Announcements

- A8 Written + WeBWork - Nov 21
  Do a total of 20 questions.

- A8 questions
  - A8: 1 4 6 8 9
  - Extra Curve Sketching: 19 21 22 33 36

Today: Optimization in Physical Models
**Fermat’s Principle**  A ray of light will travel along a path that minimizes the time taken.

**Goal:** Using Fermat’s Principle we want to show that Snell’s Law holds: $\frac{\sin \Theta_i}{\sin \Theta_r} = \frac{C_a}{C_w}$ speed of light in air speed of light in water.
1. Re-draw with coordinates.

P \((X_p, Y_p)\)

\(\theta_i\)

\(Y_p\)

\(X_p\)

\(X_p - x\) < 0

So \(X_p - x < 0\)

Q \((X_q, Y_q)\)

\((x, 0)\)

\((X_q - x)\)

\((x, 0)\)

\((X_q, 0)\)

Clicker Check-in:

A - OK.

B - Bored.

C - Confused.

θ / \parallel equal angles

θ / \perp parallel
Total time

\[ T = \frac{lp}{Ca} + \frac{la}{Cw} \]

\[ \frac{lp}{Ca} \text{ has units } \frac{m}{m/\text{sec}} = \text{Sec.} \]

- \( lp = \text{ is the distance from } (X_p, Y_p) \text{ to } (X, 0) \).

\[ lp = \sqrt{(X_p - x)^2 + Y_p^2} \]

- \( la = \sqrt{(X_q - x)^2 + Y_q^2} \)

- \( T(x) = \frac{1}{Ca} \sqrt{(X_p - x)^2 + Y_p^2} + \frac{1}{Cw} \sqrt{(X_q - x)^2 + Y_q^2} \)

\[(Ca, X_p, Y_p \text{ etc. all constants.})\]
\[ O = T'(x) \]
\[ O = \frac{1}{C_a} \frac{-2(x_p-x)}{\sqrt{(x_p-x)^2 + y_p^2}} + \frac{1}{C_w} \frac{-2(x_q-x)}{\sqrt{(x_q-x)^2 + y_q^2}} \]

\[ \sin \Theta_i = \frac{(x_p-x)}{\sqrt{(x_p-x)^2 + y_p^2}} \]

\[ O = \frac{1}{C_a} \sin \Theta_i - \frac{1}{C_w} \sin \Theta_r \]

\[ \frac{1}{C_w} \sin \Theta_r = \frac{1}{C_a} \sin \Theta_i \]

\[ \frac{C_a}{C_w} = \frac{\sin \Theta_i}{\sin \Theta_r} \]

Snell's Law.
Calculus: Find the minimum of $T(x)$:

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$$

$$T'(x) = \frac{1}{Ca} \frac{-2(X_p-x)}{2\sqrt{(X_p-x)^2 + Y_p^2}} + \frac{1}{Cw} \frac{-2(X_q-x)}{2\sqrt{(X_q-x)^2 + Y_q^2}}$$

$T(x) \rightarrow +\infty$ as $x \rightarrow -\infty$ or $x \rightarrow +\infty$.

If $T'(x) = 0$ at only one $x$-value, then this $x$-value is a minimum because $T(x) \rightarrow +\infty$ as $x \rightarrow \pm\infty$.

(CLD Text: Thm. 3.5.17)
Models with Parameters: Example

(WebWork 6) The illumination of an object is proportional to the strength of the source and inversely proportional to the square of the distance of the object to the source.

\[ I(x) = \frac{S}{x^2} \]

\( I(x) \) = Illumination is proportional to strength & inversely prop. to the distance squared.
Two light sources: 1 m apart

Bright 3S

Dim S

Where in between, is the darkest spot?

Darkest spot will occur at the x-location of minimum intensity $I(x)$.

$I(x) = \frac{3S}{x^2} + \frac{S}{(1-x)^2}$. 
• Minimum of $I(x)$?

• Note that as $x \to 1$, $I(x) \to +\infty$.

(why: $I(x) = \frac{3S}{x^2} + \frac{S}{(1-x)^2}$)

as $x \to 0$, $I(x) \to +\infty$

because

• If $I'(x) = 0$ only once, then that C.P. is a minimum.

• Find $I'(x)$. Find where $I'(x) = 0$.

Hint: $I(x) = 3Sx^{-2} + S(1-x)^{-2}$

As $x \to 1$, $(1-x)^2 \to 0$
and so $\frac{S}{(1-x)^2} \to +\infty$.

(limitation of physical model).
\[
I'(x) = -6Sx^{-3} + 2S(1-x)^{-3} \cdot (-1)
\]
\[
\implies 0 = -6Sx^{-3} + 2S(1-x)^{-3}
\]
\[
\frac{6S}{x^3} = \frac{2S}{(1-x)^3}
\]
\[
(1-x)^3 \cdot 3 = x^3
\]
\[
(1-x)^{3^{1/3}} = x^{3^{1/3}}
\]
\[
3^{1/3} - 3^{1/3} x = x
\]
\[
3^{1/3} = \left(1 + 3^{1/3}\right)x
\]
\[
\frac{1}{(\frac{1}{3})^{3/2} + 1} = \frac{3^{1/3}}{1 + 3^{1/3}} = x
\]
MODELS WITH PARAMETERS

- Birthday is even:
  Find the maximum of
  \[ P(I) = \frac{cI}{I^2 + I + 1}, \quad c > 0 \]
  (photosynthesis as a function of light intensity)

- Birthday is odd:
  Find the maximum of
  \[ Y(N) = \frac{kN}{1 + N^2}, \quad k > 0 \]
  (yield as a function of nitrogen)
Summary + Reflection

• Used Fermat's Principle to derive Snell's Law.
  • Optimized total time (like in the boat-shark-beach problem).
  • Limit Argument + only one CP ⇒ existence of minimum.
• Optimization with the Inverse-Square Law.
  → See Workshop Q3, WebWork A8 Question 6.
• Optimization with models that have parameters.
  → treat the parameters as constants!

1. The thing I find the easiest about optimization problems is __________.

2. The thing I find the most difficult and I will practice the most is __________.