First Name: ______________________  Last Name: ______________________

Student-No: ______________________  Section: ______________________

Grade: ______________________

The remainder of this page has been left blank for your workings.
Very short answer questions

1. 2 marks Each part is worth 1 mark. Please write your answers in the boxes.

   (a) Find all the points at which the function \( f(x) = \frac{1}{1-|x|} \) is continuous.

   \[
   \text{Answer: } (-\infty, -1) \cup (-1, 1) \cup (1, +\infty)
   \]

   (Equivalently, \( f(x) \) is continuous everywhere except at \( x = -1 \) and \( x = 1 \))

   \[
   \text{Solution: } \text{The discontinuities of } f(x) \text{ are vertical asymptotes which occur when the denominator } 1-|x| \text{ is equal to } 0. \text{ Solving } 1-|x| = 0 \text{ we get } |x| = 1. \text{ Therefore } x = -1 \text{ and } x = 1 \text{ are the only discontinuities of } f(x). \text{ The function } f(x) \text{ is continuous on the intervals } (-\infty, -1), (-1, 1) \text{ and } (1, +\infty). \text{ (In other words, } f(x) \text{ is continuous everywhere except at } x = -1 \text{ and } x = 1.\)

   (b) Compute the derivative of \( y = \frac{x^2 - 1}{x^2 + 1} \). Simplify your answer.

   \[
   \text{Answer: } y' = \frac{4x}{(x^2 + 1)^2}
   \]

   \[
   \text{Solution: } \text{We use the quotient rule:}
   \]

   \[
   \frac{d}{dx} \left( \frac{x^2 - 1}{x^2 + 1} \right) = \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}
   \]
Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.
   (a) Find the equation of the tangent line to \( y = 1 + 2x - x^2 \) at \( x = -2 \).
   
   **Answer:** \( y = 6x + 5 \)

   **Solution:** The derivative of \( y = 1 + 2x - x^2 \) is \( y' = 2 - 2x \) and so the slope of the tangent line at \( x = -2 \) is \( 2 - 2(-2) = 6 \). When \( x = -2 \), the \( y \) value of the curve is \( y = 1 + 2(-2) - (-2)^2 = -7 \). The equation of the tangent line is \( y = 6(x - (-2)) + (-7) \) which is \( y = 6x + 5 \).

   (b) Find all values of \( c \) such that the following function is continuous at \( x = c \):

   \[
   f(x) = \begin{cases} 
   cx - 2 & \text{if } x \leq c \\
   -x^2 & \text{if } x > c 
   \end{cases}
   \]

   Justify your answer using the definition of continuity.
   
   **Answer:** \( c = \pm 1 \)

   **Solution:** The function is continuous for \( x \neq c \) since each of those two branches are polynomials. So, the only question is whether the function is continuous at \( x = c \); for this we need

   \[
   \lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x).
   \]

   We compute

   \[
   \lim_{x \to c^-} f(x) = \lim_{x \to c^-} (cx - 2) = c^2 - 2;
   \]

   \[
   f(c) = c^2 - 2 \text{ and}
   \]

   \[
   \lim_{x \to c^+} f(x) = \lim_{x \to c^+} (-x^2) = -c^2.
   \]

   So, we need \( c^2 - 2 = -c^2 \), which yields \( c^2 = 1 \), i.e. \( c = -1 \) or \( c = 1 \).
Long answer question — you must show your work

3. [4 marks] Show that there is at least one real number \( x \) satisfying

\[
\sin(x) + \frac{1}{x} = 1
\]

Solution:

- Let \( f(x) = \sin(x) + \frac{1}{x} - 1 \). It suffices to show that this function has a zero.
- Note that the function is continuous for all \( x \neq 0 \), so we may use the IVT on any interval that avoids 0.
- When \( x = \pi/2 \), \( f(\pi/2) = \sin(\pi/2) + 2/\pi - 1 = 1 + 2/\pi - 1 = 2/\pi > 0 \).
- When \( x = 3\pi \), \( f(3\pi) = \sin(3\pi) + 1/3\pi - 1 = 1/3\pi - 1 < 0 \).
- Since the function is negative at \( x = 3\pi \) and positive at \( x = \pi/2 \), by the IVT the function must be zero at some \( x \) between \( \pi/2 \) and \( 3\pi \).
- It actually occurs at \( q = 2.4984097 \ldots \)
The remainder of this page has been left blank for your workings.
Very short answer questions

1. [2 marks] Each part is worth 1 marks. Please write your answers in the boxes.

   (a) Find all the points at which the function \( f(x) = \frac{1}{\sqrt{(x - 2)(x + 3)}} \) is continuous.

   Answer: \((-\infty, -3) \cup (2, \infty)\)

   Solution: The function is continuous when \((x - 2)(x + 3) > 0\) which yields the intervals \((-\infty, -3) \cup (2, \infty)\).

   (b) Compute the derivative of \(\left(\frac{x + 1}{2x + 3}\right)\). Simplify your answer.

   Answer: \(\frac{1}{(2x + 3)^2}\)

   Solution: We use the quotient rule:

   \[
   \frac{d}{dx} \left(\frac{x + 1}{2x + 3}\right) = \frac{1 \cdot (2x + 3) - (x + 1) \cdot 2}{(2x + 3)^2}.
   \]

   \[
   = \frac{1}{(2x + 3)^2}.
   \]
2. [4 marks] Each part is worth 2 marks.

(a) Find the equation of the tangent line to the graph of \( y = x^3 - 2x \) at \( x = -1 \).

\[
\text{Answer: } y - 1 = x + 1.
\]

**Solution:** We compute the derivative of \( x^3 - 2x \) as being \( 3x^2 - 2 \), which evaluated at \( x = -1 \) yields 1. At \( x = -1 \), \( y = -1 + 2 = 1 \) so the equation of the tangent line is

\[
y - 1 = 1 \cdot (x + 1).
\]

(b) Find all values of \( c \) such that the following function is continuous at \( x = c \):

\[
f(x) = \begin{cases} 
3x^2 & \text{if } x \leq c \\
 cx + 2 & \text{if } x > c 
\end{cases}
\]

Justify your answer using the definition of continuity.

\[
\text{Answer: } c = \pm 1
\]

**Solution:** The function is continuous for \( x \neq c \) since each of those two branches are polynomials. So, the only question is whether the function is continuous at \( x = c \); for this we need

\[
\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x).
\]

We compute

\[
\lim_{x \to c^-} f(x) = \lim_{x \to c^-} 3x^2 = 3c^2;
\]

\[
f(c) = 3c^2 \text{ and }
\]

\[
\lim_{x \to c^+} f(x) = \lim_{x \to c^+} cx + 2 = c \cdot c + 2 = c^2 + 2.
\]

So, we need \( 3c^2 = c^2 + 2 \), which yields \( c^2 = 1 \), i.e. \( c = -1 \) or \( c = 1 \).
Long answer question — you must show your work

3. [4 marks] Show that there is at least one real number \( q \) so that

\[
\frac{1}{q} = 3 \sin(\pi q)
\]

Solution:

- Let \( f(x) = \frac{1}{x} - 3 \sin(\pi x) \). It suffices to show that this function has a zero.
- Note that the function is continuous for \( x > 0 \), so we may use the IVT on that interval.
- When \( x = 1/2 \), \( f(x) = \frac{1}{1/2} - 3 \sin(\pi/2) = 2 - 3 = -1 < 0 \).
- When \( x = 1 \), \( f(x) = \frac{1}{1} - 3 \sin(\pi) = 1 - 0 = 1 > 0 \).
- Since the function is negative at \( x = 1/2 \) and positive at \( x = 1 \), by the IVT the function must be zero at \( x = q \) for some \( q \) between \( 1/2 \) and 1.
- It actually occurs at \( q = 0.8757011855 \ldots \) (but also at other points).
The remainder of this page has been left blank for your workings.
Very short answer questions

1. [2 marks] Each part is worth 1 marks. Please write your answers in the boxes.

(a) Find all the points at which the function \( f(x) = \frac{1}{\sqrt{x^2 - 1}} \) is continuous.

Answer: \((-\infty, -1) \cup (1, \infty)\)

Solution: The function is continuous when \( x^2 - 1 > 0 \). ie when \((x+1)(x-1) > 0\) which yields the intervals \((-\infty, -1) \cup (1, \infty)\).

(b) Compute the derivative of \( \frac{x}{5x + 2} \). Simplify your answer.

Answer: \(\frac{2}{(5x + 2)^2}\)

Solution: We use the quotient rule:

\[
\frac{d}{dx} \left( \frac{x}{5x + 2} \right) = \frac{(5x + 2) \cdot 1 - 5x}{(5x + 2)^2}
= \frac{2}{(5x + 2)^2}
\]
2. [4 marks] Each part is worth 2 marks.

(a) Find the equation of the tangent line to the graph of \( y = 4x^2 + x - 1 \) at \( x = 1 \).

**Answer:** \( y - 4 = 9 \cdot (x - 1) \).

**Solution:** We compute the derivative of \( 4x^2 + x - 1 \) as being \( 8x + 1 \), which evaluated at \( x = 1 \) yields 9. At \( x = 1 \), \( y = 4 + 1 - 1 = 4 \) so the equation of the tangent line is

\[
y = 4 + 9 \cdot (x - 1), \quad \text{or equivalently, } y - 4 = 9 \cdot (x - 1), \quad \text{that is, } y = 9x - 5.
\]

(b) Find all values of \( c \) such that the following function is continuous at \( x = c \):

\[
f(x) = \begin{cases} 
  x^2 - 1 & \text{if } x \leq c \\
  1 - cx & \text{if } x > c 
\end{cases}
\]

Justify your answer using the definition of continuity.

**Answer:** \( c = \pm 1 \)

**Solution:** The function is continuous for \( x \neq c \) since each of those two branches are polynomials. So, the only question is whether the function is continuous at \( x = c \); for this we need

\[
\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x).
\]

We compute

\[
\lim_{x \to c^-} f(x) = \lim_{x \to c^-} (x^2 - 1) = c^2 - 1; \quad f(c) = c^2 - 1 \quad \text{and} \\
\lim_{x \to c^+} f(x) = \lim_{x \to c^+} (1 - c \cdot c) = 1 - c^2.
\]

So, we need \( c^2 - 1 = 1 - c^2 \), which yields \( c^2 = 1 \), i.e. \( c = -1 \) or \( c = 1 \).
Long answer question — you must show your work

3. 4 marks Show that there is at least one real number $z$ so that

$$\cos(\pi z) = \sqrt{z} - 1/2.$$

Solution:

- Let $f(x) = \cos(\pi x) - \sqrt{x} + 1/2$. It suffices to show that this function has a zero.
- Note that the function is continuous for $x \geq 0$, so we may use the IVT on that interval.
- When $x = 0$, $f(0) = \cos 0 - \sqrt{0} + 1/2 = 1 + 1/2 > 0$.
- When $x = 1$, $f(1) = \cos \pi - \sqrt{1} + 1/2 = -1 - 1 + 1/2 = -3/2 < 0$.
- Since the function, which is continuous on the interval $[0,1]$, is positive at $x = 0$ and negative at $x = 1$, by the IVT the function must be zero at some $z$ between 0 and 1.