1. An individual of blood type O and Rhesus negative (denoted by “O-”) are universal donors. If 46% of individuals have blood type O and 16% of individuals are Rhesus negative, and the two blood types are independent of one another. Calculate the following probabilities.

(a) Probability of finding a universal donor, i.e. an “O-” individual.
(b) Probability of finding an individual that is “O+”.
(c) Probability of finding an individual that is not O but is Rhesus negative.
(d) Probability of finding an individual that is not O and is Rhesus positive.

Answer:
(a) \( P(O-) = P(O)P(-) = 0.46 \times 0.16 = 0.0736 \).
(b) \( P(O+) = P(O)P(+) = P(O)(1 - P(-)) = 0.46 \times (1 - 0.16) = 0.3864 \).
(c) \( P(O^c-) = P(O^c)P(-) = (1 - P(O))P(-) = (1 - 0.46) \times 0.16 = 0.0864 \).
(d) \( P(O^c+) = P(O^c)P(+) = (1 - P(O))(1 - P(-)) = (1 - 0.46)(1 - 0.16) = 0.4536 \).

2. The probability of a white male having blond hair is 10%. The probability of a white male becoming bald is 18%. The probability of a blond white male who becomes bald is 5%. \( P(B) = 0.1, \ P(b) = 0.18, \ P(B \cap b) = 0.05, \ B=\text{blond}, \ b=\text{bald} \). Calculate the following probabilities.

(a) Probability of a white male being blond but does not become bald.
(b) Probability of a white male being blond and/or become bald.
(c) Probability of a white male becoming bald if he is blond.
(d) Probability of a white male being originally blond if he later becomes bald.

Answer:
(a) \( P(B \cap b^c) = P(B) - P(B \cap b) = 0.1 - 0.05 = 0.05 \).
(b) \( P(B \cup b) = P(B) + P(b) - P(B \cap b) = 0.1 + 0.18 - 0.05 = 0.23 \).
(c) \( P(b|B) = \frac{P(B \cap b)}{P(B)} = \frac{0.05}{0.1} = 0.5 \).
(d) \( P(B|b) = \frac{P(B \cap b)}{P(b)} = \frac{P(b)P(B)}{P(b)} = \frac{0.05}{0.18} = 0.277 \ldots \).

3. Ability to taste phenylthiocarbamide (PTC) is thought to be determined by a single dominant gene with incomplete penetrance. Among North American Caucasians, there is a 70% chance of being able to taste PTC (i.e. \( P(\text{taster}) = 0.7 \)). If everybody who tastes PTC is a carrier (i.e. \( P(\text{carrier}|\text{taster}) = 1 \)) and if 80% of the population carries the gene (i.e. \( P(\text{carrier}) = 0.8 \),
what is the penetrance of the gene? That is, what is the probability of tasting PTC if you are a carrier?

**Answer:**

\[
P(taster|carrier) = \frac{P(carrier|taster)P(taster)}{P(carrier)} = \frac{1 \times 0.7}{0.8} = 0.875.
\]

4. Imagine doing the single Bernoulli experiment twice independently with a probability of yielding 1 (or moving 1 step to the right) being \( p \). Then the total outcome could be 0, 1, or 2 (successes or steps to the right). Determine the probability of each outcome. Calculate the mean and variance.

**Answer:** This is a binomial distribution.

\[
p(X^{2,p} = 0) = p((0, 0)) = (1 - p)^2.
\]

\[
p(X^{2,p} = 1) = p((1, 0) \text{ or } (0, 1)) = 2p(1 - p).
\]

\[
p(X^{2,p} = 2) = p((1, 1)) = p^2.
\]

\[
E(X^{2,p}) = 2p.
\]

\[
Var(X^{2,p}) = 2p(1 - p).
\]

5. Statistics show that in a certain country, the probability of a new born being male is 0.49 and being female is 0.51. Suppose that these probabilities apply to each individual family. Calculate the probability of a family of 9 children having 6 daughters.

**Answer:** This is a binomial distribution.

\[
p(X^{9,0.51} = 6) = \frac{9!}{(9 - 6)!6!} \times 0.51^6 \times 0.49^3 \approx 84 \times 0.00207 \approx 0.1739.
\]

6. **The geometric distribution** is associated with the experiment of tossing an unfair coin with a probability \( p \) of getting a head. \( G_k \) is defined as the probability of yielding the first head at the \( k^{th} \) toss. Let \( X^G \) be the number of tosses required to yield the first head in each experiment. Then, the random variable \( X^G = \{1, 2, 3, \ldots\} \) is the set of all positive integers. Answer the following questions.

(a) What is the probability of tossing the coin \( k - 1 \) times with no head?

(b) What is the probability of tossing the coin \( k - 1 \) times with no head but then getting a head at the \( k^{th} \) toss (i.e. \( G_k \))?

(c) Write down a sigma sum that represents the expected value of the random variable \( X^G \), i.e. \( E(X^G) \). Do not try to calculate it yet.

(d) The follow series

\[
\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1 - x}, \quad (|x| < 1)
\]

is called the **geometric series**. Differentiate both sides with respect to (wrt) \( x \), then in the resulting equation substitute each \( x \) by \( 1 - p \). Use the equation obtained to calculate \( E(X^G) \) given in (c).
(e) Differentiate both sides of the geometric series wrt to $x$ as we did in (c). Multiply both sides by $x$ and then differentiate both sides again wrt $x$. Show that the resulting equation is
\[ \sum_{k=1}^{\infty} k^2 x^{k-1} = \frac{1 + x}{(1 - x)^3}. \]

Use this equation and the same technique used in (d), calculate $Var(X^G)$.

**Answer:**

(a) $P(T^{k-1}) = (1 - p)^{k-1}$.
(b) $G_k = P(T^{k-1}H) = p(1 - p)^{k-1}$.
(c) $E(X^G) = \sum_{k=1}^{\infty} kG_k = \sum_{k=1}^{\infty} kp(1 - p)^{k-1}$.

(d) The differentiation yields
\[ \sum_{k=0}^{\infty} kx^k = \sum_{k=1}^{\infty} kx^k = \frac{1}{(1 - x)^2}. \]

Let $x = 1 - p$, thus $p = 1 - x$. Substitute into the equation above
\[ \sum_{k=1}^{\infty} k(1 - p)^{k-1} = \frac{1}{p^2}. \]

Multiply both sides by $p$, one yields
\[ \sum_{k=1}^{\infty} kp(1 - p)^{k-1} = \frac{1}{p}. \]

Therefore,
\[ E(X^G) = \sum_{k=1}^{\infty} kp(1 - p)^{k-1} = \frac{1}{p}. \]

(e) Multiply $x$ to both sides of the equation
\[ \sum_{k=1}^{\infty} kx^k = \frac{1}{(1 - x)^2}, \]
one gets
\[ \sum_{k=1}^{\infty} k^2 x^k = \frac{x}{(1 - x)^2}, \]

Differentiate both sides, one obtains
\[ \sum_{k=1}^{\infty} k^2 x^{k-1} = \frac{(1 - x)^2 - 2x(1 - x)(-1)}{(1 - x)^4} = \frac{1 - x^2}{(1 - x)^4} = \frac{1 + x}{(1 - x)^3}. \]
Let $x = 1 - p$, thus $p = 1 - x$. Substitute into the equation above

$$\sum_{k=1}^{\infty} k^2 (1 - p)^{k-1} = \frac{2 - p}{p^3}.$$ 

Multiply both sides by $p$,

$$\sum_{k=1}^{\infty} k^2 p(1 - p)^{k-1} = \frac{2 - p}{p^2} = E((X^G)^2).$$

Therefore,

$$Var((X^G) = E((X^G)^2) - (E(X^G))^2 = \frac{2 - p}{p^2} - \frac{1}{p^2} = \frac{1 - p}{p^2}.$$