\right\}=e^{5 x+11}
$$

This tells us that $\frac{1}{5} e^{5 x+11}$ is an antiderivative of $e^{5 x+11}$. Therefore, the most general antiderivative of $e^{5 x+11}$ is $F(x)=\frac{1}{5} e^{5 x+11}+C$.
4.1.2.8. Solution. We know the derivatives of sine and cosine. We'll work from there to build a function whose derivative is $f(x)$. We'll start by finding an antiderivative of $7 \cos (13 x)$.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{\sin x\} & =\cos x \\
\frac{\mathrm{~d}}{\mathrm{~d} x}\{\sin (13 x)\} & =13 \cos (13 x) \\
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{1}{13} \sin (13 x)\right\} & =\frac{13}{13} \cos (13 x)=\cos (13 x) \\
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{7}{13} \sin (13 x)\right\} & =7 \cos (13 x)
\end{aligned}
$$

So, an antiderivative of $7 \cos (13 x)$ is $\frac{7}{13} \sin (13 x)$.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{\cos x\} & =-\sin x \\
\frac{\mathrm{~d}}{\mathrm{~d} x}\{-\cos x\} & =\sin x \\
\frac{\mathrm{~d}}{\mathrm{~d} x}\{-\cos (5 x)\} & =5 \sin (5 x) \\
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{-\frac{3}{5} \cos (5 x)\right\} & =\left(\frac{3}{5}\right) 5 \sin (5 x)=3 \sin (5 x)
\end{aligned}
$$

So, an antiderivative of $3 \sin (5 x)$ is $-\frac{3}{5} \cos (5 x)$.
The most general antiderivative of $3 \sin (5 x)+7 \cos (13 x)$ is

$$
F(x)=-\frac{3}{5} \cos (5 x)+\frac{7}{13} \sin (13 x)+C .
$$

4.1.2.9. Solution. We know the derivative of $\tan x$ is $\sec ^{2} x$. Modifying this
slightly, we see (using the chain rule)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{\tan (x+1)\} & =\sec ^{2}(x+1) \cdot \frac{\mathrm{d}}{\mathrm{~d} x}\{x+1\} \\
& =\sec ^{2}(x+1)
\end{aligned}
$$

So, $\tan (x+1)$ is an antiderivative of $\sec ^{2}(x+1)$. Therefore, the most general antiderivative of $\sec ^{2}(x+1)$ is $F(x)=\tan (x+1)+C$.
4.1.2.10. Solution. We note that $f(x)$ looks similar to $\frac{1}{x}$.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\{\log |x|\} & =\frac{1}{x} \\
\frac{\mathrm{~d}}{\mathrm{~d} x}\{\log |x+2|\} & =\frac{1}{x+2}
\end{aligned}
$$

The most general antiderivative of $f(x)$ is $F(x)=\log |x+2|+C$.
4.1.2.11. Solution. Our function $f(x)$ bears some resemblance to the derivative of arcsine, $\frac{1}{\sqrt{1-x^{2}}}$ :

$$
\begin{aligned}
f(x)=\frac{7}{\sqrt{3-3 x^{2}}} & =\frac{7}{\sqrt{3}}\left(\frac{1}{\sqrt{1-x^{2}}}\right) \\
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{7}{\sqrt{3}} \arcsin (x)\right\} & =\frac{7}{\sqrt{3}}\left(\frac{1}{\sqrt{1-x^{2}}}\right)=f(x)
\end{aligned}
$$

So, the most general antiderivative of $f(x)$ is $F(x)=\frac{7}{\sqrt{3}} \arcsin (x)+C$.
4.1.2.12. Solution. We notice that $f(x)$ looks similar to the derivative of the arctangent function, $\frac{1}{1+x^{2}}$.

$$
f(x)=\frac{1}{1+25 x^{2}}=\frac{1}{1+(5 x)^{2}}
$$

This gives us a first guess for our antiderivative: perhaps $\arctan (5 x)$ will work. We test it by differentiating, making sure we don't forget the chain rule.

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\{\arctan (5 x)\}=\frac{1}{1+(5 x)^{2}} \cdot 5
$$

We're close to $f(x)$, but we've multiplied by 5 . That's easy to take care of: we can divide our guess by 5 .

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{1}{5} \arctan (5 x)\right\} & =\frac{1}{5}\left(\frac{1}{1+(5 x)^{2}}\right) \cdot 5 \\
& =\frac{1}{1+25 x^{2}}=f(x)
\end{aligned}
$$

So, the most general antiderivative of $f(x)$ is $F(x)=\frac{1}{5} \arctan (5 x)+C$.
4.1.2.13. Solution. First, let's find the antiderivative of $f^{\prime}(x)$. It's a polynomial, so we can use the observation from the text that an antiderivative of $x^{n}$, for any constant $n \neq 1$, is $\frac{1}{n+1} x^{n+1}$. Remember that the most general antiderivative will have an added constant.

$$
\begin{aligned}
f(x) & =\frac{3}{3} x^{3}-\frac{9}{2} x^{2}+4 x+C \\
& =x^{3}-\frac{9}{2} x^{2}+4 x+C
\end{aligned}
$$

Use the fact that $f(1)=10$ to solve for $C$.

$$
\begin{aligned}
10 & =1-\frac{9}{2}+4+C \\
C & =\frac{19}{2}
\end{aligned}
$$

All together,

$$
f(x)=x^{3}-\frac{9}{2} x^{2}+4 x+\frac{19}{2} .
$$

4.1.2.14. Solution. First, let's find the antiderivative of $f^{\prime}(x)$. We know that one antiderivative of $\cos (x)$ is $\sin x$. We might guess that an antiderivative of $\cos (2 x)$ is $\sin (2 x)$. Check by differentiating:

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\{\sin (2 x)\}=2 \cos (2 x)
$$

This is close to $f^{\prime}(x)$, but we need to divide by 2 .

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{1}{2} \sin (2 x)\right\}=\cos (2 x)
$$

So, $f(x)=\frac{1}{2} \sin (2 x)+C$ for some constant $C$. We can find $C$ using the given information $f(\pi)=\pi$.

$$
\begin{aligned}
\pi=f(\pi) & =\frac{1}{2} \sin (2 \pi)+C \\
\pi & =C
\end{aligned}
$$

Therefore, $f(x)=\frac{1}{2} \sin (2 x)+\pi$.
4.1.2.15. Solution. Looking at the table in the notes, we see the antiderivative of $\frac{1}{x}$ is $f(x)=\log |x|+C$. The given information $f(-1)=0$ lets us find $C$ :

$$
0=f(-1)=\log |-1|+C
$$

$$
\begin{aligned}
& 0=\log (1)+C \\
& 0=C
\end{aligned}
$$

So, $f(x)=\log |x|$.
Remark: it is true that $\log x$ is an antiderivative of $\frac{1}{x}$, since the derivative of $\log x$ is $\frac{1}{x}$. However, $\log x$ is only defined for positive values of $x$. Since the given information tells you that $f(x)$ is defined when $x=-1$, you need to use a more general antiderivative of $\frac{1}{x}: \log |x|+C$.
4.1.2.16. Solution. An antiderivative of 1 is $x$, and an antiderivative of $\frac{1}{\sqrt{1-x^{2}}}$ is $\arcsin (x)$. So, $f(x)=\arcsin x+x+C$. The given information lets us find $C$.

$$
\begin{aligned}
-\frac{\pi}{2}=f(1) & =\arcsin (1)+1+C \\
-\frac{\pi}{2} & =\frac{\pi}{2}+1+C \\
C & =-\pi-1
\end{aligned}
$$

So, $f(x)=\arcsin x+x-\pi-1$.
4.1.2.17. Solution. If $P(t)$ is the population at time $t$, then the information given in the problem is $P^{\prime}(t)=100 e^{2 t}$. Antidifferentiating, we see $P(t)=50 e^{2 t}+C$, where $C$ is some constant. We want to know for what value of $t$ we get $P(t)=P(0)+300$.

$$
\begin{aligned}
P(t) & =P(0)+300 \\
50 e^{2 t}+C & =50 e^{0}+C+300 \\
50 e^{2 t} & =350 \\
e^{2 t} & =7 \\
2 t & =\log (7) \\
t & =\frac{1}{2} \log (7)
\end{aligned}
$$

It takes $\frac{1}{2} \log 7$ hours (about 58 minutes) for the initial colony to increase by 300 individuals.
4.1.2.18. Solution. If $A(t)$ is the amount of money in your account at time $t$, then the given information is

$$
A^{\prime}(t)=1500 e^{\frac{t}{50}}
$$

Antidifferentiating,

$$
A(t)=75000 e^{\frac{t}{50}}+C
$$

for some constant $C$.
That is, at time $t$, the amount of money in your bank account is $75000 e^{\frac{t}{50}}+C$ dollars, for some constant $C$.
4.1.2.19. Solution. Let $P(t)$ be the amount of power your house has used since time $t=0$. If $t$ is measured in hours, and $P(t)$ in kWh (kilowatt-hours), then $P^{\prime}(t)$ is the rate at which your house is consuming power, in kW . So, the given information is that

$$
P^{\prime}(t)=0.5 \sin \left(\frac{\pi}{24} t\right)+0.25
$$

Antidifferentiating,

$$
P(t)=-\frac{12}{\pi} \cos \left(\frac{\pi}{24} t\right)+0.25 t+C
$$

Since $P(0)$ is the amount of energy consumed after 0 hours, $P(0)=0$, so

$$
\begin{aligned}
0=P(0) & =-\frac{12}{\pi}+C \\
C & =\frac{12}{\pi} \\
P(t) & =\frac{12}{\pi}\left[1-\cos \left(\frac{\pi}{24} t\right)\right]+0.25 t
\end{aligned}
$$

After 24 hours, your energy consumed is

$$
\begin{aligned}
P(24) & =\frac{12}{\pi}[1-\cos (\pi)]+0.25(24) \\
& =\frac{24}{\pi}+6 \approx 13.6 \mathrm{kWh}
\end{aligned}
$$

## Exercises - Stage 3

4.1.2.20. *. Solution. We differentiate $f(x)$ and $g(x)$ using the chain rule. Recall $\frac{\mathrm{d}}{\mathrm{d} x}\{\arcsin (x)\}=\frac{1}{\sqrt{1-x^{2}}}$.

$$
\begin{aligned}
f^{\prime}(x) & =\frac{2}{\sqrt{1-\sqrt{x}^{2}}} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x} \sqrt{x} \\
& =\frac{2}{\sqrt{1-x}} \cdot \frac{1}{2 \sqrt{x}} \\
& =\frac{1}{\sqrt{x-x^{2}}} \\
g^{\prime}(x) & =\frac{1}{\sqrt{1-(2 x-1)^{2}}} \cdot \frac{\mathrm{~d}}{\mathrm{~d} x}\{2 x-1\} \\
& =\frac{2}{\sqrt{1-4 x^{2}+4 x-1}} \\
& =\frac{1}{\sqrt{x-x^{2}}}
\end{aligned}
$$

So, the derivative of $f(x)-g(x)$ is 0 , which implies that $f(x)$ and $g(x)$ differ by a
constant-perhaps a surprising result!
Remark: we don't need calculus to show that $f(x)$ and $g(x)$ only differ by a constant. Define $\theta=\sin ^{-1} \sqrt{x}$, so that $\sin \theta=\sqrt{x}$ and $f(x)=2 \theta$, and then

$$
\begin{aligned}
\sin \left[f(x)-\frac{\pi}{2}\right] & =-\cos f(x)=-\cos 2 \theta=-\left[1-2 \sin ^{2} \theta\right] \\
& =2 \sin ^{2} \theta-1=2 x-1
\end{aligned}
$$

Our goal is to take the arcsine of the first and last expressions and conclude $f(x)-$ $\frac{\pi}{2}=\arcsin (2 x-1)=g(x)$. However, before we can say $\arcsin \left(\sin \left[f(x)-\frac{\pi}{2}\right]\right)=$ $f(x)-\frac{\pi}{2}$, we have to check that $-\frac{\pi}{2} \leq f(x)-\frac{\pi}{2} \leq \frac{\pi}{2}$. That is, we need to show that $0 \leq f(x) \leq \pi$.
As $0 \leq \theta \leq \frac{\pi}{2}$ (since $\sin ^{-1}$ always takes values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and since $\sqrt{x} \geq 0$ ), we have that $0 \leq f(x)=2 \theta \leq \pi$. So,

$$
f(x)-\frac{\pi}{2}=\arcsin \left(\sin \left[f(x)-\frac{\pi}{2}\right]\right)=\arcsin (2 x+1)=g(x)
$$

4.1.2.21. Solution. The derivative of $\sin (2 x)$ (which occurs in the second term) is $2 \cos (2 x)$ (which occurs in the first term). Similarly, the derivative of $\cos (3 x)$ (which occurs in the first term) is $-3 \sin (3 x)$ (which occurs in the second term). So, $f(x)$ seems to have come from the product rule.

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\{\sin (2 x) \cos (3 x)\}=2 \cos (2 x) \cos (3 x)-3 \sin (2 x) \sin (3 x)
$$

Then the antiderivative of $f(x)$ is $F(x)=\sin (2 x) \cos (3 x)+C$.
4.1.2.22. Solution. The function $f(x)$ looks like perhaps it came from the quotient rule. Recall

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{u(x)}{v(x)}\right\}=\frac{v(x) u^{\prime}(x)-u(x) v^{\prime}(x)}{(v(x))^{2}}
$$

Then, since the denominator of $f(x)$ is $\left(x^{2}+1\right)^{2}$, we might guess $v(x)=\left(x^{2}+1\right)$. That leaves $u(x)=e^{x}$.

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{e^{x}}{x^{2}+1}\right\}=\frac{\left(x^{2}+1\right) e^{x}-e^{x}(2 x)}{\left(x^{2}+1\right)^{2}}=f(x)
$$

So, the antiderivative of $f(x)$ is $F(x)=\frac{e^{x}}{x^{2}+1}+C$.
4.1.2.23. Solution. We know that $e^{x}$ is its own antiderivative. The derivative of $e^{g(x)}$, for some function $g(x)$, is $g^{\prime}(x) e^{g(x)}$. Since $3 x^{2}$ is the derivative of $x^{3}, f(x)$ fits this pattern. We guess the antiderivative of $f(x)$ is $F(x)=e^{x^{3}}+C$.
We check by differentiating.

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{e^{x^{3}}+C\right\}=3 x^{2} e^{x^{3}}=f(x)
$$

Indeed, the antiderivative of $f(x)$ is $F(x)=e^{x^{3}}+C$.
4.1.2.24. Solution. We know the antiderivative of $\sin x$ is $-\cos x$. Since $\sin \left(x^{2}\right)$ appears in our function, let's investigate the derivative of $-\cos \left(x^{2}\right)$.

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{-\cos \left(x^{2}\right)\right\}=\sin \left(x^{2}\right) \cdot 2 x
$$

This differs from $f(x)$ only by a constant multiple.

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{-\frac{5}{2} \cos \left(x^{2}\right)\right\}=\frac{5}{2} \sin \left(x^{2}\right) \cdot 2 x=5 x \sin \left(x^{2}\right)=f(x)
$$

So, the antiderivative of $f(x)$ is $F(x)=-\frac{5}{2} \cos \left(x^{2}\right)+C$
Remark: as in Question 4.1.2.23, our function $f(x)$ involved some function $g(x)$ as well as $g^{\prime}(x)$. This pattern is the basis of an important method of antidifferentiation, called the Substitution Rule.
4.1.2.25. Solution. For any $x$ in the domain of $\log (x), e^{\log x}=x$. So, $f(x)=x$ for every $x$ in its domain. Then its antiderivative is $F(x)=\frac{1}{2} x^{2}+C$.
4.1.2.26. Solution. As in Question 4.1.2.11, we notice that our function is similar to $\frac{1}{\sqrt{1-x^{2}}}$, but in this case it doesn't factor quite as nicely.

$$
\begin{aligned}
f(x)=\frac{7}{\sqrt{3-x^{2}}} & =\frac{7}{\sqrt{3\left(1-\frac{x^{2}}{3}\right)}} \\
& =\frac{7}{\sqrt{3}}\left(\frac{1}{\sqrt{1-\frac{x^{2}}{3}}}\right)
\end{aligned}
$$

What we really want under that square root, instead of $\frac{x^{2}}{3}$, is simply $x^{2}$. We can get close: we can get something squared.

$$
f(x)=\frac{7}{\sqrt{3}}\left(\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^{2}}}\right)
$$

Now, the thing that's squared isn't $x$, it's $\frac{x}{\sqrt{3}}$. This gives us a first guess for an antiderivative: perhaps $F(x)=\frac{7}{\sqrt{3}} \arcsin \left(\frac{x}{\sqrt{3}}\right)$ will work. Let's try it! Remember to use the chain rule when you differentiate.

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{\frac{7}{\sqrt{3}} \arcsin \left(\frac{x}{\sqrt{3}}\right)\right\}=\frac{7}{\sqrt{3}}\left(\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^{2}}}\right) \cdot \frac{1}{\sqrt{3}}
$$

We're very close! We're only off by a constant, and those are easy to fix. We're dividing by $\sqrt{3}$ when we differentiate, so let's multiply our function by $\sqrt{3}$.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left\{7 \arcsin \left(\frac{x}{\sqrt{3}}\right)\right\} & =7\left(\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^{2}}}\right) \cdot \frac{1}{\sqrt{3}} \\
& =\frac{7}{\sqrt{3} \sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^{2}}} \\
& =\frac{7}{\sqrt{3-x^{2}}}=f(x)
\end{aligned}
$$

Therefore, the most general antiderivative of $f(x)$ is $F(x)=7 \arcsin \left(\frac{x}{\sqrt{3}}\right)+C$.
4.1.2.27. Solution. Following Example 4.1.7, let $V(H)$ be the volume of the solid formed by rotating the segment of the parabola from $x=-H$ to $x=H$. Our plan is to evaluate $V^{\prime}(H)=\lim _{h \rightarrow H} \frac{V(H)-V(h)}{H-h}$ and then antidifferentiate $V^{\prime}(H)$ to find $V(H)$. Since we don't know $V(H)-V(h)$ (yet) we first find upper and lower bounds on it when $h<H$.
For a constant $h<H, V(H)-V(h)$ is the volume of the solid inside the larger object (with length $2 H$ ) and outside the smaller object (with length $2 h$, shown below in blue). There are two regions inside the larger object and outside the smaller.


One of the regions inside the larger object and outside the smaller (blue) object looks like this:


Remember we formed this solid by rotating the curve $y=x^{2}+1$ about the $x$-axis. So, the cross-section of this solid is a circle, and the radius of the circle when we are $x$ units from the origin is $x^{2}+1$. So, the largest radius of the little shape shown above, which occurs at the right end, is $H^{2}+1$, and the smallest radius (at the left end) is $h^{2}+1$.
The volume of the shape shown above is less than a cylinder of radius $H^{2}+1$ and height $H-h$, and it is more than the volume of a cylinder of radius $h^{2}+1$ and height $H-h$. So, the volume of the shape shown above is between $(H-h) \pi\left(h^{2}+1\right)^{2}$ and $(H-h) \pi\left(H^{2}+1\right)^{2}$ cubic units.
Recall that the volume inside the object of length $2 H$ and outside the object of length $2 h$ consists of two copies of the shape shown above. Therefore:

$$
2(H-h) \pi\left(h^{2}+1\right)^{2}<V(H)-V(h)<2(H-h) \pi\left(H^{2}+1\right)^{2}
$$

Now that we have upper and lower bounds for $V(H)-V(h)$, we can find $V^{\prime}(H)$.

$$
\left.\begin{array}{rl}
\frac{2(H-h) \pi\left(h^{2}+1\right)^{2}}{H-h} & <\quad \frac{V(H)-V(h)}{H-h}
\end{array}<\frac{2(H-h) \pi\left(H^{2}+1\right)^{2}}{H-h}, ~<\frac{V(H)-V(h)}{H-h}<2 \pi\left(H^{2}+1\right)^{2}\right)
$$

Since the limits on both ends are simply $2 \pi\left(H^{2}+1\right)^{2}$, by the Squeeze Theorem,

$$
V^{\prime}(H)=\lim _{h \rightarrow H} \frac{V(H)-V(h)}{H-h}=2 \pi\left(H^{2}+1\right)^{2}
$$

Now that we know $V^{\prime}(H)$, we antidifferentiate to find $V(H)$.

$$
V^{\prime}(H)=2 \pi\left(H^{4}+2 H^{2}+1\right)
$$

$$
V(H)=2 \pi\left(\frac{1}{5} H^{5}+\frac{2}{3} H^{3}+H\right)+C
$$

When $H=0$, there is no solid, so $V(0)=0$. Therefore,

$$
V(H)=2 \pi\left(\frac{1}{5} H^{5}+\frac{2}{3} H^{3}+H\right) .
$$


[^0]:    2 If you let 1 be a prime number then you have to treat $1 \times 2 \times 3$ and $2 \times 3$ as different factorisations of the number 6. This causes headaches for mathematicians, so they don't let 1 be prime.
    3 Some schools (and even some provinces!!) may use " $I$ " for integers, but this is extremely nonstandard and they really should use correct notation.

[^1]:    1 Perhaps one of the most famous experiments in all of physics is Galileo's leaning tower of Pisa experiment, in which he dropped two balls of different masses from the top of the tower and observed that the time taken to reach the ground was independent of their mass. This disproved Aristotle's assertion that heavier objects fall faster. It is quite likely that Galileo did not actually perform this experiment. Rather it was a thought-experiment. However a quick glance at Wikipedia will turn up some wonderful footage from the Apollo 15 mission showing a hammer and feather being dropped from equal height hitting the moon's surface at the same time. Finally, Galileo determined that the speed of falling objects increases at a constant rate, which is equivalent to the formula stated here, but it is unlikely that he wrote down an equation exactly as it is here.

[^2]:    1 We have used this term in an imprecise way, but it does have a precise mathematical meaning.
    2 Though it lies outside the scope of the course, you can find the formal $\epsilon-\delta$ proof of this result at the end of Section 1.7.

[^3]:    1 A quick google will turn up many articles on the development and history of calculus. Wikipedia has a good one.

[^4]:    $a \quad$ Also take a look at "logarithmic differentiation" in Section 2.10.

[^5]:    5 You really should. Look this up in Appendix A. 8 if you have forgotten.

[^6]:    $\uparrow \quad a \quad$ It's probably a good moment to go back and look at Example 2.2.10.

[^7]:    $a \quad$ Take another look at Appendix A.14.

[^8]:    1 Willard Libby, of Chicago University was awarded the Nobel Prize in Chemistry in 1960, for developing radiocarbon dating.
    2 A good question to ask yourself is "How can a scientist (who presumably doesn't live 60 centuries) measure this quantity?" One way exploits the little piece of calculus we are about to discuss.

[^9]:    4 We are using the fact that the logarithm is a continuous function and Theorem 1.6.10.

[^10]:    1 Plug "Fibonacci sequence in nature" into your search engine of choice.

