1. The Beverton-Holt model describes the growth in a given population from one generation to the next. It is given by

\[ f(x) = \frac{\nu K x}{K + (\nu - 1)x}, \]

where \( x \) is the population in a given generation, \( f(x) \) is the population in the next, and \( \nu > 1 \) and \( K > 0 \) are constants. Because \( x \) represents a population, the domain is restricted to \( x \geq 0 \).

(a) Find the horizontal asymptote of \( f(x) \).

(b) Describe in a few sentences the physical interpretation of the horizontal asymptote found in part (a)—what is the significance of the asymptote with respect to the population?

(c) Solve the equation \( f(x) = x \).

(d) In physical terms, what do the solutions in part (c) represent? (Hint: \( K \) is sometimes defined to be the carrying capacity of the population.)

(a) We have

\[
\lim_{x \to \infty} \frac{\nu K x}{K + (\nu - 1)x} = \lim_{x \to \infty} \frac{\nu K}{K + (\nu - 1)} = \frac{\nu K}{\nu - 1} = \frac{K}{1 - \frac{1}{\nu}}.
\]

Thus \( f(x) \) has a horizontal asymptote \( y = \frac{K}{1 - \frac{1}{\nu}} \).

(b) The function \( f(x) \) returns the population in the next generation, given the current population \( x \). The horizontal asymptote represents the next generation population given a very large current population: if \( x \) is very large, the next generation population will be very close to \( \frac{K}{1 - \frac{1}{\nu}} \).

The fact that an asymptote exists means that the Beverton-Holt model predicts that the effect of overpopulation is an immediate, next-generation return to a population just below the carrying capacity (see part (d), below). In contrast, models like the Ricker model have a horizontal asymptote \( y = 0 \); that is, they predict that the effect of overpopulation is an immediate, next-generation return to a population close to zero.

(c) To solve \( f(x) = x \) we cross-multiply, then group and factor:

\[
\frac{\nu K x}{K + (\nu - 1)x} = x
\]

\[
\nu K x = K x + (\nu - 1)x^2
\]

\[
0 = K x - \nu K x + (\nu - 1)x^2
\]

\[
0 = (1 - \nu)x(K - x).
\]

There are two solutions: \( x = 0, K \).

(d) The solutions in part (c) are populations \( x \) predicted by the model to return the exact same population in the following generation. The solution \( x = 0 \) represents the fact that a population of 0 will neither grow nor shrink. The solution \( x = K \), the carrying capacity, represents a population that reproduces its numbers exactly.

Both 0 and \( K \) are sustainable populations. Whether or not they are stable is an interesting and separate question.
2. Let

\[ a_1 < a_2 < \cdots < a_{k-1} < a_k < a_{k+1} < \cdots < a_n \]

be numbers where \( k \) is odd and \( n \) is even. Describe the vertical asymptote \( x = a_k \) of the function

\[ f(x) = \frac{1}{(x-a_1)(x-a_2)\cdots(x-a_{k-1})(x-a_k)(x-a_{k+1})\cdots(x-a_n)}. \]

In other words, determine \( \lim_{x \to a_k^-} f(x) \) and \( \lim_{x \to a_k^+} f(x) \).

First, we note that when \( x \) is close to \( a_k \),

\( (x-a_1), (x-a_2), \ldots, (x-a_{k-1}) > 0. \)

Therefore, only the sign of the remaining factors \( (x-a_k), (x-a_{k+1}), \cdots, (x-a_n) \) affects the sign of the given limits.

Similarly, when \( x \) is close to \( a_k \),

\( (x-a_{k+1}), \cdots, (x-a_n) < 0. \)

There are an odd number \( n-k \) of these factors, so their product is also negative.

All that remains is to determine the sign of the vanishing factor \( (x-a_k) \). When \( x < a_k \), \( (x-a_k) < 0 \), so

\[ \lim_{x \to a_k^-} f(x) = \lim_{x \to a_k^-} \left( \frac{1}{(x-a_1)(x-a_2)\cdots(x-a_{k-1})}\right) \left( \frac{1}{x-a_{k+1}} \right) \left( \frac{1}{x-a_n} \right) \left( \frac{1}{x-a_k} \right) = \infty, \]

since the first factor is positive, the second factor is negative, and the third factor is negative (and very large). When \( x > a_k \), \( (x-a_k) > 0 \), so

\[ \lim_{x \to a_k^+} f(x) = \lim_{x \to a_k^+} \left( \frac{1}{(x-a_1)(x-a_2)\cdots(x-a_{k-1})}\right) \left( \frac{1}{x-a_{k+1}} \right) \left( \frac{1}{x-a_n} \right) \left( \frac{1}{x-a_k} \right) = -\infty, \]

since the first factor is positive, the second factor is positive, and the third factor is negative (and very large).