First Name: ____________________________  Last Name: ____________________________

Student-No: ____________________________  Section: ____________________________

Grade:

The remainder of this page has been left blank for your workings.
Very short answer questions

1. **2 marks** Each part is worth 1 mark. Please write your answers in the boxes.

(a) Scientists have isolated 16 grams of a strange radioactive element at 12AM. At 4AM only 4 grams of the element is left. What is the half-life of this strange new substance?

<table>
<thead>
<tr>
<th>Answer: 2 hours</th>
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</table>

**Solution:** Initially there is 16g and after 4 hours there is 4g. So the half-life is 2 hours.

(b) Consider a function, $h(x)$, whose fourth-degree Taylor polynomial around $x = 1$ is $2016 - 28(x - 1)^2 + 10(x - 1)^4$. What is $h'(1)$?

<table>
<thead>
<tr>
<th>Answer: 0</th>
</tr>
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</table>

**Solution:** Since $T_4(x) = h(1) + h'(1) \cdot (x - 1) + \frac{1}{2} h''(1) \cdot (x - 1)^2 + \frac{1}{6} h'''(1) \cdot (x - 1)^3 + \frac{1}{24} h^{(4)}(1) \cdot (x - 1)^4$, we must have that $h'(1) = 0$. 

Quiz 4.4: Page 2 of 4
Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

(a) Suppose a particle’s position is given by \( s(t) = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t + 2016 \). Over what time interval is the particle moving in the negative direction?

Answer: \( t \in (1, 2) \)

Solution: The velocity is given by

\[ v(t) = t^2 - 3t + 2 = (t - 1)(t - 2) \]

Hence the velocity is negative when \( 1 < t < 2 \).

(b) Estimate \( \tan^2(0.01) + 0.01 \) using a linear approximation.

Answer: 0.01

Solution: We use the function \( f(x) = \tan^2(x) + x \) and point \( a = 0 \) as the centre of our approximation since we know that

\[ f(a) = f(0) = \tan^2(0) + 0 = \left( \frac{\sin(0)}{\cos(0)} \right)^2 + 0 = 0 \]

We compute \( f'(x) = 2\tan(x)\sec^2(x) + 1 \)

\[ f'(0) = 1 \]

So, a linear approximation of \( \tan^2(0.01) + 0.01 \) is

\[ T_1(0.01) = f(0) + f'(0) \cdot 0.01 = 0.01 \]
Long answer question — you must show your work

3. 4 marks You place a cone-shaped bottle on the table. It is 6cm high. Its circular base (which is resting on the table) has radius 3cm and its volume is $18\pi \text{cm}^3$. You fill it with water at rate of $1 \text{cm}^3$ per second. What is the rate at which the depth of the water changes when it is 3cm deep?

HINT: Think about the shape of the air in bottle.

Solution: The hardest bit of this question is realising that cone is not the usual orientation in these related rates questions.

- Let $h$ be the depth of water in the cone, and $r$ be the radius of the water at that height.
- Additionally let $a = 6 - h$, which is the distance from the tip of the cone to the water-surface. It is also the height of the cone formed by the air in the bottle.
- Similar triangles shows that $a/r = 6/3$. So $r = a/2$.
- Now let $V$ be the volume of water and $A$ be the volume of air in the bottle

$$V = 18\pi - A \quad A = \frac{\pi}{3} ar^2 = \frac{\pi}{12} a^3 = \frac{\pi}{12} (6 - h)^3$$

- Now

$$\frac{dV}{dt} = -\frac{dA}{dt} = +\frac{\pi}{4} (6 - h)^2 \frac{dh}{dt}$$

- We know $h = 3$, $\frac{dV}{dt} = 1$, so

$$1 = \frac{\pi}{4} (6 - 3)^2 \frac{dh}{dt} = \frac{9\pi}{4} \frac{dh}{dt} \quad \frac{dh}{dt} = \frac{4}{9\pi}$$
First Name: ___________________________ Last Name: ___________________________

Student-No: ___________________________ Section: ___________________________

Grade:

The remainder of this page has been left blank for your workings.
Very short answer questions

1. [2 marks] Each part is worth 1 mark. Please write your answers in the boxes.

(a) Scientists have isolated 12 grams of a strange radioactive element at 1PM. At 7PM only 3 grams of the element is left. What is the half-life of this strange new substance?

Answer: 3 hours

Solution: Initially there is 12g and after 6 hours there is 3g. So the half-life is 3 hours.

(b) Consider a function, \( h(x) \), whose third-degree Maclaurin polynomial is \( 5 - x - 3x^2 + 4x^3 \). What is \( h''(0) \)?

Answer: −6

Solution: Since \( T_3(x) = h(0) + h'(0) \cdot x + \frac{1}{2} h''(0) \cdot x^2 + \frac{1}{6} h'''(0) \cdot x^3 \), we must have that \( h''(0) = -6 \).
Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.
   (a) Suppose a particle’s position is given by $s(t) = 2t^3 - 9t^2 + 12t$. Over what time interval is the particle moving in the positive direction?
   Answer: $t < 1$ or $t > 2$
   
   Solution: The velocity is given by
   
   $$v(t) = 6t^2 - 18t + 12 = 6(t^2 - 3t + 2) = 6(t - 2)(t - 1)$$
   
   Hence the velocity is positive when $t < 1$ or $t > 2$.

   (b) Estimate $\arctan(1.1)$ using a linear approximation. Leave your answer in terms of $\pi$.
   Answer: $\frac{\pi}{4} + \frac{1}{20}$
   
   Solution: We use the function $f(x) = \arctan x$ and point $a = 1$ as the centre of our approximation since we know that
   
   $$f(a) = f(1) = \arctan(1) = \frac{\pi}{4}$$
   
   We compute $f'(x) = \frac{1}{1+x^2}$
   
   $$f'(1) = \frac{1}{2}$$
   
   So, a linear approximation of $\arctan(1.1)$ is
   
   $$T_1(x) = f(1) + f'(1) \cdot (x - 1) = \frac{\pi}{4} + \frac{x - 1}{2}$$
   
   $$T_1(1.1) = \frac{\pi}{4} + \frac{0.1}{2} = \frac{\pi}{4} + \frac{1}{20}$$
Long answer question — you must show your work

3. 4 marks You place a cone-shaped bottle on the table. It is 12cm high and its circular base (which is resting on the table) has radius 4cm and its volume is $64\pi cm^3$. You fill it with water a rate of $3 cm^3$ per second. What is the rate at which the depth of the water changes when it is 5cm deep?

**HINT:** Think about the shape of the air in bottle.

**Solution:** The hardest bit of this question is realising that cone is not the usual orientation in these related rates questions.

- Let $h$ be the depth of water in the cone, and $r$ be the radius of the water at that height.
- Additionally let $a = 12 - h$, which is the distance from the tip of the cone to the water-surface. It is also the height of the cone formed by the air in the bottle.
- Similar triangles shows that $a/r = 12/4$. So $r = a/3$.
- Now let $V$ be the volume of water and $A$ be the volume of air in the bottle

$$V = \text{Constant} - A \quad A = \frac{\pi}{3}ar^2 = \frac{\pi}{27}a^3 = \frac{\pi}{27}(12 - h)^3$$

- Now

$$\frac{dV}{dt} = -\frac{dA}{dt} = +\frac{\pi}{9}(12 - h)^2 \frac{dh}{dt}$$

- We know $h = 5, \frac{dV}{dt} = 3$, so

$$3 = \frac{\pi}{9}(12 - 5)^2 \frac{dh}{dt}$$

$$= \frac{49\pi}{9} \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{27}{49\pi}$$
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Grade:

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Very short answer questions

1. 2 marks Each part is worth 1 mark. Please write your answers in the boxes.
   
   (a) Scientists have isolated 12 grams of a strange radioactive element at 2PM. At 10PM only 3 grams of the element is left. What is the half-life of this strange new substance?

   Answer: 4 hours

   Solution: Initially there is 12g and after 8 hours there is 3g=12g×\frac{1}{4}. So the half-life is 4 hours.

   (b) Consider a function, \( h(x) \), whose third-degree Maclaurin polynomial is \( x - \frac{3}{2}x^2 + \frac{5}{3}x^3 \). What is \( h''(0) \)?

   Answer: -3

   Solution: Since \( T_3(x) = h(0) + h'(0) \cdot x + \frac{1}{2}h''(0) \cdot x^2 + \frac{1}{6}h'''(0) \cdot x^3 \), we must have that \( \frac{1}{2}h''(0) = -\frac{3}{2} \) and \( h''(0) = -3 \).
Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

(a) Suppose a particle’s position is given by \(s(t) = t^3 - 9t^2 + 15t\). Over what time interval is the particle moving in the negative direction?

**Answer:** \(t \in (1, 5)\)

**Solution:** The velocity is given by

\[v(t) = 3t^2 - 18t + 15 = 3(t^2 - 6t + 5) = 3(t - 1)(t - 5)\]

Hence the velocity is negative when \(1 < t < 5\).

(b) Estimate \(\cot\left(\frac{\pi}{4} + \frac{1}{20}\right)\) using a linear approximation.

**Answer:** 0.9

**Solution:** We use the function \(f(x) = \cot x\) and point \(a = \frac{\pi}{4}\) as the centre of our approximation since we know that

\[f(a) = \cot(\pi/4) = 1\]

We compute \(f'(x) = -1/\sin(x)^2\)

\[f'(\pi/4) = -\frac{1}{(1/\sqrt{2})^2} = -2\]

So, a linear approximation of \(\cot\left(\frac{\pi}{4} + \frac{1}{20}\right)\) is

\[T_1\left(\frac{\pi}{4} + \frac{1}{20}\right) = f(\pi/4) + f'(\pi/4) \cdot \frac{1}{20} = 1 - 2 \times \frac{1}{20} = 0.9\]
Long answer question — you must show your work

3. [4 marks] You place a cone-shaped bottle on the table. It is 20cm high and its circular base (which is resting on the table) has radius 5cm and its volume is $\frac{500}{3}\pi cm^3$. You fill it with water at a rate of $4cm^3$ per second. What is the rate at which the depth of the water changes when it is 4cm deep?

HINT: Think about the shape of the air in bottle.

Solution: The hardest bit of this question is realising that cone is not the usual orientation in these related rates questions.

- Let $h$ be the depth of water in the cone, and $r$ be the radius of the water at that height.
- Additionally let $a = 20 - h$, which is the distance from the tip of the cone to the water-surface. It is also the height of the cone formed by the air in the bottle.
- Similar triangles shows that $a/r = 20/5$. So $r = a/4$.
- Now let $V$ be the volume of water and $A$ be the volume of air in the bottle

$$V = \text{Constant} - A$$

$$A = \frac{\pi ar^2}{3} = \frac{\pi}{3 \times 16} a^3 = \frac{\pi}{3 \times 16} (20 - h)^3$$

- Now

$$\frac{dV}{dt} = - \frac{dA}{dt} = + \frac{\pi}{16} (20 - h)^2 \frac{dh}{dt}$$

- We know $h = 4$, $\frac{dV}{dt} = 4$, so

$$4 = \frac{\pi}{16} (20 - 4)^2 \frac{dh}{dt}$$

$$= 16\pi \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{4\pi} \approx 0.0796 \text{ cm/s}$$