Continuous-Time Dynamical Systems: Sketching solution curves and bifurcations

(Due in one week! Thursday Nov 9, 2017)

Materials contained in this assignment are of great importance for this course and will definitely be tested in the final exam!

1. Consider the equation for a population with Allee effect,
\[
\frac{dN}{dt} = N[r - a(N - b)^2],
\]
where \( r, a, b > 0 \) are constants. The goal is to scale the equation and obtain a dimensionless equation that also minimizes the number of parameters. Substitute \( n = N/c \) and \( \tau = t/d \) into the equation and scale it into the dimensionless form
\[
\frac{dn}{d\tau} = n[1 - \alpha(n - 1)^2].
\]
Express \( c, d, \alpha \) in terms of the original model parameters \( r, a, b \).

**Answer:** \( c = b, d = 1/r, \) and \( \alpha = ab^2/r. \)

2. Modeling an autocatalytic chemical reaction (modified from Strogatz 2.3.2). According to the law of mass action, the rate of an elementary reaction is proportional to the product of the concentrations of the reactants. For example, a reaction in which chemical \( C \) is produced by \( A \) reacting with \( B \) with rate constant \( k \) is represented by

\[
k \quad A + B \longrightarrow C
\]
Then, \( d[C]/dt = k[A][B] \), where \([X]\) denotes the concentration of \( X \). Notice that the reaction only goes one way, thus the equation contains only one term on the right-hand-side.

(a) Consider the following reversible chemical reactions (that goes both ways).

\[
\begin{align*}
k_+ \\
A + X & \quad \Rightarrow \quad X + X,
\end{align*}
\]
where \( k_+, k_- \) are rate constants. When considering reversible reactions, the rate of change should include the reaction leading to the production of a chemical positive and the reaction that is removing a chemical negative. Such a reaction is called autocatalytic since one of the product is a reactant which helps turning an \( A \) into an \( X \) while itself remains unchanged. Let \( x = [X] \) and \( a = [A] \). Assume that there is an enormous surplus of \( A \) such that \( a \) remains constant through out the reaction. Write down the differential equation for \( x \).
(b) Find all fixed points (an alternative term often used to refer to steady states in differential equations) for the equation of \( x \) obtained previously and determine their stability.

(c) Sketch the phase portrait for \( x \).

(d) Sketch the graph of \( x(t) \) for some representative choices of initial values for \( x(0) \).

Answer:

(a) \( \dot{x} = k_+ ax - k_- x^2 = f(x) \).

(b) \( x_s = 0 \), \( \frac{ak_+}{k_-} \). \( f'(x) = k_+ a - 2k_- x \).

\( f'(0) = k_+ a > 0 \Rightarrow x_s = 0 \) is unstable.

\( f'(\frac{ak_+}{k_-}) = k_+ a - 2k_+ a = -k_+ a < 0 \Rightarrow x_s = \frac{ak_+}{k_-} \) is stable.

(c) (see figure)

(d) (see figure)

3. Tumor growth model (modified based on Strogatz 2.3.3). The growth of cancerous tumors had been modeled by the Gompertz law

\[
\frac{dN}{dt} = -aN \ln(bN), \quad (a, b > 0),
\]
where $N(t)$ represents the number of cells in the tumor.

(a) Express the right-hand-side (rhs) of the equation into the sum of two parts.

(b) Explain the effects of changing $b$ (i.e., $0 < b < 1$, $b = 1$, $b > 1$) on the linear term.

(c) Explain the effects of changing $N$ (i.e., $0 < N < 1$, $N = 1$, $N > 1$) on the nonlinear term.

(d) Show that if $n = aN$, $\tau = at$, $\rho = b/a$, then the equation becomes $\frac{dn}{d\tau} = -n \ln(\rho n)$. What does it mean to the model as a dynamical system?

(e) For the simplified system $\frac{dn}{d\tau} = -n \ln(\rho n) = f(n)$, graph $n(\tau)$ for various initial values and for two values of $\rho = 1/2$, 2 based on the graph of $f(n)$ as a function of $n$.

(f) Find the reference and read about the good agreement between this model and data. (Aroesty, Lincoln, Shapiro, Boccia. (1973) Math. Biosci.: 17, p.243). (No need to hand in this part of the assignment.)

**Answer:**

(a) $f(N) = -aN \ln(bN) = -a [\ln b + \ln N] = -(a \ln b)N - aN \ln N$.

(b) The linear term is $-(a \ln b)N$ is positive if $0 < b < 1$, and is negative when $b > 1$. For $b = 1$, it vanishes and the linear term no longer exists.

(c) The nonlinear term is $-aN \ln N$ is positive if $0 < N < 1$, and is negative when $N > 1$. It vanishes for $N = 1$.

(d) Plug $n = aN$, $\tau = at$, $\rho = b/a$ into the equation,

$$\frac{dN}{dt} = \frac{dn}{d\tau} = -a \frac{n}{a} \ln(\frac{bN}{a}) = -n \ln(\rho n).$$

(e) The graph of $f(n)$ for $\rho = 1/2$, 2 are given below. The solution curves $n(\tau)$ are given in the sketches.
4. Consider the following fishery model (Strogatz 3.7.3 modified).

\[
\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H,
\]

where \( H (\geq 0) \) represents a constant (density independent) rate of harvest. When \( H = 0 \), \( N \) follows logistic growth.

(a) Express the equation in dimensionless form

\[
\frac{dx}{d\tau} = x(1 - x) - h.
\]

Express \( x, \tau, \) and \( h \) in terms of \( N, t, r, K, \) and \( H \). Use the dimensionless equation to answer all questions below.

(b) Plot the phase portrait for a few representative values of \( h \).

(c) Show that a bifurcation occurs at a certain value \( h_c \). Determine its type and the value of \( h_c \).

(d) Discuss the long term behaviour of the population for \( h < h_c \) and \( h > h_c \). Explain/justify your result using qualitative analysis.

(e) Sketch the bifurcation diagram (i.e. \( x_s \) as a function of \( h \)).
(f) Sketch another bifurcation diagram on which all possible qualitatively distinct phase portraits are stacked.

(g) Identify a severe problem of the model when the solution becomes meaningless.

Solution:

(a) \( x = N/K, \tau = rt, h = H/(rK) \).

(b) \( x = N/K, \tau = rt, h = H/(rK) \).

(c) Solve \( x(1 - x) - h = 0 \) for steady states and yield \( x_s = \frac{1}{2}[1 \pm \sqrt{1 - 4h}] \). We can see that there is no root for \( h > 1/4 \) and there are two roots for \( h < 1/4 \). Thus, \( h_c = 1/4 \) is a SN bifurcation. We can also use the criteria the verify this.

(d) For \( h < h_c \), there exist two fixed points \( x_s^- = \frac{1}{2}[1 - \sqrt{1 - 4h}] \) which is unstable and \( x_s^+ = \frac{1}{2}[1 - \sqrt{1 - 4h}] \) which is stable. Thus, the basin of attraction for \( x_s^+ \) is \( (x_s^-, \infty) \). If the system starts from any point within this basin, it will eventually approach to \( x_s^+ \). Otherwise, it approaches \( -\infty \).

(e) (see the figure above!)

(f) (see the figure on the right!)
(g) A severe problem is that for $h > h_c$, $N(t) \to -\infty$ as $t \to \infty$ even when we start at $N(0) > 0$. This could even happen when $h < h_c$ and $0 < N(0) < x^-$. In realistic systems, when a population size is zero, it should stay there because it is already extinct.

5. Consider a fishery model in which the problem identified in Question 1(f) is fixed (Strogatz 3.7.4 modified).

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) - H \frac{N}{A + N},$$

where $H, A > 0$ are constants and the harvest rate is density-dependent (it increases as $N$ increases) but limited by the value of $H$.

(a) Give a biological interpretation of the parameter $A$; what does it measure? Explain why $H$ sets the limit (i.e. maximum) for the rate of harvest.

(b) Express the equation in dimensionless form

$$\frac{dx}{d\tau} = x(1 - x) - h \frac{x}{a + x}.$$  

Express $x, \tau, h$ and $a$ in terms of $N, t, r, K, H$ and $A$. Use the dimensionless equation to answer all questions below.

(c) Show that the system can have one, two, or three fixed points, depending on the values of $a$ and $h$. Determine the stability of each possible fixed point.

(d) Analyze the dynamics near $x = 0$ and show that a bifurcation occurs when $h_c = a$ (consider $a$ a known constant). Determine its type.

(e) Show that another bifurcation occurs when $h_c = (a + 1)^2/4$ for values of $a < a^*$, where the value of $a^*$ is to be determined. Determine the type of this bifurcation point.

(f) Draw the 2D bifurcation diagram in which each set of the two types of bifurcation points are plotted as a curve in the 2-D parameter space $(a, h)$. Then, sketch a 1D bifurcation diagram against the parameter $h$ while choosing the value of $a$ to be a fixed number in the interval $(0, 1)$. Is it possible to achieve more than one stable fixed points (a case often referred to as hysteresis)?

(g) Sketch another copy of the 1D bifurcation diagram obtained above, on which all possible qualitatively distinct phase portraits are stacked.

(h) Explain if and why the problem discussed in Question 4(g) is fixed in this model.

Solution:
(a) Parameter \( A \) defines a special value of \( N \) such that when \( N = A \), the harvest rate is \( H/2 \). Because the value of the function \( N/(A + N) \) is an increasing function of \( N \) and saturates at 1 for infinity large values of \( N \), therefore \( HN/(A+N) \) saturates at the value of \( H \).

(b) \( x = N/K, \, \tau = rt, \, h = H/(rK) \), and \( a = A/K \).

(c) We need to solve

\[
f(x) = x(1 - x) - h \frac{x}{a + x} = 0 \quad \Rightarrow \quad x(1 - x) = h \frac{x}{a + x}.
\]

Now, we see that \( x_s = 0 \) is always a steady state. For non-zero steady states, we divide both sides by \( x \) to yield

\[
1 - x = \frac{h}{a + x} \quad \Rightarrow \quad x^2 - (1 - a)x + h - a = 0.
\]

The latter yield

\[
x_s = \frac{1}{2} \left[ 1 - a \pm \sqrt{(1 + a)^2 - 4h} \right].
\]

Therefore, we conclude: (i) if \( 4h > (1+a)^2 \), there is only one s.s. \( x_s = 0 \); for \( 4h = (1+a)^2 \), \( x_s = 0 \) and \( x_s = \frac{1-a}{2} \) are the two fixed points; (iii) for \( 4h < (1 + a)^2 \), there exists three s.s.: \( x_s = 0 \) and \( x_s^\pm = \frac{1-a}{2} \pm \sqrt{(1+a)^2 - h} \).

To determine the stability, we can either use the plot of \( f(x) \) (which is the best way in this case) or use XPPAUT to get the bifurcation diagram showing the stability. Stability of the steady states in this system can also be determined graphically. See the figure.

We can also do it analytically as follows (more challenging for this example):

\[
f'(x) = 1 - 2x - \frac{ah}{(a + x)^2} = 1 - 2x - \frac{a}{h} \left[ \frac{h}{a + x} \right]^2.
\]
Thus, $f'(0) = 1 - \frac{h}{a}$. Therefore, it is unstable if $h < a$ and stable if $h > a$, which implies that $h_c = a$ is a bifurcation point (a TC point in particular). For $x_s \neq 0$, notice that

$$1 - x_s = \frac{h}{a + x_s}.$$ 

Thus,

$$f'(x_s) = 1 - 2x_s - \frac{a}{h}(1 - x_s)^2 = 2(1 - x_s) - \frac{a}{h}(1 - x_s)^2 - 1 = -\frac{a}{h} \left[ (1 - x_s)^2 - 2\frac{h}{a}(1 - x_s) + \frac{h}{a} \right].$$

Therefore, if $f'(x_s) = 0$ happens, it can only happen at

$$1 - x_s = \frac{h}{a} \pm \sqrt{\frac{h}{a} \left( \frac{h}{a} - 1 \right)}.$$ 

If $h < a$, $f'(x_s) = 0$ is not possible, $f'(x_s) < 0$ for all $x_s \neq 0$, i.e. both $x_s^\pm$ are stable. It is not easy to determine the sign of $f'(x_s^\pm)$ for $h > a$. However, based on the stability change at the TC point ($h = a$) and/or the fact that neighbouring steady states have alternating stability properties, we can figure out that $x_s^-$ is unstable and $x_s^+$ is stable for $a < h < \frac{(1+a)^2}{4}$.

(d) It is a TC point. One can either verify this by checking all criteria TC1 – TC4 (I’m not going to type the details here), or just use the bifurcation diagram to show it. Stability analysis in (c) also indicates that.

(e) Based on the result in (c), $h_c = \frac{(1+a)^2}{4}$ is where two new steady states emerge. Thus, it is a SN point.

(f) See figure for the 2D bifurcation diagram and a 1D bifurcation diagram for values of $0 < a < 1$ fixed and treating $h$ as the parameter. Yes, bistability does occur in this system.
(g) (See figure above right!)

(h) Because now $x_s = 0$ is always a steady state, the population size $x$ will stay non-negative if $x(0)$ is non-negative. Once it reaches zero, it will stay there.