(1) Put your name, student number, and signature above.
(2) In all questions, you must show work — i.e. display intermediate results, to get full credit.
(3) Be neat! I will not attempt to decipher messy calculations.
(4) All work you wish to be graded must be placed on this sheet. Scrap work on additional papers/work books is allowed but will not be graded.
(5) Making use of any book, lecture notes, or previous home work answers is NOT ALLOWED.
(6) CAUTION – Candidates guilty of any of the following or similar practises shall be immediately dismissed and shall be liable to disciplinary action: (a) Making use of unauthorized books, papers, notes, etc; (b) Speaking or communicating with other candidates; (c) Purposely exposing written papers to the view of other candidates.
I. Answer “True” or “False” to the statements below. **Put your answers in the boxes.** (2 mark each)

(a) For the initial value problem $\frac{dx}{dt} = x(4 - x^2)$ and $x(0) = 1$, $x(t) \to 0$ as $t \to \infty$.

False.

(b) A pendulum of mass $m$ and length $L$, in the absence of damping and external driving, is described by the second order equation $\frac{d^2 \theta}{dt^2} + \frac{g}{L} \sin \theta = 0$.

True.

(c) Following an appropriate scaling (i.e. dimensional analysis), the equation in (b) can further be transformed into the following system of two 1st order equations: $\frac{d\theta}{d\tau} = v$, $\frac{dv}{d\tau} = -\sin \theta$.

True.

(d) Consider the system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = x^3 - x$. $E(x, y) = \frac{1}{2} y^2 + \frac{1}{2} x^2 - \frac{1}{4} x^4$ is a conserved quantity.

True.
II. These questions do not require lengthy calculations. **Put your answers in the boxes.** (4 marks each)

(a) A fish population in a lake under a constant rate of harvest is described by the differential equation:
\[
\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) - H,
\]
where the unit for \(N\) and \(K\) is million, for \(t\) is year, for \(r\) is year\(^{-1}\), and for \(H\) is million/year. A non-dimensionalized form of the equation is given by
\[
\frac{dx}{d\tau} = x(1-x) - h.
\]
Give the expressions that define \(x\), \(\tau\), \(h\) in terms of \(N\), \(t\), \(r\), \(K\), \(H\).

\[
x = N/K, \quad \tau = rt, \quad h = H/(rK).
\]

(b) Show that \(x_s = \frac{1}{2}\) and \(h_c = \frac{1}{4}\) is a saddle-node bifurcation point for
\[
\frac{dx}{d\tau} = f(x, h) = x(1-x) - h.
\]

SN1: \(f(x_s, h_c) = f\left(\frac{1}{2}, \frac{1}{4}\right) = 0\); SN1: \(f_h = -1 \neq 0\); SN3: \(f_x(x_s, h_c) = 1 - 2x_s = 0\); SN4: \(f_{xx} = -2 \neq 0\).

(c) The normal form of a saddle-node bifurcation is expressed as \(\dot{x} = a(h - h_c) + b(x - x_s)^2\). For the saddle-node point at \((x_s, h_c) = \left(\frac{1}{2}, \frac{1}{4}\right)\) of the equation \(\frac{dx}{d\tau} = x(1-x) - h\), calculate the values of \(a\) and \(b\) and express this equation in its normal form near this saddle-node point.

\[
a = -1, \quad b = -1, \quad \dot{x} = -(h - \frac{1}{4}) - (x - \frac{1}{2})^2.
\]

(d) Determine if \((x_s, r_c) = (0, 0)\) is a bifurcation point of the equation \(\frac{dx}{d\tau} = x(r - x)\). If yes, determine the type (SN or TC or PF) and write down the normal form.

Yes, it is a TC bifurcation. \(\dot{x} = rx - x^2\). 

3
III. Consider the following system of nonlinear differential equations in a rectangular area of the phase plane $-2 \leq x \leq 2$ and $-2 \leq y \leq 1$. 

\[ \frac{dx}{dt} = y - x, \]
\[ \frac{dy}{dt} = x^2 + y. \] 

(a) Draw the nullclines and find all the fixed points. Represent these points with a black dot in the phase plane provided below.

(b) For each fixed point, discuss its linear stability and classify its type based on the linearization of the system. Sketch a portrait of the trajectories near each critical point. Indicate whether it is hyperbolic or non-hyperbolic.

(c) Draw a rough phase portrait for the system using the provided phase plane. Clearly indicate the flow direction in each area separated by the nullclines and on the nullclines. (Try it out on a separate draft before putting down the final picture).

(d) Show that this system of two 1st order equations can be transformed into the following 2nd equation:
\[ \frac{d^2x}{dt^2} = x + x^2 \] (hint: by calculating $\frac{d^2x}{dt^2}$). Show that the system is a conservative system with a potential function $V(x) = -\frac{1}{2}x^2 - \frac{1}{3}x^3$. 

Solution:

(a) In order to find the critical points, we shall find the x- and y-nullclines.
The x-nullcline is: $y = x$ which is the 45 degree line going through (0, 0).
The y-nullcline is: $y = -x^2$ which is parabola that opens downward and passes through (0, 0).
Substitute $y = x$ into $y = -x^2$, we obtain $x(x + 1) = 0$ which yields two critical points (0, 0) and
$(-1, -1)$. See the intersecting points of the two nullclines in the figure.

(b) Let’s begin first by finding the Jacobian matrix:

$$J(x, y) = \begin{pmatrix} -1 & 1 \\ 2x & 1 \end{pmatrix}$$

(i) At (0, 0):

$$J_{(0,0)} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$ The eigenvalues are $\lambda_{1,2} = \pm 1$

Therefore (0, 0) is a saddle point which is unstable. For $\lambda_1 = 1$, $\vec{v}_1(1, 2)^T$ which is an unstable
invariant set. For $\lambda_2 = -1$, $\vec{v}_2(1, 0)^T$ which is the stable invariant set.

(ii) At $(-1, -1)$:

$$J_{(-1,-1)} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}.$$ The eigenvalues are $\lambda_{1,2} = \pm i$.

Therefore $(-1, -1)$ is a center which is neutrally stable. The trajectories near this critical point
rotate around it in a clockwise direction.

(c) See the sketch of phase portrait in the figure.

(d) Differentiate both sides of the first equation $\dot{x} = y - x$ with respect to time, we obtained $\ddot{x} = \dot{y} - \dot{x} =
(x^2 + y) - (y - x) = x^2 + x$.

For a conservative system, $\ddot{x} = -\frac{dV}{dx} = -(\frac{1}{2}x^2 - \frac{1}{3}x^3) = x + x^2$.

(More space for Question III !)
Figure 1: Phase portrait of the system.