Announcements

- Written and WeBWorK 7 and Quiz 7 Nov 14 — I’ll post this later today
- Midterm Corrections Assignment due Nov 14
  - Quiz Questions TBA

Today

1. Curve Sketching Part I Quiz
2. Introduction to Optimization Problems — Xinyu Cheng
Midterm Chat

- *Experiential Learning Cycle:*
  1. An experience (e.g. midterm)
  2. What happened?
  3. Why did it happen?
  4. What next?
Midterm Chat

- Midterm Corrections Assignment (due Nov 14)
- Regrade requests only before Nov 14
- To your studying add:
  1. reflective practice (write study notes as you go! take-away messages)
  2. Study sessions?

Thu 9:30–11am, ORCH 3058
Thu 2–3:30pm ORCH 3072
Thu 6–7:30pm ORCH 4016
Sun 3:30–5pm ORCH 4072

The materials of the sessions are posted on this webpage, feel free to include this information in your message

http://blogs.ubc.ca/shborouchaki/teaching/math-180/
10.1. Intro to Optimization Problems

Review local maxima/minima

\[ f'(c) = 0 \rightarrow \text{critical point} \]

1st Derivative Tests

2nd Derivative Tests

Goal: Optimization Problems.

E.g. Xinyu's M.Sc. Thesis

\[ f(x) = x^2 e^{-x^2}, \quad x > 0. \]

\[ f'(x) = 2xe^{-x^2} + x^2 e^{-x^2} (-2x) = e^{-x^2} (2x - 2x^3). \]

\[ f'(c) = 0 \text{ when } 2c - 2c^3 = 0 \Rightarrow c = 1. \]

\[ 2c(1-c^2) = 0 \]

so \( c = 0, \pm 1 \),

but \( x > 0 \)

maximum value \( f(1) = 1^2 e^{-1^2} = 1 \cdot e^{-1} = 1/e. \)
Eq. A farmer has 200 ft to build a rectangular enclosure with that fencing provides for the maximal area. What dimensions provide the maximal area?

\[ 2x + 2y = 200 \quad \text{from the perimeter.} \]

\[ A = xy \]

\[ 2x + 2y = 200 \quad \Rightarrow \quad 2y = 200 - 2x \]

\[ y = 100 - x. \]

\[ A = x \cdot y = x(100 - x). \]

\[ A(x) = x(100 - x) = 100x - x^2. \]

\[ A'(x) = 100 - 2x \]

\[ A'(x) = 0 \quad \Rightarrow \quad c = 50. \quad \text{ok! But let's check to} \]

\[ \text{see if we've found a max or min.} \]
1st derivative test:

<table>
<thead>
<tr>
<th></th>
<th>&lt;50</th>
<th>50</th>
<th>&gt;50</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A')</td>
<td>+</td>
<td>0</td>
<td>-</td>
</tr>
</tbody>
</table>

\(A'' = (A')' = (100 - 2x)' = -2 < 0\).

Concave down \(\Rightarrow\) local maximum.

\(c = 50 \rightarrow\) optimal choice of \(x\)

\(\rightarrow\) corresponding \(y = 100 - x = 100 - 50 = 50\).

The optimal dimensions are \(x = 50\), \(y = 50\) and maximal area is \(50^2\).
Ex. Surface area of a can without top is $100 \pi \text{ cm}^2$. Can you arrange the can to maximise its volume?

Surface area $= \pi r^2 + 2\pi rh$.

$= 100\pi$.

Volume $= \pi r^2 h$.

- $100\pi = \pi r^2 + 2\pi rh$

  $h = \frac{100\pi - \pi r^2}{2\pi r}$.

- $V = \pi r^2 h = \pi r^2 \cdot \frac{(100\pi - \pi r^2)}{2\pi r}$.

  $= \frac{r (100\pi - \pi r^2)}{2}$

- $V(r) = \frac{r}{2} (100\pi - \pi r^2) = 50\pi r - \frac{\pi r^3}{2}$
\[ \sqrt{r} = 50\pi - \frac{3\pi c^2}{2} \]

- Solve \( \sqrt{c} = 0 \)

\[ 50\pi - \frac{3\pi c^2}{2} = 0 \cdot \frac{50 - 3c^2}{2} = 0 \]

\[ 50 = 3c^2 \]

\[ \frac{100}{3} = c^2 \]

\( c = \pm \frac{10}{\sqrt{3}} = \pm \sqrt{\frac{100}{3}} = c \)

So only need \( c = \sqrt{\frac{100}{3}} \) so that the radius is positive.

\[ \sqrt{3} \]

\[ \text{WHY DO WE IGNORE } c = -\sqrt{\frac{100}{3}}? \]

\[ V''(c) = -3\pi c \]

\( c = 0 \)

\[ V''(c) = -3\pi c < 0 \]

So concave down at \( \text{MAXIMUM} \): \( c = \sqrt{\frac{100}{3}} \)

\( \rightarrow c = \sqrt{\frac{100}{3}} \) is a local maximum.
Maximum volume:

\[ h = \frac{100\pi - \pi c^2}{2\pi c} = \frac{100 \pi - \pi \left(\sqrt{\frac{100}{3}}\right)^2}{2\pi \sqrt{\frac{100}{3}}} \]

\[ = \frac{100 - \left(\sqrt{\frac{100}{3}}\right)^2}{2\pi \sqrt{\frac{100}{3}}} \]

\[ V_{\text{max}} = \pi r^2 h \text{ with } r = \sqrt{\frac{100}{3}} \text{ and } h \uparrow. \]
Recipe for solving optimization problems.

1. Identify the key quantity \{object to be optimized\}, information unknown.
2. Set up eqn \{related to surface, area, shape, etc\}.
3. Draw a picture if needed.
4. Reduce variables.
5. Set up target function.
6. Find derivative → critical point.
7. Perform tests (1st & 2nd derivative test).
Exercise: The strength $S$ of a wooden beam is proportional to its cross-sectional area width $w$ and the square of its height $h$, i.e.

$$S = kwh^2$$

for some constant $k$. Given a circular log with diameter 12 inches, what sized beam can be cut from the log with max $S$?

\[ \begin{align*}
\text{Thales' Theorem} & \quad \text{a triangle inscribed in a semi-circle with one side the diameter is a right triangle.}
\end{align*} \]

\[ \begin{align*}
\Rightarrow w^2 + h^2 &= 12^2 \\
\Rightarrow h &= \sqrt{144-w^2}.
\end{align*} \]
\[ S = k \cdot w^2 \]

\[ \implies S'(w) = k \cdot 144 - 2kww' \]

\[ \implies S'(w) = 144k - 3kw^2 \]

\[ S'(w) = 0 \quad \text{when} \quad 144k = 3kw^2 \]

\[ \implies \frac{144}{3} = c^2 \]

\[ \implies c = \frac{12}{\sqrt{3}} \]

\[ c > 0 \quad \text{since} \quad c \text{ is the width} \]

\[ \therefore c = \frac{12}{\sqrt{3}} \]

\[ h^2 = 144 - \left( \frac{144}{3} \right)^2 = 144 - \frac{144}{3} \]

\[ \text{Later do a first/second derivative test to check.} \]

\[ \text{Reflection :} \]

\[ \text{Reflection :} \]