Nonlinear Maps in Discrete-Time Dynamical Systems

(Due in One Week! Thursday Oct 5, 2017)

0.1

A nonlinear map describing the time evolution of an insect population is given by

\[ x_{t+1} = -\frac{1}{8}x_t(x_t^2 - 7x_t + 2), \quad (0 \leq x_t \leq \infty). \]

(a) Plot the graphs of the functions on both sides of the equation above. Hand in a copy of the graph.

(b) Based on the graph, determine how many fixed points are there for this map. And find these fixed points.

(c) Determine the linear stability of each fixed point found in (b) based on the graph.

(d) Determine the long term behaviour of the map (i.e. as \( t \to \infty \)), if (i) \( x_0 = 1 \); (ii) \( x_0 = 2 \); (iii) \( x_0 = 3 \). (Assume that there is neither noise nor other random perturbations in the system.)

Solution:

(a) See figure on the right.

(b) Based on the graph, \( x^* = 0, 2, 5 \) are the 3 fixed points of the map.

(c) Based on the slope of the graph at these fixed points, \( x^* = 0, 5 \) are stable, \( x^* = 2 \) is unstable.

(d) (i) For (i) \( x_0 = 1 \), \( x_t \to 0 \) as \( t \to \infty \). (ii) For \( x_0 = 2 \), \( x_t = x_0 = 2 \) for all \( t > 0 \). (iii) For \( x_0 = 3 \), \( x_t \to 5 \) as \( t \to \infty \).
0.2

**One-locus diploid selection model.** Consider the map that describes the time evolution of the frequency of finding $A$ allele in a diploid population with selection

$$p_{t+1} = \frac{sp_t^2 + p_t(1-p_t)}{sp_t^2 + 2p_t(1-p_t) + t(1-p_t)^2}, \quad (0 \leq p_t \leq 1)$$

where $s = V_{AA}/V_{Aa}$ and $t = V_{aa}/V_{Aa}$ are the survival probabilities of homozygote individuals of $AA$ and $aa$ types, respectively, relative to the survival probability of the heterozygote individuals.

(a) If $s = 1.2$ and $t = 0.8$ (i.e. $V_{AA} > V_{Aa}$ and $V_{aa} < V_{Aa}$), use any method that you have learned to find out all fixed points of the map. Then, determine the stability of each one of them. Predict the long term behaviour of the map for any initial frequency $0 < p_0 < 1$.

(b) If $s = 0.7$ and $t = 1.3$ (i.e. $V_{AA} < V_{Aa}$ and $V_{aa} > V_{Aa}$), use any method that you have learned to find out all fixed points of the map. Then, determine the stability of each one of them. Predict the long term behaviour of the map for any initial frequency $0 < p_0 < 1$.

(c) If $s = 0.7$ and $t = 0.6$ (i.e. $V_{AA}, V_{aa} < V_{Aa}$), use any method that you have learned to find out all fixed points of the map. Then, determine the stability of each one of them. Predict the long term behaviour of the map for any initial frequency $0 < p_0 < 1$.

(d) If $s = 1.5$ and $t = 1.5$ (i.e. $V_{AA}, V_{aa} > V_{Aa}$), use any method that you have learned to find out all fixed points of the map. Then, determine the stability of each one of them. Predict the long term behaviour of the map for initial frequencies (i) $p_0 = 0.2$; (ii) $p_0 = 0.7$.

**Solution:**

(a) There exist two fixed points $p^* = 0$ which is unstable and $p^* = 1$ which is stable. For any initial value $0 < p_0 < 1$, $p_t \to 1$ as $t \to \infty$. (See figure.)

(b) There exist two fixed points $p^* = 0$ which is stable and $p^* = 1$ which is unstable. For any
initial value $0 < p_0 < 1$; $p_t \to 0$ as $t \to \infty$. (See figure.)

(c) There exist three fixed points $p^* = 0$, $1$ which are both unstable and $p^* \approx 0.571428$ (based on iteration equation solver result) which is stable. For any initial value $0 < p_0 < 1$, $p_t \to 0.571428$ as $t \to \infty$. (See figure.)

(d) There exist three fixed points $p^* = 0$, $1$ which are both stable and $p^* = 0.5$ which is unstable. (i) For initial value $p_0 = 0.2$, $p_t \to 0$ as $t \to \infty$. (ii) For initial value $p_0 = 0.7$, $p_t \to 1$ as $t \to \infty$. (See figure.)

0.3

**Newton’s Iteration Equation** Newton’s method is one brilliant method for solving an algebraic equation numerically with a desired accuracy that will be acceptable for most applied problems in science and engineering. The equations to be solved are of the form

$$f(x) = 0,$$

i.e. we want to find all roots of the function $f(x)$. Newton proposed that one can solve all roots of
\( f(x) \) by solving the following iteration equation

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = F(x_n), \quad (f'(x_n) \neq 0 \text{ for all } n),
\]

for an adequately chosen initial value \( x_0 \), i.e. if multiple fixed points exist, an initial value close to one specific fixed point will often (but not necessarily) yield that particular fixed point.

(a) Show that each fixed point of Newton’s map \( x_{n+1} = F(x_n) \) is a solution of the equation \( f(x) = 0 \) (i.e. a root of \( f(x) \)).

(b) Show that any fixed point of Newton’s map is super stable, i.e. if \( x^* = F(x^*) \), then \( F'(x^*) = 0 \).

(c) Now consider finding the roots of the following \( 5^{th} \) degree polynomial function, i.e. solving the equation

\[
f(x) = x^5 - 9x^3 + x^2 + 14x + 1 = 0.
\]

Plot the graph of \( f(x) \) and determine how many roots it has, i.e. how many times the graph of \( f(x) \) crosses the horizontal axis.

(d) Now show that one can turn the equation in (c) into the following form

\[
x = -0.1(x^5 - 9x^3 + x^2 + 4x + 1).
\]

Therefore, we can try to solve the equation by finding the fixed points of the map derived from this equation without using Newton’s method

\[
x_{n+1} = h(x_n) = -0.1(x^5 - 9x^3 + x^2 + 4x + 1).
\]

(e) Plot graphs of the functions on both sides of the map in (d) and determine the number of fixed points of this map and the stability of them. Based on the results, use one sentence to explain why the map that we obtained in (d) is not good for solving the equation (c).

(f) Now, derive Newton’s map for the equation in (c):

\[
x_{n+1} = F(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} = ?
\]

(g) Plot the graphs of both sides of Newton’s map obtained in (f). Determine the number of fixed points and the stability of each one of them. Based on the results, use one sentence to explain why Newton’s map is good for solving the equation in (c).

(h) Solve all fixed points of Newton’s map obtained in (f) with an accuracy of up to 5 digits after the decimal point. These are actually all the 5 solutions of the equation in (c).

Solution:

(a) If \( x^* \) is a fixed point of Newton’s map, then

\[
x^* = F(x^*) = x^* - \frac{f(x^*)}{f'(x^*)} \iff \frac{f(x^*)}{f'(x^*)} = 0 \iff f(x^*) = 0.
\]
(b) Differentiate the right-hand-side of Newton’s map, one obtains

\[ F'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = 1 - 1 + \frac{f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2} \]

Since the only requirement for Newton’s method to work is that \( f'(x) \neq 0 \), therefore,

\[ F'(x^*) = \frac{f(x^*)f''(x^*)}{(f'(x^*))^2} = 0 \] because \( f(x^*) = 0 \).

(c) See figure.

Based on the graph, it crosses the horizontal axis 5 times, thus \( f(x) \) has 5 roots.

(d) Note that

\[ x^5 - 9x^3 + x^2 + 14x + 1 = 0 \quad \Rightarrow \quad 10x = -(x^5 - 9x^3 + x^2 + 4x + 1) \quad \Rightarrow \quad x = -0.1(x^5 - 9x^3 + x^2 + 4x + 1). \]

(e) See figure.

Based on the graph, there are 5 fixed points but only the one near zero is stable and all the other 4 are unstable. Since 4 out of 5 fixed points are unstable, this map will only help solve one root and thus is not a good map for solving the equation in (c).
(f)  
\[ f'(x) = 5x^4 - 27x^2 + 2x + 14. \]

Thus,
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^5 - 9x_n^3 + x_n^2 + 14x_n + 1}{5x_n^4 - 27x_n^2 + 2x_n + 14} \]

(g) See figure.

Based on the graph, there are 5 fixed points and each one of them is stable (actually super stable!) Therefore, Newton’s map is a good map that guarantees to yield all 5 fixed points thus helping us find all 5 roots of the equation in (c).

(h) Under the guidance of the graph in (g):
starting from \( x_0 = -3 \), one finds \( x^*_1 = -2.74953 \);
starting from \( x_0 = -1.5 \), one finds \( x^*_2 = -1.27186 \);
starting from \( x_0 = -0.1 \), one finds \( x^*_3 = -0.0720395 \);
starting from \( x_0 = 1.5 \), one finds \( x^*_4 = 1.57812 \);
starting from \( x_0 = 3 \), one finds \( x^*_5 = 2.51531 \).