Long Answer Questions

Quiz 6: Find the global maximum and the global minimum for \( f(x) = x^3 - 6x^2 + 2 \) on the interval \([3, 5]\).

Quiz 5: Two particles move in the cartesian plane. Particle A travels on the x-axis starting at \((10, 0)\) and moving towards the origin with a speed of 2 units per second. Particle B travels on the y-axis starting at \((12, 0)\) and moving towards the origin with a speed of 3 units per second. What is the rate of change of the distance between the two particles when particle A reaches the point \((4, 0)\)?

Quiz 4: If \( x^2 \cos(y) + 2xe^y = 8 \), then find \( y' \) at the points where \( y = 0 \). You must justify your answer.

Quiz 3: Determine whether the derivative of following function exists at \( x = 0 \)

\[
f(x) = \begin{cases} 
2x^3 - x^2 & \text{if } x \leq 0 \\
x^2 \sin \left( \frac{1}{x} \right) & \text{if } x > 0
\end{cases}
\]

You must justify your answer using the definition of a derivative.

Quiz 2: Show that there exists at least one real number \( c \) such that \( 2 \tan(c) = c + 1 \).

Quiz 1: Compute the limit \( \lim_{x \to 1} \frac{\sqrt{x + 2} - \sqrt{4 - x}}{x - 1} \).
Quizzes 4-6 Short Answer Questions

Quiz 6:
Find the intervals where \( f(x) = \frac{\sqrt{x}}{x+6} \) is increasing.

Let \( f(x) = x^2 - 2\pi x - \sin(x) \). Show that there exists a real number \( c \) such that \( f'(c) = 0 \).

Quiz 5:
Estimate \( \sqrt{35} \) using a linear approximation

Consider a function \( f(x) \) which has \( f'''(x) = \frac{x^3}{10 - x^2} \). Show that when we approximate \( f(1) \) using its second Maclaurin polynomial, the absolute error is less than \( \frac{1}{50} = 0.02 \).

Quiz 4:
Find \( f'(x) \) if \( f(x) = (x^2 + 1)^{\sin(x)} \).

Consider a function of the form \( f(x) = Ae^{kx} \) where \( A \) and \( k \) are constants. If \( f(0) = 3 \) and \( f(2) = 5 \), find the constants \( A \) and \( k \).
Quiz 3: Find the equation of the tangent line to the graph of \( y = \cos(x) \) at \( x = \frac{\pi}{4} \).

For what values of \( x \) does the derivative of \( \frac{\sin(x)}{x^2 + 6x + 5} \) exist?

Quiz 2: Compute

\[
\lim_{x \to -\infty} \frac{3x + 5}{\sqrt{x^2 + 5 - x}}
\]

Find all values of \( c \) such that the following function is continuous:

\[
f(x) = \begin{cases} 
8 - cx & \text{if } x \leq c \\
x^2 & \text{if } x > c
\end{cases}
\]

Use the definition of continuity to justify your answer.

Quiz 1: Find all solutions to \( x^3 - 3x^2 - x + 3 = 0 \)

Compute the limit \( \lim_{x \to 2} \frac{x - 2}{x^2 - 4} \)
Solutions
Quiz 6 Long Answer

Find the global maximum and the global minimum for $f(x) = x^3 - 6x^2 + 2$ on the interval $[3, 5]$.

We compute $f'(x) = 3x^2 - 12x$, which means that $f(x)$ has no singular points (i.e., it is differentiable for all values of $x$), but it has two critical points obtained by solving $f'(x) = 0$, i.e. $3x(x - 4) = 0$ which yields the two critical points $x = 0$ and $x = 4$. In order to compute the global maximum and the global minimum for $f(x)$ on the interval $[3, 5]$, we compute

$$f(3) = -25, \ f(4) = -30 \text{ and } f(5) = -23.$$ 

So, the global maximum is $f(5) = -23$ while the global minimum is $f(4) = -30$. 
Quiz 5 Long Answer

Two particles move in the cartesian plane. Particle A travels on the $x$-axis starting at $(10, 0)$ and moving towards the origin with a speed of 2 units per second. Particle B travels on the $y$-axis starting at $(12, 0)$ and moving towards the origin with a speed of 3 units per second. What is the rate of change of the distance between the two particles when particle A reaches the point $(4, 0)$?

The position of particle A along the $x$ axis starts at 10, and decreases 2 units per second, so its position is given by $10 - 2t$, where $t$ is measured in seconds. Similarly, the position of $B$ along the $y$ axis is given by $y = 12 - 3t$. The distance $z$ between the two particles satisfies $z^2 = x^2 + y^2$. When $x = 4$, we solve $4 = 10 - 2t$ for $t$ and find $t = 3$, so $y = 12 - 3(3) = 3$. Then $z = 5$ when $t = 2$.

Differentiating implicitly, $z^2 = x^2 + y^2$ tells us

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

so, when $t = 3$,

$$2(5) \frac{dz}{dt} = 2(4)(-2) + 2(3)(-3)$$

Then the distance between the two particles is changing at $-\frac{17}{5}$ units per second.
Quiz 4 Long Answer

If \( x^2 \cos(y) + 2xe^y = 8 \), then find \( y' \) at the points where \( y = 0 \).

You must justify your answer.

- First we find the \( x \)-ordinates where \( y = 0 \).

\[
x^2 \cos(0) + 2xe^0 = 8
\]
\[
x^2 + 2x - 8 = 0
\]
\[
(x + 4)(x - 2) = 0
\]

So \( x = 2, -4 \).

- Now we use implicit differentiation to get \( y' \) in terms of \( x, y \):

\[
x^2 \cos(y) + 2xe^y = 8 \quad \text{differentiate both sides}
\]
\[
x^2 \cdot (-\sin y) \cdot y' + 2x \cos y + 2xe^y \cdot y' + 2e^y = 0
\]

- Now set \( y = 0 \) to get

\[
x^2 \cdot (-\sin 0) \cdot y' + 2x \cos 0 + 2xe^0 \cdot y' + 2e^0 = 0
\]
\[
0 + 2x + 2xy' + 2 = 0
\]
\[
y' = -\frac{2 + 2x}{2x} = -\frac{1 + x}{x}
\]

So at \( (x, y) = (2, 0) \) we have \( y' = -\frac{3}{2} \),

and at \( (x, y) = (-4, 0) \) we have \( y' = -\frac{3}{4} \).
Determine whether the derivative of following function exists at $x = 0$

$$f(x) = \begin{cases} 
2x^3 - x^2 & \text{if } x \leq 0 \\
 x^2 \sin \left( \frac{1}{x} \right) & \text{if } x > 0 
\end{cases}$$

You must justify your answer using the definition of a derivative.
The function is differentiable at $x = 0$ if the following limit:

$$
\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f(x) - 0}{x} = \lim_{x \to 0} \frac{f(x)}{x}
$$

exists (note that we used the fact that $f(0) = 0$ as per the definition of the first branch which includes the point $x = 0$). We compute left and right limits; so

$$
\lim_{x \to 0^-} \frac{f(x)}{x} = \lim_{x \to 0^-} \frac{2x^3 - x^2}{x} = \lim_{x \to 0^-} 2x^2 - x = 0
$$

and

$$
\lim_{x \to 0^+} \frac{x^2 \sin \left( \frac{1}{x} \right)}{x} = \lim_{x \to 0^+} x \cdot \sin \left( \frac{1}{x} \right).
$$

This last limit equals 0 by Squeeze Theorem since

$$
-1 \leq \sin \left( \frac{1}{x} \right) \leq 1
$$

and so,

$$
-x \leq x \cdot \sin \left( \frac{1}{x} \right) \leq x,
$$

where in these inequalities we used the fact that $x \to 0^+$ yields positive values for $x$. Finally, since $\lim_{x \to 0^+} -x = \lim_{x \to 0^+} x = 0$, Squeeze Theorem yields that also $\lim_{x \to 0^+} x \sin \left( \frac{1}{x} \right) = 0$, as claimed. Since the left and right limits match (they’re both equal to 0), we conclude that indeed $f(x)$ is differentiable at $x = 0$ (and its derivative at $x = 0$ is actually equal to 0).
Quiz 2 Long Answer

Show that there exists at least one real number \( c \) such that \( 2 \tan(c) = c + 1 \).

We let \( f(x) = 2 \tan(x) - x - 1 \). Then \( f(x) \) is a continuous function on the interval \((-\pi/2, \pi/2)\) since \( \tan(x) = \sin(x)/\cos(x) \) is continuous on this interval, while \( x + 1 \) is a polynomial and therefore continuous for all real numbers.

We find a value \( a \in (-\pi/2, \pi/2) \) such that \( f(a) < 0 \). We observe immediately that \( a = 0 \) works since

\[
f(0) = 2 \tan(0) - 0 - 1 = 0 - 1 = -1 < 0.
\]

We find a value \( b \in (-\pi/2, \pi/2) \) such that \( f(b) > 0 \). We see that \( b = \pi/4 \) works since

\[
f(\pi/4) = 2 \tan(\pi/4) - \pi/4 - 1 = 2 - \pi/4 - 1 = 1 - \pi/4 = (4 - \pi)/4 > 0,
\]

because \( 3 < \pi < 4 \).

So, because \( f(x) \) is continuous on \([0, \pi/4]\) and \( f(0) < 0 \) while \( f(\pi/4) > 0 \), then the Intermediate Value Theorem guarantees the existence of a real number \( c \in (0, \pi/4) \) such that \( f(c) = 0 \).
Quiz 1 Long Answer

Compute the limit \( \lim_{x \to 1} \frac{\sqrt{x + 2} - \sqrt{4 - x}}{x - 1} \).

If we try to do the limit naively we get \(0/0\). Hence we must simplify.

\[
\frac{\sqrt{x + 2} - \sqrt{4 - x}}{x - 1} = \frac{\sqrt{x + 2} - \sqrt{4 - x}}{x - 1} \cdot \frac{\sqrt{x + 2} + \sqrt{4 - x}}{\sqrt{x + 2} + \sqrt{4 - x}}
\]

\[
= \frac{(x + 2) - (4 - x)}{(x - 1)(\sqrt{x + 2} + \sqrt{4 - x})}
\]

\[
= \frac{2x - 2}{(x - 1)(\sqrt{x + 2} + \sqrt{4 - x})}
\]

\[
= \frac{2}{\sqrt{x + 2} + \sqrt{4 - x}}
\]
Quiz 6 Short Answer

Find the intervals where \( f(x) = \frac{\sqrt{x}}{x+6} \) is increasing.

\((0, \infty)\)

Let \( f(x) = x^2 - 2\pi x - \sin(x) \). Show that there exists a real number \( c \) such that \( f'(c) = 0 \).

We note that \( f(x) \) is continuous and differentiable over all real numbers. Since \( f(0) = f(2\pi) = 0 \), by Rolle's Theorem (also by the Mean Value Theorem) there exist some \( c \) between 0 and \( 2\pi \) such that \( f'(c) = 0 \).
Estimate $\sqrt{35}$ using a linear approximation

$L(x) = f(a) + f'(a)(x - a)$. If $f(x) = \sqrt{x}$ and $a = 36$, then $f(a) = 6$ and $f'(a) = \frac{1}{12}$. So,

$L(x) = 6 + \frac{1}{12}(x - 36)$. Then: $\sqrt{35} = f(35) \approx L(35) = 6 + \frac{1}{12}(35 - 36) = 6 - \frac{1}{12} = \frac{71}{12}$

Consider a function $f(x)$ which has $f'''(x) = \frac{x^3}{10 - x^2}$. Show that when we approximate $f(1)$ using its second Maclaurin polynomial, the absolute error is less than $\frac{1}{50} = 0.02$.

For some $c$ between 0 and 1:

$$|f(1) - T_2(1)| = \left| \frac{f'''(c)}{3!} (1 - 0)^3 \right| = \frac{1}{6} \left| \frac{c^3}{10 - c^2} \right|$$

Since $c$ is between 0 and 1, we note $0 < c^3 < 1$ and $9 < 10 - c^2 < 10$, so:

$$|f(1) - T_2(1)| < \frac{1}{6} \left| \frac{1}{9} \right| = \frac{1}{54} < \frac{1}{50}$$
Quiz 4 Short Answer

Find \( f'(x) \) if \( f(x) = (x^2 + 1)^\sin(x) \).

\[
f'(x) = (x^2 + 1)^\sin x \left[ \frac{2x \sin x}{x^2 + 1} + \cos x \cdot \log(x^2 + 1) \right]
\]

Consider a function of the form \( f(x) = Ae^{kx} \) where \( A \) and \( k \) are constants. If \( f(0) = 3 \) and \( f(2) = 5 \), find the constants \( A \) and \( k \).

\[
A = 3, \quad k = \frac{\log(5/3)}{2}
\]
Find the equation of the tangent line to the graph of $y = \cos(x)$ at $x = \frac{\pi}{4}$.

\[ y = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \left(x - \frac{\pi}{4}\right) \]

For what values of $x$ does the derivative of \( \frac{\sin(x)}{x^2 + 6x + 5} \) exist? Explain your answer.

\[ x \neq -1, -5 \]
Quiz 2 Short Answer

Compute

\[ \lim_{x \to -\infty} \frac{3x + 5}{\sqrt{x^2 + 5} - x} \]

\(-\frac{3}{2}\) (Remember: when \(x\) is negative, \(\sqrt{x^2} = |x| = -x\).

Find all values of \(c\) such that the following function is continuous:

\[ f(x) = \begin{cases} 8 - cx & \text{if } x \leq c \\ x^2 & \text{if } x > c \end{cases} \]

Use the definition of continuity to justify your answer.

When \(x \neq c\), \(f(x)\) is locally a polynomial, so it is continuous. The only difficult spot is when \(x = c\). Note:

- \(f(c) = 8 - c^2\)
- \(\lim_{x \to c^-} f(x) = \lim_{x \to c^-} (8 - cx) = 8 - c^2\)
- \(\lim_{x \to c^+} f(x) = \lim_{x \to c^+} (x^2) = c^2\)

Since \(f(x)\) is continuous at \(c\) only if \(f(c) = \lim_{x \to c} f(x)\), we see the only values of \(c\) that make \(f\) continuous are those that satisfy \(c^2 = 8 - c^2\). That is, \(\pm 2\).
**Quiz 1 Short Answer**

Find all solutions to \( x^3 - 3x^2 - x + 3 = 0 \)

\[
x^3 - 3x^2 - x + 3 = x^2(x - 3) - (x - 3) = (x^2 - 1)(x - 3) = (x + 1)(x - 1)(x - 3)
\]

So, the solutions are \( x = 1, x = 3, \) and \( x = -1. \)

Compute the limit \( \lim_{x \to 2} \frac{x - 2}{x^2 - 4} \)

\[
\lim_{x \to 2} \frac{x - 2}{x^2 - 4} = \lim_{x \to 2} \frac{x - 2}{(x - 2)(x + 2)} = \lim_{x \to 2} \frac{1}{x + 2} = \frac{1}{4}
\]