First Name: ________________________ Last Name: ________________________

Student-No: ________________________ Section: ________________________

Grade:

The remainder of this page has been left blank for your workings.
Very short answer questions

1. [2 marks] Each part is worth 1 mark. Please write your answers in the boxes.

(a) Evaluate the following limit by interpreting it as a derivative:

\[
\lim_{x \to \pi/4} \frac{\tan(x) - 1}{x - \pi/4}
\]

You may use the derivative rules we have learned so far in class — but do not use L’Hôpital’s rule.

Answer: 2

Solution: This limit represents the derivative computed at \(x = \pi/4\) of the function \(f(x) = \tan(x)\). Since the derivative of \(f(x)\) is \(\sec^2(x)\), then its value at \(x = \pi/4\) is exactly 2.

(b) Let \(y = \arctan(\log(x))\). Compute \(\frac{dy}{dx}\).

(Recall: \(\log x = \log_e x = \ln x\).)

Answer: \(\frac{1}{x(1 + \log(x))^2}\)

Solution:

\[
\frac{dy}{dx} = \frac{d}{dx} \arctan(\log x) = \frac{1}{1 + (\log x)^2} \cdot \frac{1}{x}
\]
Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.
   (a) Find \( f'(x) \) if \( f(x) = (x^2 + 1)^{\sin(x)} \).

**Solution:** We use logarithmic differentiation; so
\[
\log f(x) = \log(x^2 + 1) \cdot \sin x
\]
Then differentiate to obtain
\[
\frac{f'(x)}{f(x)} = \frac{d}{dx} \left[ \log(x^2 + 1) \cdot \sin x \right] = \cos x \cdot \log(x^2 + 1) + \frac{2x \sin x}{x^2 + 1}
\]
In conclusion:
\[
f'(x) = f(x) \cdot \left( \cos x \cdot \log(x^2 + 1) + \frac{2x \sin x}{x^2 + 1} \right) \quad \text{or equivalently}
\]
\[
= (x^2 + 1)^{\sin(x)} \cdot \left( \cos x \cdot \log(x^2 + 1) + \frac{2x \sin x}{x^2 + 1} \right)
\]

(b) Let \( f(x) \) be a function differentiable at \( x = 3 \) and let \( g(x) = x \cdot f(x) \). The line tangent to the curve \( y = f(x) \) at \( x = 3 \) has slope 2 while the line tangent to the curve \( y = g(x) \) at \( x = 3 \) has slope 5. What is \( f(3) \)?

**Solution:**
\begin{itemize}
  \item We know \( f'(3) = 2 \) and \( g'(3) = 5 \) and \( g'(x) = xf'(x) + f(x) \)
  \item This means \( 5 = g'(3) = 3f'(3) + f(3) = 6 + f(3) \)
  \item Hence \( f(3) = -1. \)
\end{itemize}
Long answer question — you must show your work

3. [4 marks] If \(x^2y + 4 = 4y^2e^x\), then find \(\frac{dy}{dx}\) at all points where \(x = 0\). You must justify your answer.

Solution:

• First find the \(y\)-coordinates where \(x = 0\):

\[
0 + 4 = 4y^2e^0 \\
1 = y^2
\]

Hence \(y = \pm 1\).

• Now we use implicit differentiation to get \(y'\) in terms of \(x, y\):

\[
x^2y' + 2xy = 8yy'e^x + 4y^2e^x
\]

• Now set \(x = 0\) to get

\[
0 = 8yy' + 4y^2 \\
y' = -y/2
\]

• So at \((x, y) = (0, 1)\) we have \(y' = -1/2\),

• and at \((x, y) = (0, -1)\) we have \(y' = 1/2\).
The remainder of this page has been left blank for your workings.
Very short answer questions

1. [2 marks] Each part is worth 1 marks. Please write your answers in the boxes.

   (a) Evaluate the following limit by interpreting it as a derivative:

   \[
   \lim_{x \to 4} \left( \frac{x^{3/2} - 4^{3/2}}{x - 4} \right)
   \]

   You may use the derivative rules we have learned so far in class — but do not use L’Hopital’s rule.

   **Answer:** 3

   **Solution:** This limit represents the derivative computed at \( x = 4 \) of the function \( f(x) = x^{3/2} \). Since the derivative of \( f(x) \) is \( (3/2) \cdot x^{1/2} \), then its value at \( x = 4 \) is exactly \( (3/2) \cdot 4^{1/2} = 3 \).

   (b) Let \( y = \arcsin (e^x) \). Compute \( \frac{dy}{dx} \).

   **Answer:** \( \frac{e^x}{\sqrt{1 - e^{2x}}} \)

   **Solution:**

   \[
   \frac{dy}{dx} = \frac{d}{dx} \arcsin (e^x) \\
   = \frac{1}{\sqrt{1 - e^{2x}}} \cdot e^x
   \]
2.  
(a) Find \( f'(x) \) if \( f(x) = (x^4 + 1)\cos(x) \).

Solution: We use logarithmic differentiation; so

\[
\log f(x) = \log(x^4 + 1) \cdot \cos x
\]

Then differentiate to obtain

\[
\frac{f'(x)}{f(x)} = \frac{d}{dx} \left[ \log(x^4 + 1) \cdot \cos x \right] = -\sin x \cdot \log(x^4 + 1) + \frac{4x^3 \cos x}{x^4 + 1}
\]

In conclusion:

\[
f'(x) = f(x) \cdot \left( -\sin x \cdot \log(x^4 + 1) + \frac{4x^3 \cos x}{x^4 + 1} \right)
\]

or equivalently

\[
= (x^4 + 1)\cos(x) \cdot \left( -\sin x \cdot \log(x^4 + 1) + \frac{4x^3 \cos x}{x^4 + 1} \right)
\]

(b) Let \( f(x) \) be a function differentiable at \( x = 1 \) and let \( g(x) = x \cdot f(x) \). The line tangent to the curve \( y = f(x) \) at \( x = 1 \) has slope 3 while the line tangent to the curve \( y = g(x) \) at \( x = 1 \) has slope 4. What is \( f(1) \)?

Solution:

- We know \( f'(1) = 3 \) and \( g'(1) = 4 \) and \( g'(x) = xf'(x) + f(x) \)
- This means \( 4 = g'(1) = f'(1) + f(1) = 3 + f(1) \)
- Hence \( f(1) = 1 \).
Long answer question — you must show your work

3. 4 marks If $xy^2 + 8 = 2y^2 e^x$, then find $\frac{dy}{dx}$ at all points where $x = 0$. You must justify your answer.

Solution:

- First find the $y$-coordinates where $x = 0$:

$$0 + 8 = 2y^2 e^0$$
$$4 = y^2$$

Hence $y = \pm 2$.

- Now we use implicit differentiation to get $y'$ in terms of $x, y$:

$$x(2yy') + y^2 = 4yy'e^x + 2y^2 e^x$$

- Now set $x = 0$ to get

$$y^2 = 4yy' + 2y^2$$
$$y' = -y/4$$

- So at $(x, y) = (0, 2)$ we have $y' = -1/2$,

- and at $(x, y) = (0, -2)$ we have $y' = 1/2$. 
Very short answer questions

1. [2 marks] Each part is worth 1 marks. Please write your answers in the boxes.

(a) Evaluate the following limit by interpreting it as a derivative:

\[ \lim_{x \to 10} \left( \frac{e^{2x} - e^{20}}{x - 10} \right) \]

You may use the derivative rules we have learned so far in class — but do not use L’Hopital’s rule.

Answer: \(2e^{20}\)

**Solution:** This limit represents the derivative computed at \(x = 10\) of the function \(f(x) = e^{2x}\). Since the derivative of \(f(x)\) is \(2e^{2x}\), then its value at \(x = 10\) is exactly \(2e^{20}\).

(b) Let \(y = \arctan(\sqrt{x})\). Compute \(\frac{dy}{dx}\).

Answer: \(\frac{1}{2\sqrt{x}(1 + x)}\)

**Solution:**

\[
\frac{dy}{dx} = \frac{d}{dx} \arctan(\sqrt{x}) \\
= \frac{1}{1 + (\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} \\
= \frac{1}{2\sqrt{x}(1 + x)}
\]
Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.

(a) Find \( f'(x) \) if \( f(x) = (\cos x)^{x^4+1} \).

**Solution:** We use logarithmic differentiation; so

\[
\log f(x) = (x^4 + 1) \log \cos x
\]

Then differentiate to obtain

\[
\frac{f'(x)}{f(x)} = \frac{d}{dx} \left\{ (x^4 + 1) \log \cos x \right\} = 4x^3 \log \cos x + (x^4 + 1) \frac{-\sin x}{\cos x}
\]

\[
= 4x^3 \log \cos x - \frac{(x^4 + 1) \sin x}{\cos x}
\]

In conclusion:

\[
f'(x) = f(x) \cdot \left( 4x^3 \log \cos x - \frac{(x^4 + 1) \sin x}{\cos x} \right) \text{ or equivalently}
\]

\[
= (\cos x)^{x^4+1} \cdot \left( 4x^3 \log \cos x - \frac{(x^4 + 1) \sin x}{\cos x} \right)
\]

(b) Let \( f(x) \) be a function differentiable at \( x = 2 \) and let \( g(x) = x \cdot f(x) \). The line tangent to the curve \( y = f(x) \) at \( x = 2 \) passes through the point \((2, 3)\). The line tangent to the curve \( y = g(x) \) at \( x = 2 \) has slope 5. What is \( f'(2) \)?

**Solution:**

- We know \( f(2) = 3 \), \( g'(2) = 5 \), and \( g'(x) = xf'(x) + f(x) \).
- This means \( 5 = g'(2) = 2f'(2) + f(2) = 2f'(2) + 3 \)
- Hence \( f'(2) = 1 \).
Long answer question — you must show your work

3. [4 marks] If $8 \log x + 2xy = y^2$, then find $\frac{dy}{dx}$ at all points where $x = 1$. (Recall: $\log x = \log_e x = \ln x$.) You must justify your answer.

Solution:

- First find the $y$-coordinates where $x = 1$:

\[
8 \log(1) + 2(1)y = y^2
\]
\[
0 + 2y = y^2
\]
\[
2y - y^2 = 0
\]
\[
y(2 - y) = 0
\]

Hence $y = 0$ or $y = 2$.

- Now we use implicit differentiation to get $y'$ in terms of $x, y$:

\[
\frac{8}{x} + 2x \cdot y' + 2y = 2y \cdot y'
\]

- Now set $x = 1$ and $y = 0$ to get

\[
\frac{8}{1} + 2y' + 0 = 0
\]
\[
2y' = -8
\]
\[
y' = -4
\]

So at $(x, y) = (1, 0), y' = -4$

- Now set $x = 1$ and $y = 2$ to get

\[
\frac{8}{1} + 2y' + 2(2) = 2(2)y'
\]
\[
12 + 2y' = 4y'
\]
\[
12 = 2y'
\]
\[
y' = 6
\]

So at $(x, y) = (1, 2), y' = 6$