Math 345 Final Exam (April 2010)

Last Name: ____________________________ First name: ____________________________

Student #: ____________________________ Signature: ____________________________

Circle your section #:________________

I have read and understood the instructions below:

Please sign: __________________________

Instructions:

1. No notes or books except those mentioned below are to be used in this exam.

2. You are allowed to bring a letter-sized formula sheet and a small-screen, non-graphic, non-programmable calculator.

3. Justify every answer whenever is necessary, and show your work. Unsupported answers will receive no credit.

4. You will be given 2.5 hrs to write this exam. Read over the exam before you begin. You are asked to stay in your seat during the last 5 minutes of the exam, until all exams are collected.

5. At the end of the hour you will be given the instruction “Put away all writing implements and remain seated.” Continuing to write after this instruction will be considered as cheating.

6. Academic dishonesty: Exposing your paper to another student, copying material from another student, or representing your work as that of another student constitutes academic dishonesty. Cases of academic dishonesty may lead to a zero grade in the exam, a zero grade in the course, and other measures, such as suspension from this university.

<table>
<thead>
<tr>
<th>Question</th>
<th>grade</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>24</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>24</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>100</td>
</tr>
</tbody>
</table>
Question 1: [24 marks]

Consider the one-dimensional differential equation

\[ \dot{x} = f(x, r) = x(r + 2x - x^2), \]

where the variable \( x(t) \) and the parameter \( r \) both belong to the real line \( \mathbb{R} \).

(a) Note that \( x_s = 0 \) is a fixed point for all \( r \). (i) Use linear stability analysis to determine the value \( r_c \) at which its stability changes. (ii) Determine the intervals on the \( r \)-axis on which \( x_s = 0 \) is either stable or unstable. (iii) Determine the type of this bifurcation point \((0, r_c)\) (SN, TC, or PF). (iv) Find its normal form.

(b) (i) Find all fixed points for all possible values of \( r \) (clearly state the conditions under which they exist). (ii) Plot the set \( \mathcal{Z} = \{(x, r) : f(x, r) = 0\} \) (horizontal axis \( r \), vertical axis \( x \)). (iii) Sketch all the possible qualitatively different one-dimensional phase portraits for different values of \( r \). (No need to sketch the graph of \( \dot{x} \) versus \( x \)).

(c) Based on results obtained in (b)(iii) or any other method(s), determine the stability of each fixed point in the plot obtained in (b)(ii). Sketch a bifurcation diagram (horizontal axis \( r \), vertical axis \( x \)), showing all fixed points. Draw the stable branch(s) of fixed points with solid curves and the unstable branch(s) with dashed curves. (This diagram is identical in shape as that in (b)(ii) but is different only in separating the stable from the unstable fixed points).

(d) There is another bifurcation point in addition to that in (a). (i) Determine the values of \( x_s \) and \( r_c \) for this fixed point. (ii) Determine its type. (iii) Find its normal form.
Question 2: [16 marks]

Answer “True” or “False” to the statements below. **Put your answers in the boxes.** (4 mark each)

(a) \((x_s, r_c) = (0, 1)\) is a transcritical bifurcation point for the equation \(\dot{x} = r - x - e^{-x}\).

(b) The system \(\dot{x} = -x + 4y\) and \(\dot{y} = -x - y^3\) has no closed orbits.

(c) Consider a map \(x_{n+1} = f(x_n)\). A period-2 cycle (characterized by \(f(p) = q\) and \(f(q) = p\)) is stable if the multiplier \(\lambda = f'(p)f'(q) < 1\).

(d) For the system \(\dot{x} = x(3 - x - 2y)\) and \(\dot{y} = y(2 - x - y)\), the fixed point \((0, 0)\) is a hyperbolic unstable node.
Question 3: [16 marks]

These questions do not require lengthy calculations. **Put your answers in the boxes.** (4 marks each)

(a) The first-order differential equation \( \frac{dz}{dt} = \lambda [N - z - x_0 \exp(-kz/\lambda)] \) can be non-dimensionalized to \( \frac{dx}{d\tau} = a - bx - e^{-x} \) by introducing the non-dimensional variables \( x = z/Z \) and \( \tau = t/T \). Determine the expressions for \( Z, T, a, b \) in terms of \( \lambda, N, x_0, k \).

(b) For the doubling map \( x_{n+1} = f(x_n) \) where \( f(x) = \begin{cases} 2x, & 0 \leq x < 1/2 \\ 2x - 1, & 1/2 \leq x < 1 \end{cases} \). Find out its orbit if \( x_0 = 57/60 \). On which periodic orbit does it evolve to? Determine the stability of this periodic orbit.

(c) Suppose \( \dot{x} = f(x, y), \dot{y} = g(x, y) \). Then, \( \dot{x} = -\nabla V \) (where \( \mathbf{x} = (x, y)^T \) and \( \nabla V = (\partial V/\partial x, \partial V/\partial y)^T \)) implies \( f(x, y) = -\partial V/\partial x \) and \( g(x, y) = -\partial V/\partial y \). These can be “partially integrated” to find \( V \) for gradient systems. Find \( V \) for the system \( \dot{x} = y + 2xy, \dot{y} = x + x^2 - y^2 \).

(d) For the system \( \dot{x} = -xy \) and \( \dot{y} = xy - y \) (\( x, y > 0 \)). Find a quantity \( V(x, y) \) that is conserved (Hint: Form a differential equation for \( dy/dx \). Solve it using separation of variables).

(More space for Question 3)
Consider the quadratic map \( x_{n+1} = f(x_n) = x_n^2 + r \), with \( x, r \in \mathbb{R} \), \( r \) is the parameter.

(a) Find all the fixed points and determine for which values of \( r \) do they exist.

(b) Determine the stability of the fixed points found in (a) as a function of \( r \). Find the value of \( r \) at which the fixed point is superstable (i.e. \( \lambda = f'(x^*) = 0 \)).

(c) Find all the period-2 cycles. (Hint: Suppose \( f(p) = q \) and \( f(q) = p \). Show that \( p, q \) are roots of the equation \((x^2 - x + r)(x^2 + x + r + 1) = 0\)).

(d) Determine the stability of the period-2 cycles as a function of \( r \). Find the value of \( r \) at which the 2-cycle solution is superstable (i.e. \( \lambda = f'(p)f'(q) = 0 \)).

(e) Plot a bifurcation diagram based on the information obtained above for \(-2 \leq r \leq 1\).
(More space for Question 4)
Question 5: \[24 \text{ marks}\]

The Brusselator is a simple model of hypothetical chemical oscillator, named after the home of scientists who proposed it.

\[
\begin{align*}
\dot{x} & = 1 - (b + 1)x + ax^2y, \\
\dot{y} & = bx - ax^2y,
\end{align*}
\]

where \(x, y \geq 0\) are the dimensionless concentrations of two chemicals and \(a, b > 0\) are parameters.

(a) Find all fixed point(s). Use the Jacobian to classify each one and determine for which values of \(a, b\) it is hyperbolic and for which value(s) it is not.

(b) Sketch the nullclines in the first quadrant \(x, y \geq 0\). Determine the direction of the flow in each separate region and on each section of the nullclines.

(c) Construct a trapping region for the flow in the first quadrant \(x, y \geq 0\).

(d) Show that a Hopf bifurcation occurs at some parameter value \(b = b_c\) and express \(b_c\) in terms of parameter \(a\).

(e) Does the limit cycle exist for \(b > b_c\) or \(b < b_c\)? Explain, using the Poincaré-Bendixson theorem.

(f) Find the approximate period of the limit cycle for \(b \approx b_c\) using the imaginary part of the eigenvalues at \(b = b_c\).
(More space for Question 5.)