# Math 215/255 Lecture Notes 2019

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## Contents

# 1 Introduction to Differential Equations

## 1.1 What is a differential equation (DE)?

**A Differential Equation (DE):** An equation that relates one derivative of an unknown function, $y(t)$, to other quantities that often include $y$ itself and/or its other derivatives.

$$y^{(n)} = F(t,\ y,\ y',\ \dots\ ,\ y^{(n-1)}), \tag{1}$$

where $t$ is the independent/free variable (many textbooks prefer to use $x$), $y^{(n)} \equiv \frac{d^n y}{dt^n}$, $(n \geq 1)$ is the $n^{th}$ derivative of the unknown function.

- In (1), we call $y^{(n)}$ the LHS of the equation and $F(t,\ y,\ y',\ \dots\ ,\ y^{(n-1)})$ the RHS of the equation.
- A solution of the DE (1) is a function $y(t)$ that satisfy the DE. In other words, when $y(t)$ is plugged in the DE, it will make LHS=RHS.
- Solving a differential equation is to find all possible functions $y(t)$ that satisfy the equation.

**Example 1.1.1:**

$$N'(t) = kN(t), \qquad (N'(t) \text{ for } \frac{dN(t)}{dt}), \tag{2}$$

models the change of the size of a population $N(t)$ when the average net growth rate is $k$ (typically a known constant).

**Example 1.1.2:**

$$my'' + cy' + ky = F(t), \qquad (y',\ y'' \text{ for } \frac{dy}{dt},\ \frac{d^2y}{dt^2}) \tag{3}$$

models the displacement $y(t)$ of a mass suspended by a spring, subject to an external force $F(t)$. $m,\ c,\ k$ are known constants.

**Example 1.1.3:**

$$c_t = D(c_{xx} + c_{yy} + c_{zz}) \quad \text{also expressed as} \quad \frac{\partial c}{\partial t} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right), \tag{4}$$

models the diffusion of a pollutant in the atmosphere in which $c(t, x, y, z)$ (a multivariable function) describes the concentration of the pollutant at time $t$ and space location $(x, y, z)$. $D$ $(> 0)$ is the diffusion constant of the pollutant.

**Example 1.1.4:**

$$c_x + c_{xx} = c \quad \text{also expressed as} \quad \frac{\partial c}{\partial x} + \frac{\partial^2 c}{\partial x^2} = c \tag{5}$$

where $c(t, x, y, z)$ is a multivariable function.

**Example 1.1.5:**

$$ty^{(4)} - y^2 = 0, \qquad (y^{(4)} \text{ for } \frac{d^4y}{dt^4}), \tag{6}$$

does not necessarily model anything.

## 1.2 Ordinary DE (ODE) versus partial DE (PDE)

**Ordinary Differential Equation (ODE):** a DE in which the unknown is a single-variable function or in case the unknown is a multivariable function but all derivatives are taken w.r.t. the same independent variable.

Among the previous examples, (2), (3), (5), and (6) are ODEs.

**Partial Differential Equation (PDE):** a DE in which the unknown is a multi-variable function and its derivatives w.r.t. more than one independent variables appear in the equation.

Among the previous examples, (4) is a PDE.

**Focus of the present course:** ODEs.

## 1.3 Order of a DE

**Order of a DE:** the order of the highest derivative of the unknown that appears in the equations.

Among the previous examples, (2) is 1st order, (3), (4), and (5) are 2nd order, (6) is 4th order.

**Remark:** An $n^{th}$ $(n > 1)$ order ODE can always be reduced to a system of $n$ 1st order ODEs.

**Example 1.3.1:**

$$my''(t) = -mg \qquad \text{or} \qquad y''(t) = -g \tag{7}$$

models the displacement of a free-falling mass in a frictionless medium on the surface of the earth. $y(t)$ is the vertical displacement of the mass at time $t$. It is a 2nd order ODE.

By introducing a second unknown function $v(t) = y'(t)$ (i.e. its velocity), we turn the above ODE into the following system of two 1st order ODEs involving two unknowns $y(t)$ and $v(t)$:

$$\begin{cases} y'(t) = v(t), \\ v'(t) = -g. \end{cases} \tag{8}$$

Systems of linear ODEs will be covered later in this course.

## 1.4 Linear versus nonlinear ODEs

**Linear vs nonlinear:** An ODE is linear if it can be expressed in the following form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t), \tag{9}$$

where $a_0, a_1, ..., a_{n-1}, f(t)$ are known functions of $t$ but often are known constants, i.e. if only linear combinations of the unknown and its derivatives appear in the equation. Otherwise, it is nonlinear.

Among the previous examples, all but (6) are linear.

**Remarks:**

- Linear ODEs can often be solved in closed forms. We shall devote more than two third of the time in this course on solving linear ODEs.
- Nonlinear ODEs are often too difficult to solve in closed forms. Numerical, qualitative, and approximation methods are often required in solving nonlinear ODEs. We shall only touch a little bit on nonlinear ODEs.

## 1.5 Why should we learn ODEs?

The answers to many social, economic, scientific, and engineering problem are achieved in solving ODEs that correctly model/describe the corresponding problem.

## 1.6 Solutions and solution curves of an ODE

**Solution of an ODE:** *A function $y(t)$ is a solution of an ODE if $y$ and its derivatives are continuous and satisfy the ODE on an interval $a < t < b$.*

**Solution curve:** *The graph of a solution $y(t)$ in the ty-plan is called the solution curve.*

At this point, a few questions should be asked (but not answered). First, does every ODE have a solution? Second, if so, is it the only solution? Last, what is the interval in which the solution is valid?

**Existence and uniqueness:** In eq.(1), if $F$ and its partial derivatives w.r.t. its dependent variables are continuous in the closed $ty$-domain of our interest, the existence and uniqueness of a solution can be guaranteed for any initial condition $(t_0, y(t_0))$ located in the interior of the domain. We shall not get deeper into this in the lecture here. Read Ch.1.2.2 of the online text and/or Ch.2.8 of the Boyce-DiPrima textbook for more details.

**Remark:** Before we learn any of the techniques for solving ODEs, we can always try to guess what the solution could be. Fortunately, we can always verify if a guessed solution is indeed a solution by substituting it into the ODE (by verifying it).

**Example 1.6.1:** Find one solution and then all possible solutions of the following ODE.

$$y'(t) = y(t) \tag{10}$$

**Answer:**

Obviously, $y(t) = 0$ is one but a trivial solution.

Notice that we are looking for a function whose derivative is equal to itself. Only exponential functions have this property. Thus, we realize $y(t) = e^t$ is one non-trivial solution.

A more careful inspection reveals that $y(t) = 2e^t$ is also a solution, further more

$$y(t) = Ce^t$$

for an arbitrary constant $C$. This form actually represents all possible solutions of the ODE including the above two special solutions (for $C = 0$ and $C = 1$, respectively).

**Special solution vs general solution:** Any function that satisfies the ODE is one *special solution*. The expression that represents all possible solutions of an ODE is called the *general solution*. It must contain at least one arbitrary constant.

**Remark:** An ODE usually has a family of infinitely many solutions that differ by one or more constants. The number of arbitrary constants is equal to its order and are often referred to as integration constants since they arise each time we integrate.

## 1.7 Slope fields and solution curves

Consider the following first-order ODE

$$y'(t) = f(t, y). \tag{11}$$

Since its right-hand-side (rhs) is known, we can always explicitly calculate the slope of $y(t)$ (i.e. $y'(t)$) for any given pair of values $(t,\ y)$. In other words, the slope of the solution can always be predetermined.

**Slope field** of the ODE is obtained by the plotting the slope of the solution at regular intervals in the ty-plane. Since the **solution curve** must be tangent to the slope field, it can be obtained by tracing out a curving while making sure that it is tangent to the slope field at every point of its passage.

**Example 1.7.1:** Plot the slope field of the following ODE and trace out the solution curve for $y(0) = 0.5$.

$$y' = y \tag{12}$$

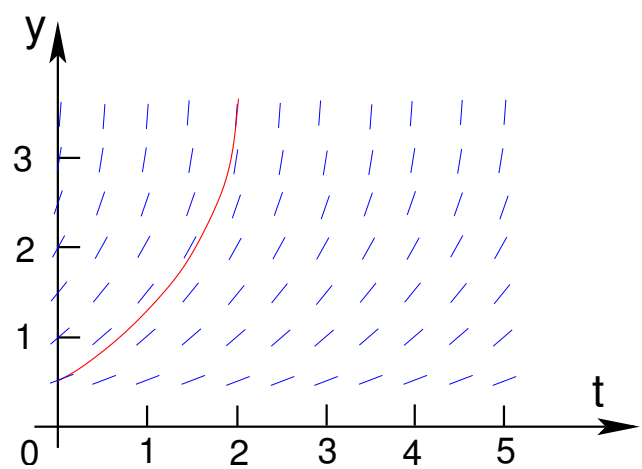



Figure 1: Sketch of the slope field and one solution curve of $y' = y$.

# 2 First-Order ODEs

All first-order ODEs can be expressed in the following 'normal' form:

$$y'(t) = f(t, y(t)). \tag{13}$$

## 2.1 Vanishing derivative or direct integration/antiderivation

**Theorem 2.1.1 (Vanishing Derivative Theorem):**

1. The general solution of $y'(t) = 0$ is $y(t) = C$, $C$ is any constant.
2. The general solution of $y'(t) = f(t)$ ($f$ is continuous) is $y(t) = F(t) + C$, where $F(t) = \int f(t)dt$, and $C$ is an arbitrary constant.

**Proof:**
Part 1. Suppose $y(t)$ is a solution of the ODE $y' = 0$ on a $t$-interval $I$. According to *Mean value Theorem*, if $t_0$ and $t$ are both on $I$, then there is a number $t*$ ($t_0 < t* < t$) such that

$$y(t) - y(t_0) = y'(t*)(t - t_0)$$

Since $y'(t*) = 0$, we see that for all $t$ in $I$,

$$y(t) = y(t_0) = C.$$

Part 2. Let $F(t)$ be an antiderivative of $f$ on $I$, i.e. $F'(t) = f(t)$. Rewrite the ODE $y' = f$ into the following form

$$y' - f = (y - F)' = 0$$

Applying the 1st part of the theorem, we see that for all $t$ in $I$,

$$y(t) - F(t) = C.$$

**Example 2.1.1:** Find the general solution of

$$y'(t) = \cos t. \tag{14}$$

**Answer:** Note that

$$y'(t) = \cos t \qquad \Rightarrow \qquad y'(t) - \cos t = 0 \qquad \Rightarrow \qquad (y(t) - \sin t)' = 0,$$

we see that

$$y(t) - \sin t = C \qquad \Rightarrow \qquad y(t) = \sin t + C.$$

Alternatively,

$$y'(t) = \cos t \qquad \Rightarrow \qquad y(t) = \int \cos t dt = \sin t + C.$$

**Example 2.1.2:** Find the general solution of the population model

$$N'(t) = -kN(t). \tag{15}$$

**Answer:** First rewrite the equation in the following form,

$$N' + kN = 0$$

Multiply both sides by the function $e^{kt}$

$$e^{kt}N' + ke^{kt}N = (e^{kt}N)' = 0.$$

According to the Vanishing Derivative Theorem,

$$e^{kt}N(t) = C \qquad or \qquad N(t) = Ce^{-kt}.$$

**Example 2.1.3:** Find the general solution of

$$3y^2y' = e^t, \qquad \text{(nonlinear!)} \tag{16}$$

**Answer:** Note that $3y^2y' = (y^3)'$, thus the ODE can be rewritten as

$$(y^3)' = e^t \qquad \Rightarrow \qquad y^3 = \int e^t dt = e^t + C \qquad \Rightarrow \qquad y(t) = \sqrt[3]{e^t + C}.$$

## 2.2 Mathematical modeling using differential equations

Modeling is a process loosely described as recasting a problem from its natural environment into a form, called a model, that can be analyzed via techniques we understand and trust (ODEs). It often involves the following steps listed in this very simple example.

**Example 2.2.1:** A simple model of free falling objects.

**Motivations:** Questions to answer: (1) If it takes 3 seconds for a piece of stone to drop from the top of a building, what is the height of the building? (2) If a cat falls from a roof of 4.9 meters high, what is the speed when she hits the ground? Assume that we can ignore air resistance.

**Determine system variables:** One single independent variable is time $t$. The unknown variables (functions) are the vertical position of the free-falling object $y(t)$ and its speed $v(t)$.

**Determine the domain of variation for all variables:** For example, $t$ varies between 0 and 3 seconds for question (1), $y(t)$ varies between 0 and 4.9 meters in question (2).

**Natural law:** *Newton's second law of motion*: A free-falling object near earth's surface moves with constant acceleration $g$ if air resistence can be ignored, where $g$ is the earth's gravitational constant.

**Write down the equation and initial conditions:**:

$$y''(t) = -g, \qquad 0 \leq t \leq T \qquad (T \; is \; the \; falling \; time) \tag{17}$$

$$y(0) = h, \qquad y'(0) = v(0) = 0 \qquad (h = height \; of \; the \; building) \tag{18}$$

**Determine system parameters:** The earth's gravitational constant $g = 9.8 \; m/s^2$, the total time $T = 3 \; s$, the initial height $h = 4.9 \; m$..

**Solve the equation:** Remember, by introducing a new function $v(t) = y'(t)$ (i.e. its speed), we turn this 2nd-order ODE in a system of two 1st order ODEs:

$$\begin{cases} y'(t) = v(t), \\ v'(t) = -g. \end{cases} \tag{19}$$

with initial conditions $y(0) = h$ and $v(0) = 0$. Note that $-gt$ is the antiderivative of $-g$,

$$v'(t) = -g \qquad \Rightarrow \qquad v(t) = -gt + C_1$$

Since $v(0) = 0$, $C_1 = 0$. Thus, $v(t) = -gt$ (unique).

Notice also $-gt^2/2$ is the antiderivative of $-gt$,

$$y'(t) = v(t) = -gt \qquad \Rightarrow \qquad y(t) = -gt^2/2 + C_2$$

Since $y(0) = h$, $C_2 = h$. Therefore, $y(t) = h - gt^2/2$ (unique).

The answer to our first question about the height of the building is found by substituting $t = 3$ $sec$, $g = 9.8$ $m/sec^2$, and $y(t) = 0$ (means object on the ground) into the solution: $h = 44.1$ m! If you feel any doubt about this result, you should blame the assumption that air resistance can be neglected in the model but NOT the math. As to the cat's impact speed, we can first find out how long does it take for her to reach the ground (1 sec for $h = 4.9$ $m$), then to find out that the speed is $v(1) = -9.8$ $m/s$ (minus sign for downward direction).

**Explanation of the results and possibly predictions.**

## 2.3 Brief summary of what we have learned

1. Solving ODEs basically involves finding a function given a constraint on its derivative(s).
2. Integration is often required in the process of solving ODEs.
3. There are usually infinitely many solutions to an ODE that differ by one or more integration constants.
4. An expression of all possible solutions to an ODE is called the *general solution*. Any other solution is called a *special/particular solution*.
5. Unique solution is obtained only when appropriate initial conditions are satisfied.
6. Modeling with ODEs usually follow some standard steps.

## 2.4 Separable Equations

**Definition:** A first-order ODE is called *separable* if it can be expressed as

$$y'(t) = g(t)h(y), \tag{20}$$

where $g(t)$ is a function of $t$ only while $h(y)$ is a function of $y$ only.

**Separation of variables:** For a separable equation,

$$\frac{dy}{dt} = g(t)h(y) \qquad \Rightarrow \qquad \frac{dy}{h(y)} = g(t)dt,$$

which is solved by integrating both sides:

$$\int \frac{dy}{h(y)} = \int g(t)dt.$$

**Example 2.4.1:** The population model

$$\frac{dN}{dt} = -kN, \tag{21}$$

is a separable equation. Solve it using separation of variables.

**Answer:** Separation of variables yields

$$\frac{dN}{N} = -kdt \qquad \Rightarrow \qquad \int \frac{dN}{N} = -\int kdt \Rightarrow \ln N = -kt + C_1 \qquad \Rightarrow \qquad N(t) = e^{C_1 - kt} = Ce^{-kt}.$$

**Example 2.4.2:** Solve the separable equation

$$\frac{dy}{dx} = \frac{x^2}{1+y^2}.$$

**Answer:** Separation of variables yields

$$(1+y^2)dy = x^2dx \qquad \Rightarrow \qquad \int (1+y^2)dy = \int x^2dx \qquad \Rightarrow \qquad y + \frac{y^3}{3} = \frac{x^3}{3} + C,$$

where the solution $y(x)$ is expressed in an implicit form.

**Example 2.4.3**

Solve $y' = 2xe^y(1+x^2)^{-1}$.

Step 1. Separate the variables and express the equation into the following form:

$$e^{-y}dy = \frac{2xdx}{1+x^2}$$

Step 2. Integrate both sides.

$$\int e^{-y}dy = \int \frac{2xdx}{1+x^2}$$

Thus (calculate the integrals $\int e^y dy = e^y$ and $\int \frac{2xdx}{1+x^2} = ln(1+x^2)$),

$$-e^{-y} = ln(1+x^2) + C \qquad or \qquad y = -ln(-ln(1+x^2) - C)$$

This solution is defined only for $-ln(1+x^2) - C > 0$ or $C < -ln(1+x^2)$. For this problem, $y$ can be expressed explicitly in terms of $x$.

**Example 2.4.4**

Solve $y' = \frac{xy}{(1+x)(1+y)}$, with $y(0) = 1$ and $x > 0$. (A problem from the final exam in 1995).

This is an *initial value problem* (IVP). First find the genral solution then determine the constant.
Step 1. Separate the variables and express the equation into

$$\frac{1+y}{y}dy = \frac{x}{1+x}dx$$

Step 2. Integrate both sides.

$$\int (1+\frac{1}{y})dy = \int (1-\frac{1}{1+x})dx$$

Thus,

$$y + \ln y = x - \ln(1+x) + C_0 \qquad or \qquad ye^y = C_1\frac{e^x}{1+x}$$

where $C_1 = e^{C_0}$. This general solution is always valid for $x > 0$ and $C_1 > 0$.
Step 3. Substitute intitial condition into the general solution to determine the value $C_0$ or $C_1$.

$$C_0 = 1 \qquad or \qquad C_1 = e$$

Therefore, the solution for the IVP is:

$$y + \ln y = x - \ln(1+x) + 1 \qquad or \qquad ye^y = \frac{e^{1+x}}{1+x}.$$

In this problem, $y$ can not be solved explicitly in terms of $x$.

**Example 2.4.5**

Solve $y' + p(t)y = 0$

This is a homogeneous, first-order, linear ODE. We can see that it is also *separable*.

$$\frac{1}{y}dy = -p(t)dt$$

Integrate both sides, we get

$$\ln y = -P(t) + C_0$$

Or by taking the exponential of both sides

$$y(t) = Ce^{-P(t)} \qquad\qquad C = e^{C_0}$$

**Summary:** Separable equations are ODEs that can be either linear or nonlinear. This a special class of ODEs that can be solved by applying the method called *separation of variables*. However, linear ODEs can readily be solved in closed forms using other methods as discussed below.

## 2.5 First-order linear ODEs

All first-order linear ODEs can be expressed in the following 'normal' form:

$$y' + p(t)y = q(t), \tag{22}$$

where $p(t)$ and $q(t)$ are known functions of $t$.

**Method of integrating factor:** Main idea is to turn the lhs of eq.(22) into the derivative of a single function $F(t, y)$ such that eq.(22) is turned into the form

$$F' = f(t),$$

which can be solved by direct integration.

**Integrating factor:** the idea expressed above is always achievable by multiplying both sides of eq.(22) by its *integrating factor*

$$e^{P(t)}, \qquad \text{with} \qquad P(t) = \int p(t)dt \qquad (i.e.\ P'(t) = p(t)),$$

which yields

$$e^{P(t)}y' + p(t)e^{P(t)}y = q(t)e^{P(t)} \qquad \Rightarrow \qquad \left[e^{P(t)}y\right]' = q(t)e^{P(t)} \qquad \Rightarrow \qquad e^{P(t)}y(t) = R(t) + C,$$

where $R(t) = \int q(t)e^{P(t)}dt$. Dividing both sides by $e^{P(t)}$, we obtain

$$y(t) = e^{-P(t)}\left[R(t) + C\right]. \tag{23}$$

**Example 2.5.1**

$$y' + y = t.$$

**Answer:** $p(t) = 1$, $q(t) = t$. The integrating factor is

$$e^{\int p(t)dt} = e^{\int dt} = e^t.$$

Multiply both sides by $e^t$ yields

$$e^t y' + e^t y = te^t \qquad \Rightarrow \qquad (e^t y)' = te^t \qquad \Rightarrow \qquad e^t y = \int te^t dt = te^t - e^t + C.$$

Thus, $y(t) = e^{-t}[te^t - e^t + C]$.

**Theorem 2.5.1 (General Solution Theorem):** The general solution of the linear ODE

$$y' + p(t)y = q(t),$$

where $p(t)$ and $q(t)$ are continuous on a $t$-interval $I$, has the form

$$y(t) = e^{-P(t)}[R(t) + C],$$

where $P(t) = \int p(t)dt$ and $R(t) = \int q(t)e^{P(t)}dt$, and $C$ is any constant.

**Proof.** Multiply both sides of the equation by the *integrating factor* $e^{P(t)}$,

$$e^{P(t)}(y' + p(t)y) = e^{P(t)}q(t)$$

This equation can be re-written in the following form

$$(e^{P(t)}y)' = e^{P(t)}q(t)$$

Applying the *Vanishing Derivative Teorem*, we get

$$e^{P(t)}y(t) = R(t) + C$$

Multiply both sides by the nonzero function $e^{-P(t)}$. End of proof.

**Remarks:** This theorem tells us that to solve a linear ODE, we only need to calculate the antiderivatives (indefinite integrals) of the two functions $p(t)$ and $e^{P(t)}q(t)$.

**Example 2.5.2:** Solve $y' - 2y = 4 - t$.

**Answer:** Notice that $p(t) = -2$ and $q(t) = 4 - t$.

$$P(t) = \int (-2)dt = -2t \qquad \Rightarrow \qquad e^{P(t)} = e^{-2t}.$$

$$R(t) = \int q(t)e^{P(t)}dt = \int (4-t)e^{-2t}dt = \frac{1}{2}(t-4)e^{-2t} + \frac{1}{4}e^{-2t}.$$

$$y(t) = e^{-P(t)}[R(t) + C] = Ce^{2t} + \frac{1}{2}t - \frac{7}{4}.$$

**Example 2.5.3:** Solve $ty' + 2y = \frac{\sin t}{t}$. (A problem from the 1995 final exam!)

**Answer:**

Step 1: Transform the equation into the normal form given by eq.(22) and identify the $p(t)$ and $q(t)$ for this problem. In this case, we have to devide both sides by $t$. (Be careful that we can do this only for $t \neq 0$! At $t = 0$ the equation is reduced to $2y(0) = 1$ (remember $lim_{t\to 0}\frac{sint}{t} = 1$)).

$$y' + (\frac{2}{t})y = \frac{\sin t}{t^2}$$

Thus, $p(t) = 2/t$ and $q(t) = \sin t/t^2$ for this equation.

Step 2: Calculate the antiderivatives.

$$P(t) = \int \frac{2}{t}dt = \ln(t^2), \qquad \text{thus the integrating factor is} \qquad e^{P(t)} = t^2.$$

$$R(t) = \int (e^{P(t)})\frac{\sin t}{t^2}dt = \int (t^2)\frac{\sin t}{t^2}dt = \int \sin t dt = -\cos t$$

Therefore, the solution for $t \neq 0$ is

$$y(t) = \frac{C - \cos t}{t^2}$$

Notice that for $t = 0$ this solution exists only when $C = 1$. In this case, $y(0) = 1/2$ (remember $cost = 1 - t^2/2! + t^4/4! - \cdots$.) This is in agreement with the original equation. Therefore, there is only one single solution (when $C = 1$) for this equation in the domain $(-\infty < t < +\infty)$. However, on either $(-\infty < t < 0)$ or $(0 < t < +\infty)$ there exists a whole class of solution for any $C$.

Step 3: Verify the result by substituting it back into the original ODE.

Note that we can always check if our calculation is correct! The special concern here is not to forget to discuss the special situations like $t = 0$.

## 2.6 Existence and uniqueness

**Thoerem 2.6.1 (for linear 1st-order ODEs):** If $p(t)$, $q(t)$ are continuous on an open interval $\alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique solution $y = \phi(t)$ that satisfies the following initial value problem (IVP):

$$\begin{cases} y' + p(t)y = q(t), \\ y(t_0) = y_0 \quad . \end{cases} \tag{24}$$

**Proof:** Solve explicitly $y = \phi(t)$ using method of integrating factor.

**Thoerem 2.6.2 (for general 1st-order ODEs):** If $f$ and $\frac{\partial f}{\partial y}$ are continuous in the rectangle: $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point $(t_0, y_0)$ in the $ty$-plane. Then, in some interval $t_0 - h < t < t_0 + h$ $(h > 0)$ contained in $\alpha < t < \beta$, there exists a unique solution $y = \phi(t)$ that satisfies the following IVP:

$$\begin{cases} y' = f(t, y), \\ y(t_0) = y_0 \quad . \end{cases} \tag{25}$$

**Proof:** Read 2.8 of Boyce and DiPrima.

**Example 2.6.1:** Find an interval on which

$$\begin{cases} ty' + 2y = 4t^2, \\ y(1) = 2 \quad . \end{cases} \tag{26}$$

has a unique solution.

**Answer:** Rewrite it in normal form: $y' + \frac{2}{t}y = 4t$. Note that

$$q(t) = 4t$$

is continuous everywhere, but

$$p(t) = \frac{2}{t}$$

is continuous in either $(-\infty,\ 0)$ or $(0,\ +\infty)$ but not at $t = 0$. But only $(0,\ +\infty)$ contains $t_0 = 1$. So, the above IVP has a unique solution on interval $(0,\ +\infty)$.

**Example 2.6.2:** Find a rectangle in the $xy$-plane in which the following IVP has a unique solution.

$$\begin{cases} y' = \frac{3x^2+4x+2}{2(y-1)}, \\ y(0) = -1 \quad . \end{cases} \tag{27}$$

**Answer:** For this 1st-order ODE, the $rhs$ is given by

$$f(x, y) = \frac{3x^2 + 4x + 2}{2(y - 1)}$$

which is continuous for all $-\infty < x < \infty$. But both $f$ and $\frac{\partial f}{\partial y}$ are discontinuous at $y = 1$. Thus, a unique solution is possible for either $-\infty < y < 1$ or $1 < y < \infty$. But only $-\infty < y < 1$ contains $y_0 = -1$. Therefore, the IVP has a unique solution in the rectangle: $-\infty < x < \infty$ and $-\infty < y < 1$.

## 2.7 Numerical methods

In many practical science, engineering, and social economical application, the ODEs are often too hard or impossible to solve exactly in closed forms. Fortunately, one can always solve an ODE using numerical methods on computers.

**Motivation/goal:** Given the following IVP defined on the interval $a \le t \le b$

$$\begin{cases} y' = f(t, y), \\ y(t_0) = y_0 \quad . \end{cases} \tag{28}$$

solve *approximately* the values of $y(t)$ on discretized points of $t$ (i.e. $a \le t_0,\ t_1,\ \dots,\ t_N \le b$) with *desired accuracy* and *explicit error estimates*.

**Standard procedure:**

- **Step 1:** Discretize the interval $[a,\ b]$ into $N$ (usually $N >> 1$ is a very big integer) subintervals often of equal length

  $$h = \frac{t_N - t_0}{N} = \frac{b-a}{N},$$

  $h$ (usually $0 < h << 1$ is a very small number) is called the *step-size*.

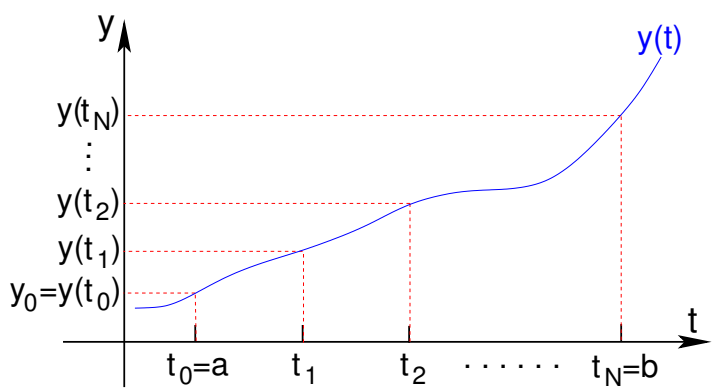



Figure 2: Discretization of the interval $[a,\ b]$.

- **Step 2:** Select one iteration scheme (the actual numerical method) from a collection of known schemes to calculate $y_i$ $(i = 1,\ 2,\ \dots,N)$ sequentially

  $$y_0 \text{ (given)} \quad \to \quad y_1 \quad \to \quad y_2 \quad \to \quad \cdots \quad \to \quad y_N$$

  such that

  $$y_i \approx y(t_i), \qquad (i = 1,\ 2,\ \dots,N).$$

- **Step 3:** Determine the upper bound of truncation errors based on the step-size and the specific iteration method used. When the iteration method is fixed, usually the smaller the step-size $h$ the better the sequence of numbers $y_i$ $(i = 1,\ 2,\ \ldots\ , N)$ approximates the exact solution $y(t)$ itself in $[a,\ b]$.

**Euler's method:** The simplest method based on tangent-line approximation.

Now we focus on how to calculate the value $y_i$ based on the previous value $y_{i-1}$.

**Basic idea behind Euler's method:** If the step-size is very small, the solution curve is segmented into $N$ very small pieces by the red dashed vertical lines (see Fig. 3) drawn at $t = t_i$ $(i = 1,\ 2,\ \ldots\ , N)$. We can approximate each segment by a straight line tangent to the curve (see Fig. 3 for a schematic diagram).

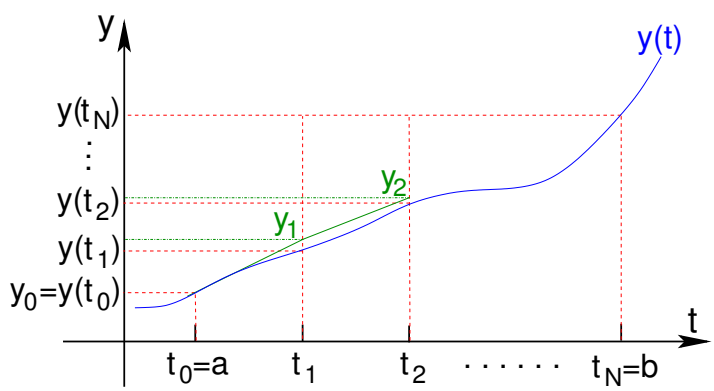



Figure 3: Euler's/tangent-line method demonstrated. Green solid lines are the tangent lines while dash-dotted green lines represent the values of $y_1$ and $y_2$.

**Formula for Euler's method:** can be derived using the discretized version of the ODE

$$\frac{y_{i+1} - y_i}{t_{i+1} - t_i} \approx f(t_i,\ y_i) \qquad \Rightarrow \qquad \frac{y_{i+1} - y_i}{h} \approx f(t_i,\ y_i) \qquad \Rightarrow \qquad y_{i+1} - y_i \approx hf(t_i,\ y_i)$$

Therefore,

$$y_{i+1} = y_i + hf(t_i,\ y_i), \qquad (i = 0,\ 1,\ \ldots\ , N-1) \tag{29}$$

is Euler's iteration scheme (*Euler's method*). Given the value of $y_i$ we can use it to calculate the value of $y_{i+1}$. Thus, given the initial value of $y_0$, we can calculate $y_1,\ y_2,\ \ldots\ ,\ y_N$ sequentially.

**Sources of error:** There are two major sources of error in such numerical methods:

1. *truncation error* that arises when we approximated the curve by its tangent line and it propagates as we iterate further from the initial point (see Fig. 3). This error can be limited by choosing a better iteration scheme and/or reducing the step-size $h$.

2. *round-off error* that occurs when we calculate real numbers with infinitely many decimal points by computer numbers with finite decimal points. This error can be reduced by using computers that allow float numbers with longer digits.

**Truncation error for Euler's method:**

1. *Local truncation error* is the error that occurs at a single iteration step.

$$E_{local} = Kh^2$$

   for Euler's method, where $K$ is a constant.

2. *Global truncation error* is the error that accumulated after $N$ steps of iteration.

$$E_{global} = Mh$$

   for Euler's method, where $M$ is a constant.

**Euler's method is first-order** because its $E_{global}$ is proportional to the first power of step-size $h$. For an $n$-th order method, usually $E_{global} = Mh^n$. For an $n$-th order method, $E_{global}$ is reduced by $n$ orders of magnitude when $h$ is reduced by 10 fold.

**Example 2.7.1:** Use Euler's method with step-size $h = 0.1$ to solve the following IVP on interval $0 \le t \le 1$.

$$\begin{cases} y' + 2y = t^3 e^{-2t}, \\ y(0) = 1 \qquad . \end{cases} \tag{30}$$

**Answer:** For this ODE, $f(t, y) = -2y + t^3 e^{-2t}$, $t_0 = 0$, $y_0 = 1$, $t_{10} = 1$.

$$\begin{aligned} y(0.1) &\approx y_1 = y_0 + hf(t_0,\ y_0) = 1 + 0.1(-2 + 0) = 0.8; \\ y(0.2) &\approx y_2 = y_1 + hf(t_1,\ y_1) = 0.8 + 0.1(-2(0.8) + 0.1^3 e^{-0.2}) = 0.640082; \\ y(0.3) &\approx y_3 = y_2 + hf(t_2,\ y_2) = 0.512602; \\ \cdots & \quad \cdots \\ y(1) &\approx y_{10} = y_9 + hf(t_9,\ y_9) = 0.139779. \end{aligned} \tag{31}$$

Actually, the exact solution of this IVP can be found

$$y(t) = \frac{e^{-2t}}{4}(t^4 + 4)$$

which yields $y(1) = 0.169169....$ The error is quite significant since $h = 0.1$ is not small enough. However, if we made $h = 0.0000001$, we would have to make 1 million times more calculations. This is beyond doing by hand, but fortunately for modern computers this only requires tiny, negligible computation time.

**Convergence of numerical solution to exact solution:**

**Example 2.7.2:** Given the IVP

$$\begin{cases} y' = -2y, \\ y(0) = 1 \quad , \end{cases} \tag{32}$$

of which we know the exact solution is $y(t) = e^{-2t}$. Suppose there is no round-off error, show that the theoretical approximate solution using Euler's method converges to this exact solution as the step-size $h$ approaches zero.

**Answer:** Divide the interval $[0,\ t]$ into $N$ subintervals of length $h$:

$$h = \frac{t - 0}{N} = \frac{t}{N}.$$

Using Euler's iteration scheme:

$$y_{i+1} = y_i + hf(t_i,\ y_i) = y_i + h(-2y_i) = (1 - 2h)y_i, \qquad (i = 0,\ 1,\ \ldots\ , N - 1)$$

Therefore,

$$y_1 = (1 - 2h)y_0 = (1 - 2h),$$

$$
\begin{aligned}
y_2 &= (1-2h)y_1 = (1-2h)^2, \\
y_3 &= (1-2h)y_2 = (1-2h)^3, \\
\cdots &= \cdots, \\
y_N &= (1-2h)y_{N-1} = (1-2h)^N.
\end{aligned}
$$

Thus,

$$
y(t) \approx y_N = (1-2h)^N = (1-\frac{2t}{N})^N = \left[1+\frac{(-2t)}{N}\right]^N \xrightarrow{N\to\infty,\ h=\frac{t}{N}\to 0} e^{-2t},
$$

where the famous limit

$$
\lim_{x\to\infty}(1+\frac{a}{x})^x = e^a
$$

was used.

## 2.8 Exact Equations

To understand this subsection, one needs to know the Chain rule for differentiating a multivariable function.

$$\frac{dz(x(t), y(t))}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

**Example 2.8.1**

Solve $2x + y^2 + 2xyy' = 0$.

This equation is neither linear nor separable. The methods for sovling those types of ODEs can't apply here. Yet, as we will show in the following, this equation can also be solved in closed form. Why? Because it belongs to another special class of equations that happens to be solvable: **exact equations**.

Let's first express it in the following "standard" form:

$$M(x, y) + N(x, y)y' = 0 \tag{33}$$

In this example, $M(x, y) = 2x + y^2$ and $N(x, y) = 2xy$. The special feature of this equation is

$$M_y = N_x (= 2y) \tag{34}$$

This is actually the *definition* of an exact equation. According to this definition, separable equations are also exact because for separable equations $M_y(x) = N_x(y)(= 0)$.

What is good about this feature? It allows us to find a function $\Psi(x, y)$ with its derivative with respect to $x$ equals the lhs of eq.(33), thus having a vanishing derivative.

For this to be true, the following equation should hold.

$$\frac{d\Psi}{dx} \overbrace{=}^{chain\ rule} \Psi_x \frac{dx}{dx} + \Psi_y \frac{dy}{dx} = \Psi_x + \Psi_y y' = M + Ny' = 0 \qquad \Rightarrow \qquad \Psi(x, y(x)) = C. \tag{35}$$

The problem is how to find the function $\Psi$. According to eq.(35),

$$\Psi_x = M \qquad\qquad \Psi_y = N$$

To solve the particular problem in this example, we can first solve $\Psi_y = N$ by integrating both sides with respect to $y$ (treating $x$ as constant while doing this):

$$\int \Psi_y dy = \int N dy = \int 2xydy \;\Rightarrow \qquad \Psi = xy^2 + h(x)$$

where $h(x)$ is an unknown function of $x$. Now substitute the $\Psi$ expression into $\Psi_x = M$, we obtain

$$y^2 + h'(x) = M = 2x + y^2 \;\Rightarrow \qquad h'(x) = 2x \;\Rightarrow \qquad h(x) = x^2 + C$$

Therefore, $\Psi = x^2 + xy^2 + C$ for this example problem. The solution now becomes

$$x^2 + xy^2 = C \qquad (only\ one\ constant\ is\ needed!)$$

Actually the smae method could be used to calculate $\Psi$ for general exact equations.
Step 1. Integrate $\Psi_y = N$ to obtaine $\Psi(x, y) = \int N(x, y)dy + h(x)$.
Step 2. Substitue $\Psi(x, y) = \int N(x, y)dy + h(x)$ into $\Psi_x = M$ to get $h'(x) = M(x, y) - \int N_x(x, y)dy$.
Step 3. Integrate the equation for $h'(x)$. Before that, we have to check if the rhs is indenpendent of $y$ by differnetiating it with respect to $y$

$$\frac{\partial}{\partial y}[M(x, y) - \int N_x(x, y)dy] = M_y - N_x = 0$$

Therefore, the special feature $M_y = N_x$ guarantees that $h(x)$ can be determined by solving $h'(x) = M(x, y) - \int N_x(x, y)dy$. We have proved the following theorem.

**Theorem 2.8.1** Let $M, N, M_y, N_x$ be continuous in the rectangular region $R$ in the $xy$-plane. Then

$$M(x, y) + N(x, y)y' = 0$$

is an exact equation in $R$ iff (if and only if)

$$M_y(x, y) = N_x(x, y)$$

That is, there exists a function $\Psi$ satisfying

$$\Psi_x(x, y) = M(x, y) \qquad \Psi_y(x, y) = N(x, y)$$

Since $\frac{d\Psi}{dx} = M(x, y) + N(x, y)y' = 0$. The general solution of $M(x, y) + N(x, y)y' = 0$ is implicitly defined by $\Psi(x, y) = C$ (C is any constant).

**Example 2.8.2**

Solve $(ycosx + 2xe^y) + (sinx + x^2e^y - 1)y' = 0$.

Step 1. Check if it is exact. $M(x, y) = ycosx + 2xe^y$, $N(x, y) = sinx + x^2e^y - 1$.

$$M_y(x, y) = cosx + 2xe^y = N_x(x, y)$$

Step 2. Start calculating the function $\Psi(x, y)$ with $\Psi_x = M$ or $\Psi_y = N$ which ever is simpler.

$$\Psi_x = M(x, y) = ycosx + 2xe^y$$

Integrating with respect to $x$, we obtain

$$\Psi(x,y) = ysinx + x^2e^y + h(y)$$

Step 3. Substitute $\Psi(x,y) = ysinx + x^2e^y + h(y)$ into $\Psi_y = N$.

$$sinx + x^2e^y + h'(y) = sinx + x^2e^y - 1$$

Thus,

$$h'(y) = -1 \qquad \Rightarrow \qquad h(y) = -y + C_0$$

Step 4. Write down the solution in implicit form.

$$\Psi(x,y) = ysinx + x^2e^y - y + C_0 = C_1 \Rightarrow \qquad ysinx + x^2e^y - y = C \ (C = C_1 - C_0)$$

Therefore, the constant in $h(y)$ can always be obsorbed in the constant on the rhs of the final solution. We do not have to put a constant when solving the ODE for $h$.

**Example 2.8.3.**

Solve $(3xy + y^2) + (x^2 + xy)y' = 0$.

Step 1. Check if it is exact.

$$M_y(x,y) = 3x + 2y \neq N_x(x,y) = 2x + y$$

This is not an exact equation.

However, some equations that are not exact (including this particular one given above), can be transformed into an exact equation by multiplying both sides of the equation by an function $\mu(x,y)$ (also called *integrating factor*). Unfortunately, to find the integrating factor is often as difficult as solving the original ODE. A detail discussion is out of the scope of this course.

However, the the above equation. It is not so hard to find that $x$ is an integrating factor. Since the ODE $(3x^2y + xy^2) + (x^3 + x^2y)y' = 0$ is exact.

$$M_y = 3x^2 + 2xy = N_x$$

Thus,

$$\Psi_y = x^3 + x^2y \Rightarrow \qquad \Psi = x^3y + x^2y^2/2 + h(x)$$

And,

$$3x^2y + xy^2 + h'(x) = 3x^2y + xy^2 \Rightarrow \qquad h'(x) = 0 \Rightarrow \qquad h(x) = C_0$$

Therefore the solution is $x^3y + x^2y^2/2 = C$.

Notice that the same equation can be solved by using other integrating factors. For exampl, $\mu(x,y) = 1/(xy(2x+y))$ is also an integrating factor.

## 2.9 Autonomous equations and phase space analysis

**Def:** The first-order ODE $y' = f(t, y)$ is called an autonomous equation if

$$f(t, y) = f(y)$$

i.e. if the RHS does not explicitly depend on the independent variable $t$. The space (i.e. a line here) span by the unknown function (i.e. the $y$-axis) is call the *phase space* or *state space*.

- The slope/vector filed of an autonomous 1st order ODE does not change as a function of time (i.e. on each line parallel to the $t$-axis, the vectors are all parallel.)
- The direction of *phase flow* is fixed at every point in phase space (i.e it does not change with $t$).
- Non autonomous ODEs can be transformed into autonomous ODEs by introducing one additional variable.

**Example 2.9.1: The logistic equation.**

$y' = y(1 - y)$ is an autonomous equation and is separable. Solve the equation and sketch the solutions for a few representative initial values $y(0) = y_0$.

**Answer:** Separation of variables yields

$$\frac{dy}{y(y-1)} = -dt \qquad \Rightarrow \qquad \int \frac{dy}{y(y-1)} = -\int dt = -t + C_1.$$

The integral on the lhs requires integration by partial fractions:

$$\int \frac{dy}{y(y-1)} = \int \left( \frac{1}{y-1} - \frac{1}{y} \right) dt = \ln(y-1) - \ln y = \ln \frac{y-1}{y}.$$

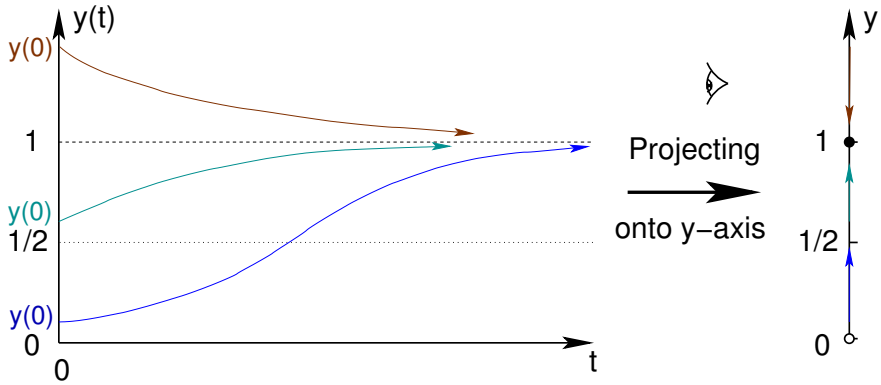
Thus,

$$\ln \frac{y-1}{y} = -t + C_1 \qquad \Rightarrow \qquad \frac{y-1}{y} = Ce^{-t} \qquad \Rightarrow \qquad y(t) = \frac{1}{1 - Ce^{-t}}.$$

Substitute the initial condition $y(0) = y_0$ into the solution, one finds

$$y(t) = \frac{y_0}{y_0 - (y_0 - 1)e^{-t}}.$$

One finds two special solutions: (i) for $y_0 = 0$, $y(t) = 0$ for all $t \geq 0$; (ii) for $y_0 = 1$, $y(t) = 1$ for all $t \geq 0$. One needs a graph plotter to find out how the solution curves looks like for other initial conditions (see figure).




The image on the solutions projected onto the phase space on the right is called the *phase portrait* of the equation. In this phase portrait, the direction of time evolution is represented by arrow directions which are often referred to as *flow direction* or *phase trajectories*.

**Phase space analysis** is a qualitative method we often employ in the study of autonomous ODEs so that we can achieve the same results presented in the figure above without having to solve the equation and/or use a graph plotter to find the solution curves and the phase portrait.

**Step 1.** Finding the *steady states* (also called fixed points, critical points, equilibriums, ...)

**Def:** A **steady state** of the autonomous equation $y' = f(y)$ is a state at which $y' = 0$, i.e. it is a special solution of the equation that remains constant (steady) for all $t \geq 0$. All steady states are found by solving

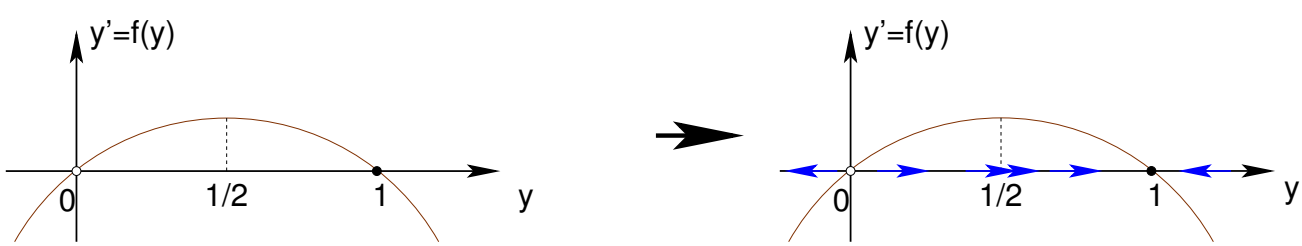
$$f(y) = 0.$$

**Example:** For the logistic equation $f(y) = y(1-y)$.

$$f(y) = y(1-y) = 0 \quad \Rightarrow \quad y_s = 0 \text{ and } y_s = 1$$

are the two steady states.

**Step 2.** Sketch the curve of $f(y)$ and determine the phase flow directions.



Notice that if $f'(y) > 0$ at a steady state, the trajectories flow away in both directions; however, if $f'(y) < 0$, the trajectories flow toward it in both directions.

**Step 3.** Determine the stability of the steady states using the graph of $f(y)$.

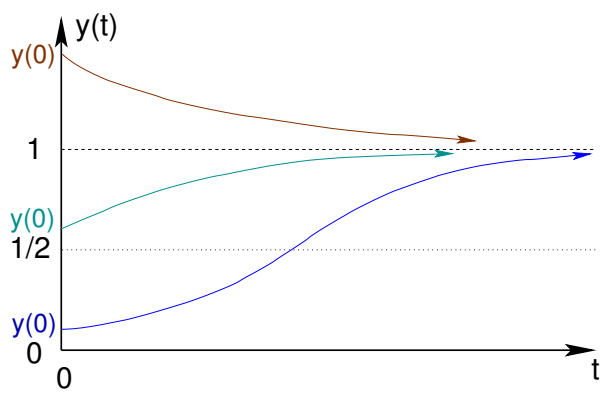
**Def: Stability of a steady state.** A steady state $y_s$ is stable, if trajectories flow toward it from all possible directions; it is unstable if trajectories flow away from it in at least one direction.

In real world, only stable steady states can be observed in experimental observations due to the existence of noises.

For the logistic model $y' = y(1-y)$, plot of $f(y)$ vs $y$ shows clearly that $y_s = 0$ is unstable because the trajectories point away from it. However, $y_s = 1$ is stable because trajectories from all possible directions point toward it.

**Step 4.** Sketch the solution curves $y(t)$ using the graph of $f(y)$.

For an autonomous equation, the plot of $f(y)$ uniquely defines the slope of $y(t)$, i.e. $y'(t)$, at each value of $y$. Therefore, one can always trace out the concavity of the solution $y(t)$ and determine the point(s) of inflection in $y(t)$. This allows us the sketch the shape of the solution curve for each initial condition we choose. The result is the following qualitative graph of the solutions for a few representative initial conditions.



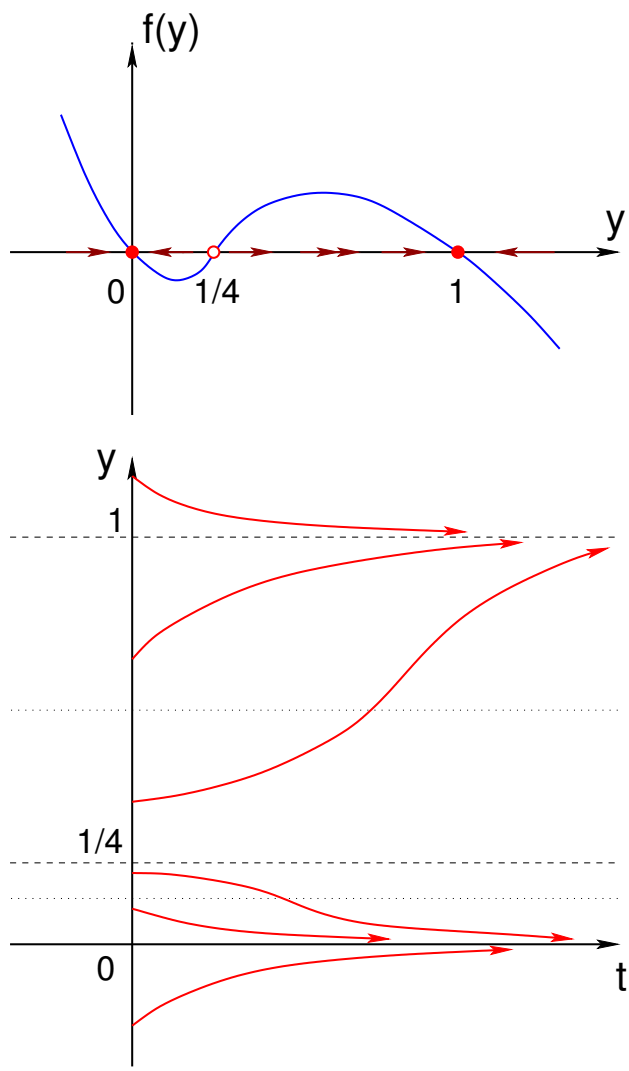

For $y(0) > 0$ close to zero (blue curve in the figure above), the solution is "sigmoidal" because the curve is initially concave up as the slope increases. It reaches maximum slope at $y = 0.5$, after that the slope decreases as it turns into concave down shape. The point of inflection happens at the local maximum of $f(y)$.

**Example 2.9.2:**

Find all steady states of the autonomous ODE $y' = y(y-0.25)(1-y)$ and determine their stability. Sketch the solution curve for each representative choice of the initial value $y(0)$.

**Ans:** It is clear that $y_s = 0,\ 0.25,\ 1$ are the values that make $f(y) = y(y-0.25)(1-y) = 0$. Thus, there are 3 steady states. A sketch of $f(y)$ is given below which clearly show the 3 zeros/steady states. The flow directions show that $y_s = 0$ and 1 are two stable steady states but $y_s = 0.25$ is unstable. This is a system that show the phenomenon of *bistability*.

Based on the sketch of the $f(y)$, one can sketch the solutions curves for 6 representative initial value of $y$ (see the figure below.)




Therefore, phase space analysis allows us to qualitatively find the solutions of the nonlinear ODE without having to solve the equation analytically in closed form or numerically using a computer.

**Linear stability analysis.**

Often $f(y)$ is a function that is not easy to sketch the curve. In this case, we still can determine the stability of the steady states by using analytical method.

Consider an ODE $y' = f(y)$. Let $y_s$ be a steady state, i.e. $f(y_s) = 0$. The stability of $y_s$ is determined by the behaviour of the system near $y_s$. Let $\delta(t)$ ($|\delta(t)| << 1$) be a small difference between $y(t)$ and $y_s$, thus

$$\delta(t) = y(t) - y_s \qquad \Rightarrow \quad y(t) = y_s + \delta(t).$$

Substitute into the ODE:

$$\frac{d(y_s + \delta)}{dt} = f(y_s + \delta), \quad \Rightarrow \quad \delta' = f(y_s) + f'(y_s)\delta + O(\delta^2).$$

Since $|\delta|^2 << |\delta|$, we can ignore higher order terms in $\delta$ and obtain the following *linearized* ODE

$$\boxed{\delta' = f'(y_s)\delta, \qquad \Rightarrow \qquad \delta(t) = \delta_0 e^{f'(y_s)t}.}$$

Therefore,

- If $f'(y_s) > 0$, $\delta(t) \to \infty$ as $t \to \infty$, $y_s$ is unstable.
- If $y_s$ is unstable, phase trajectories (flows) move away from it in at least one directions.
- If $f'(y_s) < 0$, $\delta(t) \to 0$ as $t \to \infty$, $y_s$ is stable.
- If $y_s$ is stable, phase trajectories (flows) converge (move) toward it from all possible directions.
- If $f'(y_s) = 0$, neutral or stability determined by higher order terms (if exist).
- Unstable steady states, under normal conditions, cannot be detected in experimental settings or numerical simulations.

**Application to Example 2.9.1:** For logistic equation, $f(y) = y(1-y)$, $y_s = 0,\ 1$ are the steady states.

$$f'(y) = 1 - y + y(-1) = 1 - 2y \qquad f'(0) = 1 > 0 \quad \text{and} \quad f'(1) = -1 < 0.$$

Thus, $y_s = 0$ is unstable and $y_s = 1$ is stable.

**Application to Example 2.9.2:** For this example, $f(y) = y(y-0.25)(1-y)$, $y_s = 0,\ 0.25,\ 1$ are the steady states.

$$f'(y) = (y-0.25)(1-y) + y(1-y) + y(y-0.25)(-1)$$
$$f'(0) = -0.25 < 0 \text{ (stable)}; \quad f'(0.25) = 0.25 \times 0.75 > 0 \text{ (unstable)}; \quad f'(1) = -0.75 < 0 \text{ (stable)}.$$

## 2.10 Applications of first-order ODEs

**Example 2.10.1: Terminal speed problem.**

A paratrooper jumps out of an airplane at altitude $h$. With a fully open parachute, the air resistance is proportional to the speed with a constant $mk$, where $m$ is the mass of the trooper and constant $k > 0$ is related to the size, shape, and other properties of the parachute. Assume that initially vertical speed of the trooper is zero and that $h$ is large enough so that a terminal constant speed is reached long before the trooper hits the ground. Find:

(a) the speed of the trooper as a function of time $v(t)$;

(b) the terminal speed $v_t = v(\infty)$.

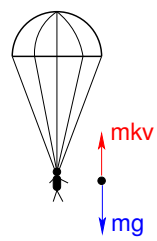



Figure 4: Forces experienced by a paratrooper.

**Answer:** Based on Newton's law of motion

$$my'' = mv' = -mg - mkv \qquad \Rightarrow \qquad v' = -g - kv \qquad \Rightarrow \qquad v' + kv = -g,$$

where "-" of the first term implies the positive direction of the $y$ coordinate is upward, "-" of the second term means the air resistance is alway opposite to the direction of the speed $v$. The initial condition is $v(0) = 0$.

This is a first-order, linear ODE with $p(t) = k$ and $q(t) = -g$. Thus, the integrating factor $e^{P(t)}$ and $R(t)$ are

$$e^{\int p(t)dt} = e^{kt}, \qquad \Rightarrow \qquad R(t) = \int (-g)e^{kt}dt = -\frac{g}{k}e^{kt}.$$

Therefore the general solution is

$$v(t) = e^{-kt}[R(t) + C] = Ce^{-kt} - \frac{g}{k}.$$

Using the initial condition $v(0) = 0$, we obtain $C = \frac{g}{k}$. Thus

$$v(t) = \frac{g}{k}[e^{-kt} - 1] \qquad \xrightarrow{t\to\infty} \qquad -\frac{g}{k}.$$

The terminal speed is $v(\infty) = -\frac{g}{k}$, where the "-" sign implies it is downward.

**Example 2.10.2: Mixing problem.**

Salt solution of concentration $s$ ([kg]/[L]) is continuously flown into the mixing tank at a rate $r$ ([L]/[min]). Solution in the tank is continuously stirred and well mixed. The rate for the outflow of the well-mixed solution is also $r$. Initially (i.e., at $t = 0$), the volume of the solution inside the tank is $V$ ([L]) and the total amount of salt is $Q_0$ ([kg]). Find:

(a) the total amount of salt in the tank as a function of time $Q(t)$;

(b) the amount of salt a very long time after the initiation of the experiment, $Q(\infty)$.

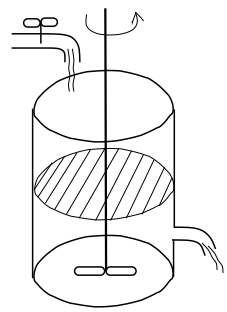


Figure 5: Mixing salt solution in a stirred tank.

**Answer:** Because the rates of in and out flows are identical, the volume remains unchanged in the experiment, $V = const$.

$$\frac{dQ}{dt} = \text{rate in } - \text{ rate out} = rs - r\frac{Q}{V}, \qquad \Rightarrow \qquad Q' + \frac{r}{V}Q = rs.$$

This is a first-order, linear ODE with $p(t) = \frac{r}{V}$ and $q(t) = rs$. Thus, the integrating factor $e^{P(t)}$ and $R(t)$ are

$$e^{\int p(t)dt} = e^{\frac{r}{V}t}, \qquad \Rightarrow \qquad R(t) = \int rse^{\frac{r}{V}t}dt = sVe^{\frac{r}{V}t}.$$

Therefore the general solution is

$$Q(t) = e^{-\frac{r}{V}t}[R(t) + C] = sV + Ce^{-\frac{r}{V}t}.$$

Using the initial condition $Q(0) = Q_0$, we obtain $C = Q_0 - sV$. Thus

$$Q(t) = sV + (Q_0 - sV)e^{-\frac{r}{V}t}$$

is the amount of salt in the tank as a function of time. Since

$$Q(t) = sV + (Q_0 - sV)e^{-\frac{r}{V}t} \qquad \xrightarrow{t\to\infty} \qquad sV,$$

after a very long time the total amount of salt in the tank is $Q(\infty) = sV$.

**Example 2.10.3: Escape speed.**

A bullet is shot vertically into the sky. Assume that air resistance can be ignored. Answer the following questions:

1. If the initial speed is $v_0$, at what height, $h_{max}$, it reverses direction and starts to fall?
2. What should the value of $v_0$ be in order to make $h_{max} = \infty$? (i.e., the escape speed $v_e$.)

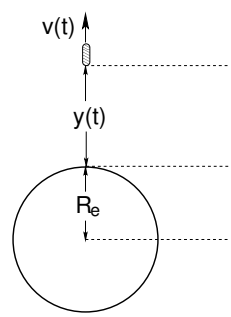



Figure 6: Escaping bullet.

**The known related quantities:**

- $m$=mass of the bullet.
- $G$=universal gravitational constant.
- $R_e$=radius of the earth.
- $M_e$=mass of the earth.
- $g = \frac{GM_e}{R_e^2}$=gravitational constant on earth's surface.

**Answer:** based on Newton's law

$$my'' = F \qquad \Rightarrow \qquad mv' = -\frac{mM_eG}{(R_e+y)^2} \qquad \Rightarrow \qquad \frac{dv}{dt} = -\frac{M_eG}{(R_e+y)^2}.$$

Note that the ODE as it is contains two unknown functions $y(t)$ and $v(t)$. Using the chain rule

$$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}.$$

This expression allows us to solve $v$ as a function of $y$ since $t$ is not explicitly involved in the equation. Now, we have

$$v\frac{dv}{dy} = -\frac{M_eG}{(R_e+y)^2}.$$

This is a separable equation. Separation of variables yields

$$vdv = -\frac{M_e G}{(R_e + y)^2} dy \qquad \Rightarrow \qquad \int vdv = -\int \frac{M_e G}{(R_e + y)^2} dy$$

which leads to

$$\frac{1}{2}v^2 = \frac{M_e G}{R_e + y} + C = \frac{M_e G}{R_e^2} \frac{R_e^2}{R_e + y} + C = \frac{gR_e^2}{R_e + y} + C.$$

Using the initial condition $v(0) = v_0$, we obtain $C = \frac{1}{2}v_0^2 - gR_e$. Thus,

$$v^2 = \frac{2gR_e}{1 + y/R_e} + v_0^2 - 2gR_e.$$

1. To find $h_{max}$, plug in $v(h_{max}) = 0$. We obtain $h_{max} = \frac{R_e v_0^2}{2gR_e - v_0^2}$ and $v_0^2 = \frac{2gR_e h_{max}}{R_e + h_{max}}$.

2. To find the escape speed $v_e$, plug in $h_{max} = \infty$. We obtain $v_e^2 = 2gR_e \Rightarrow v_e = \sqrt{2gR_e} \approx 11.1\ km/s$.

# 3 Second-Order Linear ODEs

## 3.1 Introduction

A second-order ODE can be generally expressed as

$$y'' = f(t,\ y,\ y'). \tag{36}$$

If $f(t,\ y,\ y') = f(y,\ y')$ (i.e. its rhs is explicitly independent of $t$), it is called an *autonomous* ODE. Otherwise, it is *non-autonomous*.

A second-order *linear* ODE is often expressed in the following "normal" form

$$y'' + p(t)y' + q(t)y = g(t), \tag{37}$$

where $p(t),\ q(t),\ g(t)$ are known functions of $t$. Basically, if these three functions are continuous in an interval containing $t_0$, a unique solution exists for proper initial conditions $y(t_0) = y_0,\ y'(t_0) = y_1$ in the neighbourhood of $t_0$ (read Boyce and DiPrima for more details).

**Homogeneous vs nonhomogeneous:** If $g(t) = 0$ in eq.(37), it is *homogeneous*; otherwise, it is *nonhomogeneous*.

**Linear operator and simplified expression for 2nd-order linear ODEs:**
Define a linear operator

$$L \equiv \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t) \tag{38}$$

such that

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = y'' + p(t)y' + q(t)y. \tag{39}$$

Thus, the 2nd-order linear ODE in eq.(37) is shortened to

$$L[y] = g(t). \tag{40}$$

**Theorem 3.1.1 Principle of superposition:** If $y_1(t)$ and $y_2(t)$ are both solutions of the homogeneous equation

$$L[y] = 0,$$

then a linear combination (superposition) of the two

$$y(t) = C_1y_1(t) + C_2y_2(t)$$

is also a solution of the equation, where $C_1,\ C_2$ are arbitrary constants.

**Proof:**
$y_1$ is a solution $\Rightarrow L[y_1] = 0$.
$y_2$ is a solution $\Rightarrow L[y_2] = 0$.
Now,

$$L[C_1y_1 + C_2y_2] \overset{\text{substitute in \& reorganize}}{=} C_1L[y_1] + C_2L[y_2] = C_1 \times 0 + C_2 \times 0 = 0,$$

therefore, $y(t) = C_1y_1 + C_2y_2$ is also a solution.

**Definition: linear dependence (LD) vs linear independence (LI).** On a given interval $I = (a,\ b)$, tow functions $f(t)$ and $g(t)$ are *linearly dependent* (LD) if

$$k_1f(t) + k_2g(t) = 0 \qquad (\text{for all } t \in I)$$

can be satisfied by two constants $k_1$ and $k_2$ that are not both zero (i.e., one is the constant multiple of the other). Otherwise, if it is satisfied only when $k_1 = k_2 = 0$ (i.e., one is not the constant multiple of the other), then the two are *linearly independent* (LI).

**Example 3.1.1:** Determine whether the following pairs of functions are LI?

1. $f(t) = e^{2t}$ and $g(t) = e^{-t}$.
2. $f(t) = e^{3t}$ and $g(t) = e^{3(t+1)}$.

**Answer:**

1. $k_1e^{2t} + k_2e^{-t} = e^{-t}[k_1e^{3t} + k_2] = 0$ if and only if (iff) $k_1 = k_2 = 0$. So, they are LI.
2. $(-e^3)e^{3t} + e^{3(t+1)} = 0$ for $k_1 = -e^3$ and $k_2 = 1$, they are LD.

Later, we shall introduce a more straight forward method for verifying the linear independence of two functions.

**Theorem 3.1.2 The general solution of** $L[y] = 0$ is

$$y(t) = C_1y_1(t) + C_2y_2(t), \tag{41}$$

where $C_1,\ C_2$ are arbitrary constants, $y_1(t),\ y_2(t)$ are two LI solutions of $L[y] = 0$ and are referred to as a *fundamental set* of solutions.

**Remark:** Based on this theorem, finding the general solution of $L[y] = 0$ is reduced to finding two LI solutions.

## 3.2 Homogeneous, 2nd-order, linear ODEs with constant coefficients

If in the homogeneous 2nd-order linear ODE $L[y] = 0$, both $p(t)$ and $q(t)$ are constants, the equation becomes an ODE with constant coefficients

$$ay'' + by' + cy = 0, \tag{42}$$

where $a\ (\neq 0),\ b,\ c$ are constants.

**Question:** How to solve eq.(42)?

**Surprising insight** can be achieved by a verbal interpretation of eq.(42):

- Eq.(42) says a linear combination of $y'',\ y',\ y$ is equal to zero $\Longleftrightarrow$
- $y'',\ y',\ y$ are LD on each other $\Longleftrightarrow$
- $y'',\ y',\ y$ are constant multiples of each other $\Longleftrightarrow$
- $y'',\ y',\ y$ only differ by a multiplicative factor $\Longleftrightarrow$
- $y'',\ y',\ y$ are all exponential functions of the form $e^{rt}$.

**Conclusion:** For eq.(42), we always look for solutions of the form $y(t) = e^{rt}$.

Substitute $y(t) = e^{rt}$ into eq.(42), we obtain

$$(ar^2 + br + c)e^{rt} = 0 \qquad \Longleftrightarrow \qquad ar^2 + br + c = 0, \tag{43}$$

often referred to as the *characteristic equation* of eq.(42).

The quadratic characteristic equation eq.(43) often yields two roots:

$$r_{1,\ 2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Both $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ are solutions to eq.(42). If $r_1 \neq r_2$, the two are LI and form a fundamental set. In this case the general solution to eq.(42) is

$$y(t) = c_1 y_1 + c_2 y_2 = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \tag{44}$$

where $c_1,\ c_2$ are arbitrary constants.

**Example 3.2.1** Find the general solution of

$$y'' + 5y' + 6y = 0.$$

**Answer:** Let $y(t) = e^{rt}$. Plug into the ODE, we obtain the following characteristic equation

$$r^2 + 5r + 6 = 0 \qquad \Longrightarrow \qquad (r+3)(r+2) = 0,$$

which yields $r_1 = -3$ and $r_2 = -2$. Thus, both

$$y_1(t) = e^{r_1 t} = e^{-3t} \qquad \text{and} \qquad y_2(t) = e^{r_2 t} = e^{-2t}$$

are solutions to the ODE and are LI of each other (because $r_1 \neq r_2$). Thus, they form a fundamental set. The general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{-3t} + c_2 e^{-2t}.$$

**Example 3.2.2** Solve the following IVP

$$\begin{cases} 4y'' - 8y' + 3y = 0, \\ y(0) = 2, \ y'(0) = \frac{1}{2}. \end{cases}$$

**Answer:** The characteristic equation is

$$4r^2 - 8r + 3 = 0$$

which yields

$$r_{1,\ 2} = \frac{8 \pm \sqrt{64 - 48}}{8} = 1 \pm \frac{1}{2} = \frac{3}{2}, \ \frac{1}{2}.$$

Thus, the general solution is

$$y(t) = c_1 e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t}.$$

Using initial conditions

$$y(0) = 2 \qquad \Longrightarrow \qquad c_1 + c_2 = 2;$$

$$y'(0) = \frac{1}{2} \qquad \Longrightarrow \qquad \frac{3}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2} \qquad \Longrightarrow \qquad 3c_1 + c_2 = 1.$$

Solve the two algebraic equations for $c_1 = -\frac{1}{2}$, $c_2 = \frac{5}{2}$. Therefore

$$y(t) = -\frac{1}{2}e^{\frac{3}{2}t} + \frac{5}{2}e^{\frac{1}{2}t}.$$

Everything seems easy and moves smoothly until we encounter some suprprises.

**Special case I: repeated roots.**

**Example 3.2.3** Find the general solution of

$$y'' - 4y' + 4y = 0.$$

**Answer:** The characteristic equation is

$$r^2 - 4r + 4 = 0 \qquad \Longrightarrow \qquad (r-2)^2 = 0$$

which yields

$$r = r_1 = r_2 = 2,$$

This means that there is only one value $r = 2$ that allows $y_1(t) = e^{rt} = e^{2t}$ to satisfy the ODE. Without a second LI solution, we cannot write down the general solution!

**Question:** In this case, how can we find a 2nd LI solution $y_2(t)$? How to find the general solution?

**Answer: Reduction of order (Due to D'Alembert)** The general solution can be expressed as $y(t) = v(t)e^{2t}$ (an educated guess!)

To determine the function $v(t)$, we substitute the guessed solution into the ODE:

$$y'(t) = v'e^{2t} + 2ve^{2t} = (v' + 2v)e^{2t}.$$
$$y''(t) = \left[(v' + 2v)e^{2t}\right]' = (v'' + 4v' + 4v)e^{2t}.$$

Substitute $y,\ y',\ y''$ into the ODE, we obtain

$$\left[(v'' + 4v' + 4v) - 4(v' + 2v) + 4v\right]e^{2t} = 0 \overset{\text{simplify}}{\Longrightarrow} v'' = 0.$$

Thus, $v(t) = c_1 + c_2 t$.

**The general solution is:**

$$y(t) = v(t)e^{2t} = (c_1 + c_2 t)e^{2t} = c_1 e^{2t} + c_2 t e^{2t}.$$

We notice that this general solution is a linear combination of two terms $y_1(t) = e^{2t}$ and $y_2(t) = te^{2t}$. Indeed, we can easily verify that $y_2(t)$ is another solution of the ODE that is LI of $y_1(t)$.

**Theorem 3.2.1** If the characteristic equation of $ay'' + by' + c = 0$ yields repeated root $r_1 = r_2 = r$, then

$$y_1(t) = e^{rt}$$

is one solution and

$$y_2(t) = te^{rt}$$

is a second solution that is LI of $y_1(t)$. The two form a fundamental set of solutions.

**Proof:**

We know $y_1(t) = e^{rt}$ is a solution since $r$ is the root of the characteristic equation $ar^2 + br + c = 0$. The two roots are given by

$$r_{1,\ 2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$r_1 = r_2$ only occurs when $b^2 - 4ac = 0$ which yields

$$r = r_1 = r_2 = \frac{-b}{2a} \qquad \Longrightarrow \qquad 2ar + b = 0.$$

Notice that:

$$y_2'(t) = \left(te^{rt}\right)' = e^{rt} + rte^{rt} = (1 + rt)e^{rt},$$

$$y_2''(t) = \left((1 + rt)e^{rt}\right)' = re^{rt} + (r(1 + rt)e^{rt} = (2r + r^2t)e^{rt}.$$

Plug into the ODE:

$$a(2r + r^2t)e^{rt} + b(1 + rt)e^{rt} + cte^{rt} = 0,$$

which simplifies to

$$(ar^2 + br + c)te^{rt} + (2ar + b)e^{rt} = 0.$$

Therefore, $y_2(t) = te^{rt}$ is a solution. We shall learn a technique later to verify that it is LI of $y_1(t) = e^{rt}$.

**Example 3.2.4** Solve the IVP

$$\begin{cases} y'' + 6y' + 9y = 0, \\ y(0) = 3,\ y'(0) = -1. \end{cases}$$

**Answer:** Ch. eq. is $r^2 + 6r + 9 = 0 \Longrightarrow (r + 3)^2 = 0 \Longrightarrow r = r_1 = r_2 = -3$.

Based on Theorem 3.2.1, $y_1(t) = e^{-3t}$ and $y_2(t) = te^{-3t}$ form a fundamental set.

Thus, the general solution is

$$y(t) = c_1 e^{-3t} + c_2 t e^{-3t}.$$

Its derivative is

$$y'(t) = -3c_1 e^{-3t} + c_2 e^{-3t} - 3c_2 t e^{-3t}$$

Using initial condition,

$$y(0) = 3 = c_1, \qquad y'(0) = -1 = -3c_1 + c_2 = -9 + c_2,$$

which yield $c_1 = 3$ and $c_2 = 8$. Therefore, the solution to the IVP is

$$y(t) = 3e^{-3t} + 8te^{-3t} = (3 + 8t)e^{-3t}.$$

**Special case II: complex roots.**

**Example 3.2.5** Find the general solution of

$$y'' + 2y' + 10y = 0.$$

**Answer:** Ch. eq. is $r^2 + 2r + 10 = 0 \Longrightarrow$

$$r_{1,\ 2} = \frac{-2 \pm \sqrt{4 - 40}}{2} = -1 \pm \sqrt{1 - 10} = -1 \pm \sqrt{-9}.$$

No real-valued root!!!

**Remark:** We can no longer proceed without appropriate knowledge of complex numbers.

## 3.3 Introduction to complex numbers and Euler's formula

### 3.3.1 Definitions and basic concepts

**The imaginary number $i$:**

$$i \equiv \sqrt{-1} \qquad \Longleftrightarrow \qquad i^2 = -1. \tag{45}$$

**Every imaginary number is expressed as a real-valued multiple of $i$:**

$$\sqrt{-9} = \sqrt{9}\sqrt{-1} = \sqrt{9}i = 3i.$$

**A complex number:**

$$z = a + bi, \tag{46}$$

where $a,\ b$ are real, is the sum of a real and an imaginary number.

**The real part of z=a+bi:** $Re\{z\} = a$ is a real number.

**The imaginary part of z=a+bi:** $Im\{z\} = b$ is a also a real number.

**A complex number z=a+bi** represents a point $(a,\ b)$ in a 2D space, called the complex space.

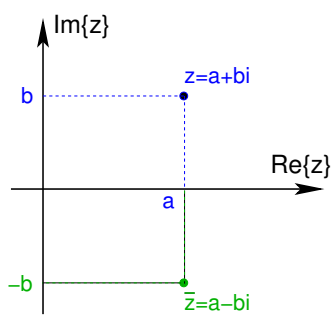



Figure 7: A complex number $z$ and its conjugate $\bar{z}$ in complex space. Horizontal axis contains all real numbers, vertical axis contains all imaginary numbers.

**The complex conjugate of z=a+bi:** is $\bar{z} = a - bi$ (i.e., reversing the sign of $Im\{z\}$ changes $z$ into $\bar{z}$!)

Notice that:

$$Re\{z\} = \frac{z + \bar{z}}{2} = a, \qquad Im\{z\} = \frac{z - \bar{z}}{2i} = b.$$

### 3.3.2 Basic complex computations

**Addition/subtraction:** If $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ ($a_1,\ a_2,\ b_1,\ b_2$ are real), then

$$z_1 \pm z_2 = (a_1 + b_1 i) \pm (a_2 + b_2 i) = (a_1 \pm a_2) + (b_1 \pm b_2)i.$$

Or, real parts plus/minus real parts, imaginary parts plus/minus imaginary parts.

**Multiplication by a real number c:**

$$cz_1 = ca_1 + cb_1 i.$$

**Multiplication between complex numbers:**

$$z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 i^2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i.$$

All rules are identical to those for multiplication between real numbers, just remember $i^2 = -1$.

**Length** of a complex number $z = a + bi$

$$|z| = \sqrt{z\bar{z}} = \sqrt{(a + bi)(a - bi)} = \sqrt{a^2 + b^2},$$

which is dentical to the length of a 2D vector $(a,\ b)$.

**Division between complex numbers:**

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{(a_1 + b_1 i)(a_2 - b_2 i)}{|z_2|^2} = \frac{(a_1 a_2 + b_1 b_2) - (a_1 b_2 + a_2 b_1)i}{a_2^2 + b_2^2}.$$

**Example 3.3.1:** Given that $z_1 = 3 + 4i$, $z_2 = 1 - 2i$, calculate

1. $z_1 - z_2$;
2. $\frac{z_1}{2}$;
3. $|z_1|$;
4. $\frac{z_2}{z_1}$.

**Answer:**

1. $z_1 - z_2 = (3 - 1) + (4 - (-2))i = 2 + 6i$;
2. $\frac{z_1}{2} = \frac{3}{2} + \frac{4}{2}i = 1.5 + 2i$;
3. $|z_1| = \sqrt{z_1 \bar{z}_1} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$;
4. $\frac{z_2}{z_1} = \frac{z_2 \bar{z}_1}{z_1 \bar{z}_1} = \frac{(1-2i)(3-4i)}{5^2} = \frac{11-10i}{25} = \frac{11}{25} - \frac{2}{5}i$.

### 3.3.3 Back to Example 3.2.5

**Example 3.2.5** Find the general solution of

$$y'' + 2y' + 10y = 0.$$

**Answer:** Ch. eq. is $r^2 + 2r + 10 = 0 \Longrightarrow$

$$r_{1,\ 2} = \frac{-2 \pm \sqrt{4 - 40}}{2} = -1 \pm \sqrt{1 - 10} = -1 \pm \sqrt{-9} = -1 \pm 3i.$$

Since $r_1 \neq r_2$ (actually $r_1 = \bar{r}_2$, i.e. they are a pair of complex conjugates), the general solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^{(-1+3i)t} + c_2 e^{(-1-3i)t} = e^{-t}[c_1 e^{3it} + c_2 e^{-3it}].$$

**Remarks:**

- The solution contains exponential functions with complex-valued exponents.
- The ODE is real-valued and models a spring-mass problem.
- The solution should also be real-valued and with clear physical meanings.

**Question:** How does $e^{3it}$ relate to real-valued functions?

### 3.3.4 Complex-valued exponential and Euler's formula

**Euler's formula:**

$$e^{it} = \cos t + i \sin t. \tag{47}$$

Based on this formula and that $e^{-it} = \cos(-t) + i\sin(-t) = \cos t - i \sin t$:

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \qquad \sin t = \frac{e^{it} - e^{-it}}{2i}. \tag{48}$$

**Why?** Here is a way to gain insight into this formula.

Recall the Taylor series of $e^t$:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

We assume that this series holds when the exponent is complex-valued.

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n \text{ even}}^{\infty} \frac{(it)^n}{n!} + \sum_{n \text{ odd}}^{\infty} \frac{(it)^n}{n!} = \sum_{m=0}^{\infty} \frac{(it)^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(it)^{2m+1}}{(2m+1)!}$$

$$= \sum_{m=0}^{\infty} \frac{i^{2m} t^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{i^{2m+1} t^{2m+1}}{(2m+1)!} = \sum_{m=0}^{\infty} \frac{(i^2)^m t^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{i(i^2)^m t^{2m+1}}{(2m+1)!}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m+1}}{(2m+1)!} = \cos t + i \sin t.$$

### 3.3.5 Back again to Example 3.2.5

**Example 3.2.5** Find the general solution of

$$y'' + 2y' + 10y = 0.$$

**Answer:** Ch. eq. is $r^2 + 2r + 10 = 0 \Longrightarrow$

$$r_{1,\ 2} = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm \sqrt{1-10} = -1 \pm \sqrt{-9} = -1 \pm 3i.$$

Since $r_1 \neq r_2$ (actually $r_1 = \bar{r}_2$, i.e. they are a pair of complex conjugates), the general solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^{(-1+3i)t} + c_2 e^{(-1-3i)t} = e^{-t}[c_1 e^{3it} + c_2 e^{-3it}].$$

Using Euler's formula

$$e^{(-1+3i)t} = e^{-t-3it} = e^{-t} e^{3it} = e^{-t}[\cos 3t + i \sin 3t].$$

We now demonstrate that the two real-valued functions

$$y_1(t) = Re\{e^{(-1+3i)t}\} = e^{-t}\cos 3t \qquad \text{and} \qquad y_2(t) = Im\{e^{(-1+3i)t}\} = e^{-t}\sin 3t$$

are also solutions to the ODE and they are LI.

Let's verify $y_1(t)$ only (the verification of $y_2(t)$ is very similar).

$$y_1'(t) = \left[e^{-t}\cos 3t\right]' = -e^{-t}\cos 3t - 3e^{-t}\sin 3t = -e^{-t}[\cos 3t + 3\sin 3t];$$

$$y_1''(t) = e^{-t}[\cos 3t + 3\sin 3t] - e^{-t}[-3\sin 3t + 9\cos 3t] = e^{-t}[-8\cos 3t + 6\sin 3t].$$

Substitute into the ODE, we obtain

$$e^{-t}[-8\cos 3t + 6\sin 3t] - 2e^{-t}[\cos 3t + 3\sin 3t] + 10e^{-t}\cos 3t = 0,$$

which shows that $y_1(t)$ is indeed a real-valued solution. Similarly, $y_2(t)$ is also a solution. We shall demonstrate later that they are LI. Thus, the real-valued general solution is:

$$y(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t.$$

**Theorem 3.3.1** If the characteristic equation of the ODE $ay'' + by' + cy = 0$ ($a,\ b,\ c$ are constants) yields a pair of complex conjugates $r,\ \bar{r} = \alpha \pm \beta i$ (complex-valued roots alway occur in pairs of conjugates), then

$$y_c(t) = e^{rt} = e^{(\alpha+\beta i)t} \qquad \text{and} \qquad \bar{y}_c(t) = e^{\bar{r}t} = e^{(\alpha-\beta i)t}$$

form a fundamental set of complex-valued solutions (complex-valued solutions are also conjugates of each other).

$$y_1(t) = Re\{y_c\} = \frac{1}{2}(y_c + \bar{y}_c) = e^{\alpha t}\cos(\beta t) \qquad \text{and} \qquad y_2(t) = Im\{y_c\} = \frac{1}{2i}(y_c - \bar{y}_c) = e^{\alpha t}\sin(\beta t)$$

form a fundamental set of real-valued solutions.

**Proof:** $y_c(t)$ and $\bar{y}_c(t)$ are obviously solutions to the ODE since $r$ and $\bar{r}$ are roots of the characteristic equation. Later we shall demonstrate using Wronskian that they are indeed LI.

Since $y_1(t)$ and $y_2(t)$ are both linear combinations of $y_c(t)$ and $\bar{y}_c(t)$. Based on the Principle of Superposition, they must also be solutions of the ODE. We shall demonstrate later that they are LI.

### 3.3.6 One more example

**Example 3.2.5** Solve the IVP

$$\begin{cases} y'' + 4y' + 13y = 0, \\ y(0) = 2, \ y'(0) = -3. \end{cases}$$

**Answer:** Ch. eq. is $r^2 + 4r + 13 = 0 \Longrightarrow$

$$r_{1,\ 2} = \frac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm \sqrt{-9} = -2 \pm \sqrt{-9} = -2 \pm 3i.$$

Therefore,

$$y_c(t) = e^{(-2+3i)t} = e^{-2t}e^{3it} = e^{-2t}[\cos 3t + i \sin 3t].$$

is the complex-valued solution. Thus,

$$y_1(t) = Re\{y_c(t)\} = e^{-2t} \cos 3t, \qquad y_2(t) = Im\{y_c(t)\} = e^{-2t} \sin 3t$$

form a fundamental set. So, the real-valued general solution is

$$y(t) = c_1 e^{-2t} \cos 3t + c_2 e^{-2t} \sin 3t = e^{-2t}[c_1 \cos 3t + c_2 \sin 3t].$$

Using initial condition $y(0) = 2$, we obtain $c_1 = 2$. Note that,

$$y'(t) = -2e^{-2t}[c_1 \cos 3t + c_2 \sin 3t] + e^{-2t}[-3c_1 \sin 3t + 3c_2 \cos 3t].$$

The initial condition $y'(0) = -3$ yields $-3 = -2c_1 + 3c_2 \Rightarrow c_2 = \frac{1}{3}$.

The solution to the IVP is

$$y(t) = e^{-2t}[2 \cos 3t + \frac{1}{3} \sin 3t].$$

## 3.4 Applications: mechanical and electrical vibrations

### 3.4.1 Harmonic vibration of a spring-mass system in frictionless medium

Consider the spring-mass system in frictionless medium as shown in Fig. 8. $u = 0$ represents the equilibrium position where the spring length is equal to its unstressed natural length. Let the spring constant be $k$ ($> 0$), which measures the hardness of the spring. Assume that the spring obeys Hook's law: $F = -ku$, where $u$ the displacement of the mass w.r.t. the equilibrium position.

1. Find the general solutions.
   (a) $u(0) = u_0,\ u(0) = 0$ (initial displacement but no initial speed);
   (b) $u(0) = 0,\ u'(0) = v_0$ (initial speed but no initial displacement);
   (c) $u(0) = u_0,\ u'(0) = v_0$ (both initial displacement and speed).

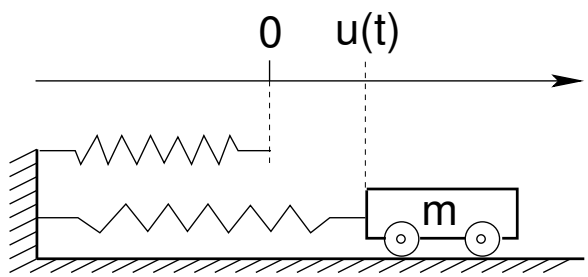



Figure 8: Spring-mass system in frictionless medium giving rise to sustained harmonic oscillations.

**Answer:** Based on Newton's second law,

$$mu'' = F \qquad \Longrightarrow \qquad mu'' = -ku \qquad \Longrightarrow \qquad u'' + \frac{k}{m}u = 0 \qquad \Longrightarrow \qquad u'' + \omega_0^2 u = 0,$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is often referred to as the intrinsic frequency of the system.

The characteristic equation is

$$r^2 + \omega_0^2 = 0 \qquad \Longrightarrow \qquad r^2 = -\omega_0^2 \qquad \Longrightarrow \qquad r_{1,\,2} = \pm\sqrt{-\omega_0^2} = \pm\omega_0 i.$$

Thus, a complex-valued solution is:

$$y_c(t) = e^{\omega_0 i t} = \cos(\omega_0 t) + i\sin(\omega_0 t).$$

Therefore,

$$y_1(t) = Re\{y_c(t)\} = \cos(\omega_0 t), \qquad y_2(t) = Im\{y_c(t)\} = \sin(\omega_0 t)$$

form a fundamental set of real-valued solutions. Thus,

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

Notice that

$$y'(t) = -c_1 \omega_0 \sin(\omega_0 t) + c_2 \omega_0 \cos(\omega_0 t).$$

For initial conditions (a):

$$y(0) = u_0 \qquad \Longrightarrow \qquad c_1 = u_0; \qquad y'(0) = 0 \qquad \Longrightarrow \qquad c_2 = 0.$$

Thus,

$$y(t) = u_0 \cos(\omega_0 t).$$

For initial conditions (b):

$$y(0) = 0 \qquad \Longrightarrow \qquad c_1 = 0; \qquad y'(0) = v_0 \qquad \Longrightarrow \qquad c_2 = \frac{v_0}{\omega_0}.$$

Thus,

$$y(t) = \frac{v_0}{\omega_0} \sin(\omega_0 t).$$

For initial conditions (c):

$$y(0) = u_0 \qquad \Longrightarrow \qquad c_1 = u_0; \qquad y'(0) = v_0 \qquad \Longrightarrow \qquad c_2 = \frac{v_0}{\omega_0}.$$

Thus,

$$y(t) = u_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t).$$

**Question:** How does this solution relate to a single sin or cos function?

Using the following trig-identity

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta,$$

combined with the following triangle of the coefficients,

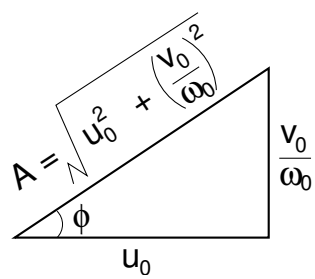



Figure 9: Triangle of the coefficients.

we can turn the solution

$$y(t) = u_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t) = A \left[ \cos(\omega_0 t) \frac{u_0}{A} + \sin(\omega_0 t) \frac{\frac{v_0}{\omega_0}}{A} \right]$$

$$= A \left[ \cos(\omega_0 t) \cos\phi + \sin(\omega_0 t) \sin\phi \right] = A \cos(\omega_0 t - \phi),$$

where

$$A = \sqrt{u_0^2 + \left( \frac{v_0}{\omega_0} \right)^2}, \qquad \phi = \tan^{-1} \frac{\frac{v_0}{\omega_0}}{u_0} = \tan^{-1} \frac{v_0}{u_0 \omega_0}.$$

Here are a few important concepts related to harmonic oscillations:

- $A =$ amplitude. $-A \leq y(t) \leq A$ for all $t$.
- $A$ depends on both initial displacement $u_0$ and initial speed $v_0$. If $v_0 = 0$, $A = u_0$; if $u_0 = 0$, $A = \frac{v_0}{\omega_0}$.
- $\phi$ is the initial phase, which is determined by initial conditions.
- The angular frequency (in $rad/s$) is: $\omega_0 = \sqrt{k/m}$.
- The frequency (in $Hz$) is: $f_0 = \frac{\omega_0}{2\pi}$.
- Period (in $s$) is: $T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$.

### 3.4.2 Spring-mass-damper system: a simple model of suspension systems

See Fig. 10. A mass $m$ is supported by a Hookian spring with a spring constant $k$. The damper provides a damping force that is proportional to the speed with a damping constant $\gamma$. All constants are positive valued. At equilibrium, the length of the spring is shrunk by $s$, i.e. $ks = mg$. This equilibrium position is chosen to be $y = 0$.

1. Write down the ODE that describes the motion of the mass.

2. Find the general solution.

3. Characterize the properties of the motion under three damping conditions:
   (a) Over damped;
   (b) Critically damped;
   (c) Under damped.

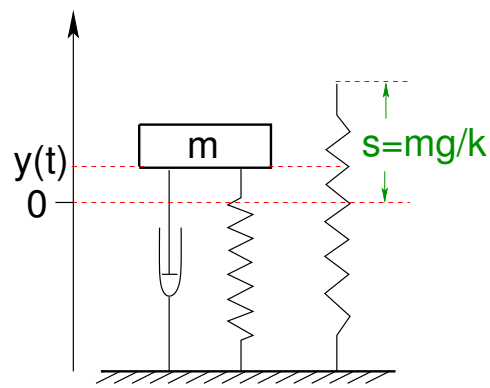



Figure 10: Simplified quarter car model.

**Answer:** Based on Newton's second law,

$$my'' = F \qquad \Longrightarrow \qquad my'' = -mg - k(y - s) - \gamma y' \qquad \Longrightarrow \qquad my'' = -ky - \gamma y',$$

where condition $sk = mg$ was used. This results in

$$my'' + \gamma y' + ky = 0, \tag{49}$$

which is a homogeneous, 2nd-order, linear ODE with constant coefficients.

The ch. eq. is given by

$$mr^2 + \gamma r + k = 0,$$

which yields

$$r_{1,\ 2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \frac{\gamma}{2m}\left[-1 \pm \sqrt{\delta}\right], \tag{50}$$

where

$$\delta = 1 - \frac{4mk}{\gamma^2}.$$

**(1) Over damped condition:** When damping is big, $\gamma^2 > 4mk$ such that

$$0 < \delta = 1 - \frac{4mk}{\gamma^2} < 1 \qquad \Longrightarrow \qquad \left[-1 \pm \sqrt{\delta}\right] < 0.$$

Under this condition, $r_1 \neq r_2$ and $r_1,\ r_2 < 0$. The general solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \qquad \stackrel{t \longrightarrow \infty}{\longrightarrow} \qquad 0, \tag{51}$$

which means the displacement is damped to zero exponentially irrespective of initial conditions.

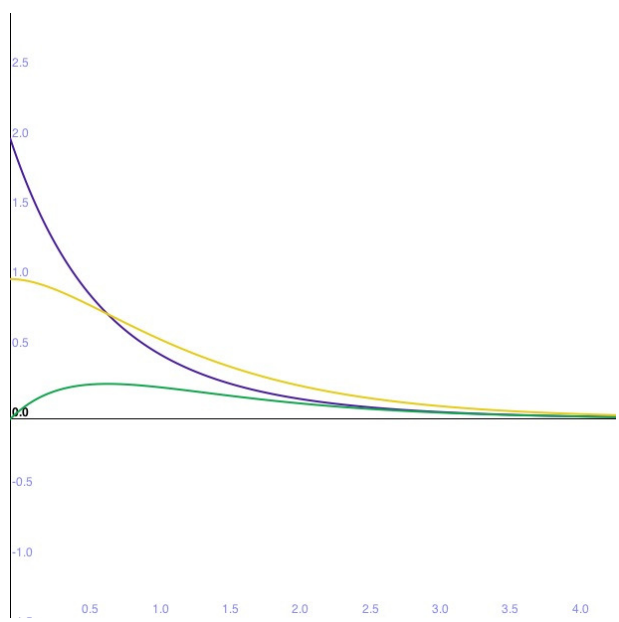



Figure 11: Over damped solutions for three different initial conditions.

**(2) Critically damped condition:** This happens when $\gamma^2 = 4mk$ such that $\delta = 0$.

Now, we have repeated root for the ch. eq.

$$r = r_1 = r_2 = -\frac{\gamma}{2m}.$$

Under this condition, the general solution is

$$y(t) = [c_1 + c_2 t]e^{rt} = [c_1 + c_2 t]e^{-\frac{\gamma}{2m}t} \qquad \overset{t\longrightarrow\infty}{\longrightarrow} \qquad 0. \tag{52}$$

Now the decay to zero is often slower than a simple exponential function because of the term $c_2 t e^{-\frac{\gamma}{2m}t}$.

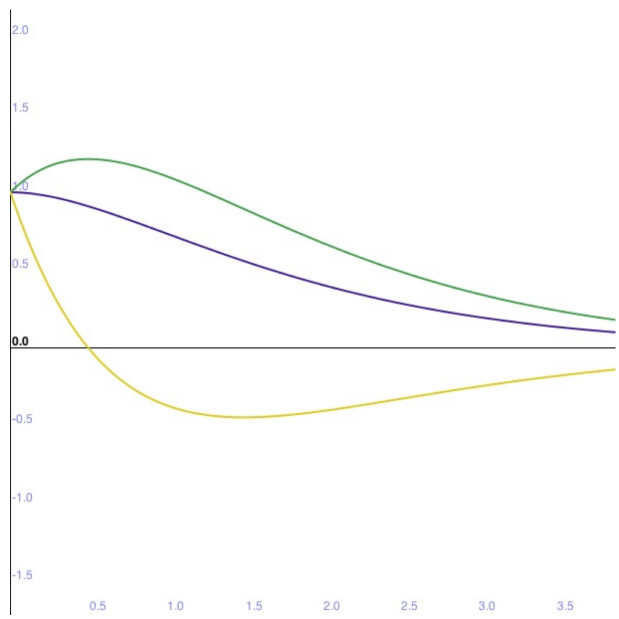



Figure 12: Critically damped solutions for three different initial conditions.

**(3) Under damped condition:** This occurs when $\gamma^2 < 4mk$ such that $\delta < 0$. Now, the ch. eq. gives a pair of complex roots.

$$r,\ \bar{r} = \frac{\gamma}{2m}\left[-1 \pm \sqrt{\delta}\right] = \frac{\gamma}{2m}\left[-1 \pm i\sqrt{-\delta}\right] = -\frac{\gamma}{2m} \pm \omega i,$$

where

$$\omega = \frac{\gamma}{2m}\sqrt{-\delta} = \frac{\gamma}{2m}\sqrt{\frac{4mk}{\gamma^2} - 1} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} = \sqrt{\frac{k}{m}}\sqrt{1 - \frac{\gamma^2}{4mk}} = \omega_0\left(1 - \frac{\gamma^2}{8mk} + \cdots\right)$$

where

$$\omega_0 \equiv \sqrt{\frac{k}{m}}$$

is the intrinsic frequency of the harmonic oscillator for $\gamma = 0$ (i.e., in frictionless medium). Note that, $\omega < \omega_0$. Now, the general solution is

$$y(t) = e^{-\frac{\gamma}{2m}t}\left[c_1 \cos(\omega t) + c_2 \sin(\omega t)\right] = Ae^{-\frac{\gamma}{2m}t}\cos(\omega t - \phi),$$

where

$$A = \sqrt{c_1^2 + c_2^2}, \qquad \phi = \tan^{-1}\frac{c_2}{c_1}.$$

When the long term behaviour is concerned,

$$y(t) = Ae^{-\frac{\gamma}{2m}t}\cos(\omega t - \phi) \qquad \overset{t\longrightarrow\infty}{\longrightarrow} \qquad 0.$$

where $\omega$ is referred to as the *quasi-frequency* since the vibration is not exactly periodic.

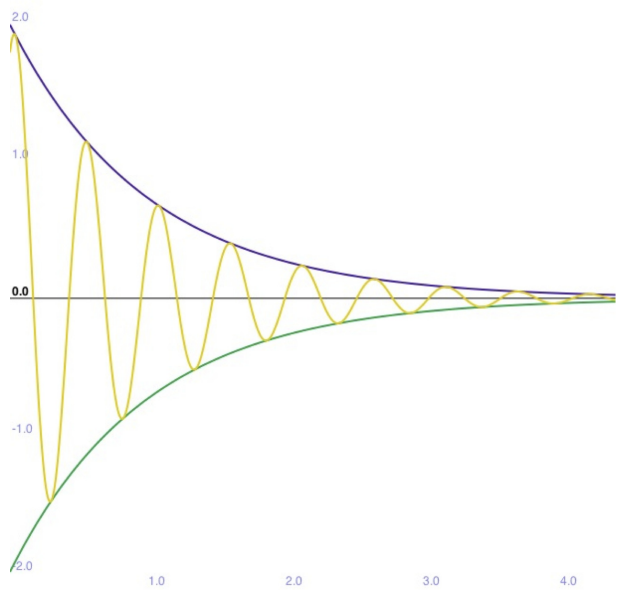



Figure 13: Under damped solution for one set of initial conditions.

**Conclusion:** In the presence of friction the motion of the mass will eventually stop under all conditions, but the approach to equilibrium is different. In under damped condition, the approach is oscillatory with exponentially decreasing amplitude.

**Example 3.4.1:** In a spring-mass-damper system, $m = 2\ kg$. The spring is known to give a force of $F_s = 3\ N$ when stretched/compressed by $l = 10\ cm$. The damper is known to provide a drag force of $F_d = 3\ N$ at a speed of $v_d = 5\ m/s$. Let $y(t)$ be the displacement of the mass w.r.t. the equilibrium position at time $t$. Initial conditions are: $y(0) = 5\ cm$, $y'(0) = 10\ cm/s$.

1. Find $y(t)$.
2. If the motion is under damped, find the ratio between the quasi-frequency of the motion and its intrinsic frequency, $\frac{\omega}{\omega_0} =?$

**Answer:** First, we need to unify the units by adopting the $m - kg - s$ system.

$$l = 0.1\ m, \qquad y(0) = 0.05\ m, \qquad y'(0) = 0.1\ m/s.$$

The spring constant is obtained by solving $F_s = kl$,

$$k = \frac{F_s}{l} = \frac{3}{0.1} = 30\ (N/m).$$

The damping coefficient is obtained by solving $F_d = \gamma v_d$,

$$\gamma = \frac{F_d}{v_d} = \frac{3}{5} = 0.6\ (sN/m).$$

The ODE that describes our spring-mass-damper system is

$$my'' + \gamma y' + ky = 0 \qquad \Longrightarrow \qquad 2y'' + 0.6y' + 30y = 0.$$

Roots for the corresponding ch. eq. are:

$$r_1,\ r_2 = \frac{-0.6 \pm \sqrt{0.36 - 240}}{4} = -0.15 \pm 3.87008i.$$

Thus, the quasi-frequency is $\omega = 3.87008$. It is obviously under damped, with the following general solution

$$y(t) = e^{-0.15t}\left[c_1 \cos(3.87008t) + c_2 \sin(3.87008t)\right].$$

Using initial condition, we obtain

$$y(0) = 0.05 \qquad \Longrightarrow \qquad c_1 = 0.05; \qquad\qquad y'(0) = 0.1 \qquad \Longrightarrow \qquad c_2 = 0.02778.$$

Therefore,

$$y(t) = e^{-0.15t}\left[0.05 \cos(3.87008t) + 0.2778 \sin(3.87008t)\right].$$

The ratio between the two frequencies is

$$\frac{\omega}{\omega_0} = \frac{3.87008}{\sqrt{\frac{k}{m}}} = \frac{3.87008}{\sqrt{\frac{30}{2}}} = \frac{3.87008}{\sqrt{15}} \approx \frac{3.87008}{3.87298} \approx 0.99925.$$

### 3.4.3 Oscillations in a RLC circuit

A typical RLC circuit is composed of a resistor (R), an inductor (L), a capacitor (C) combined with a power source $E(t)$ and a switch (see Fig. 14).

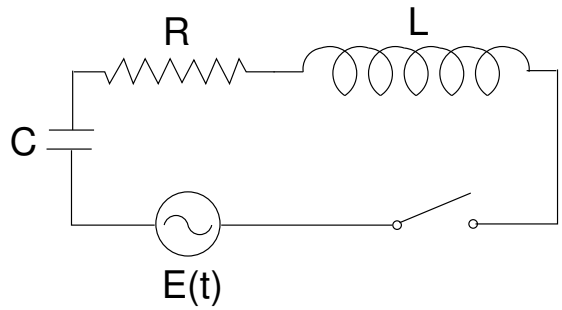



Figure 14: A typical RLC circuit.

Let $Q(t)$ = the amount of electric charge (measured in units of Coulomb) on each plate of the capacitor at time $t$. Then,

$$I = \frac{dQ}{dt}$$

(measured in units of Amperes) is the current that flows through the circuit.

Voltage drop across a resistor with resistance $R$ (measured in units of Ohm) is

$$V_R = RI = R\frac{dQ}{dt} \qquad \text{(Ohm's law).}$$

Voltage drop across a capacitor with capacitance $C$ (measured in units of Faraday) is

$$V_C = \frac{Q}{C} \qquad \text{(Definition of capacitance : } C = \frac{Q}{V_C}\text{).}$$

Voltage drop across an inductor with inductance $L$ (measured in units of Henry) is

$$V_L = L\frac{dI}{dt} = L\frac{d^2Q}{dt^2} \qquad \text{(Definition of inductance).}$$

**Kirchhoff's 2nd law:** In a closed RLC circuit, the impressed voltage is equal to the sum of voltage drops across all elements in the circuit.

Thus,

$$V_L + V_R + V_C = E(t) \qquad \Longrightarrow \qquad LQ'' + RQ' + \frac{1}{C}Q = E(t). \tag{53}$$

When the impressed voltage $E(t)$ is turned to zero, the equation becomes homogeneous

$$LQ'' + RQ' + \frac{1}{C}Q = 0. \tag{54}$$

Interestingly, if we differentiate both sides w.r.t $t$ and remembering that $\frac{dQ}{dt} = Q' = I$, we obtain

$$LI'' + RI' + \frac{1}{C}I = 0.$$

Therefore, the current $I(t)$ obeys the same ODE as the charge $Q(t)$.

Mathematically, this equation is identical to that of the spring-mass-damper system. Now,

- Charge/current plays the role of displacement: $Q(t),\ I(t) \leftrightarrow y(t)$;
- Inductor plays the role of the load/mass: $L \leftrightarrow m$;
- Resistor plays the role of the damper (dissipation of energy): $R \leftrightarrow \gamma$;
- Capacitor plays the role of the spring (storage of energy): $\frac{1}{C} \leftrightarrow k$.

**In the absence of resistor:** $R = 0$, the circuit is reduced to an LC circuit which gives rise to harmonic oscillations

$$LQ'' + \frac{1}{C}Q = 0 \qquad \Longrightarrow \qquad Q'' + \frac{1}{LC}Q = 0 \qquad \Longrightarrow \qquad Q'' + \omega_0^2 Q = 0,$$

where the intrinsic frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}}. \tag{55}$$

**In the presence of resistor:** The ch. eq. yields the following roots

$$r_1,\ r_2 = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L} = -\frac{R}{2L}\left[1 \pm \sqrt{1 - \frac{4L}{CR^2}}\right]. \tag{56}$$

Over, critical, and under damped conditions can occur depending on magnitude of the term $\frac{4L}{CR^2}$.

In under damped condition, the roots are

$$r_1,\ r_2 = -\frac{R}{2L}\left[1 \pm i\sqrt{\frac{4L}{CR^2} - 1}\right] = -\frac{R}{2L} \pm i\omega,$$

where the quasi-frequency $\omega$ is

$$\omega = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{CR^2}{4L}} = \omega_0(1 - \frac{CR^2}{8L} + \cdots). \tag{57}$$

**Example 3.4.2:** Consider a RLC circuit with $C = 10^{-5}\ F$, $R = 200\ \Omega$, $L = 0.5\ H$ with initial conditions: $Q(0) = Q_0 = 10^{-6}\ C$, $Q'(0) = I_0 = -2 \times 10^{-4}\ A$.

1. Find $\omega_0$ (i.e. the vibration frequency when $R = 0$.)
2. Find $Q(t)$, and determine its long time behaviour.
3. Determine if the system is over, critical, or under damped.

**Answer:**

1. $$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.5 \times 10^{-5}}} \approx 447.2\ s^{-1}$$

2. The roots to the ch. eq. $Lr^2 + Rr + \frac{1}{C} = 0$ are

$$r_1,\ r_2 = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L} = -200 \pm \sqrt{40000 - 200000} = -200 \pm \sqrt{-160000} = -200 \pm 400i,$$

where the quasi-freqyency $\omega = 400 < \omega_0$. So, the general solution is

$$Q(t) = e^{-200t}[c_1 \cos(400t) + c_2 \sin(400t)].$$

Using initial conditions,

$$Q(0) = 10^{-6} \qquad \Longrightarrow \qquad c_1 = 10^{-6}; \qquad\qquad Q'(0) = -2 \times 10^{-4} \qquad \Longrightarrow \qquad c_2 = 0.$$

Therefore,

$$Q(t) = 10^{-6} e^{-200t} \cos(400t) \qquad \overset{t \to \infty}{\longrightarrow} \qquad 0.$$

3. The system is under damped.

## 3.5 Linear independence, Wronskian, and fundamental solutions

**Theorem 3.5.1:** If $r_1 \neq r_2$, then $y_1(t) = e^{r_1 t},\ y_2(t) = e^{r_2 t}$ are two linearly independent functions.

**Proof:** The goal is to show that

$$k_1 e^{r_1 t} + k_2 e^{r_2 t} = 0$$

is satisfied only when $k_1 = k_2 = 0$.

Pick two arbitrary points in $t$: $t_0$ and $t_1$ ($t_0 \neq t_1$).

Substitute each value of $t$ into the equation above:

$$\begin{cases} k_1 e^{r_1 t_0} + k_2 e^{r_2 t_0} = 0, \\ k_1 e^{r_1 t_1} + k_2 e^{r_2 t_1} = 0. \end{cases}$$

Question: what are $k_1,\ k_2$ that satisfy both equations?

Turn this system of algebraic equations in matrix form:

$$\begin{bmatrix} e^{r_1 t_0} & e^{r_2 t_0} \\ e^{r_1 t_1} & e^{r_2 t_1} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad A\vec{x} = \vec{0}, \tag{58}$$

where

$$A = \begin{bmatrix} e^{r_1 t_0} & e^{r_2 t_0} \\ e^{r_1 t_1} & e^{r_2 t_1} \end{bmatrix}, \qquad \vec{x} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, \qquad \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Based on linear algebra

1. If $\det A \neq 0$, then $A$ is invertible, $\vec{x} = A^{-1}\vec{0} = \vec{0}$ is the only solutions.
2. If $\det A = 0$, then $A$ is not invertible, there exist infinitely many solutions $\vec{x} \neq \vec{0}$.

Let's evaluate $\det A$:

$$\det A = \begin{vmatrix} e^{r_1 t_0} & e^{r_2 t_0} \\ e^{r_1 t_1} & e^{r_2 t_1} \end{vmatrix} = e^{r_1 t_0} e^{r_2 t_1} - e^{r_2 t_0} e^{r_1 t_1} = e^{r_1 t_0 + r_2 t_1} - e^{r_2 t_0 + r_1 t_1}$$

$$= e^{r_1 t_0 + r_2 t_1}[1 - e^{r_2 t_0 + r_1 t_1 - (r_1 t_0 + r_2 t_1)}] = e^{r_1 t_0 + r_2 t_1}[1 - e^{(r_2 - r_1)(t_0 - t_1)}] \neq 0,$$

because $r_1 \neq r_2$, $t_0 \neq t_1$. Therefore,

$$\vec{x} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is the only possible solution for arbitrary choices of $t_0$ and $t_1$ ($t_0 \neq t_1$). Thus, $e^{r_1 t}$ and $e^{r_2 t}$ are LI provided $r_1 \neq r_2$.

**Theorem 3.5.2 Wronskian and LI:** If $f(t)$, $g(t)$ are differentiable functions in an open interval $I$, the *Wronskian* of the two is defined by

$$W[f,\ g](t) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g.$$

If $W[f,\ g](t) \neq 0$ for some $t = t_0 \in I$, then $f$ and $g$ are LI in $I$.

Often, if $W[f,\ g](t) = 0$ for every $t \in I$, then $f$ and $g$ are LD. This result should be used with **caution** although it applies to almost all cases that we encounter in this course. However, there has been controversy and counter examples on this result. For example, it was reported that Peano pointed out (1889) that $f(t) = t^2$ and $g(t) = |t|t$ are both differentiable. But

$$W[t^2,\ |t|t] = t^2(|t|t)' - (t^2)'|t|t = t^2(2|t|) - 2t|t|t = 0 \quad \text{for all t!}$$

It is true that they are LD on $(-\infty,\ 0)$ and $(0,\ \infty)$ because they are constant multiples of each other. However, in any open interval containing $t = 0$, e.g. $(-a,\ a)$ $(a > 0)$, they are LI. A number of other conditions should be satisfied to ensure that vanishing Wronskian means linear dependence.

**Proof:** Start with the equation

$$k_1 f + k_2 g = 0.$$

Differentiate both sides, we obtain

$$k_1 f' + k_2 g' = 0.$$

The two form a system of algebraic equations

$$\begin{cases} k_1 f + k_2 g = 0, \\ k_1 f' + k_2 g' = 0. \end{cases}$$

Evaluate the determinant of its coefficient matrix

$$\det A = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g = W[f,\ g](t).$$

Thus, $W[f,\ g](t) \neq 0$ for any one $t = t_0 \in I$, $k_1 = k_2 = 0$ is the only solution. Then, $f$ and $g$ are LI in $I$.

**Example 3.5.1:** When repeated roots occur, the ODE $ay'' + by' + cy = 0$ has two solutions $y_1 = e^{rt}$, $y_2 = te^{rt}$. Use Wronskian to show that they are LI.

**Answer:**

$$W[y_1,\ y_2](t) = y_1 y_2' - y_1' y_2 = e^{rt}(e^{rt} + rte^{rt}) - re^{rt}te^{rt} = e^{2rt} \neq 0.$$

Thus, $y_1$ and $y_2$ are LI.

**Example 3.5.2:** When complex-valued roots occur, the ODE $ay''+by'+cy=0$ has two real-valued solutions $y_1 = e^{rt}\cos\omega t$, $y_2 = e^{rt}\sin\omega t$. Use Wronskian to show that they are LI.

**Answer:**

$$W[y_1,\ y_2](t) = y_1y_2' - y_1'y_2 = e^{rt}\cos\omega t[re^{rt}\sin\omega t + \omega e^{rt}\cos\omega t] - e^{rt}\sin\omega t[re^{rt}\cos\omega t - \omega e^{rt}\sin\omega t]$$

$$= \omega e^{2rt}[\cos^2\omega t + \sin^2\omega t] = \omega e^{2rt} \neq 0 \text{ for all } t.$$

Thus, $y_1$ and $y_2$ are LI.

**Wronskian of two solutions of $L[y] = 0$ can be calculated BEFORE they are solved!**

**Theorem 3.5.3 (Abel's theorem):** If $y_1(t),\ y_2(t)$ are two solutions of the following ODE

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where $p(t),\ q(t)$ are continuous in an open interval $I$, then the Wronskian between the two is totally determined by $p(t)$ (*even before the solutions are found!*)

$$W[y_1,\ y_2](t) = Ce^{-\int p(t)dt},$$

where $C$ is a constant that depends on $y_1,\ y_2$ but not on $t$. If $p(t) = p_0 = const.$, then $W[y_1,\ y_2](t) = Ce^{-p_0 t}$. In this case, only two possibilities can occur: If $y_1,\ y_2$ are two solutions of $L[y] = 0$ (where $L = \frac{d^2}{dt^2} + p_0\frac{d}{dt} + q_0$ ($p_0,\ q_0$ are constants)), then (i) $W[y_1,\ y_2](t) \neq 0$ for all $t$ (when they are LI); (ii) $W[y_1,\ y_2](t) = 0$ for all $t$ (when they are LD). In this case, It is impossible for $W[y_1,\ y_2](t)$ to be zero for some values of $t$ but not for other values of $t$! $W[y_1,\ y_2](t) \neq 0$ when $y_1(t),\ y_2(t)$ are LI and form a fundamental set of solutions.

**Proof:**

$$\begin{aligned}
&y_1 \text{ is a solution} && \Longrightarrow && y_1'' + p(t)y_1' + q(t)y_1 = 0. && (a)\\
&y_2 \text{ is a solution} && \Longrightarrow && y_2'' + p(t)y_2' + q(t)y_2 = 0. && (b)\\
&y_1 \times (b) && \Longrightarrow && y_1y_2'' + p(t)y_1y_2' + q(t)y_1y_2 = 0, && (I)\\
&(a) \times y_2 && \Longrightarrow && y_1''y_2 + p(t)y_1'y_2 + q(t)y_1y_2 = 0. && (II)
\end{aligned}$$

Do subtraction $(I) - (II)$:

$$[y_1y_2'' - y_1''y_2] + p(t)[y_1y_2' - y_1'y_2] = 0. \qquad (c)$$

Note that:

$$W \equiv W[y_1,\ y_2](t) = y_1y_2' - y_1'y_2,$$
$$W' = [y_1y_2' - y_1'y_2]' = y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2' = y_1y_2'' - y_1''y_2.$$

Substitute into eq.(c), we obtain

$$W' + p(t)W = 0 \qquad \Longrightarrow \qquad W' = -p(t)W \qquad \Longrightarrow \qquad \frac{dW}{W} = -p(t)dt$$

$$\int \frac{dW}{W} = \int -p(t)dt \qquad \Longrightarrow \qquad \ln W = -\int p(t)dt + C_1 \qquad \Longrightarrow \qquad W = Ce^{-\int p(t)dt}.$$

**Example 3.5.3:** If $p(t)$ is differentiable and $p(t) > 0$, then for two solutions $y_1(t),\ y_2(t)$ of the following ODE

$$[p(t)y']' + q(t)y = 0,$$

the Wronskian

$$W[y_1,\ y_2](t) = \frac{C}{p(t)}.$$

**Answer:** Rewrite the ODE into the 'normal' form

$$[p(t)y']' + q(t)y = p(t)y'' + p'(t)y' + q(t)y = 0 \qquad \Longrightarrow \qquad y'' + \frac{p'(t)}{p(t)}y' + \frac{q(t)}{p(t)}y = 0.$$

Apply Abel's theorem:

$$W[y_1,\ y_2](t) = Ce^{-\int \frac{p'(t)}{p(t)}dt} = Ce^{-\int \frac{dp}{p}} = Ce^{-\ln p} = \frac{C}{e^{\ln p}} = \frac{C}{p}.$$

**Theorem 3.5.4 (Uniqueness of solution):** Suppose that $y_1(t),\ y_2(t)$ are solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and that the Wronskian

$$W[y_1,\ y_2](t) = y_1y_2' - y_1'y_2,$$

is non-zero at $t = t_0$ where the initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_0'$$

are assigned. There exists a unique choice of $c_1,\ c_2$ for which

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

is the unique solution of the IVP above.

**Proof:** Based on Principle of Superposition

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

is a solution of the ODE. To determine the values of $c_1,\ c_2$, we use ICs:

$$y(t_0) = y_0 \qquad \Longrightarrow \qquad c_1y_1(t_0) + c_2y_2(t_0) = y_0,$$

$$y'(t_0) = y_0' \qquad \Longrightarrow \qquad c_1y_1'(t_0) + c_2y_2'(t_0) = y_0'.$$

Turn this system in matrix form:

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix} \qquad \Longrightarrow \qquad A\vec{x} = \vec{b}.$$

Since

$$\det A = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = W[y_1,\ y_2](t_0) \neq 0,$$

there exists a unique solution $\vec{x} = A^{-1}\vec{b}$ which can be expressed as

$$c_1 = \frac{1}{\det A} \begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}, \qquad c_2 = \frac{1}{\det A} \begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}$$

based on Cramer's rule from linear algebra.

**Example 3.5.4:** For the following ODEs:

(a) $y'' + \omega^2 y = 0$,

(b) $y'' - \omega^2 y = 0$.

Find a fundamental set of real-valued solutions $y_1(t),\ y_2(t)$ that satisfies

(i) $y_1(0) = 1,\ y_2(0) = 0$;

(ii) $y_1'(0) = 0,\ y_2'(0) = 1$.

**Answer:**

(a) $y'' + \omega^2 y = 0 \Longrightarrow$ ch. eq. $r^2 + \omega^2 = 0 \Longrightarrow r_{1,\ 2} = \pm\omega i \Longrightarrow y_c(t) = e^{i\omega t} = \cos\omega t + i\sin\omega t$. Therefore,

$$y_1(t) = Re\{y_c(t)\} = \cos\omega t, \qquad y_2(t) = Im\{y_c(t)\} = \sin\omega t$$

form a fundamental set. It is easy to verify that $y_1(0) = 1,\ y_2(0) = 0$, and $y_1'(0) = 0,\ y_2'(0) = \omega$. To make $y_2'(0) = 1$, we choose

$$y_2(t) = \omega^{-1}\sin\omega t,$$

while leaving $y_1(t)$ as expressed above.

(b) $y'' - \omega^2 y = 0 \Longrightarrow$ ch. eq. $r^2 - \omega^2 = 0 \Longrightarrow r_{1,\ 2} = \pm\omega$. $r_1 \neq r_2$ are both real. Therefore

$$y_i(t) = e^{\omega t}, \qquad y_{ii}(t) = e^{-\omega t}$$

form a fundamental set. But $y_i(0) = y_{ii}(0) = 1$, $y_i'(0) = -y_{ii}'(0) = \omega$. They do not satisfy the initial conditions listed above.

Introduce two new functions: *hyperbolic sine* and *hyperbolic cosine*

$$\sinh\omega t = \frac{1}{2}[e^{\omega t} - e^{-\omega t}], \qquad \cosh\omega t = \frac{1}{2}[e^{\omega t} + e^{-\omega t}]. \tag{59}$$

Notice that both are linear combinations of $y_i(t)$ and $y_{ii}(t)$.

We notice the striking similarity between these definitions and those of *sine* and *cosine*:

$$\sin\omega t = \frac{1}{2i}[e^{i\omega t} - e^{-i\omega t}], \qquad \cos\omega t = \frac{1}{2}[e^{i\omega t} + e^{-i\omega t}].$$

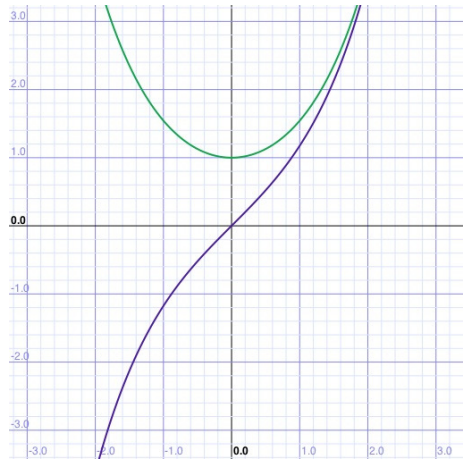



Figure 15: Graphs of $\sinh x = \frac{1}{2}(e^x - e^{-x})$ (purple) and $\cosh x = \frac{1}{2}(e^x + e^{-x})$ (green) .

Now,

$$y_1(t) = \cosh \omega t, \qquad y_2(t) = \sinh \omega t$$

also form a fundamental set since both are also solutions of the ODE based on the Principle of Superposition.

**Catenary:** The shape of the curve of $\cosh(ax)$ is often referred to as as catenary - the curve that an idealized hanging chain or cable assumes under its own weight when supported only at its ends. To learn more about catenary check here: http://en.wikipedia.org/wiki/Catenary.

Figure 16: A hanging chain with short links forms a catenary.

More similarities between hyperbolic trig and trig functions (not a complete list):

- $\sinh \omega t$ is odd, $\cosh \omega t$ is even.
- $\sinh(0) = 0,\ \cosh(0) = 1.$
- $\sinh' \omega t = \omega \cosh \omega t$ , $\cosh' \omega t = \omega \sinh \omega t$ (no change in sign!)
- $\cosh^2 \omega t - \sinh^2 \omega t = 1$ (subtraction not addition!)
- $\tanh \omega t = \sinh \omega t / \cosh \omega t.$

For a fundamental set of $y'' - \omega^2 y = 0$ that satisfies the above listed conditions, we choose

$$y_1(t) = \cosh \omega t, \qquad y_2(t) = \omega^{-1} \sinh \omega t.$$

It is easy to check that $y_1(0) = 1,\ y_2(0) = 0$ and $y_1'(0) = 0,\ y_2'(0) = 1.$

**Example 3.5.5:** Some advantage of using the hyperbolic trig functions as fundamental set. For the ODE

$$4y'' - y = 0.$$

(a) Find the general solution expressed in terms of exponential functions.

(b) Find the general solution expressed in terms of hyperbolic trig functions obeying the conditions in Example 3.5.4.

(c) In each case, determine the constants by using the ICs: $y(0) = 1,\ y'(0) = -1$.

**Answer:** In 'normal' form, the ODE reads $y'' - \frac{1}{4}y = 0 \Rightarrow$ ch. eq.: $r^2 - \frac{1}{4} = 0 \Rightarrow r_1,\ r_2 = \pm\frac{1}{2} = \pm\omega$.

(a) The general solution expressed in terms of exponential functions:

$$y(t) = c_1 e^{\frac{1}{2}t} + c_2 e^{-\frac{1}{2}t}. \qquad (a)$$

(b) The general solution expressed in terms of hyperbolic trig functions:

$$y(t) = c_1 \cosh(\omega t) + c_2 \omega^{-1} \sinh(\omega t) = c_1 \cosh(\frac{1}{2}t) + 2c_2 \sinh(\frac{1}{2}t). \qquad (b)$$

(c) Using ICs in solution (a), we obtain

$$y(0) = 1 = c_1 + c_2, \qquad y'(0) = -1 = \frac{1}{2}c_1 - \frac{1}{2}c_2.$$

Solve this system, we obtain $c_1 = -\frac{1}{2}$ and $c_2 = \frac{3}{2}$. Thus,

$$y(t) = -\frac{1}{2}e^{\frac{1}{2}t} + \frac{3}{2}e^{-\frac{1}{2}t}. \qquad (A)$$

Using ICs in solution (b), we obtain

$$y(0) = 1 = c_1, \qquad y'(0) = -1 = c_2.$$

There is no need to solve an algebraic system for $c_1,\ c_2$! Therefore,

$$y(t) = \cosh(\frac{1}{2}t) - 2\sinh(\frac{1}{2}t). \qquad (B)$$

It is easy to verify with a little bit algebra that (A) and (B) are identical but expressed in different forms.

**Example 3.5.6:** Solve the followin ODEs

(a) $y'' + \omega^2 y = 0,$

(b) $y'' - \omega^2 y = 0,$

with the initial conditions:

$$y(t_0) = A, \qquad y'(t_0) = B.$$

**Answer:**

(a) The general solution expressed in terms of trig functions:

$$y(t) = c_1 \cos \omega(t - t_0) + c_2 \omega^{-1} \sin \omega(t - t_0), \qquad (a)$$

Using the ICs:

$$y(t_0) = c_1 \cos 0 + c_2 \omega^{-1} \sin 0 = c_1 = A, \qquad y'(t_0) = -\omega c_1 \sin 0 + c_2 \cos 0 = c_2 = B.$$

Therefore,

$$y(t) = A \cos \omega(t - t_0) + B\omega^{-1} \sin \omega(t - t_0).$$

(b) The general solution expressed in terms of hyperbolic trig functions:

$$y(t) = c_1 \cosh \omega(t - t_0) + c_2 \omega^{-1} \sinh \omega(t - t_0). \qquad (b)$$

Using the ICs

$$y(t_0) = c_1 \cosh 0 + c_2 \omega^{-1} \sinh 0 = c_1 = A, \qquad y'(t_0) = \omega c_1 \sinh 0 + c_2 \cosh 0 = c_2 = B.$$

Therefore,

$$y(t) = A \cosh \omega(t - t_0) + B\omega^{-1} \sinh \omega(t - t_0).$$

## 3.6 Nonhomogeneous, 2nd-order, linear ODEs

Nonhomogeneous linear 2nd-order ODEs of the form

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \tag{60}$$

where $g(t) \neq 0$ also occurs often in a wide variety of applied problems.

**A model of suspension systems:** When a car runs on bumpy roads, the suspension system undergoes a sustained oscillatory force. To model this situation, we consider a modified version of our quarter car model. See Fig. 17. For this system, the ODE becomes

$$my'' + \gamma y' + ky = F_f \cos \omega_f t, \tag{61}$$

where $F_f$ is the amplitude of the external forcing, $\omega_f$ is the forcing frequency. Both $F_f$, $\omega_f$ are supposed to be known constants.

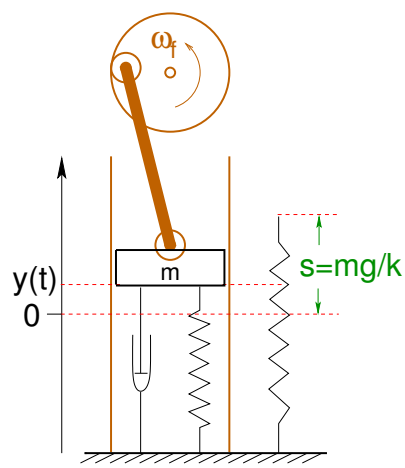



Figure 17: Periodically forced spring-mass-damper system: a model of forced suspension systems.

**A RLC circuit powered by an alternative voltage source:** When a closed RCL circuit is powered by an alternative voltage source, See Fig. 18. For this system, the ODE becomes

$$LQ'' + RQ' + \frac{1}{C}Q = F_f \cos \omega_f t, \tag{62}$$

where $F_f$ is the amplitude of the external forcing, $\omega_f$ is the forcing frequency. Both $F_f$, $\omega_f$ are supposed to be known constants.

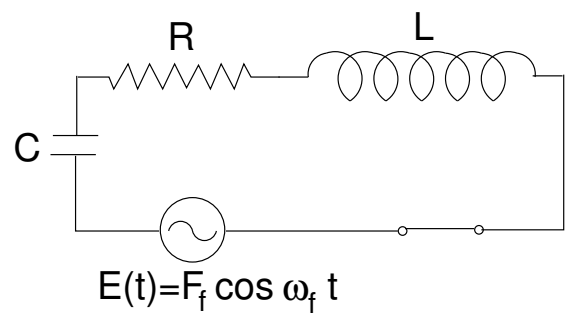



Figure 18: Periodically forced RLC circuit

**Questions :** How to solve these nonhomogeneous ODEs?

**Theorem 3.6.1:** If $Y_1$ and $Y_2$ are solutions of the nonhomogeneous ODE $L[y] = g(t)$, then the difference between the two $Y_1 - Y_2$ is a solution of the homogeneous ODE $L[y] = 0$.

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2$$

where $y_1,\ y_2$ form a fundamental set of $L[y] = 0$, $c_1,\ c_2$ are constants.

**Proof:** Since $L[Y_1] = g(t)$, $L[Y_2] = g(t)$, and remember $L$ is linear

$$L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(t) - g(t) = 0.$$

This means that $Y_1 - Y_2$ is a solution of $L[y] = 0$. Any solution of $L[y] = 0$ can be expressed as

$$c_1 y_1 + c_2 y_2$$

provided $y_1,\ y_2$ form a fundamental set.

**Theorem 3.6.2 Solution structure of $L[y] = g(t)$:** The general solution of $L[y] = g(t)$ can be expressed in the following form

$$y(t) = c_1 y_1 + c_2 y_2 + Y(t) = y_h(t) + y_p(t)$$

where $y_1,\ y_2$ form a fundamental set of $L[y] = 0$, $c_1,\ c_2$ are arbitrary constants, $y_h(t) = c_1 y_1 + c_2 y_2$ is the general solution of the corresponding homogeneous ODE $L[y] = 0$ ($h$ for homogeneous), $y_p(t) = Y(t)$ is one particular solution of $L[y] = g(t)$ ($p$ for particular).

**Proof:** Follow theorem 3.6.1, treating $y(t)$ as $Y_1$ and $Y(t)$ as $Y_2$.

**Remark:** The general solution of $L[y] = g(t)$ is the sum of the general solution $y_h(t)$ of the homogeneous ODE $L[y] = 0$ and a particular solution $y_p(t)$ of $L[y] = g(t)$. We know how to solve $y_h(t)$.

**Question:** How to find one $y_p(t)$? (Usually takes more time and effort to solve than $y_h(t)$!).

### 3.6.1 Method 1: Undetermined coefficients

**Basic idea:** Based on the nature of $g(t)$, we generate an educated guess of $y_p(t)$, leaving some constants to be determined by plugging it into the ODE.

**Example 3.6.1:** Find one particular solution $y_p(t)$ for

$$2y'' + 3y' + y = t^2.$$

**Answer:** For this example, $g(t) = t^2$ is a 2nd degree polynomial. It is natural to assume that $y_p(t)$ is also a polynomial of degree 2:

$$y_p(t) = At^2 + Bt + C,$$

where $A,\ B,\ C$ are coefficients to be determined.

Notice that

$$y_p'(t) = 2At + B, \qquad y_p''(t) = 2A.$$

Substitute $y_p''(t),\ y_p'(t),\ y_p(t)$ into the ODE, we obtain

$$2(2A) + 3(2At + B) + At^2 + Bt + C = t^2 \qquad \Longrightarrow$$

$$At^2 + (6A + B)t + (4A + 3B + C) = t^2.$$

Coefficients of equal power of $t$ should be identical on both sides:

$$A = 1;$$

$$6A + B = 0 \qquad \Longrightarrow \qquad B = -6A = -6;$$

$$4A + 3B + C = 0 \qquad \Longrightarrow \qquad C = -4A - 3B = -4 + 18 = 14.$$

Therefore,

$$y_p(t) = t^2 - 6t + 14,$$

is one particular solution.

**Example 3.6.2:** Find one particular solution $y_p(t)$ for

$$2y'' + 3y' + y = 3\sin t.$$

**Answer:** An educated guess is: $y_p(t) = A\cos t + B\sin t$. Note that

$$y_p'(t) = -A\sin t + B\cos t,$$

$$y_p''(t) = -A\cos t - B\sin t.$$

Substitute $y_p''(t),\ y_p'(t),\ y_p(t)$ into the ODE:

$$lhs = 2(-A\cos t - B\sin t) + 3(-A\sin t + B\cos t) + A\cos t + B\sin t = (3B - A)\cos t - (B + 3A)\sin t$$

$$rhs = 3\sin t.$$

Coefficients on both sides must be identical:

$$3B - A = 0, \qquad -(B + 3A) = 3 \qquad \Longrightarrow \qquad A = 3B = -\frac{9}{10}, \qquad B = -\frac{3}{10}.$$

Therefore,

$$y_p(t) = -\frac{3}{10}\left[3\cos t + \sin t\right].$$

**Example 3.6.3:** Find the general solution for

$$2y'' + 3y' + y = t^2 + 3\sin t.$$

**Answer:** Observation:

$$g(t) = g_1(t) + g_2(t),$$

with $g_1(t) = t^2$ and $g_2(t) = 3\sin t$, each was solved separately in the previous two examples.

**Theorem 3.6.3 Principle of Superposition:** For the nonhomogeneous ODE

$$L[y] = y'' + p(t)y' + q(t)y = g(t).$$

If

$$g(t) = g_1(t) + g_2(t),$$

and that

$$L[y_{p1}] = g_1(t), \qquad L[y_{p2}] = g_2(t),$$

then

$$y_p(t) = y_{p1}(t) + y_{p2}(t)$$

is a particular solution of $L[y] = g(t)$.

**Proof:** Since $L$ is linear $\Rightarrow$

$$L[y_p] = L[y_{p1} + y_{p2}] = L[y_{p1}] + L[y_{p2}] = g_1(t) + g_2(t) = g(t).$$

**Back to Example 3.6.3:** Based on the Principle of Superposition and results from the two previous examples, a particular solution is:

$$y_p(t) = y_{p1}(t) + y_{p2}(t) = t^2 - 6t + 14 - \frac{3}{10}\left[3\cos t + \sin t\right].$$

To find $y_h(t)$, we solve the ch. eq.: $2r^2 + 3r + 1 = 0 \Rightarrow r_1 = -\frac{1}{2}$, $r_2 = -1$. Thus,

$$y_h(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{-t}.$$

The general solution is

$$y(t) = y_h(t) + y_p(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{-t} + t^2 - 6t + 14 - \frac{3}{10}\left[3\cos t + \sin t\right].$$

Notice that

$$y_h(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{-t} \qquad \overset{t\to\infty}{\longrightarrow} \qquad 0!$$

Therefore,

$$y(t) = y_h(t) + y_p(t) \qquad \overset{t\to\infty}{\longrightarrow} \qquad y_p(t) = t^2 - 6t + 14 - \frac{3}{10}\left[3\cos t + \sin t\right].$$

$y_p$ **cannot be identical to either solution in the fundamental set:**

**Example 3.6.3:** Find the general solution for

$$2y'' + 3y' + y = e^{-t}.$$

**Answer:** The homogeneous part is identical to the previous example. Thus,

$$y_h(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{-t}.$$

Since $g(t) = e^{-t}$, it seems reasonable to assume

$$y_p(t) = Ae^{-t}.$$

It is not Ok here!!! Because $Ae^{-t}$ is already contained in the second term of $y_h(t)$.

In this case, a correct educated guess is

$$y_p(t) = Ate^{-t}.$$

Notice that

$$y_p'(t) = Ae^{-t} - Ate^{-t} = A(1-t)e^{-t},$$
$$y_p''(t) = -Ae^{-t} - A(1-t)e^{-t} = A(-2+t)e^{-t}.$$

Substitute $y_p''(t),\ y_p'(t),\ y_p(t)$ into the ODE:

$$lhs = 2A(-2+t)e^{-t} + 3A(1-t)e^{-t} + Ate^{-t} = -Ae^{-t} = e^{-t} = rhs \qquad \Longrightarrow \qquad A = -1.$$

Thus, $y_p(t) = -te^{-t}$ and the general solution is

$$y(t) = y_h(t) + y_p(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{-t} - te^{-t}.$$

**A list of frequently encountered $g(t)$ and the educated guess of $y_p(t)$:**

Table 1: Educated guess of $y_p(t)$ for $L[y] = g(t)$. Here, $s = 0,\ 1,$ or 2 is the integer that ensures no term in $y_p(t)$ is identical to either terms of $y_h(t)$.

| $g(t)$ | $y_p(t)$ |
|---|---|
| $P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_{n-1}t + a_n$ | $t^s(A_0t^n + A_1t^{n-1} + \cdots + A_{n-1}t + A_n)$ |
| $P_n(t)e^{\alpha t}$ | $t^s(A_0t^n + A_1t^{n-1} + \cdots + A_{n-1}t + A_n)e^{\alpha t}$ |
| $P_n(t)e^{\alpha t}\cos\beta t, \quad P_n(t)e^{\alpha t}\sin\beta t$ | $t^s[(A_0t^n + A_1t^{n-1} + \cdots + A_{n-1}t + A_n)\cos\beta t +$ $(B_0t^n + B_1t^{n-1} + \cdots + B_{n-1}t + B_n)\sin\beta t]e^{\alpha t}$ |

**Example 3.6.4:** Use the table above to determine the appropriate guess of $y_p(t)$ for the following ODE

$$y'' + 2y' + y = t^2e^{-t}.$$

**Answer:** The ch. eq. is: $r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = r_1 = r_2 = -1$ (repeated root!)

$$y_h(t) = (c_1 + c_2t)e^{-t}.$$

Based on the table,

$$y_p(t) = t^s(At^2 + Bt + C)e^{-t}.$$

To make sure that no term in $y_p(t)$ is identical to any term in $y_h(t)$,

$$s = 2 \qquad \Longrightarrow \qquad y_p(t) = t^2(At^2 + Bt + C)e^{-t}.$$

Now, calculating the coefficients $A,\ B,\ C$ will be rather long and tedious.

**Pros and cons of the method of undetermined coefficients:**

- Straightforward, easy to understand.
- Principle of Superposition can be used to break it to smaller problems.
- Restricted to certain forms of $g(t)$ as listed in Table 3.6.1.
  E.g., it does not apply easily to: $y'' + 2y' + y = \sqrt{t}e^{-t}$ (A final exam problem in 2000).
- Does not apply to linear ODEs with varying coefficients $y'' + p(t)y' + q(t)y = g(t)$.
- Calculations are often long and tedious.

### 3.6.2 Method 2: Variation of parameters (Lagrange)

**Basic idea:** If $y_1(t),\ y_2(t)$ form a fundamental set of the homogeneous ODE $L[y]=0$, then

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t).$$

Lagrange suggested that a particular solution for the nonhomogeneous ODE $L[y]=g(t)$ can always be expressed as

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

which is obtained by allowing the parameters $c_1,\ c_2$ in $y_h(t)$ to vary as functions of $t$.

**Theorem 3.6.4:** Consider the linear, 2nd-order, nonhomogeneous ODE

$$L[y] = y'' + p(t)y' + q(t)y = g(t),$$

where $p(t),\ q(t),\ g(t)$ are continuous in an open interval $I$. $y_1(t),\ y_2(t)$ form a fundamental set of the homogeneous ODE $L[y]=0$, then one particular solution for $L[y]=g(t)$ is

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

where

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W[y_1,\ y_2](t)}dt, \qquad u_2(t) = \int \frac{y_1(t)g(t)}{W[y_1,\ y_2](t)}dt,$$

are calculated by excluding the integration constant. The general solution of $L[y]=g(t)$ is

$$y(t) = y_h(t) + y_p(t) = c_1y_1(t) + c_2y_2(t) + u_1(t)y_1(t) + u_2(t)y_2(t) = [c_1 + u_1(t)]y_1(t) + [c_2 + u_2(t)]y_2(t).$$

**Proof:** (Not required!) The goal is to solve for $u_1(t)$ and $u_2(t)$.

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 \qquad \Longrightarrow \\ y_p' &= u_1y_1' + u_2y_2' + u_1'y_1 + u_2'y_2 \qquad \Longrightarrow \\ y_p' &= u_1y_1' + u_2y_2', \end{aligned}$$

if we force the following constraint on $u_1(t)$ and $u_2(t)$ such that

$$u_1'y_1 + u_2'y_2 = 0. \qquad (a)$$

Now,

$$y_p'' = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2'.$$

Substitute $y_p'',\ y_p',\ y_p$ into the $L[y]=g(t)$ and reorganize and simplify the terms, we obtain

$$u_1L[y_1] + u_2L[y_2] + u_1'y_1' + u_2'y_2' = g(t).$$

Notice that $L[y_1] = 0$ and $L[y_2] = 0$ , we obtain

$$u_1'y_1' + u_2'y_2' = g(t). \qquad (b)$$

We now have the system of two equations

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0, & (a) \\ u_1'y_1' + u_2'y_2' = g(t). & (b) \end{cases}$$

Turn this system of algebraic equations in matrix form:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix} \qquad \Longrightarrow \qquad A\vec{x} = \vec{b}, \tag{63}$$

where

$$A = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}, \qquad \vec{x} = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}.$$

Important note: Since the formula given in **Theorem 3.6.4** was derived using the ODE of the "standard form"

$$y'' + p(t)y' + q(t)y = g(t),$$

it is important to change your equation into this standard form before using this formula. Atually, it is always a good habit to change your ODE into this form before you try to solve it.

Since

$$\det A = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W[y_1,\ y_2] \neq 0,$$

there exists a unique solution (based on Cramer's rule):

$$u_1' = \frac{1}{\det A} \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = -\frac{y_2 g}{W[y_1,\ y_2]}, \qquad u_2' = \frac{1}{\det A} \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = \frac{y_1 g}{W[y_1,\ y_2]}.$$

Integrate both expressions to obtain

$$u_1 = -\int \frac{y_2 g}{W[y_1,\ y_2]} dt, \qquad u_2 = \int \frac{y_1 g}{W[y_1,\ y_2]} dt.$$

Since we only need one particular solution, we do not need to include the integration constant in these integrals.

**Back to Example 3.6.4:** Find the general solution for the following ODE.

$$y'' + 2y' + y = t^2 e^{-t}.$$

**Answer:** The ch. eq. is: $r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = r_1 = r_2 = -1$ (repeated root!)

$$y_h(t) = (c_1 + c_2 t)e^{-t}.$$

Notice that $y_1(t) = e^{-t}$, $y_2(t) = te^{-t}$. The Wronskian

$$W[y_1,\ y_2] = y_1 y_2' - y_1' y_2 = e^{-t}(1-t)e^{-t} - (-e^{-t})te^{-t} = e^{-2t}.$$

Based on the method of undetermined coefficient,

$$y_p(t) = t^2(At^2 + Bt + C)e^{-t}.$$

Calculating these coefficients $A,\ B,\ C$ will be rather long and tedious.

Now, let us use "variation of parameters".

$$u_1(t) = -\int \frac{y_2 g}{W[y_1,\ y_2]}dt = -\int \frac{te^{-t}t^2e^{-t}}{e^{-2t}}dt = -\int t^3 dt = -\frac{1}{4}t^4.$$

$$u_2(t) = \int \frac{y_1 g}{W[y_1,\ y_2]}dt = \int \frac{e^{-t}t^2e^{-t}}{e^{-2t}}dt = \int t^2 dt = \frac{1}{3}t^3.$$

Thus,

$$y_p(t) = u_1 y_1 + u_2 y_2 = -\frac{1}{4}t^4 e^{-t} + \frac{1}{3}t^3(te^{-t}) = \left[\frac{1}{3} - \frac{1}{4}\right] t^4 e^{-t} = \frac{1}{12}t^4 e^{-t}.$$

The general solution is

$$y(t) = y_h(t) + y_p(t) = [c_1 + c_2 t]\, e^{-t} + \frac{1}{12}t^4 e^{-t}.$$

**Example 3.6.5:** Find the general solution for the following ODE (taken from 2000 final exam).

$$y'' + 2y' + y = \sqrt{t}e^{-t}.$$

**Answer:** Because $g(t) = \sqrt{t}e^{-t}$ is not included in the in Table 3.6.1, we do not have an educated guess of $y_p(t)$. We use the "variation of parameters" method.

Ch. eq. $r^2 + 2r + 1 = 0 \Longrightarrow (r+1)^2 = 0 \Longrightarrow r_1 = r_2 = r = -1 \Longrightarrow y_1(t) = e^{-t},\ y_2(t) = te^{-t}$.

$$y_h(t) = (c_1 + c_2 t)e^{-t}.$$

The Wronskian is

$$W[y_1,\ y_2] = y_1 y_2' - y_1' y_2 = e^{-t}(1-t)e^{-t} - (-e^{-t})te^{-t} = e^{-2t}.$$

Thus,

$$u_1(t) = -\int \frac{y_2 g}{W[y_1,\ y_2]}dt = -\int \frac{te^{-t}\sqrt{t}e^{-t}}{e^{-2t}}dt = -\int t^{\frac{3}{2}}dt = -\frac{2}{5}t^{\frac{5}{2}}.$$

$$u_2(t) = \int \frac{y_1 g}{W[y_1,\ y_2]}dt = \int \frac{e^{-t}\sqrt{t}e^{-t}}{e^{-2t}}dt = \int t^{\frac{1}{2}}dt = \frac{2}{3}t^{\frac{3}{2}}.$$

Thus,

$$y_p(t) = u_1 y_1 + u_2 y_2 = -\frac{2}{5}t^{\frac{5}{2}}e^{-t} + \frac{2}{3}t^{\frac{3}{2}}(te^{-t}) = \left[\frac{2}{3} - \frac{2}{5}\right]t^{\frac{5}{2}}e^{-t} = \frac{4}{15}t^{\frac{5}{2}}e^{-t}.$$

The general solution is

$$y(t) = y_h(t) + y_p(t) = [c_1 + c_2 t]\,e^{-t} + \frac{4}{15}t^{\frac{5}{2}}e^{-t}.$$

**Example 3.6.6:** Given that $y_1(t) = t,\ y_2(t) = te^t$ form a fundamental set of the homogeneous ODE corresponding to the nonhomgeneous ODE

$$t^2y'' - t(t+2)y' + (t+2)y = 2t^3,$$

find the general solution to this nonhomgeneous ODE. (Note that for ODEs with non-constant coefficients, we do not yet know how to solve for $y_1(t)$, $y_2(t)$.)

**Answer:** First turn the equation into the 'normal' form

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 2t = g(t), \qquad (t > 0).$$

The Wronskian

$$W[y_1,\ y_2] = \begin{vmatrix} t & te^t \\ 1 & (1+t)e^t \end{vmatrix} = t(1+t)e^t - te^t = t^2e^t.$$

Now,

$$u_1 = -\int \frac{y_2 g}{W[y_1,\ y_2]}dt = -\int \frac{te^t(2t)}{t^2e^t}dt = -\int 2dt = -2t.$$

$$u_2 = \int \frac{y_1 g}{W[y_1,\ y_2]}dt = \int \frac{t(2t)}{t^2e^t}dt = \int 2e^{-t}dt = -2e^{-t}.$$

Thus,

$$y_p(t) = u_1y_1 + u_2y_2 = (-2t)t + (-2e^{-t})(te^t) = -2t^2 - 2t.$$

Therefore,

$$y(t) = y_h(t) + y_p(t) = c_1t + c_2te^t - 2t^2 - 2t = (c_1 - 2)t + c_2e^t - 2t^2 = c_1t + c_2te^t - 2t^2.$$

**Example 3.6.7:** For the nonhomgeneous ODE

$$t^2y'' - 2ty' + 2y = 4t^2, \qquad (t > 0)$$

find its general solution given that $y_1(t) = t$ is one solution to the corresponding homogeneous ODE $t^2y'' - 2ty' + 2y = 0$.

**Answer:** First turn the equation into the 'normal' form

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 4.$$

Thus, $p(t) = -\frac{2}{t}$ and $g(t) = 4$.

To find both $y_h(t)$ and $y_p(t)$, we need a second LI solution $y_2(t)$ for the homogeneous equation $L[y] = 0$.

**To solve for $y_2(t)$:** Method I: Abel's theorem.

Assuming that $y_2(t)$ is LI of $y_1(t)$, we can use Abel's theorem to calculate the Wronskian (by picking the constant $C = 1$):

$$W[y_1,\ y_2] = y_1y_2' - y_1'y_2 = e^{-\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{\ln t^2} = t^2.$$

This yields the following 1st-order, linear ODE for $y_2$:

$$ty_2' - y_2 = t^2 \qquad \Longrightarrow \qquad y_2' - \frac{1}{t}y_2 = t.$$

The integrating factor is $e^{\int p(t)dt} = e^{-\int \frac{1}{t}dt} = \frac{1}{t}$, and $R(t) = \int e^{\int p(t)dt}tdt = \int dt = t + C = t$. Thus,

$$y_2(t) = e^{-\int p(t)dt}[R(t) + C] = t[t + C] = t^2.$$

Therefore,

$$y_h(t) = c_1y_1(t) + c_2y_2(t) = c_1t + c_2t^2.$$

The Wronskian obtained by using Abel's theorem is indeed the one we are looking for

$$W[y_1,\ y_2] = y_1y_2' - y_1'y_2 = t(t^2)' - (t)'(t^2) = 2t^2 - t^2 = t^2 \qquad (> 0,\ \text{since } t > 0).$$

**To find $y_p(t)$:** we use "variation of parameters". Let

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

where

$$u_1(t) = -\int \frac{gy_2}{W[y_1,\ y_2]}dt = -\int \frac{4t^2}{t^2}dt = -\int 4dt = -4t,$$

$$u_2(t) = \int \frac{gy_1}{W[y_1,\ y_2]}dt = \int \frac{4t}{t^2}dt = \int \frac{4}{t}dt = \ln t^4.$$

Thus,

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = -4t^2 + t^2 \ln t^4.$$

The general solution to the nonhomogeneous ODE is

$$y(t) = y_h(t) + y_p(t) = c_1 t + c_2 t^2 - 4t^2 + t^2 \ln t^4 = c_1 t + c_2 t^2 + t^2 \ln t^4.$$

**To solve for $y_2(t)$:** Method II: "reduction of order". Let

$$y_2(t) = u(t)y_1(t).$$

$$y_2' = u'y_1 + uy_1', \qquad y_2'' = u''y_1 + 2u'y_1' + vy_1''.$$

Substitute $y_2'',\ y_2',\ y_2$ into the ODE $L[y] = 0$, we obtain

$$L[y_2] = uL[y_1] + u'(py_1 + 2y_1') + u''y_1 = 0.$$

The first term is zero and the remaining terms give

$$u'' + \frac{py_1 + 2y_1'}{y_1}u' = 0 \qquad \xrightarrow{v=u'} \qquad v' + \frac{py_1 + 2y_1'}{y_1}v = 0 \qquad \Longrightarrow \qquad v' = -\frac{py_1 + 2y_1'}{y_1}v.$$

Separation of variables

$$\frac{dv}{v} = -\frac{py_1 + 2y_1'}{y_1}dt \qquad \Longrightarrow \qquad \int \frac{dv}{v} = -\int \frac{py_1 + 2y_1'}{y_1}dt \qquad \Longrightarrow \qquad \ln v = -\int \frac{py_1 + 2y_1'}{y_1}dt.$$

Therefore,

$$u'(t) = v(t) = Cexp\left[-\int \frac{py_1 + 2y_1'}{y_1}dt\right] \overset{C=1}{=} exp\left[-\int \frac{py_1 + 2y_1'}{y_1}dt\right] \qquad \Longrightarrow$$

$$u(t) = \int exp\left[-\int \frac{py_1 + 2y_1'}{y_1}dt\right]dt \qquad \Longrightarrow \qquad u(t) = \int exp\left[-\int \left(p + 2\frac{y_1'}{y_1}\right)dt\right]dt$$

Substitute $p(t) = -\frac{2}{t}$ and $y_1(t) = t$ into the formula,

$$-\int \left(p + 2\frac{y_1'}{y_1}\right) dt = -\int \left(-\frac{2}{t} + 2\frac{(t)'}{t}\right) dt = -\int \frac{-2+2}{t} dt = -\int 0 dt = C_1 \qquad \Longrightarrow$$

$$u(t) = \int e^{C_1} dt = \int C_2 dt = C_2 t + C_3 \overset{C_2=1,\ C_3=0}{=} t.$$

Therefore

$$y_2(t) = u(t) y_1(t) = t^2.$$

## 3.7 Applications to forced vibrations: beats and resonance

### 3.7.1 Forced vibration in the absence of damping: $\gamma = 0$

(idealized situation)

$$\begin{cases} my'' + ky = F_f \cos(\omega_f t), \\ y(0) = 0, \quad y'(0) = 0. \end{cases} \qquad \Rightarrow \qquad y'' + \omega_0^2 y = F\cos(\omega_f t), \qquad (\omega_0 = \sqrt{\tfrac{k}{m}},\ F = \tfrac{F_f}{m})$$

**Case I:** $\omega_f \neq \omega_0$. We know

$$y_h(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

Since only even ordered derivatives occur on the lhs,

$$y_p(t) = B\cos(\omega_f t) \qquad \Longrightarrow \qquad y_p''(t) = -B\omega_f^2 \cos(\omega_f t).$$

Substitute $y_p,\ y_p''$ into the ODE:

$$-B\omega_f^2 \cos(\omega_f t) + B\omega_0^2 \cos(\omega_f t) = F\cos(\omega_f t) \qquad \Longrightarrow \qquad B = \frac{F}{\omega_0^2 - \omega_f^2}.$$

Therefore, the general solutions is

$$y(t) = y_h(t) + y_p(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F}{\omega_0^2 - \omega_f^2} \cos(\omega_f t).$$

Using initial conditions,

$$y(0) = 0 = c_1 + \frac{F}{\omega_0^2 - \omega_f^2} \qquad \Longrightarrow \qquad c_1 = -\frac{F}{\omega_0^2 - \omega_f^2};$$

$$y'(0) = 0 = \omega_0 c_2 \qquad \Longrightarrow \qquad c_2 = 0.$$

Thus,

$$y(t) = \frac{F}{\omega_0^2 - \omega_f^2} \left[\cos(\omega_f t) - \cos(\omega_0 t)\right].$$

**Question:** How to combine the sum of two trig functions with different frequencies into a product between two trig functions?

$$\text{Let} \qquad \omega_+ = \frac{1}{2}(\omega_0 + \omega_f), \qquad \omega_- = \frac{1}{2}(\omega_0 - \omega_f); \qquad \Longrightarrow \qquad \omega_0 = \omega_+ + \omega_-, \qquad \omega_f = \omega_+ - \omega_-.$$

Using the trig identities

$$\cos(\alpha \mp \beta) = \cos\alpha\cos\beta \pm \sin\alpha\sin\beta,$$

we obtain

$$\cos(\omega_f t) = \cos(\omega_+ - \omega_-)t = \cos\omega_+ t\cos\omega_- t + \sin\omega_+ t\sin\omega_- t,$$
$$\cos(\omega_0 t) = \cos(\omega_+ + \omega_-)t = \cos\omega_+ t\cos\omega_- t - \sin\omega_+ t\sin\omega_- t.$$

Therefore.

$$y(t) = \frac{F}{\omega_0^2 - \omega_f^2}\left[\cos(\omega_f t) - \cos(\omega_0 t)\right] = \frac{2F}{\omega_0^2 - \omega_f^2}\sin\omega_+ t\sin\omega_- t = \frac{2F}{\omega_0^2 - \omega_f^2}\sin\frac{\omega_0 + \omega_f}{2}t\sin\frac{\omega_0 - \omega_f}{2}t.$$

Assuming that $\bar{\omega} = \frac{\omega_0 + \omega_f}{2}$ is the average between $\omega_0$ and $\omega_f$, we can write the solution in the following form

$$y(t) = A(t)\sin\bar{\omega}t, \qquad \text{where} \qquad A(t) = \frac{2F}{\omega_0^2 - \omega_f^2}\sin\frac{\omega_0 - \omega_f}{2}t.$$

$A(t)$ is the periodically modulated amplitude, thus $y(t)$ shows the phenomenon of "beats" with a beat frequency

$$\omega_{beat} = \frac{\omega_0 - \omega_f}{2}.$$

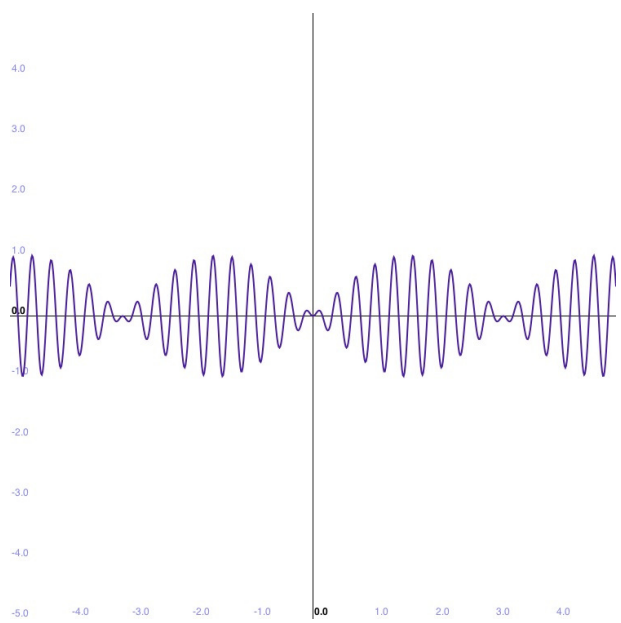



Figure 19: Beats (amplitude modulation) of $\sin(20x)\sin(x)$.

**Case II:** $\omega_f = \omega_0 = \omega$. Again,

$$y_h(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

but

$$y_p(t) = Bt \sin \omega t.$$

Notice that

$$y_p''(t) = B(2\omega \cos \omega t - \omega^2 t \sin \omega t)$$

Substitute $y_p,\ y_p''$ into the ODE:

$$B(2\omega \cos \omega t - \omega^2 t \sin \omega t) + \omega^2 Bt \sin \omega t = F \cos(\omega t),$$

we obtain

$$B = \frac{F}{2\omega}, \qquad \Longrightarrow \qquad y_p(t) = \frac{F}{2\omega} t \sin \omega t.$$

The general solution is

$$y(t) = y_h(t) + y_p(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{F}{2\omega} t \sin \omega t.$$

Using initial conditions,

$$\begin{aligned} y(0) = 0 = c_1 \qquad &\Longrightarrow \qquad c_1 = 0; \\ y'(0) = 0 = \omega c_2 \qquad &\Longrightarrow \qquad c_2 = 0. \end{aligned}$$

Therefore,

$$y(t) = \frac{F}{2\omega} t \sin \omega t.$$

**Resonance:** The amplitude of this vibration increases to infinity as time approaches infinity.

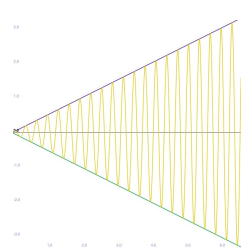


Figure 20: Resonance in the absence of damping. $f(x) = (x/2)sin(20x)$.

### 3.7.2 Forced vibration in the presence of damping: $\gamma \neq 0$

$$my'' + \gamma y' + ky = F_f \cos(\omega_f t) \qquad \Longrightarrow \qquad y'' + 2\Gamma y' + \omega_0^2 y = F\cos(\omega_f t), \tag{64}$$

where $\Gamma = \gamma/(2m)$, $\omega_0 = \sqrt{k/m}$, and $F = F_f/m$.

**Characteristic equation:**

$$r^2 + 2\Gamma r + \omega_0^2 = 0.$$

Consider under damped condition,

$$r_1,\ r_2 = -\Gamma \pm \sqrt{\Gamma^2 - \omega_0^2} = -\Gamma \pm i\sqrt{\omega_0^2 - \Gamma^2} = -\Gamma \pm i\omega,$$

where

$$\omega = \sqrt{\omega_0^2 - \Gamma^2} = \omega_0\sqrt{1 - \frac{\Gamma^2}{\omega_0^2}}.$$

Thus,

$$y_h(t) = Ae^{-\Gamma t}\cos(\omega t - \phi) \qquad \overset{t\to\infty}{\longrightarrow} \qquad 0!$$

Therefore,

$$y(t) = y_h(t) + y_p(t) \qquad \overset{t\to\infty}{\longrightarrow} \qquad y_p(t),$$

which means that the long term behaviour is determined by $y_p(t)$.

Since the nonhomogeneous term $g(t) = F\cos(\omega_f t)$ (no exponential factor $e^{-\Gamma t}$!), we can safely assume

$$y_p(t) = C\cos(\omega_f t) + D\sin(\omega_f t) = A\cos(\omega_f t - \phi),$$

where $A = \sqrt{C^2 + D^2}$, $\phi = \tan^{-1}\frac{D}{C}$.

Plug $y_p(t)$ into the ODE and after lengthy calculations, we obtain

$$C = \frac{F(\omega_0^2 - \omega_f^2)}{4\Gamma^2\omega_f^2 + (\omega_0^2 - \omega_f^2)}, \qquad D = \frac{2F\Gamma\omega_f}{4\Gamma^2\omega_f^2 + (\omega_0^2 - \omega_f^2)}.$$

Now, substitute back the original model parameters, we obtain

$$A = A(\omega_f^2) = \frac{F_f}{\sqrt{\gamma^2\omega_f^2 + m^2(\omega_0^2 - \omega_f^2)^2}}, \qquad \phi = \tan^{-1}\frac{\gamma\omega_f}{m(\omega_0^2 - \omega_f^2)}.$$

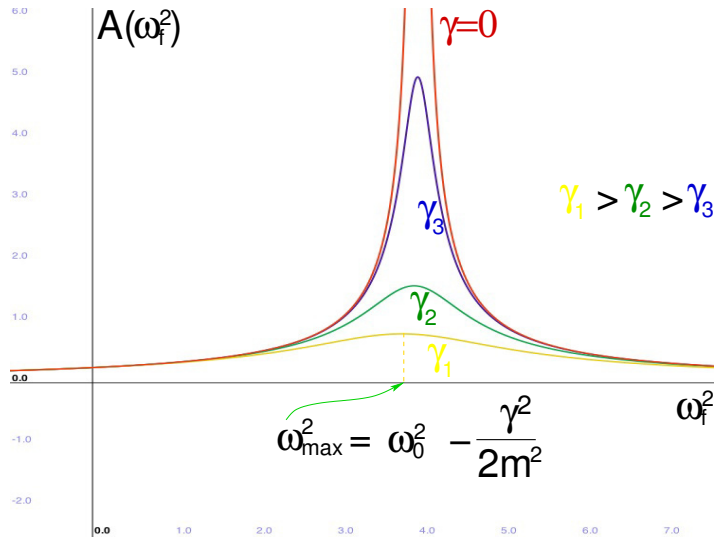



Figure 21: Resonant amplitude (expressed as multiples of $F_f$) as a function of forcing frequency $\omega_f^2$. $\gamma$ changes the steepness of the curve, steeper for smaller values of $\gamma$. When $\omega_f^2 = \omega_{max}^2$, maximum amplitude is achieved for that value of $\gamma$.

In Fig. 21, the amplitude of the vibration $A(\omega_f^2)$ is plotted as a function of $\omega_f^2$ for four different values of $\gamma$. The smaller the value of $\gamma$, the steeper the curve. The location of $\omega_{max}^2$ where the maximum amplitude is achieved also shift to the right as the value of $\gamma$ decreases.

By requiring

$$A' = \frac{dA}{d\omega_f^2} = -\frac{F_f}{2}\frac{\gamma^2 - 2m^2(\omega_0^2 - \omega_f^2)}{(\gamma^2\omega_f^2 + m^2(\omega_0^2 - \omega_f^2)^2)^{\frac{3}{2}}} = 0,$$

we find the maximum amplitude occurs when

$$\omega_f^2 = \omega_{max}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} \approx \omega_0^2 \qquad (\text{if } \gamma << 1!).$$

At this "resonant" frequency, the oscillation amplitude is

$$A_{max} = A(\omega_{max}^2) = \frac{F_f}{\gamma\sqrt{\omega_0^2 - \frac{\gamma^2}{4m^2}}}.$$

**Conclusion:** Resonance in forced vibration of a spring-mass-damper system can be understood completely by studying the properties of $y_p(t)$ of the solution. To avoid resonance, one needs to make the $\omega_f$ as different from $\omega_0$ as possible or/and to make the damping as large as possible.

# 4 Systems of linear first-order ODEs

In this lecture, we study systems of linear, first-order ODEs with constant coefficients of the form

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 + b_1(t), \\ y_2' = a_{21}y_1 + a_{22}y_2 + b_2(t), \end{cases} \tag{65}$$

where $y_1(t)$, $y_2(t)$ are the two unknown functions, $b_1(t)$, $b_2(t)$ are known functions, $a_{ij}$ $(i,\ j = 1,\ 2)$ are constants. As long as $a_{12}$ and $a_{21}$ are not both zero, the two unknown functions are interdependent and cannot be solved independently.

**Remarks:**

(1) A second-order ODE $ay'' + by' + cy = g(t)$, $(a \neq 0)$ can be reduced to a system of two first-order ODEs as follows

$$\begin{cases} y' = & z, \\ z' = -\frac{c}{a}y - \frac{b}{a}z + \frac{1}{a}g(t), \end{cases}$$

which is a special case of eq.(65) where $a_{11} = 0$ and $b_1(t) = 0$. Solving eq.(65) automatically solves $ay'' + by' + cy = g(t)$ as a special case.

(2) An $n - th$ order linear ODE can be reduced to a system of $n$ first order linear ODEs.

(3) Although many results obtained here can apply to systems of $n$ $(n > 2)$ first order ODEs, we shall focus mostly on a system of two ODEs as given in eq.(65).

**Definition of $\frac{d\mathbf{y}}{dt}$ and $\int \mathbf{y}dt$:** To express eq.(65) in matrix form, we introduce the definition of the derivation and integration of a vector/matrix of functions. Let

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}.$$

$$\text{Then,} \quad \mathbf{y}' = \frac{d\mathbf{y}}{dt} = \frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}' = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix},$$

$$\text{and} \quad \int \mathbf{y}dt = \int \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} dt = \begin{bmatrix} \int y_1(t)dt \\ \int y_2(t)dt \end{bmatrix}.$$

In these notes, we use the bold lower case letters for vectors and capital letters for matrices.

Therefore, in matrix form eq.(65) reads

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix},$$

which simplifies to

$$\mathbf{y}' = A\mathbf{y} + \mathbf{b}, \tag{66}$$

which is a nonhomogeneous system of linear, 1st-order ODEs with constant coefficients. Notice that

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}.$$

When $\mathbf{b} = \mathbf{0}$, the system becomes homogeneous

$$\mathbf{y}' = A\mathbf{y}.$$

## 4.1 Solving the homogeneous system $\mathbf{y}' = A\mathbf{y}$

Similar to the fact that the solution to the scalar ODE $y' = ay$ is $y(t) = e^{at}y(0)$, the solution to

$$\mathbf{y}' = A\mathbf{y}$$

can be generally expressed as

$$\mathbf{y}(t) = e^{At}\mathbf{y}(0),$$

where

$$e^{At} = 1 + At + \frac{1}{2!}A^2t^2 + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{n!}A^n, \qquad A^n = \overbrace{AA\cdots A}^{\text{n times}}.$$

This solution, being simple and beautiful, does not provide much help unless we can find a way to evaluate the infinite series of matrix products given by $e^{At}$ in a finite form (we shall learn how to do that later!)

**Theorem 5.1.1 Principle of Superposition:** If $\mathbf{y}_1(t),\ \mathbf{y}_2(t)$ are two solution of $\mathbf{y}' = A\mathbf{y}$, where $A$ is $n \times n$ $(n \geq 2)$, then

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2, \qquad (c_1,\ c_2,\ \text{are constants})$$

is also a solution. If $\mathbf{y}_1(t),\ \mathbf{y}_2(t)$ are linearly independent, then it is the general solution for the case when $n = 2$.

**Proof:** Notice that $\mathbf{y}' = A\mathbf{y}$ is linear and homogeneous.

**An educated guess of the solution:** Based on a very similar argument as was given in Section 3.2, we look for solutions of the form

$$\mathbf{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} v_1e^{\lambda t} \\ v_2e^{\lambda t} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t} = \mathbf{v}e^{\lambda t}, \tag{67}$$

where $v_1, v_2$ are $t$-independent constants (i.e., $\mathbf{v}$ is a constant vector). Notice that both $\lambda$ and $\mathbf{v}$ are to be determined.

Substitute it into the system $\mathbf{y}' = A\mathbf{y}$, we obtain

$$\left(\mathbf{v}e^{\lambda t}\right)' = A\mathbf{v}e^{\lambda t} \quad \Longrightarrow \quad \mathbf{v}(\lambda e^{\lambda t}) = A\mathbf{v}e^{\lambda t} \quad \overbrace{\Longrightarrow}^{\text{divide by } e^{\lambda t}}$$

$$A\mathbf{v} = \lambda\mathbf{v}, \tag{68}$$

which defines the eigenvalue $\lambda$ and eigenvector $\mathbf{v}$ of the matrix $A$.

**Theorem 5.1.2:** For a given system $\mathbf{y}' = A\mathbf{y}$, where $A$ is $2 \times 2$. If $\lambda_1,\ \lambda_2$ are the two eigenvalues of $A$ with corresponding eigenvectors $\mathbf{v}_1,\ \mathbf{v}_2$, then

$$\mathbf{y}_1 = \mathbf{v}_1 e^{\lambda_1 t}, \quad \mathbf{y}_2 = \mathbf{v}_2 e^{\lambda_2 t},$$

are both solutions of the system. If $\lambda_1 \neq \lambda_2$, then $\mathbf{y}_1,\ \mathbf{y}_2$ are linearly independent and form a fundamental set. The general solution is

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}.$$

**Proof:** It follows from the argument outlined above and the fact about eigenvalues and eigenvectors in linear algebra. The linear independence shall be verified with the introduction of the Wronskian.

**Remark:** Solving the system $\mathbf{y}' = A\mathbf{y}$ is reduced to solving $A\mathbf{v} = \lambda\mathbf{v}$ for the eigenvalues and eigenvectors of $A$. This result holds for a system of $n$ ($n > 2$) equations where the matrix $A$ is $n \times n$.

**Example 5.1.1:** Find the general solution for the ODE $y'' + 3y' + 2y = 0$ by solving the corresponding linear system.

**Answer:** Let $z = y'$ and turn the ODE into the following system

$$\begin{cases} y' = & z, \\ z' = -2y - 3z. \end{cases}$$

In matrix form, it reads

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}, \quad \Longrightarrow \quad \mathbf{y}' = A\mathbf{y},$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}.$$

Based on **Theorem 5.1.2**, we know that the general solution is

$$\mathbf{y} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t},$$

where $\lambda_1,\ \lambda_2$ are the eigenvalues of $A$ and $\mathbf{v}_1,\ \mathbf{v}_2$ are the corresponding eigenvectors.

**Question:** How to calculate eigenvalues and eigenvectors?

**Review of matrix algebra:** Finding eigenvalues and eigenvectors of a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$?

(1) Starting with the equation that defines eigenvectors and eigenvalues

$$A\mathbf{v} = \lambda\mathbf{v} \quad \Longrightarrow \quad A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \quad \Longrightarrow \quad (A - \lambda I)\mathbf{v} = \mathbf{0}.$$

**Remarks:**

(0) Basically both the eigenvalues and eigenvectors are solved by solving this algebraic equation for non-zero $\mathbf{v}$.

(i) An eigenvector $\mathbf{v}$ is a non-zero vector that is *invariant* under the action (multiplication) of the matrix $A$. ($\mathbf{v}$ invariant under $A$ implies $A\mathbf{v} \in \mathbf{v}$.)

(ii) An eigenvector $\mathbf{v}$ represents a whole line in the direction specified by $\mathbf{v}$ . Therefore, $c\mathbf{v}$ for any constant $c \neq 0$ represents the same eigenvector as $\mathbf{v}$.

(iii) The eigenvalue $\lambda$ corresponding to the eigenvector $\mathbf{v}$ is a measure of the factor by which $\mathbf{v}$ is changed by the multiplication of $A$.

(2) Eigenvectors must be non-zero (while eigenvalues can be zero). For $(A - \lambda I)\mathbf{v} = \mathbf{0}$ to have non-zero solutions,

$$\det(A - \lambda I) = 0, \quad \Longrightarrow$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0, \quad \Longrightarrow$$

$$\lambda^2 - Tr\lambda + Det = 0 \quad (\textbf{characteristic equation!})$$

where $Tr = \mathrm{tr}A = a_{11} + a_{22}$ is the *trace* (defined as the sum of the diagonal entries of $A$), $Det = \det A = a_{11}a_{22} - a_{12}a_{21}$.

(3) After solving $\lambda^2 - Tr\lambda + Det = 0$ for the two eigenvalues $\lambda_1,\ \lambda_2$, the corresponding eigenspaces or nullspaces give the corresponding eigenvectors,

$$\mathbf{v}_j = E(\lambda_j) = \mathcal{N}(A - \lambda_j I) = \ker(A - \lambda_j I), \quad (j = 1,\ 2).$$

This is because solving $(A - \lambda I)\mathbf{v} = \mathbf{0}$ is finding any non-zero vector $\mathbf{v}$ such that $(A - \lambda I)\mathbf{v} = \mathbf{0}$. This is exactly the same problem as finding the *kernel* of $(A - \lambda I)$. We shall demonstrate below how $\ker(A - \lambda I)$ is defined and calculated!

(4) An $n \times n$ square matrix can be expressed as a row of $n$ column vectors as well as a column of $n$ row vectors. Rules of matrix multiplication hold for such expressions.

No proof is provided here but I provide an example to show why.

(i) Expressed as a row of column vectors:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = [\mathbf{c}_1 \ \mathbf{c}_2], \qquad \text{where} \quad \mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus,

$$A \begin{bmatrix} 2 \\ 3 \end{bmatrix} = [\mathbf{c}_1 \ \mathbf{c}_2] \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\mathbf{c}_1 + 3\mathbf{c}_2 = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

Let's us verify the result using the normal matrix multiplication:

$$A \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} (1)(2) + (1)(3) \\ (0)(2) + (1)(3) \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

(ii) Expressed as a column of row vectors:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}, \quad \mathbf{r}_1 = [1 \ 1], \quad \mathbf{r}_2 = [0 \ 1].$$

Thus,

$$[3 \ 4]\, A = [3 \ 4] \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} = 3\mathbf{r}_1 + 4\mathbf{r}_2 = 3\,[1 \ 1] + 4\,[0 \ 1] = [3 \ 7].$$

Let's us verify the result using the normal matrix multiplication:

$$[3 \ 4]\, A = [3 \ 4] \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = [(3)(1) + (4)(0) \quad (3)(1) + (4)(1)] = [3 \ 7].$$

(iii) Multiplication between matrices: let $B$ be another $2 \times 2$ matrix, then

$$BA = B\,[\mathbf{c}_1 \ \mathbf{c}_2] = [B\mathbf{c}_1 \quad B\mathbf{c}_2], \qquad AB = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} B = \begin{bmatrix} \mathbf{r}_1 B \\ \mathbf{r}_2 B \end{bmatrix}.$$

For example, let

$$B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

Thus,

$$BA = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (1)(1) + (3)(0) & (1)(1) + (3)(1) \\ (2)(1) + (4)(0) & (2)(1) + (4)(1) \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} (1)(1) + (1)(2) & (1)(3) + (1)(4) \\ (0)(1) + (1)(2) & (0)(3) + (1)(4) \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}.$$

Let's now use the previous results to do the same multiplication

$$BA = [B\mathbf{c}_1 \quad B\mathbf{c}_2] = \left[ \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}.$$

$$AB = \begin{bmatrix} \mathbf{r}_1 B \\ \mathbf{r}_2 B \end{bmatrix} = \begin{bmatrix} [1 \ 1] \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \\ [0 \ 1] \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}.$$

(5) The **kernel** of a matrix $B$ is defined as the collection of all vectors $\mathbf{k}$ such that

$$B\mathbf{k} = \mathbf{0}, \quad \text{(i.e., all vectors that are mapped to zero by } B.\text{)}$$

It is also called the nullspace of $B$, denoted sometimes by $\mathcal{N}(B)$ and sometimes by $\ker(B)$.

(6) **Theorem Rev1:** The kernel of a matrix $B$ is span by the *linear relations* between its column vectors.

**Proof:** Express an $n \times n$ matrix as a row of $n$ columns (column vectors) $B = [\mathbf{c}_1 \ \mathbf{c}_2 \ \cdots \ \mathbf{c}_n]$. Let

$$\mathbf{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}.$$

Therefore, any $\mathbf{k}$ in $\ker(B)$ must satisfy

$$B\mathbf{k} = [\mathbf{c}_1 \ \mathbf{c}_2 \ \cdots \ \mathbf{c}_n] \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = k_1\mathbf{c}_1 + k_2\mathbf{c}_2 + \cdots + k_n\mathbf{c}_n = \mathbf{0}.$$

Any set of numbers $\{k_1, \ k_2, \ \cdots, \ k_n\}$ that satisfies $k_1\mathbf{c}_1 + k_2\mathbf{c}_2 + \cdots + k_n\mathbf{c}_n = \mathbf{0}$ is referred to as a **linear relation** of the set of vectors $\{\mathbf{c}_1, \ \mathbf{c}_2, \ \cdots, \ \mathbf{c}_n\}$. Therefore, $\ker(B)$ contains all possible linear relations of the column vectors of $B$.

**Example Rev1:**

$$\ker \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{0\mathbf{c}_1+0\mathbf{c}_2=0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}, \qquad \text{(no linear relation, columns are LI, matrix is invertible!).}$$

**Remark:** This situation never happens in calculating eigenvectors because for the eigenvector problem $(A - \lambda I)\mathbf{v} = \mathbf{0}$, $\det(A - \lambda I) = 0$ which implies that the matrix $A - \lambda I$ can't be invertible! Whenever you see this in your eigenvector calculation, something must be wrong!

**Example Rev2:**

$$\ker \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{2\mathbf{c}_1-\mathbf{c}_2=0} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad \text{(Or any nonzero multiple of it. 1D!).}$$

**Example Rev3:**

$$\ker \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{0\mathbf{c}_1+1\mathbf{c}_2+0\mathbf{c}_3=0 \ \& \ 0\mathbf{c}_1+0\mathbf{c}_2+1\mathbf{c}_3=0} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad \text{(2D! Infinitely many choices exist!).}$$

**Back to Example 5.1.1:** Since $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, we find $Tr = -3$, $Det = 2$. The ch. eq. is

$$\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0 \qquad \Longrightarrow \qquad \lambda_1 = -1, \quad \lambda_2 = -2.$$

For the eigenvectors,

$$\mathbf{v}_1 = E(\lambda_1) = \ker(A - \lambda_1 I) = \ker \underbrace{\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}}_{\mathbf{c}_1 - \mathbf{c}_2 = 0} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$\mathbf{v}_2 = E(\lambda_2) = \ker(A - \lambda_2 I) = \ker \underbrace{\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}}_{\mathbf{c}_1 - 2\mathbf{c}_2 = 0} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Therefore, the general solutions is

$$\mathbf{y} = \begin{bmatrix} y \\ z \end{bmatrix} = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t}.$$

The first row gives the solution to the original second-order ODE, while the second row gives $z = y'(t)$. Thus,

$$y(t) = c_1 e^{-t} + c_2 e^{-2t}.$$

**Example 5.1.3:** Find the general solution of the linear system $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$.

**Answer:** $Tr = 2$, $Det = -3$. Thus, ch. eq. is

$$\lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3) = 0 \qquad \Longrightarrow \qquad \lambda_1 = -1, \quad \lambda_2 = 3.$$

For the eigenvectors,

$$\mathbf{v}_1 = E(\lambda_1) = \ker(A - \lambda_1 I) = \ker \underbrace{\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}}_{\mathbf{c}1 - 2\mathbf{c}2 = 0} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

$$\mathbf{v}_2 = E(\lambda_2) = \ker(A - \lambda_2 I) = \ker \underbrace{\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}}_{\mathbf{c}1 + 2\mathbf{c}2 = 0} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Therefore, the general solutions is

$$\mathbf{y} = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}.$$

## 4.2 Phase space, vector field, solution trajectories

Given a homogeneous linear system

$$\mathbf{y}' = A\mathbf{y} \qquad \text{or} \qquad \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}.$$

If the coefficients $a_{ij},\ (i,\ j = 1,\ 2)$ are all constants, then the system is *autonomous*. Therefore, the vector field is uniquely defined in the state space $y_1 - y_2$ plane, also referred to as the *phase space*.

**Example 5.2.1:** Turn the 2nd-order ODE for harmonic oscillations $y'' + y = 0$ into a system of 1st-order ODEs. Then, sketch its slope field and trace out one solution starting from $y(0) = 2,\ y'(0) = 0$.

**Answer:** Let $v(t) = y'(t)$. Then, the ODE is turned into the following system

$$\begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}.$$

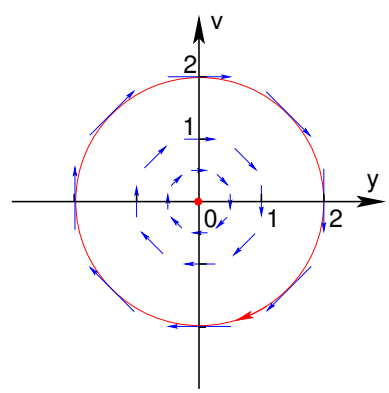



Figure 22: Vector field (blue) of the system $y' = v,\ v' = -y$ in the $y - v$ phase space. One trajectory for $y(0) = 2,\ y'(0) = 0$ (red) is traced out. It is tangent to the vector field at every point of the phase space. Variable $t$ is indirectly reflected in the (red) arrow direction traced out by the solution trajectory. The red dot represents the trivial solution $y = v = 0$.

**Example 5.2.2:** Sketch the vector field of the system $\mathbf{y}' = A\mathbf{y}$ studied in **Example 5.1.3**, where $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$. In this phase space, clearly indicate the invariant sets (i.e. the eigenvector directions) and draw the trajectory flows.

Remember that $A$ has the two eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 3$ with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

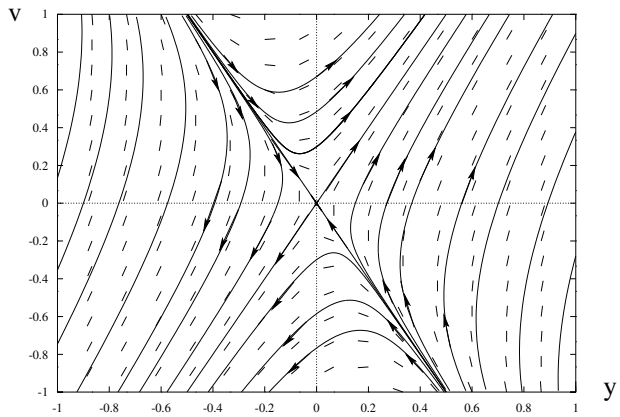



Figure 23: Vector field of the system given in Example 5.2.2. The two invariant sets $\mathbf{v}_1 = [1 \ -2]$, $\mathbf{v}_2 = [1 \ 2]$ are clearly boundaries where the trajectories can never cross. They divide the phase space into 4 distinct areas.

**Definition of invariant set:** A set of points $S$ in the phase space of the system $\mathbf{y}' = A\mathbf{y}$ is *invariant*, if for any initial condition $\mathbf{y}(0) \in S$, $\mathbf{y}(t)$ stays in $S$ for all $t \in (-\infty, \infty)$.

**Remark:** Solution trajectories in phase space can only approach an invariant set but can never cross it, making the invariant sets standout as important features of the phase diagram of a linear system.

**Theorem 5.2.1:** If $\mathbf{v}_1$, $\mathbf{v}_2$ are eigenvectors of $A$ corresponding to two distinct eigenvalues $\lambda_1 \neq \lambda_2$, then they are both invariant sets of the system $\mathbf{y}' = A\mathbf{y}$.

**Proof:** The general solution of the system is

$$\mathbf{y}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}.$$

Let's show that $\mathbf{v}_1$ is an invariant set and draw the same conclusion for $\mathbf{v}_2$. Substituting the following initial condition

$$\mathbf{y}_0 = c\mathbf{v}_1 \qquad (c \neq 0 \text{ is a constant}),$$

into the general solution, we obtain (notice that $\mathbf{v}_1,\ \mathbf{v}_2$ are linearly independent)

$$c_1 = c, \qquad c_2 = 0.$$

Therefore, the solution for this specific initial condition is

$$\mathbf{y}(t) = c\mathbf{v}_1 e^{\lambda_1 t},$$

which stays in the direction specified by $\mathbf{v}_1$ for all $t$.

Furthermore, the above result shows that the flow direction on the invariant set depends on the sign of $\lambda_1$. If $\lambda_1 < 0$, $\mathbf{y}(t) = c\mathbf{v}_1 e^{\lambda_1 t} \to \mathbf{0}$, the time arrow is pointed to the origin; if $\lambda_1 > 0$, however, $\mathbf{y}(t) = c\mathbf{v}_1 e^{\lambda_1 t} \to (\infty)c\mathbf{v}_1$, the time arrow points away from the origin.

## 4.3 How does the IC determine the solution?

**Example 5.3.1:** Solve the IVP

$$\begin{cases} \mathbf{y}' = A\mathbf{y}, \\ \mathbf{y}(0) = \mathbf{y}_0, \end{cases} \quad \text{where} \qquad A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix},$$

for the following initial conditions

$$\text{(a)} \quad \mathbf{y}_0 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}; \qquad \text{(b)} \quad \mathbf{y}_0 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}; \qquad \text{(c)} \quad \mathbf{y}_0 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

**Answer:** This system has already been solved in **Example 5.1.3**, where we found that $\lambda_1 = -1,\ \lambda_2 = 3$ and that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Therefore, the general solutions is

$$\mathbf{y}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}.$$

Using the IC(a), we obtain

$$\mathbf{y}(0) = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \Longrightarrow$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Now, we need to calculate the inverse of a $2 \times 2$ matrix.

**Review: the inverse of a $2 \times 2$ matrix.**

$$A^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}, \qquad (\det A \neq 0). \tag{69}$$

Using the formula given above, we obtain

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -4 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad c_1 = -1,\ c_2 = 0.$$

**Remark:** Had we paid attention to IC(a), this result would have been expected since $\mathbf{y}_0 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = (-1)\mathbf{v}_1$ which means the system started from a point in the *invariant set* defined by $\mathbf{v}_1$. It surely is expected to remain in it for all $t$. As a matter of fact, $c_1 = -1$ is exactly the factor in the expression of $\mathbf{y}_0$ in terms of $\mathbf{v}_1$.

Now, before we use the IC(b) to solve for $c_1$ and $c_2$, let's find out what is the relationship between $\mathbf{y}_0$ and the eigenvectors. We realize that

$$\mathbf{y}_0 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\mathbf{v}_2.$$

Thus, we expect that $c_1 = 0,\ c_2 = 2$ for this IC based on what we learned above. Let's verify it using IC(b):

$$\mathbf{y}(0) = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \Longrightarrow$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \Longrightarrow \quad c_1 = 0,\ c_2 = 2.$$

Finally, for IC(c) we notice that

$$\mathbf{y}(0) = \begin{bmatrix} 1 \\ 6 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ -2 \end{bmatrix} + (2) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (-1)\mathbf{v}_1 + 2\mathbf{v}_2.$$

Therefore, we expect $c_1 = -1,\ c_2 = 2$ for this IC. Let verify it below:

$$\mathbf{y}(0) = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} \quad \Longrightarrow$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -4 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \Longrightarrow \quad c_1 = -1,\ c_2 = 2.$$

**Theorem 5.3.1:** If $\mathbf{v}_1,\ \mathbf{v}_2$ are eigenvectors of $A$ corresponding to two distinct eigenvalues $\lambda_1 \neq \lambda_2$, then the solution to the IVP

$$\begin{cases} \mathbf{y}' = A\mathbf{y}, \\ \mathbf{y}(0) = \mathbf{y}_0, \end{cases}$$

is

$$\mathbf{y}(t) = m_1\mathbf{v}_1 e^{\lambda_1 t} + m_2\mathbf{v}_2 e^{\lambda_2 t},$$

where the constants $m_1,\ m_2$ are obtained by expressing the IC as a linear combination of $\mathbf{v}_1,\ \mathbf{v}_2$:

$$\mathbf{y}_0 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2.$$

**Proof:** In the general solution

$$\mathbf{y}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t},$$

plug in $t = 0$, we obtain

$$\mathbf{y}(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2.$$

## 4.4 Complex eigenvalues and repeated eigenvalues

**Example 5.4.1:** Find a fundamental set of the linear system $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix}$.

**Answer:** $Tr = 4$, $Det = 13$. Thus, the ch. eq. is

$$\lambda^2 - 4\lambda + 13 = (\lambda - 2)^2 + 9 = 0 \qquad \Longrightarrow \qquad \lambda,\ \bar{\lambda} = 2 \pm 3i.$$

For $\lambda = 2 + 3i$,

$$\mathbf{v} = \ker(A - (2+3i)I) = \ker \begin{bmatrix} 3-3i & -6 \\ 3 & -3-3i \end{bmatrix} \overbrace{=}^{\mathbf{r}_2 = (-1+i)\mathbf{r}_2 + \mathbf{r}_1} \ker \underbrace{\begin{bmatrix} 3-3i & -6 \\ 0 & 0 \end{bmatrix}}_{(1+i)\mathbf{c}1 + \mathbf{c}2 = 0} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}.$$

For $\bar{\lambda} = 2 - 3i$, the corresponding eigenvector should also be the complex conjugate of $\mathbf{v}$

$$\overline{\mathbf{v}} = \begin{bmatrix} \overline{1+i} \\ 1 \end{bmatrix} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}.$$

Therefore, the complex-valued solution

$$\mathbf{y_c}(t) = \mathbf{v}e^{\lambda t} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{(2+3i)t}$$

and its conjugate

$$\overline{\mathbf{y_c}(t)} = \overline{\mathbf{v}}e^{\bar{\lambda} t} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(2-3i)t}$$

form a set of complex-valued fundamental set.

Notice that

$$\mathbf{v} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{v}_r + i\ \mathbf{v}_i,$$

where its real and imaginary parts are

$$\mathbf{v}_r = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

To find real-valued fundamental set, we express $\mathbf{y_c}$ in terms of its real and imaginary parts:

$$\mathbf{y_c}(t) = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{(2-3i)t} = (\mathbf{v}_r + i\ \mathbf{v}_i)\, e^{2t}(\cos 3t + i \sin 3t)$$

$$= e^{2t}\left[\mathbf{v}_r \cos 3t - \mathbf{v}_i \sin 3t\right] + ie^{2t}\left[\mathbf{v}_r \sin 3t + \mathbf{v}_i \cos 3t\right].$$

Therefore,

$$\mathbf{y_1}(t) = Re\{\mathbf{y_c}(t)\} = e^{2t}\left[\mathbf{v}_r \cos 3t - \mathbf{v}_i \sin 3t\right] = e^{2t}\begin{bmatrix} \cos 3t - \sin 3t \\ \cos 3t \end{bmatrix},$$

$$\mathbf{y_2}(t) = Im\{\mathbf{y_c}(t)\} = e^{2t}\left[\mathbf{v}_r \sin 3t + \mathbf{v}_i \cos 3t\right] = e^{2t}\begin{bmatrix} \sin 3t + \cos 3t \\ \sin 3t \end{bmatrix}$$

form a real-valued fundamental set.

**Theorem 5.4.1:** For the linear system $\mathbf{y}' = A\mathbf{y}$, if the ch. eq. $\lambda^2 - Tr\lambda + Det = 0$ yields a pair of complex roots $\lambda$, $\overline{\lambda} = \alpha \pm i\beta$ with the corresponding eigenvectors $\mathbf{v}$, $\overline{\mathbf{v}}$, then

$$\mathbf{y_c}(t) = \mathbf{v}e^{\lambda t} \quad \text{and} \quad \overline{\mathbf{y_c}(t)} = \overline{\mathbf{v}}e^{\overline{\lambda}t}$$

form a pair of complex-valued fundamental set, while

$$\mathbf{y_1}(t) = Re\{\mathbf{y_c}(t)\} = e^{\alpha t}\left[\mathbf{v}_r \cos \beta t - \mathbf{v}_i \sin \beta t\right] \quad \text{and}$$

$$\mathbf{y_2}(t) = Im\{\mathbf{y_c}(t)\} = e^{\alpha t}\left[\mathbf{v}_r \sin \beta t + \mathbf{v}_i \cos \beta t\right]$$

form a pair of real-valued fundamental set, where

$$\mathbf{v}_r = Re\{\mathbf{v}\}, \qquad \mathbf{v}_i = Im\{\mathbf{v}\}$$

are, respectively, the real and imaginary parts of the complex-valued eigenvector $\mathbf{v}$.

**Proof:** Since

$$\mathbf{y_1}(t) = Re\{\mathbf{y_c}(t)\} = \frac{1}{2}\left(\mathbf{y_c}(t) + \overline{\mathbf{y_c}(t)}\right), \qquad \mathbf{y_2}(t) = Im\{\mathbf{y_c}(t)\} = \frac{1}{2i}\left(\mathbf{y_c}(t) - \overline{\mathbf{y_c}(t)}\right)$$

are both linear combinations of $\mathbf{y_c}(t)$ and $\overline{\mathbf{y_c}(t)}$ which are known to be solutions of the linear system $\mathbf{y}' = A\mathbf{y}$, they must also be solutions.

It is straightforward to show that they are indeed linearly independent of each other by calculating the Wronskian of the two (will be introduced later!)

**Example 5.4.2:** Find both complex- and real-valued fundamental sets of the linear system $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix}$. Then, sketch the phase digram of the system.

**Answer:** $Tr = -2$, $Det = 5$. Thus, the ch. eq. is

$$\lambda^2 + 2\lambda + 5 = (\lambda+1)^2 + 4 = 0 \qquad \Longrightarrow \qquad \lambda,\ \bar{\lambda} = -1 \pm 2i.$$

For $\lambda = -1 + 2i$,

$$\mathbf{v} = \ker(A - (-1+2i)I) = \ker \begin{bmatrix} 2-2i & 2 \\ -4 & -2-2i \end{bmatrix} \overbrace{=}^{\mathbf{r}_2=(1+i)\mathbf{r}_1+\mathbf{r}_2} \ker \underbrace{\begin{bmatrix} 2-2i & 2 \\ 0 & 0 \end{bmatrix}}_{\mathbf{c}_1+(-1+i)\mathbf{c}_2=0} = \begin{bmatrix} 1 \\ -1+i \end{bmatrix}.$$

Thus,

$$\mathbf{v} = \mathbf{v}_r + i\mathbf{v}_i = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, the complex-valued fundamental set is

$$\mathbf{y_c}(t) = \mathbf{v}e^{\lambda t} = \begin{bmatrix} 1 \\ -1+i \end{bmatrix} e^{(-1+2i)t}, \qquad \overline{\mathbf{y_c}(t)} = \overline{\mathbf{v}}e^{\bar{\lambda}t} = \begin{bmatrix} 1 \\ -1-i \end{bmatrix} e^{(-1-2i)t}.$$

The real-valued fundamental set is

$$\mathbf{y_1}(t) = Re\{\mathbf{y_c}(t)\} = e^{-t}\left[\mathbf{v}_r \cos 2t - \mathbf{v}_i \sin 2t\right] = e^{-t} \begin{bmatrix} \cos 2t \\ -\cos 2t - \sin 2t \end{bmatrix},$$

$$\mathbf{y_2}(t) = Im\{\mathbf{y_c}(t)\} = e^{-t}\left[\mathbf{v}_r \sin 2t + \mathbf{v}_i \cos 2t\right] = e^{-t} \begin{bmatrix} \sin 2t \\ -\sin 2t + \cos 2t \end{bmatrix}.$$

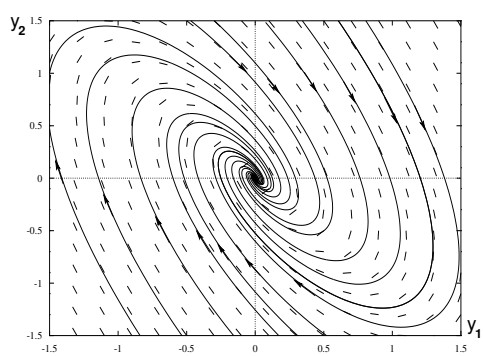



Figure 24: Phase space diagram for the system in **Example 5.4.2**. In this case, the fundamental set does not distinguish themselves from other solution trajectories. All are spiralling into the origin as time evolves.

**Example 5.4.3:** Find a fundamental set of the linear system $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$. Then, sketch the phase digram of the system.

**Answer:** $Tr = -2$, $Det = 1$. Thus, the ch. eq. is

$$\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0 \qquad \Longrightarrow \qquad \lambda_1 = \lambda_2 = -1 = \lambda.$$

For $\lambda = -1$,

$$\mathbf{v} = \ker(A - (-1)I) = \ker \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}}_{\mathbf{c}_1 + \mathbf{c}_2 = 0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus, we got one solution

$$\mathbf{y_1}(t) = \mathbf{v}e^{\lambda t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$$

How to find a second solution? Let us try

$$\mathbf{y_2}(t) = \mathbf{v}te^{\lambda t}.$$

Notice that

$$\mathbf{y_2}' = \mathbf{v}e^{\lambda t} + \lambda \mathbf{v}te^{\lambda t}.$$

Substitute both $\mathbf{y_2}$ and $\mathbf{y_2}'$ into the system $\mathbf{y}' = A\mathbf{y}$,

$$lhs = \mathbf{y_2}' = \mathbf{v}e^{\lambda t} + \lambda \mathbf{v}te^{\lambda t}, \qquad \text{while} \qquad rhs = A\mathbf{y_2} = A\mathbf{v}te^{\lambda t} = \lambda \mathbf{v}te^{\lambda t}.$$

It is obvious that $lhs \neq rhs$, the two differ by the term $\mathbf{y_1}(t) = \mathbf{v}e^{\lambda t}$.

The correct guess is

$$\mathbf{y_2}(t) = \mathbf{v}te^{\lambda t} + \mathbf{u}e^{\lambda t},$$

where the vector $\mathbf{u}$ is to be determined.

Notice that now

$$\mathbf{y_2}' = \mathbf{v}e^{\lambda t} + \lambda \mathbf{v}te^{\lambda t} + \lambda \mathbf{u}e^{\lambda t}.$$

Substitute both $\mathbf{y_2}$ and $\mathbf{y_2}'$ into the system $\mathbf{y}' = A\mathbf{y}$,

$$lhs = \mathbf{y_2}' = \mathbf{v}e^{\lambda t} + \lambda \mathbf{v}te^{\lambda t} + \lambda \mathbf{u}e^{\lambda t}, \qquad \text{while} \qquad rhs = A\mathbf{y_2} = A\mathbf{v}te^{\lambda t} + A\mathbf{u}e^{\lambda t} = \lambda \mathbf{v}te^{\lambda t} + A\mathbf{u}e^{\lambda t}.$$

By requiring $lhs = rhs$, we obtain

$$(A - \lambda I)\mathbf{u} = \mathbf{v}.$$

Since $\det(A - \lambda I) = 0$, there exist infinitely many solutions $\mathbf{u} \neq \mathbf{0}$. Any one would make the solution work.

Back to **Example 5.4.3**, $(A - \lambda I)\mathbf{u} = \mathbf{v}$ leads to

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which results in the following augmented matrix

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -1 & 1 \end{array}\right] \xrightarrow{\mathbf{r}_2 = \mathbf{r}_2 - \mathbf{r}_1} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \quad \Longrightarrow \quad u_1 - u_2 = 1 \quad \Longrightarrow \quad \mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore,

$$\mathbf{y_2}(t) = \mathbf{v}te^{\lambda t} + \mathbf{u}e^{\lambda t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} = \begin{bmatrix} t+1 \\ t \end{bmatrix} e^{-t}.$$

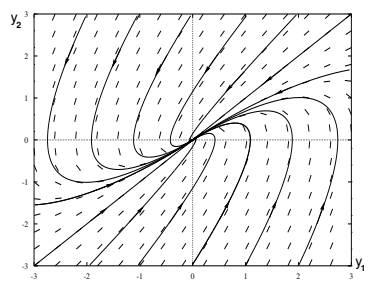



Figure 25: Phase space diagram for the system in **Example 5.4.3**. The invariant set $\mathbf{v} = [1\ 1]$ separates the phase space into two regions.

**Theorem 5.4.2:** For the linear system $\mathbf{y}' = A\mathbf{y}$, if the ch. eq. $\lambda^2 - Tr\lambda + Det = 0$ yields repeated real root $\lambda = \lambda_1 = \lambda_2$ with the corresponding eigenvector $\mathbf{v}$, then

$$\mathbf{y_1}(t) = \mathbf{v}e^{\lambda t} \quad \text{and} \quad \mathbf{y_2}(t) = \mathbf{v}te^{\lambda t} + \mathbf{u}e^{\lambda t} = [\mathbf{v}t + \mathbf{u}]\, e^{\lambda t}$$

form a fundamental set, where the constant vector $\mathbf{u}$ is any one of the infinitely many solutions of the algebraic equation

$$(A - \lambda I)\mathbf{u} = \mathbf{v}.$$

**Proof:** See arguments in **Example 5.4.3**. The linear independence of the two shall be discussed when the Wronskian is introduced.

## 4.5 Fundamental matrix and evaluation of $e^{At}$

**Definition of fundamental matrix:** If $\mathbf{y_1}(t)$, $\mathbf{y_2}(t)$ form a fundamental set for the linear system $\mathbf{y}' = A\mathbf{y}$, then its *fundamental matrix* $Y(t)$ is constructed by putting $\mathbf{y_1}(t)$ in its first column and $\mathbf{y_2}(t)$ in its second column.

$$Y(t) = [\mathbf{y_1}(t)\ \mathbf{y_2}(t)]\,.$$

(In other words, the fundamental matrix is a matrix whose columns are formed by the fundamental set!)

**Theorem 5.5.1:** Let $\mathbf{y_1}(t)$, $\mathbf{y_2}(t)$ be a fundamental set for the system $\mathbf{y}' = A\mathbf{y}$ and $Y(t) = [\mathbf{y_1}(t)\ \mathbf{y_2}(t)]$ be the fundamental matrix. Then the solution for the following IVP

$$\begin{cases} \mathbf{y}' = A\mathbf{y}, \\ \mathbf{y}(0) = \mathbf{y_0}, \end{cases}$$

is

$$\mathbf{y}(t) = e^{At}\mathbf{y_0} = Y(t)Y^{-1}(0)\mathbf{y_0}.$$

This expression implies that

$$e^{At} = Y(t)Y^{-1}(0).$$

**Remark:** Once the fundamental matrix is found, we can use this expression to evaluate $e^{At}$.

**Proof:** The general solution for the system $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y}(t) = c_1\mathbf{y_1}(t) + c_2\mathbf{y_2}(t) = [\mathbf{y_1}(t)\ \mathbf{y_2}(t)]\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Y(t)\mathbf{c}, \qquad \text{where} \qquad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Using the IC, we obtain

$$\mathbf{y}(0) = \mathbf{y_0} = Y(0)\mathbf{c} \qquad \Longrightarrow \qquad \mathbf{c} = Y(0)^{-1}\mathbf{y_0}.$$

Substitute the expression of $\mathbf{c}$ into the general solution, we obtain

$$\mathbf{y}(t) = Y(t)\mathbf{c} = Y(t)Y^{-1}(0)\mathbf{y_0}.$$

**Example 5.5.1:** Solve the following IVP

$$\begin{cases} \mathbf{y}' = A\mathbf{y}, \\ \mathbf{y}(0) = \mathbf{y_0}, \end{cases}$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}, \qquad \mathbf{y_0} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

**Answer:** Note that matrix $A$ is identical to that in **Example 5.3.1**, where we have already solved the fundamental set

$$\mathbf{y_1}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}, \qquad \mathbf{y_2}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}.$$

Thus,

$$Y(t) = [\mathbf{y_1}(t)\ \mathbf{y_2}(t)] = \begin{bmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{bmatrix}.$$

Therefore,

$$\mathbf{y}(t) = Y(t)Y^{-1}(0)\mathbf{y_0} = \begin{bmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{bmatrix} \frac{1}{4} \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{bmatrix} \frac{1}{4} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \frac{3}{4} \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} + \frac{1}{4} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} \frac{3}{4}e^{-t} + \frac{1}{4}e^{3t} \\ -\frac{3}{2}e^{-t} + \frac{1}{2}e^{3t} \end{bmatrix}.$$

From the above expression, we also found that

$$e^{At} = \exp\left( \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} t \right) = \begin{bmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{bmatrix} \frac{1}{4} \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2e^{-t} + 2e^{3t} & -e^{-t} + e^{3t} \\ -4e^{-t} + 4e^{3t} & 2e^{-t} + 2e^{3t} \end{bmatrix}.$$

**Example 5.5.2:** Solve the following IVP

$$\begin{cases} \mathbf{y}' = A\mathbf{y}, \\ \mathbf{y}(0) = \mathbf{y_0}, \end{cases}$$

where

$$A = \begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix}, \qquad \mathbf{y_0} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

**Answer:** Note that matrix $A$ is identical to that in **Example 5.4.2**, where we have already solved the fundamental set

$$\mathbf{y_1}(t) = e^{-t} \begin{bmatrix} \cos 2t \\ -\cos 2t - \sin 2t \end{bmatrix}, \qquad \mathbf{y_2}(t) = e^{-t} \begin{bmatrix} \sin 2t \\ -\sin 2t + \cos 2t \end{bmatrix}.$$

Thus,

$$Y(t) = [\mathbf{y_1}(t)\ \mathbf{y_2}(t)] = e^{-t}\begin{bmatrix} \cos 2t & \sin 2t \\ -\cos 2t - \sin 2t & -\sin 2t + \cos 2t \end{bmatrix},$$

and that

$$Y(0) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \qquad \Longrightarrow \qquad Y(0)^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Therefore,

$$\mathbf{y}(t) = Y(t)Y^{-1}(0)\mathbf{y_0} = [\mathbf{y_1}(t)\ \mathbf{y_2}(t)]\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= [\mathbf{y_1}(t)\ \mathbf{y_2}(t)]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{y_1}(t) = e^{-t}\begin{bmatrix} \cos 2t \\ -\cos 2t - \sin 2t \end{bmatrix}.$$

**Example 5.5.3:** Solve the following IVP

$$\begin{cases} \mathbf{y}' = A\mathbf{y}, \\ \mathbf{y}(0) = \mathbf{y_0}, \end{cases}$$

where

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}, \qquad \mathbf{y_0} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

**Answer:** Note that matrix $A$ is identical to that in **Example 5.4.3**, where we have already solved the fundamental set

$$\mathbf{y_1}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}, \qquad \mathbf{y_2}(t) = \begin{bmatrix} t+1 \\ t \end{bmatrix} e^{-t}.$$

Thus,

$$Y(t) = [\mathbf{y_1}(t)\ \mathbf{y_2}(t)] = e^{-t}\begin{bmatrix} 1 & t+1 \\ 1 & t \end{bmatrix},$$

and that

$$Y(0) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \Longrightarrow \qquad Y(0)^{-1} = \frac{1}{(-1)}\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

Therefore,

$$\mathbf{y}(t) = Y(t)Y^{-1}(0)\mathbf{y_0} = [\mathbf{y_1}(t)\ \mathbf{y_2}(t)]\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = [\mathbf{y_1}(t)\ \mathbf{y_2}(t)]\begin{bmatrix} -1 \\ 2 \end{bmatrix} =$$

$$= 2\mathbf{y_2}(t) - \mathbf{y_1}(t) = 2\begin{bmatrix} t+1 \\ t \end{bmatrix} e^{-t} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} = \begin{bmatrix} 2t+1 \\ 2t-1 \end{bmatrix} e^{-t}.$$

## 4.6 Wronskian and linear independence

**Definition of Wronskian:** For any two vector functions $\mathbf{y_1}(t)$, $\mathbf{y_2}(t)$ in 2D space, the Wronskian is defined as

$$W\left[\mathbf{y_1}(t),\ \mathbf{y_2}(t)\right] = \det\left[\mathbf{y_1}(t)\ \mathbf{y_2}(t)\right].$$

If $W\left[\mathbf{y_1}(t),\ \mathbf{y_2}(t)\right] \neq 0$ in an open interval of $t$ (it is still true if $W\left[\mathbf{y_1}(t),\ \mathbf{y_2}(t)\right] = 0$ for some but not all values of $t$ in the interval), then $\mathbf{y_1}(t)$, $\mathbf{y_2}(t)$ are linearly independent.

It is straightforward to verify (an exercise for you) that for each fundamental matrix we obtained in previous examples

$$\det Y(t) \neq 0.$$

Due to the result in Abel's Theorem, $\det Y(t) \neq 0$ for all values of $t$ for linear systems $\vec{y}' = A\vec{y}$ in which $A$ is a constant matrix.

## 4.7 Nonhomogeneous linear systems

**Theorem 5.7.1:** For the following nonhomogeneous linear system

$$\mathbf{y}' = A\mathbf{y} + \mathbf{b},$$

the general solution is

$$\mathbf{y}(t) = \mathbf{y_h}(t) + \mathbf{y_p}(t) = Y(t)\mathbf{c} + Y(t)\mathbf{u}(t) = Y(t)\left[\mathbf{c} + \mathbf{u}(t)\right].$$

where

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \text{is a vector of arbitrary constants, and}$$

$$\mathbf{u(t)} = \int Y(t)^{-1}\mathbf{b}(t)dt.$$

**Proof:** We only need to show that $\mathbf{y_p}(t) = Y(t)\mathbf{u}(t)$ for the vector of functions $\mathbf{u}(t)$ defined above.

Notice that

$$\mathbf{y_p}' = Y'(t)\mathbf{u(t)} + Y(t)\mathbf{u}'(t).$$

Substitute $\mathbf{y_p}(t)$ and $\mathbf{y_p}'(t)$ into the equation, we obtain

$$lhs = Y'(t)\mathbf{u(t)} + Y(t)\mathbf{u}'(t) = AY(t)\mathbf{u}(t) + \mathbf{b} = rhs$$

which results in

$$(Y'(t) - AY(t))\mathbf{u}(t) + Y(t)\mathbf{u}'(t) = \mathbf{b} \quad \overset{Y'(t)-AY(t)=\mathbf{0} \text{ or } Y'(t)=AY(t)}{\Longrightarrow} \quad Y(t)\mathbf{u}'(t) = \mathbf{b}.$$

Therefore,

$$\mathbf{u}'(t) = Y(t)^{-1}\mathbf{b}, \qquad \Longrightarrow \qquad \mathbf{u(t)} = \int Y(t)^{-1}\mathbf{b}(t)dt.$$

When deriving the previous result, we took the advantage of the fact that

$$Y'(t) = AY(t)$$

which implies that the fundamental matrix is also a solution of the linear system **y**'= A**y**. This is because $Y(t) = [\mathbf{y_1}(t)\ \mathbf{y_2}(t)]$ where both $\mathbf{y_1}(t)$, $\mathbf{y_2}(t)$ are solutions of the system, i.e.,

$$\mathbf{y_1}' = A\mathbf{y_1}, \qquad \mathbf{y_2}' = A\mathbf{y_2}.$$

Now,

$$Y'(t) = [\mathbf{y_1}(t)\ \mathbf{y_2}(t)]' = [\mathbf{y_1}'(t)\ \mathbf{y_2}'(t)] = [A\mathbf{y_1}(t)\ A\mathbf{y_2}(t)] = A\left[\mathbf{y_1}(t)\ \mathbf{y_2}(t)\right] = AY(t).$$

**Example 5.7.1:** Find the general solution for the nonhomogeneous linear system

$$\mathbf{y}' = A\mathbf{y} + \mathbf{b},$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}, \qquad \mathbf{b}(t) = \begin{bmatrix} e^t \\ 0 \end{bmatrix}.$$

**Answer:** Note that matrix $A$ is identical to that in **Example 5.3.1**, where we have already solved the fundamental set

$$\mathbf{y_1}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}, \qquad \mathbf{y_2}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}.$$

Thus,

$$Y(t) = [\mathbf{y_1}(t)\ \mathbf{y_2}(t)] = \begin{bmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{bmatrix}, \quad \Longrightarrow \quad \det Y(t) = 4e^{2t}.$$

And that,

$$Y(t)^{-1} = \frac{1}{\det Y(t)} \begin{bmatrix} 2e^{3t} & -e^{3t} \\ 2e^{-t} & e^{-t} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2e^{t} & -e^{t} \\ 2e^{-3t} & e^{-3t} \end{bmatrix}, \quad \Longrightarrow$$

$$Y(t)^{-1}\mathbf{b}(t) = \frac{1}{4} \begin{bmatrix} 2e^{t} & -e^{t} \\ 2e^{-3t} & e^{-3t} \end{bmatrix} \begin{bmatrix} e^t \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{2t} \\ e^{-2t} \end{bmatrix}.$$

Therefore,

$$\mathbf{u(t)} = \int Y(t)^{-1}\mathbf{b}(t)dt = \int \frac{1}{2} \begin{bmatrix} e^{2t} \\ e^{-2t} \end{bmatrix} dt = \frac{1}{4} \begin{bmatrix} e^{2t} \\ -e^{-2t} \end{bmatrix}.$$

Now,

$$\mathbf{y_p}(t) = Y(t)\mathbf{u}(t) = \begin{bmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{bmatrix} \frac{1}{4} \begin{bmatrix} e^{2t} \\ -e^{-2t} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 \\ -4e^{t} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^t.$$

The general solution is

$$\mathbf{y}(t) = Y(t)\mathbf{c} + \mathbf{y_p}(t) = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^t.$$

**Example 5.7.2:** Turn the nonhomogeneous second-order linear ODE $y'' + y = \cos t$ into a system of first-order ODEs. Then solve the system for $y(t)$.

**Answer:** Introduce a new variable $v(t) = y'(t)$, substitute into the equation we obtain

$$\begin{cases} y'(t) = v, \\ v'(t) = -y + \cos t \end{cases} \quad \Longrightarrow \quad \begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \cos t \end{bmatrix}.$$

Therefore, for our linear system

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \mathbf{b}(t) = \begin{bmatrix} 0 \\ \cos t \end{bmatrix}.$$

For this system, $Tr = 0$, $Det = 1$ resulting in the ch. eq. $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,\ 2} = \pm i$.

$$\mathbf{v} = \ker(A - iI) = \ker \underbrace{\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}}_{\mathbf{c}_1 + i\mathbf{c}_2 = 0} = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

The complex-valued solution is

$$\mathbf{y_c}(t) = \mathbf{v}e^{it} = \begin{bmatrix} 1 \\ i \end{bmatrix} (\cos t + i \sin t) = \begin{bmatrix} \cos t + i \sin t \\ i \cos t - \sin t \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

Thus, a set of real-valued solutions are

$$\mathbf{y_1}(t) = Re\{\mathbf{y_c}(t)\} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}, \qquad \mathbf{y_2}(t) = Im\{\mathbf{y_c}(t)\} = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

The fundamental matrix is

$$Y(t) = [\mathbf{y_1}(t)\ \mathbf{y_2}(t)] = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}, \quad \Longrightarrow \quad \det Y(t) = \cos^2 t + \sin^2 t = 1.$$

The inverse

$$Y(t)^{-1} = \frac{1}{\det Y(t)} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

Now,

$$Y(t)^{-1}\mathbf{b}(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 0 \\ \cos t \end{bmatrix} = \begin{bmatrix} -\sin t \cos t \\ \cos^2 t \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\sin 2t \\ 1 + \cos 2t \end{bmatrix}.$$

Integrate it to obtain

$$\mathbf{u(t)} = \int Y(t)^{-1}\mathbf{b}(t)dt = \int \frac{1}{2} \begin{bmatrix} -\sin 2t \\ 1 + \cos 2t \end{bmatrix} dt = \frac{1}{4} \begin{bmatrix} \cos 2t \\ 2t + \sin 2t \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \cos^2 t - \sin^2 t \\ 2t + 2 \sin t \cos t \end{bmatrix}.$$

To get the particular solution

$$\mathbf{y_p}(t) = Y(t)\mathbf{u}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \frac{1}{4} \begin{bmatrix} \cos^2 t - \sin^2 t \\ 2t + 2 \sin t \cos t \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \cos t + 2t \sin t \\ \sin t + 2t \cos t \end{bmatrix}.$$

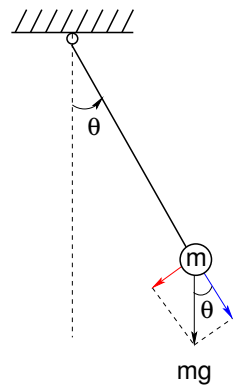
Take the result in the first row, we obtain

$$y(t) = c_1 \cos t + c_2 \sin t + \frac{1}{4}(\cos t + 2t \sin t) = c_1 \cos t + c_2 \sin t + \frac{1}{2} t \sin t.$$

# 5 Nonlinear Systems

## 5.1 Some examples of nonlinear systems

**E.g.1 The model of a pendulum.** Consider the pendulum shown in the figure below. Assume that the mass of the rod is negligible. Based on Newton's law




$$ma = F \qquad \Rightarrow \qquad mx''(t) = -mg\sin\theta(t) - cx'(t),$$

where $x(t)$ is a displacement of the mass in the direction perpendicular to the rod and $c$ is the coefficient of the air friction. But $x(t) = L\theta(t)$, where $L$ is the length of the pendulum. Substitute $x(t) = L\theta(t)$, $x'(t) = L\theta'(t)$, $x''(t) = L\theta''(t)$ into the equation, we obtain

$$mL\theta'' = -mg\sin\theta - cL\theta' \qquad \Rightarrow \qquad \theta'' = -\omega^2\sin\theta - \gamma\theta'$$

where $\omega^2 = g/L$ and $\gamma = c/(mL)$. Introduce a new variable $\phi(t) = \theta'(t)$, we obtain the following system of of two 1st-order ODEs

$$\begin{cases} \theta' = \phi, \\ \phi' = -\omega^2\sin\theta - \gamma\phi. \end{cases} \tag{70}$$

The problem is that now, this system is not a linear system because of the occurrence of the nonlinear term $\sin\theta$.

**E.g.2 A model of predator-prey interaction in population ecology.** Consider an ecosystem that contains two major interacting animal species, for example lynx and hare in Canadian north or big fish and little fish in the Mediterranean Sea. Let

x(t)=the size (i.e. the total number) of the prey population at time $t$;
y(y)=the size (i.e. the total number) of the predator population at time $t$.

Now, the time evolution of the two populations can be modelled based on the following statements:

x'(t)=net natural birth rate (birth rate-death rate)-rate of deadly encounter with predator;
y'(t)=net natural birth rate (birth rate-death rate)+rate of beneficiary encounter with prey.

It is assumed that, in the absence of the predator, the net natural birth rate of the prey is positive but the net natural birth rate of the predator is negative. Therefore, a simple model of predator-prey interaction, often called the Lotka-Volterra model, was proposed describe such kind of interaction between the two.

$$\begin{cases} x' = ax - bxy, \\ y' = cxy - dy, \end{cases} \tag{71}$$

where $a$, $b$, $c$, $d$ are all positive *parameters*. This is again a system of two 1st-order ODEs. However, this system is again nonlinear because of the appearance of the term $xy$ which describes the probability of an encounter between a predator and a prey.

In this lecture, we will learn some skills often used in the study of nonlinear systems of 1st-order ODEs of the general form

$$\begin{cases} x' = f(x, y), \\ y' = g(x, y), \end{cases} \tag{72}$$

where $f(x, y)$, $g(x, y)$ are differentiable functions of $x$ and $y$ that are nonlinear. We shall only study the case when $f$ and $g$ do not have explicit dependence on time $t$. Thus, we shall only study **autonomous** nonlinear systems of two nonlinear ODEs.

**Remarks:** Almost all systems of nonlinear ODEs cannot be solved in closed forms. Only in a few very special cases, one can solve them analytically in closed forms. The study of nonlinear ODEs is still the frontier in the study of differential equations and dynamical systems. The research remains hot and active even today and will likely remain active in the foreseeable future.

**Question:** If solutions can not be found, what can we do?

**Answer:** So far, we have tried the following approaches

(1) Qualitative methods such as phase-space analysis (partially covered here in this course).

(2) Approximation methods such as asymptotic methods (graduate level courses).

(3) Numerical and/or computational methods (widely used today in all areas).

## 5.2 Phase-plane analysis of a nonlinear system of two nonlinear ODEs

Let us introduce the basics of phase-plane analysis by using the following Lotka-Volterra model of predator-prey inteactions.

$$\begin{cases} x' = x(a-y) = f(x,y), \\ y' = sy(x-d) = g(x,y), \end{cases} \tag{73}$$

where $a,\ d,\ s > 0$ are parameters.

Here are the goals that we try to achieve with the phase-plane analysis.

(1) Understanding the behaviour of the system near some special points of interest, typically the steady states.

(2) Using linearized approximation to sketch the phase portrait(s) of the system near the steady states (local phase portraits).

(3) Combining the phase portraits near different steady states to generate a global phase portrait and use it to predict the long term behaviour of the system, i.e. the state or states the system approaches to as $t \to \infty$.

(4) Understanding the limitations of qualitative analysis.

**1. Phase-space, nullclines, steady states, and vector field.**

**Def: Phase space.** The space spanned by the unknown functions of the system. It is also referred to as the *state space*.

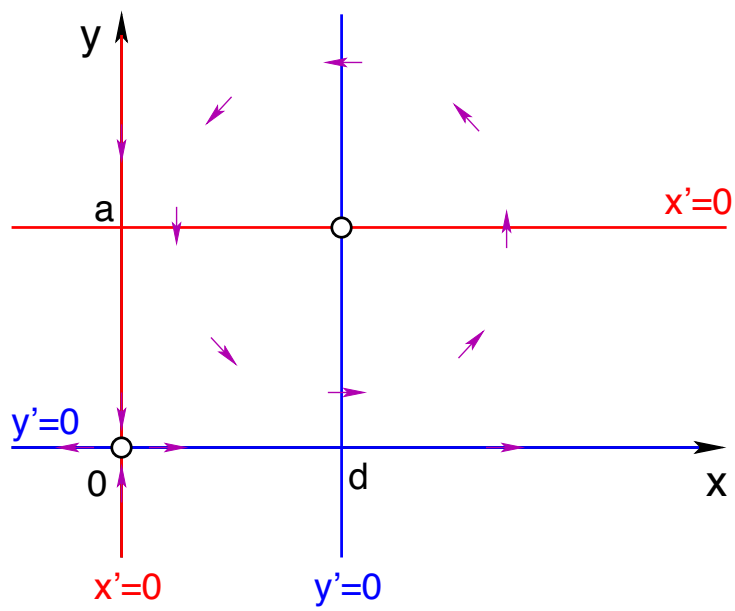
In the pendulum system, $\theta - \phi$ is the phase space; in the predator-prey system $x - y$ is the phase space.

**Def:** $x$-nullcline is the curve in phase space defined by $x' = f(x, y) = 0$ on which the rate of change in $x$ (i.e. $x'$) vanishes.
Similarly, $y$-nullcline is the curve in phase space defined by $y' = g(x, y) = 0$ on which the rate of change in $y$ (i.e. $y'$) vanishes.

In (73), the $x$- and $y$-nullclines are given by

$x$-nullclines: $x' = f(x, y) = x(a - y) = 0 \quad \Rightarrow \quad x = 0, \quad$ and $y = a$. (red in figure below.)

$y$-nullclines: $y' = g(x, y) = sy(x - d) = 0 \quad \Rightarrow \quad y = 0, \quad$ and $x = d$. (blue in figure below.)




**Notice that** The direction vectors on the $x$-nullclines are all vertical (because $x' = 0$) and those on the $y$-nullclines are all horizontal (because $y' = 0$).

**Def: Steady state (Equilibrium, Critical point, Fixed point).** A point in phase space where $x' = 0 = y'$, i.e. the rates of change in both variables are zero. That is all points in the phase space $(x_s,\ y_s)$ where

$$x' = f(x, y) = 0 \qquad \text{and} \qquad y' = g(x, y) = 0.$$

By definition, these are the points of intersection between an $x$-nullcline and a $y$-nullcline! For the example above, $(0,\ 0)$ and $(d,\ a)$ are the two steady states.

**Be careful:** Point of intersection between two $x$-nullclines (or between two $y$-nullclines) is NOT a critical point! (Check the figure above!)

**Def: Direction field.** For an autonomous system, $x' = f(x, y)$ and $y' = g(x, y)$, the value of $x'$ and $y'$ is fixed at each point in the phase space. A simple vector addition between $x'$ and $y'$ at each point determines the direction and magnitude of the vector field at each point in phase space.

**Often**, we ignore the magnitude of these vectors and pay attention only to the direction of these vectors (i.e. the scaled direction field). Any solution trajectory in phase space must be tangent to the vector field at each point in the phase space!

**2. Stability of a steady state and linear stability analysis.**

**Asymptotic stability:** A steady state is asymptotically stable if all nearby phase trajectories will move closer to and eventually approach the state as $t \to \infty$. A steady state is unstable if nearby trajectories move away from it in at least one direction.

In cases when the trajectories neither diverge nor approach a steady state (e.g. in the case of a centre), some use the word *neutral* others use the concept of Lyapunov stability (not asymptotic stability). But in our notes here, we often use the word stable to mean asymptotically stable. When we actually mean Lyapunov stable we would say that it is Lyapunov stable.

**Linearization of nonlinear systems.** To determine the stability of a steady state, sketching the vector field often is not enough. Also, the *local behaviour* (i.e. the behaviour in close neighbourhood) near a steady state of a nonlinear system is often qualitatively identical to that of a system that is a *linear approximation* of a nonlinear system. Therefore, a local phase-portrait of the linearized system often gives the correct portrait of the nonlinear system. Linearization helps both in determining the stability of a steady state and obtaining a local phase portrait of the nonlinear system.

**Three goals of linearization and stability analysis:**

- Determine the stability of a steady state.
- Characterize/classify the type of the fixed point.
- Sketch the local phase portrait(s) near the fixed point(s).

Let $(x_s,\ y_s)$ be a steady state. Thus,

$$\begin{aligned} f(x_s,\ y_s) &= 0, \\ g(x_s,\ y_s) &= 0. \end{aligned}$$

To study the behaviour near $(x_s,\ y_s)$, we express the solutions as

$$\begin{aligned} x(t) &= x_s + \alpha(t), \\ y(t) &= y_s + \beta(t), \end{aligned}$$

where $(\alpha(t),\ \beta(t))$ $(|\alpha(t)|,\ |\beta(t)| << 1)$ measures the distance between $(x(t),\ y(t))$ and $(x_s,\ y_s)$.

Substitute

$$\begin{aligned} x(t) &= x_s + \alpha(t), \\ y(t) &= y_s + \beta(t), \end{aligned}$$

into the equation the differential equation, we obtain

$$\begin{aligned} \dot{x} &= \dot{\alpha} = f(x_s + \alpha, y_s + \beta) = f(x_s,\ y_s) + f_x(x_s,\ y_s)\alpha + f_y(x_s,\ y_s)\beta + O(\alpha^2,\ \beta^2), \\ \dot{y} &= \dot{\beta} = g(x_s + \alpha, y_s + \beta) = g(x_s,\ y_s) + g_x(x_s,\ y_s)\alpha + g_y(x_s,\ y_s)\beta + O(\alpha^2,\ \beta^2). \end{aligned}$$

Ignore the higher order terms in $(\alpha(t),\ \beta(t))$, we obtain the follow system of linearized equations.

$$\begin{aligned} \dot{\alpha} &= f_x(x_s,\ y_s)\alpha + f_y(x_s,\ y_s)\beta, \\ \dot{\beta} &= g_x(x_s,\ y_s)\alpha + g_y(x_s,\ y_s)\beta. \end{aligned}$$

In matrix form $$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_{(x_s,\ y_s)} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \qquad \Rightarrow \qquad \dot{\vec{v}} = J(x_s,\ y_s)\vec{v},$$

where $\vec{v} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ and $J(x_s,\ y_s)$ is the Jacobian matrix evaluated at $(x_s,\ y_s)$. $f_x = \dfrac{\partial f}{\partial x}$, $f_y = \dfrac{\partial f}{\partial y}$, $g_x = \dfrac{\partial g}{\partial x}$, $g_y = \dfrac{\partial g}{\partial y}$ are the partial derivatives of the functions $f(x, y)$ and $g(x, y)$.

Now, let us carry out the linearization of the predator-prey example studied above, where $f(x,y) = x(a-y),\ g(x,y) = sy(x-d)$. The Jacobian matrix is

$$J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} a-y & -x \\ sy & s(x-d) \end{bmatrix}.$$

For $(x_s, y_s) = (0,0)$,

$$J(0,0) = \begin{bmatrix} a & 0 \\ 0 & -sd \end{bmatrix}.$$

Thus,

$$\lambda_1 = a,\ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \qquad \lambda_2 = -sd,\ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \text{saddle.}$$

For $(x_s, y_s) = (d,a)$,

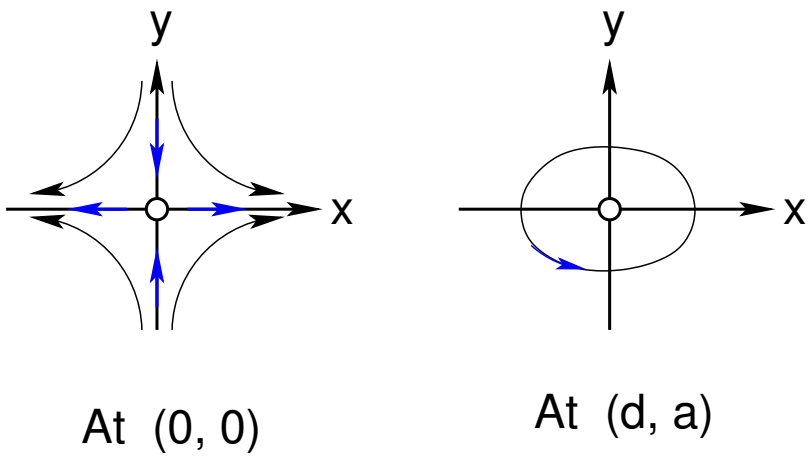
$$J(d,a) = \begin{bmatrix} 0 & -d \\ as & 0 \end{bmatrix}.$$

Therefore, $Tr = 0$ and $Det = ads > 0$.

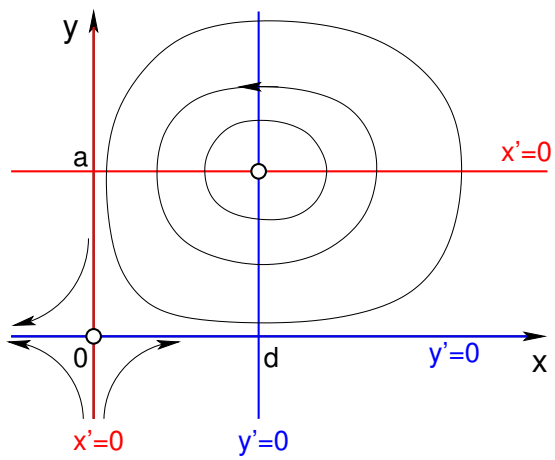
$$\lambda_1 = i\sqrt{ads}; \qquad \lambda_2 = -i\sqrt{ads} \quad \Rightarrow \quad \text{centre.}$$

## Local and Global Phase Portraits

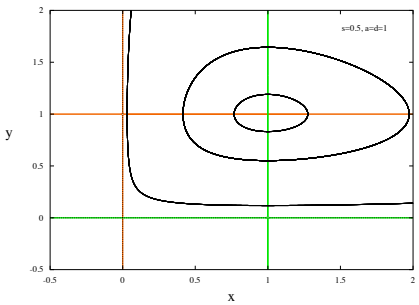
Local phase portraits:

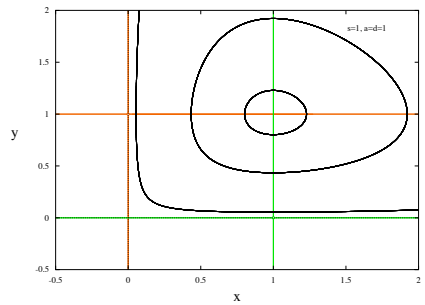


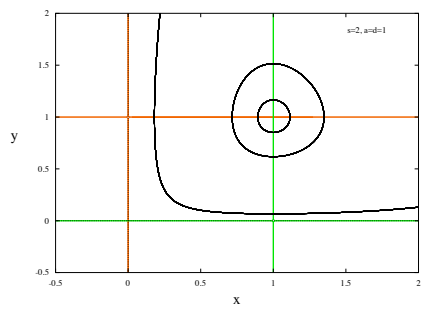

Global phase portrait:

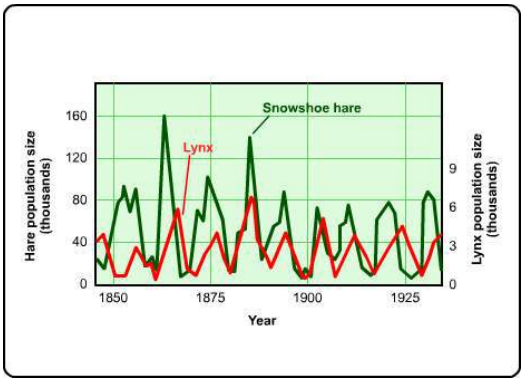


## Results obtained using a computer software



**E.g.3** For the nonlinear system,

$$\begin{cases} x' = y^2 - x, \\ y' = x^2 - y. \end{cases}$$

(a) Sketch the nullclines. Draw one arrow of the vector field in each distinct region of the phase space and on each distinct segment of the nullclines. Then, find all critical points and mark each by an open circle in the phase space.

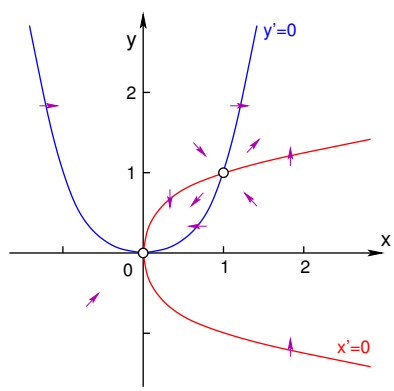
(b) Calculate the Jacobian matrix of the system.

(c) Classify each critical point found in (a) and sketch a local phase portrait near each one.

(d) Based on results above, sketch the global phase portrait. Predict the long term behaviour for two ICs: (i) $(x(0),\ y(0)) = (0.5,\ 0.5)$; (ii) $(x(0),\ y(0)) = (1.5,\ 1.5)$.

**Answer:**

(a) The critical points are $(0,\ 0)$ and $(1,\ 1)$. Nullclines and direction arrows are all sketched below.




(b)

$$J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -1 & 2y \\ 2x & -1 \end{bmatrix}.$$

(c) For $(0,\ 0)$:

$$J(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus, $\lambda = \lambda_1 = \lambda_2 = -1$ is repeated eigenvalue. Based on what we learned previously, $(0,\ 0)$ is a stable node. However, this is a special case of stable node because of the repeated eigenvalue.

We know that the corresponding eigenvalues are

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, this is a case when the algebraic multiplicity and geometric multiplicity are both equal to 2! Thus, we have repeated eigenvalue but we can find two linearly independent eigenvectors for the same eigenvalue. Actually, any nonzero vector is an eigenvector! This is because

$$E(\lambda) = \ker(A - \lambda I) = \ker \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \text{for any } a,\ b \text{ that are not both zero!}$$

Because all directions near $(0,\ 0)$ is an eigenvector direction with $\lambda = -1$, no direction is faster or slower than the others. Thus, the trajectories are not curved but strait lines. This special case of a stable node is also called a *stable star*.

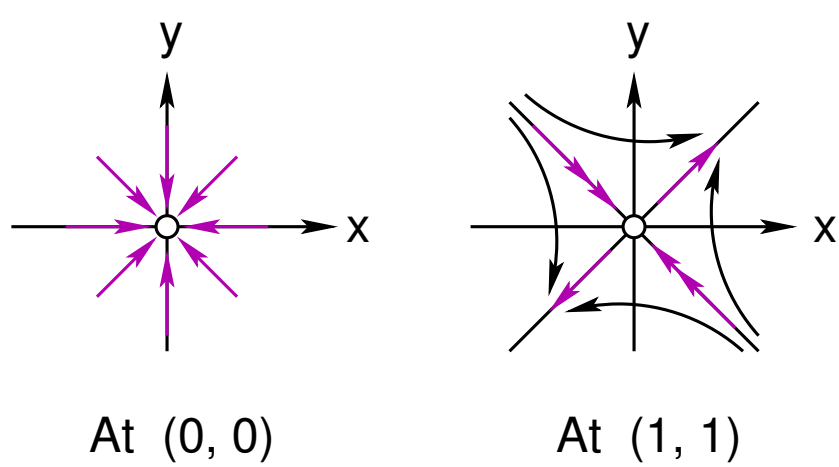
For $(1,\ 1)$:

$$J(1,1) = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}.$$

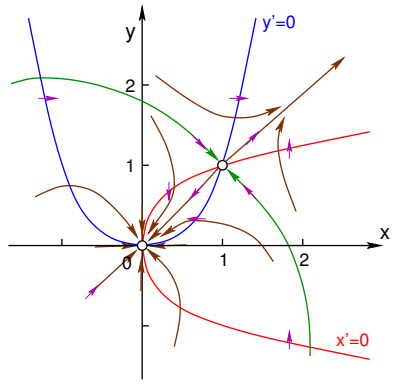
Thus, $Tr = -2,\ Det = -3$. $\lambda^2 + 2\lambda - 3 = (\lambda - 1)(\lambda + 3) = 0$ yields $\lambda_1 = 1$ and $\lambda_2 = -3$.

$$\vec{v}_1 = \ker(A - \lambda_1 I) = \ker \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \ker(A - \lambda_2 I) = \ker \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore, $(1,\ 1)$ is a saddle point $\vec{v}_2$ is the stronger unstable direction. The local phase portraits are sketched below.



(d) Global phase portrait is sketched below.

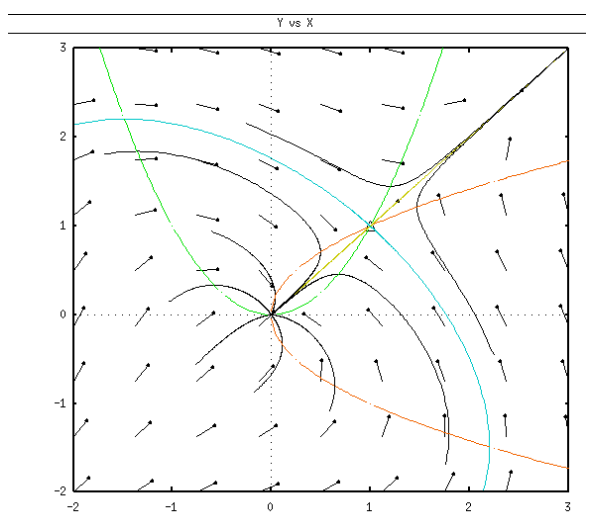



The stable invariant set of the saddle point (green) serves as a separatrix that separates the phase space into two regions.

Below it to the left, all trajectories will eventually approach the stable node (start) at $(0,\ 0)$. Thus, for IC $(x(0),\ y(0)) = (0.5,\ 0.5)$, $(x(t),\ y(t)) \to (0,\ 0)$ .

Above it to the right, all trajectories will eventually approach infinity. Thus, $(x(0),\ y(0)) = (1.5,\ 1.5)$, $(x(t),\ y(t)) \to (\infty,\ \infty)$.

Below is the global phase portrait computed by a computer software. The sketch obtained above were done before we had a chance to see the computer generated portrait. Yet the similarity is striking!



**E.g.4** A modified predator-prey interaction model in which the prey population is governed by a logistic growth model.

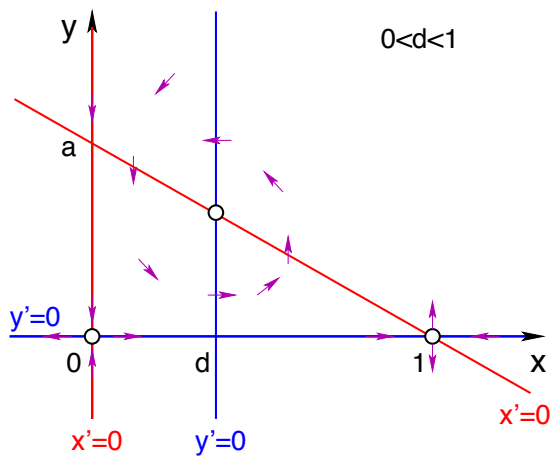
$$\begin{cases} x' = ax(1-x) - xy, \\ y' = sy(x-d), \end{cases} \qquad (a,\ s > 0,\ 0 < d < 1 \text{ are parameter}).$$

**Nullclines, Steady States, and Vector Field**

**x-nullclines:** $x = 0$ and $y = a(1-x)$.

**y-nullclines:** $y = 0$ and $x = d$.

There are 3 steady states (0,0), (1,0), and (d, a(1-d)). The vector field is given below




(0,0), (1,0) seem to be saddles and (d, a(1-d)) a centre or a spiral.

**Jacobian and Linear Stability Analysis**

$$J(x,y) = \begin{bmatrix} a(1-2x) - y & -x \\ sy & s(x-d) \end{bmatrix}.$$

For $(x_s, y_s) = (0,0)$,

$$J(0,0) = \begin{bmatrix} a & 0 \\ 0 & -sd \end{bmatrix}.$$

Thus,

$$\lambda_1 = a,\ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \lambda_2 = -sd,\ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \text{saddle.}$$

For $(x_s, y_s) = (1, 0)$,

$$J(1,0) = \begin{bmatrix} -a & -1 \\ 0 & s(1-d) \end{bmatrix}.$$

Therefore, for $0 < d < 1$

$$\lambda_1 = -a,\ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \lambda_2 = s(1-d) > 0 \quad \Rightarrow \quad \text{saddle.}$$

For $(x_s, y_s) = (d,\ a(1-d))$,

$$J(d, a(1-d)) = \begin{bmatrix} -ad & -d \\ sa(1-d) & 0 \end{bmatrix}.$$

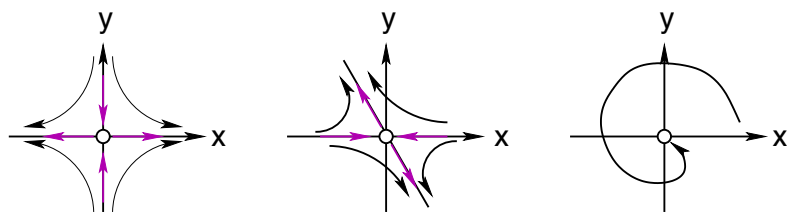
Therefore, $Tr = -ad < 0$ and $Det = ads(1-d) > 0$

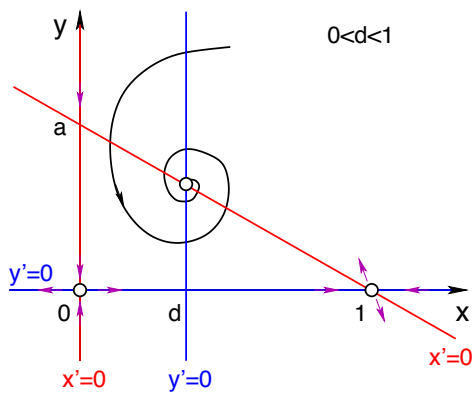
$$\Delta = \left(\frac{Tr}{2}\right)^2 - ads(1-d) = \frac{a^2d^2}{4} - ads(1-d) \quad \Rightarrow$$

$$\Delta = \frac{ad}{4}[ad - 4s(1-d)]$$

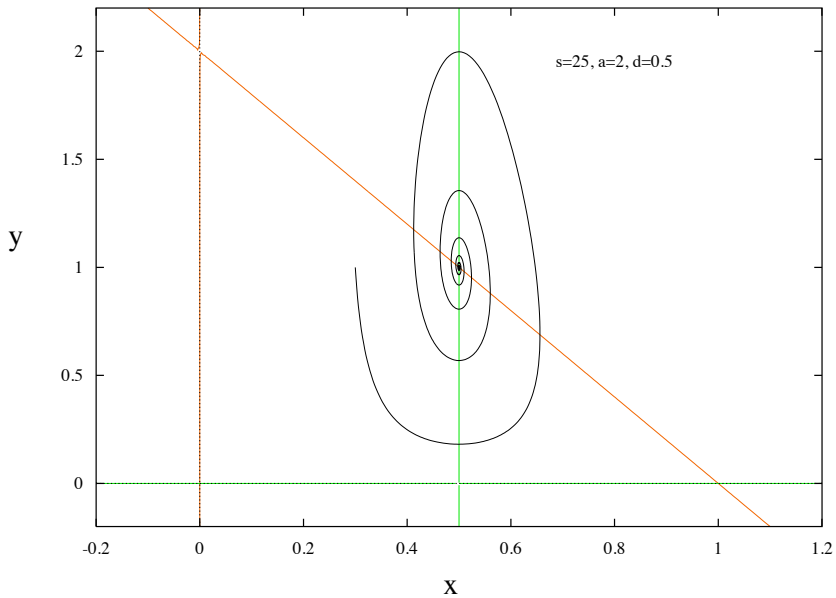
Therefore, if $\Delta > 0$, it is a stable node; if $\Delta < 0$, it is a stable spiral. When $a << s$, $\Delta < 0$ and $(d,\ a(1-d))$ is a spiral.

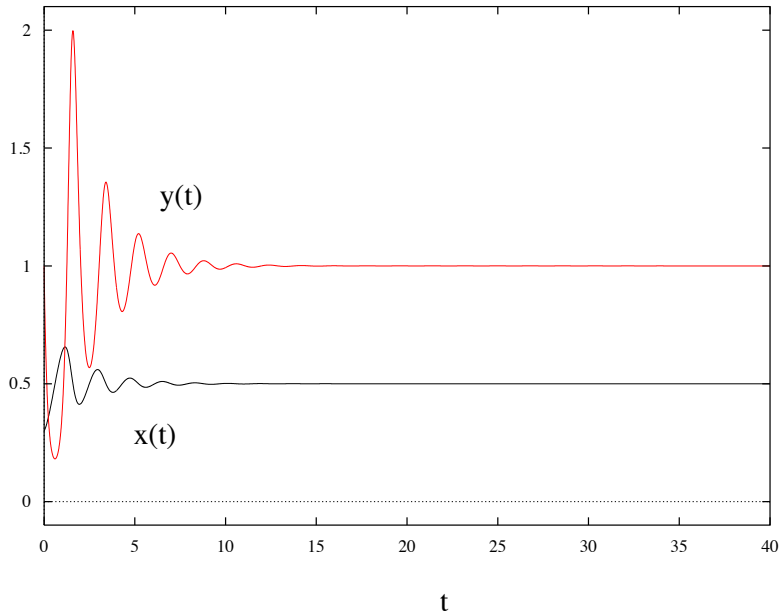
**Local and Global Phase Portraits** in case $\Delta < 0$.



**Results obtained using a computer software**



## 5.3 Classification and stability of isolated critical points

(1) Saddle: real $\lambda_1,\ \lambda_2 \neq 0$ and $\lambda_1\lambda_2 < 0$ (real and of opposite signs) (i.e. $Det < 0$). Unstable!.

(2) Node: real $\lambda_1,\ \lambda_2 \neq 0$ and $\lambda_1\lambda_2 > 0$ (real and of identical signs). Stable if $\lambda_1,\ \lambda_2 < 0$, unstable if $\lambda_1,\ \lambda_2 > 0$!

(3) Spiral/focus: When $\lambda_1,\ \lambda_2 = Tr/2 \pm \omega i$ are complex (i.e. $Det > 0$ and $Det > (Tr/2)^2$) and $Tr \neq 0$. Stable if $Tr < 0$, unstable if $Tr > 0$.

(4) Center: When $\lambda_1,\ \lambda_2 = \pm i\sqrt{Det}$, i.e. $Tr = 0$ and $Det > 0$. Neutral.

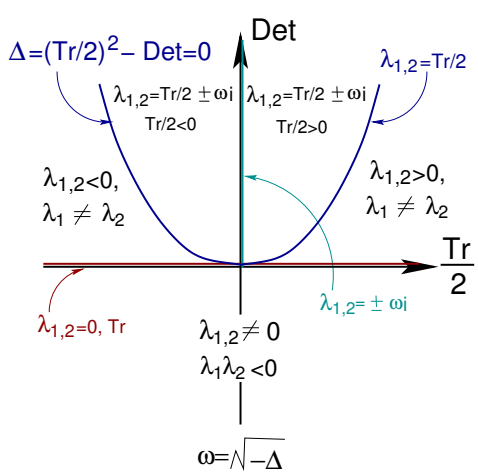
(5) Stars, degenerate nodes: When $\lambda_1 = \lambda_2 = Tr/2 \neq 0$, i.e. $Det = (Tr/2)^2 \neq 0$. Stable if $Tr < 0$, unstable if $Tr > 0$.

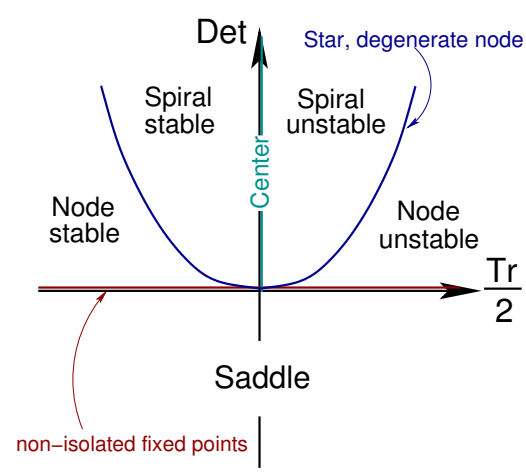
### Classification and stability in the $Det - Tr/2$ plane

Since the roots of $\lambda^2 - Tr\lambda + Det = 0$ are

$$\lambda_{1,2} = \frac{Tr}{2} \pm \sqrt{\left(\frac{Tr}{2}\right)^2 - Det} = \frac{Tr}{2} \pm \sqrt{\Delta},$$

where $Tr = \lambda_1 + \lambda_2$ and $Det = \lambda_1\lambda_2$.



## 5.4 Conservative systems that can be solved analytically in closed forms

A conservative system can be defined as a 2nd-order ODE

$$x'' + f(x) = 0 \tag{74}$$

where $x = x(t)$ and $f(x)$ is usually a nonlinear function of $x$. Such an equation typically arises when describing the Newtonian motion of a mass in the absence of friction and other dissipative forces. In this case, $x(t)$ describes the location of the mass and $x''(t)$ is the acceleration. Examples of such motions include the movement of planet under the influence of the gravitational force of the sun. Or an electrically charged particle moving in a frictionless medium under the influence of an electric potential. In case of such kind of Newtonian motion, Newton's law yields

$$ma = F \qquad \Rightarrow \qquad mx'' = -\frac{dV(x)}{dx}.$$

Assuming that $m = 1$ and $\frac{dV(x)}{dx} = f(x)$ , one obtains equation (74). Here, $V(x) = \int f(x)dx$ is often referred to as the potential function.

Multiply both sides of (74) by $x'$ and express $f(x)$ by $\frac{dV(x)}{dx}$, we obtain

$$x'x'' + \frac{dV(x)}{dx}x' = 0 \quad \Rightarrow \quad \left(\frac{1}{2}(x')^2 + V(x)\right)' = 0 \quad \Rightarrow \quad \frac{1}{2}(x')^2 + V(x) = C,$$

where $E(x, x') = \frac{1}{2}(x')^2 + V(x)$ and $C$ is an arbitrary constant. $E$ is referred to as the energy in physics. So, the energy is conserved. This is why such a system is called a *conservative* system. In this system,

$$E = \frac{1}{2}(x')^2 + V(x) = C, \qquad (V(x) = \int f(x)dx)$$

gives all possible solutions of the system in $x$ vs $x'$ space, which is actually the phase space of the corresponding system of 1st-order ODEs given below

$$\begin{cases} x' = y, \\ y' = -f(x). \end{cases} \tag{75}$$

**E.g.5** Pendulum in frictionless medium. In absence of friction, $\gamma = 0$ in (70) we previously developed for the motion of a pendulum. In this case, the equation becomes

$$\theta'' + \omega^2 \sin\theta = 0, \qquad (\omega^2 = \frac{g}{L}).$$

Or equivalently, the following system of 1st-order ODEs

$$\begin{cases} \theta' = \phi, \\ \phi' = -\omega^2 \sin\theta. \end{cases}$$

Based on the definition above, this is a conservative system of the form $\theta'' + f(\theta) = 0$. Therefore, the total energy

$$E = \frac{1}{2}(\theta')^2 + V(\theta)$$

must be conserved, where

$$V(\theta) = \int \omega^2 \sin\theta d\theta = -\omega^2 \cos\theta.$$

Therefore, all solutions are expressed as

$$\frac{1}{2}(\theta')^2 + V(\theta) = \frac{1}{2}(\theta')^2 - \omega^2 \cos\theta = C,$$

where $C$ is an arbitrary constant.

Using the initial condition $\theta(0) = \theta_0,\ \theta'(0) = 0$, we found

$$C = -\omega^2 \cos\theta_0.$$

Substitute into the solution, we obtain

$$\theta' = \phi = \pm\omega\sqrt{\cos\theta - \cos\theta_0}.$$

**Question:** How do these solutions look like in the phase space (i.e. $\theta - \phi$ space)?

**Answer:** Phase-space analysis. In the absence of a graph plotting software, we show how to sketch the phase portrait of these solutions using phase-space analysis.

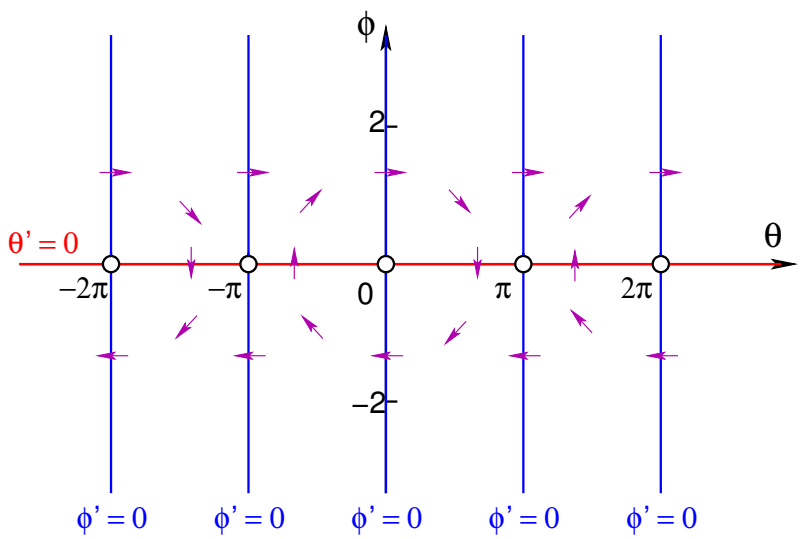
Consider the system

$$\begin{cases} \theta' = \phi, \\ \phi' = -\omega^2 \sin\theta, \qquad (\omega^2 = \frac{g}{L}.) \end{cases}$$

There is only one $\theta-$nullcline, i.e. $\phi = 0$.

$\phi-$nullclines are given by $\sin\theta = 0$, thus $\theta = 0,\ \pm\pi,\ \pm\pi + 2\pi n$ (n is any integer). Thus, there are infinitely many $\phi-$nullclines that are evenly distributed in the horizontal axis. If we focus on the interval $-\pi \le \theta\ \le \pi$, then $\theta = 0,\ \pm\pi$. Thus, there are 3 critical points in this interval: $(0,\ 0)$ and $(\pm\pi,\ 0)$.

Nullclines, critical points, and vector field are sketched as follows.




The Jacobian is

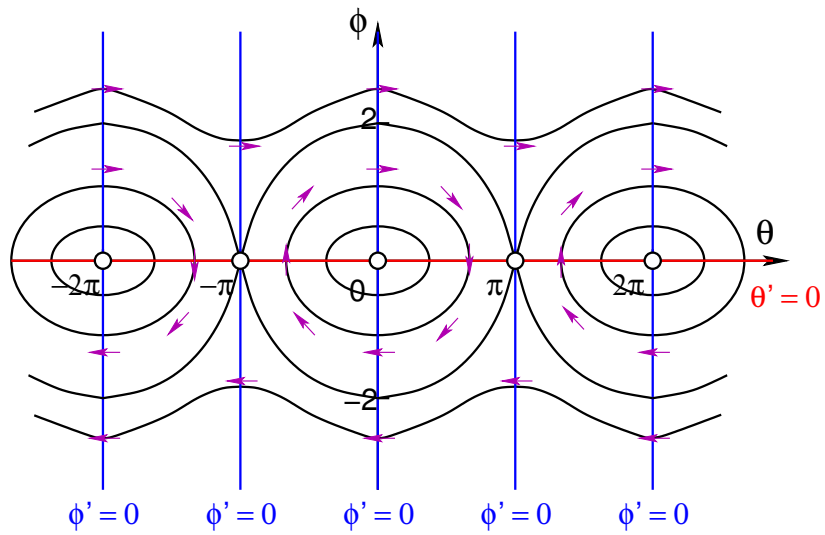
$$J(\theta,\ \phi) = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos\theta & 0 \end{bmatrix}.$$

Thus,

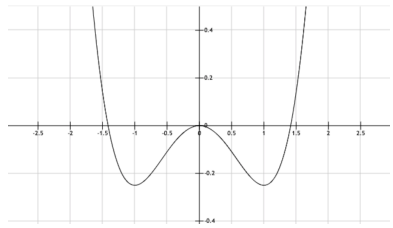
$$J(0,\ 0) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \quad \Rightarrow \quad \lambda^2 + \omega^2 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm\omega i, \qquad \text{centre!}$$

$$J(\pm\pi,\ 0) = \begin{bmatrix} 0 & 1 \\ \omega^2 & 0 \end{bmatrix} \quad \Rightarrow \quad \lambda^2 - \omega^2 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm\omega, \qquad \text{saddle!}$$

The global phase portrait is


ϕ
2
θ
−2π
−π
0
π
2π
θ' = 0
−2
ϕ' = 0
ϕ' = 0
ϕ' = 0
ϕ' = 0
ϕ' = 0

**E.g.6** Consider the conservative Newtonian motion in double-well potential is described by $x'' = x - x^3$ in which $\frac{dV(x)}{dx} = -(x - x^3)$, thus $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ whose graph has the double-well shape as in the figure below.



The ODE can be transformed into the following system of nonlinear ODEs

$$\begin{cases} x' = y, \\ y' = x - x^3. \end{cases}$$

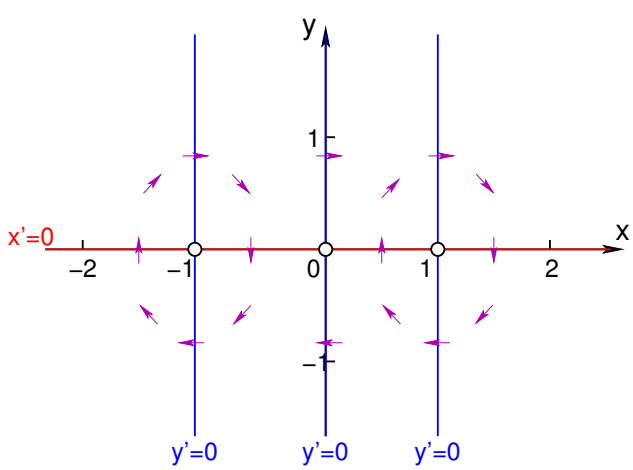
Sketch the phase portrait of representative solution curves in the phase space.

**Answer:** Based on the result discussed above,

$$E = \frac{1}{2}x' + V(x) = \frac{1}{2}y^2 + \frac{x^4}{4} - \frac{x^2}{2} = C$$

gives all solutions curves of the system. One needs a computer to help plot these curves. However, using phase-plane techniques, we can generate a sketch of the portrait.

$x$-nullcline is $y = 0$ which is the horizontal axis; $y$-nullclines are $x = -1,\ 0,\ 1$ and the . Thus, there are 3 critical points: $(0,\ 0),\ (\pm 1,\ 0)$. Nullclines, critical points, and vector field are sketched in the figure below.




The Jacobian matrix is

$$J(x, y) = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{bmatrix}.$$

For $(0,\ 0)$:

$$J(0,\ 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \Rightarrow \qquad Tr = 0,\ Det = -1 \qquad \Rightarrow \qquad \lambda^2 - 1 = 0$$

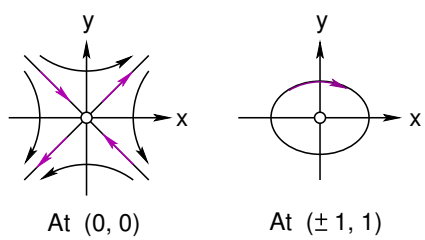
which yields two eigenvalues $\lambda_1 = -1,\ \lambda_2 = 1$. This is a saddle point. The corresponding eigenvectors are

$$\vec{v}_1 = \ker(A - \lambda_1 I) = \ker \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \ker(A - \lambda_2 I) = \ker \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $(\pm 1,\ 0)$:

$$J(\pm 1,\ 0) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \qquad \Rightarrow \qquad Tr = 0,\ Det = 2 \qquad \Rightarrow \qquad \lambda^2 + 2 = 0$$

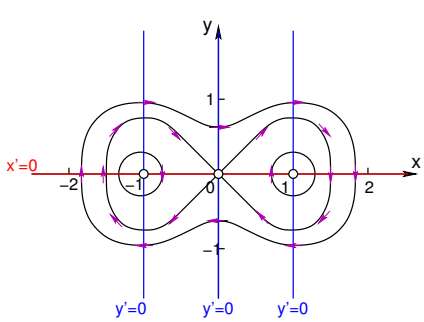
which yields two eigenvalues $\lambda_{1,\ 2} = \pm i\sqrt{2}$. These two critical points are centres. The local phase portraits are given below

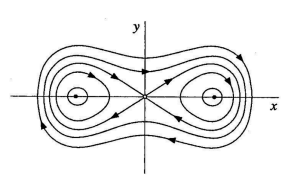


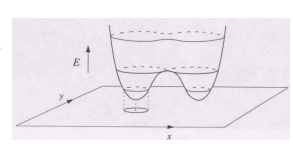

The solution curve that goes through $(0,\ 0)$ is given by

$$\frac{1}{2}y^2 + \frac{x^4}{4} - \frac{x^2}{2} = 0$$

which crosses the $x$-axis at 3 points: $x = 0$ and $x = \pm\sqrt{2}$. Combining all the results above, one can sketch the following global phase portrait.



# 6 Laplace Transform

## 6.1 Introduction - What? Why? How?

- Laplace transform (LT) is an *alternative* method for solving linear ODEs with constant coefficients.
- LT is a linear integral operation that turns an ODE into an algebraic equation.
- After solving the transformed unknown function, an inverse transform is required to obtain the desired solution.
- **Pros:**

  (i) ODE solved together with ICs, no extra step required for finding the integration constants.

  (ii) For a nonhomogeneous ODE, no need to solve $y_h(t)$ and $y_p(t)$ separately, both are solved in one step.

  (iii) Particularly good when the nonhomogeneuos term $g(t)$ is piecewise continuous, useful in engineering.

  (iv) Allows the introduction of more advanced techniques like "transfer function", ...

- **Cons:**

  The inverse Laplace transform is often technically challenging.

## 6.2 Definition and calculating the transforms using the definition

**Definition:** Let $f(t)$ be defined on $[0,\ \infty)$ and $s$ be a real number, then

$$F(s) \equiv \mathcal{L}[f(t)] \equiv \int_0^\infty e^{-st} f(t)\, dt. \tag{76}$$

is defined as the *Laplace transform* (LT) of $f(t)$ for all values of $s$ such that the *improper integral* converges. The two functions $f(t) \leftrightarrow F(s)$ form a *Lapace transform pair*, i.e. $F(s)$ is the LT of $f(t)$ and $f(t)$ is the inverse LT of $F(s)$.

**Remarks:**

1. LT is an integral operator. The transform of $f(t)$ is $F(s)$ which is a function of $s$:

$$F(s) = \mathcal{L}[f(t)] \qquad \Leftrightarrow \qquad f(t) = \mathcal{L}^{-1}[F(s)]$$

   where $\mathcal{L}^{-1}$ represents the inverse Laplace transform.

2. LT is defined by an *improper integral* - a definite integral with one or both limits being unbounded. By definition,

$$\int_0^\infty e^{-st} f(t)\, dt \equiv \lim_{T\to\infty} \int_0^T e^{-st} f(t) dt$$

   if the limit exists (i.e. if it converges).

3. The fact the limits of this integral is between 0 and $\infty$ indicates that LT emphasizes the behaviour of $f(t)$ in the future but ignores its behaviour in the past - the values of $f(t)$ for $t < 0$ plays no role in its LT.

**Example 4.2.1:** For $f(t) = 1$, find $F(s)$.

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[1] = \int_0^\infty e^{-st}(1)dt = \lim_{T\to\infty} \left[ -\frac{1}{s} e^{-st} \Big|_{t=0}^{T} \right]$$

$$= \lim_{T\to\infty} \left[ -\frac{1}{s} \right] (e^{-sT} - 1) = \begin{cases} \frac{1}{s}, & \text{if } s > 0; \\ \text{diverges}, & \text{if } s < 0. \end{cases}$$

Therefore,

$$\mathcal{L}[1] = \frac{1}{s}, \qquad (s > 0).$$

**Example 4.2.2:** For $f(t) = t$, find $F(s)$.

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[t] = \int_0^\infty e^{-st} t dt \overset{\text{integration by parts}}{=} \begin{cases} \frac{1}{s^2}, & \text{if } s > 0; \\ \text{diverges}, & \text{if } s < 0. \end{cases}$$

Therefore,

$$\mathcal{L}[t] = \frac{1}{s^2}, \qquad (s > 0).$$

**Example 4.2.3:** For $f(t) = e^{at}$, find $F(s)$.

$$F(s) = \mathcal{L}[e^{at}] = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t}\, dt = -\frac{1}{s-a} e^{-(s-a)t}\Big|_{t=0}^{\infty} = \begin{cases} \frac{1}{s-a}, & \text{if } s > a; \\ \text{diverges}, & \text{if } s < a. \end{cases}$$

Therefore,

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}, \qquad (s > a).$$

Table 2: Laplace transforms we calculated using the definition.

| $f(t) = \mathcal{L}^{-1}[F(s)]$ | $F(s) = \mathcal{L}[f(t)]$ |
|---|---|
| $1$ | $\frac{1}{s}, \quad s > 0$ |
| $t$ | $\frac{1}{s^2}, \quad s > 0$ |
| $e^{at}$ | $\frac{1}{s-a}, \quad s > a$ |
| $\sin(\omega t)$ | $\frac{\omega}{s^2+\omega^2}, \quad s > 0$ |
| $\cos(\omega t)$ | $\frac{s}{s^2+\omega^2}, \quad s > 0$ |

**Example 4.2.4:** For $f(t) = \cos \omega t$, find $F(s)$.

$$\begin{aligned} F(s) = \mathcal{L}[\cos \omega t] &= \int_0^\infty e^{-st} \cos \omega t dt = -\frac{1}{s}\left[e^{-st} \cos \omega t\Big|_{t=0}^{\infty} - \int_0^\infty e^{-st} (\cos \omega t)' dt\right] \\ &\overset{s>0}{=} -\frac{1}{s}\left[-1 + \omega \int_0^\infty e^{-st} \sin \omega t dt\right] = \frac{1}{s}\left[1 - \omega \int_0^\infty e^{-st} \sin \omega t dt\right] \\ &= \frac{1}{s}\left[1 + \frac{\omega}{s}\left(e^{-st} \sin \omega t\Big|_{t=0}^{\infty} - \omega \int_0^\infty e^{-st} \cos \omega t dt\right)\right] \overset{s>0}{=} \frac{1}{s}\left[1 + \frac{\omega}{s}\left(-\omega F(s)\right)\right] \end{aligned}$$

Now, solve this equation for $F(s)$, we obtain

$$\mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2}, \qquad (s > 0).$$

**Similarly,** for $f(t) = \sin \omega t$

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}, \qquad (s > 0).$$

## 6.3 Use a table to find Laplace transforms and the inverse

Laplace transforms and their inverses have been calculated for all frequently encountered functions. Very often, we simply need to use a table to find the transforms that we need.

The table below contains most LTs that we need for the purpose of this course. More detailed forms can be found if needed. For example, based on this table we find that

$$\mathcal{L}[t^3] = \frac{3!}{s^4} = \frac{6}{s^4}, \qquad (s > 0).$$

$$\mathcal{L}^{-1}\left[\frac{2s}{(s^2+1)^2}\right] = \mathcal{L}^{-1}\left[-\left(\frac{1}{s^2+1}\right)'\right] = -(-t)\sin t = t \sin t.$$

Table 3: Elementary Laplace Transforms.

| $f(t) = \mathcal{L}^{-1}[F(s)]$ | $F(s) = \mathcal{L}[f(t)]$ |
|---|---|
| $1$ | $\frac{1}{s}, \quad s > 0$ |
| $e^{at}$ | $\frac{1}{s-a}, \quad s > a$ |
| $t^n, \quad n \geq 0$ | $\frac{n!}{s^{n+1}}, \quad s > 0$ |
| $\sin(\omega t)$ | $\frac{\omega}{s^2+\omega^2}, \quad s > 0$ |
| $\cos(\omega t)$ | $\frac{s}{s^2+\omega^2}, \quad s > 0$ |
| $\sinh(\omega t)$ | $\frac{\omega}{s^2-\omega^2}, \quad s > \lvert\omega\rvert$ |
| $\cosh(\omega t)$ | $\frac{s}{s^2-\omega^2}, \quad s > \lvert\omega\rvert$ |
| $e^{at}\sin(\omega t)$ | $\frac{\omega}{(s-a)^2+\omega^2}, \quad s > a$ |
| $e^{at}\cos(\omega t)$ | $\frac{s-a}{(s-a)^2+\omega^2}, \quad s > a$ |
| $t^n e^{at}, \quad n \geq 0$ | $\frac{n!}{(s-a)^{n+1}}, \quad s > a$ |
| $e^{at} f(t)$ | $F(s-a)$ |
| $(-t)^n f(t)$ | $F^{(n)}(s)$ |
| $\int_0^t f(\tau)d\tau$ | $\frac{F(s)}{s}$ |
| $f^{(n)}(t)$ | $s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$ |
| $u(t-\tau)$ | $\frac{e^{-\tau s}}{s}, \quad \tau > 0, \quad s > 0$ |
| $u(t-\tau)f(t-\tau)$ | $e^{-\tau s}F(s), \quad \tau > 0, \quad s > s_0$ |
| $\delta(t-\tau)$ | $e^{-\tau s}, \quad \tau > 0$ |
| $f(at)$ | $\frac{1}{a}F(\frac{s}{a}), \quad a > 0, \quad s > as_0$ |
| $f(t) * g(t)$ | $F(s)G(s), \quad s > s_0$ |

## 6.4 More techniques involved in calculating Laplace transforms

**Theorem 4.4.1 Laplace transform is linear:** Suppose $\mathcal{L}[f_1(t)]$ and $\mathcal{L}[f_2(t)]$ are defined for $s > \alpha_j,\ (j = 1,\ 2)$. Let $s_0 = max\{\alpha_1,\ \alpha_2\}$ and let $c_1,\ c_2$ be constants. Then,

$$\mathcal{L}[c_1 f_1 + c_2 f_2] = c_1\mathcal{L}[f_1] + c_2\mathcal{L}[f_2], \qquad \text{for } s > s_0.$$

**Proof:**

$$\mathcal{L}[c_1 f_1 + c_2 f_2] \overset{\text{by def}}{=} \int_0^\infty e^{-st}[c_1 f_1 + c_2 f_2]dt = \int_0^\infty [c_1 e^{-st} f_1 + c_2 e^{-st} f_2]dt$$
$$= c_1 \int_0^\infty e^{-st} f_1 dt + c_2 \int_0^\infty e^{-st} f_2 dt = c_1\mathcal{L}[f_1] + c_2\mathcal{L}[f_2].$$

**Remark:** Laplace transform is linear because integration is linear.

**Example 4.4.1:** Use theorem 4.4.1 and the fact $\mathcal{L}[e^{at}] = 1/(s-a),\ (s > a)$, calculate $\mathcal{L}[\cosh \omega t]$ and $\mathcal{L}[\sinh \omega t]$.

**Answer:**

$$\mathcal{L}[\cosh \omega t] = \mathcal{L}[\frac{1}{2}(e^{\omega t} + e^{-\omega t})] = \frac{1}{2}\left(\mathcal{L}[e^{\omega t}] + \mathcal{L}[e^{-\omega t}]\right) = \frac{1}{2}\left(\frac{1}{s-\omega} + \frac{1}{s+\omega}\right) = \frac{s}{s^2-\omega^2},\ (s > |\omega|).$$

Similarly,

$$\mathcal{L}[\sinh \omega t] = \mathcal{L}[\frac{1}{2}(e^{\omega t} - e^{-\omega t})] = \frac{1}{2}\left(\mathcal{L}[e^{\omega t}] - \mathcal{L}[e^{-\omega t}]\right) = \frac{1}{2}\left(\frac{1}{s-\omega} - \frac{1}{s+\omega}\right) = \frac{\omega}{s^2-\omega^2},\ (s > |\omega|).$$

**Theorem 4.4.2 First shifting theorem:** If $F(s) = \mathcal{L}[f]$ for $s > s_0$, then $F(s-a) = \mathcal{L}[e^{at}f]$ for $s > s_0 + a$.

**Proof:**

$$F(s-a) \overset{\text{by definition}}{=} \int_0^\infty e^{-(s-a)t} f(t)dt = \int_0^\infty e^{-st}\left[e^{at}f(t)\right]dt \overset{\text{by definition}}{=} \mathcal{L}[e^{at}f].$$

**Example 4.4.2:** Use the first shifting theorem and the fact $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} = F(s)$, calculate $\mathcal{L}[e^{at}t^n]$.

$$\mathcal{L}[e^{at}t^n] = F(s-a) = \frac{n!}{(s-a)^{n+1}}, \qquad (s > a).$$

**Example 4.4.3:** Use the first shifting theorem and the fact $F(s) = \mathcal{L}[\cos \omega t] = \frac{s}{s^2+\omega^2}$, calculate $\mathcal{L}[e^{at}\cos \omega t]$.

$$\mathcal{L}[e^{at}\cos \omega t] = F(s-a) = \frac{s-a}{(s-a)^2+\omega^2}, \qquad (s > a).$$

## 6.5 Existence of Laplace transform

**Definition of jump discontinuity:** If $f(t)$ is discontinuous at $t_0$ but

$$f(t_0^+) - f(t_0^-) \quad \text{is finite,}$$

then $f(t)$ has a *jump discontinuity* at $t = t_0$.

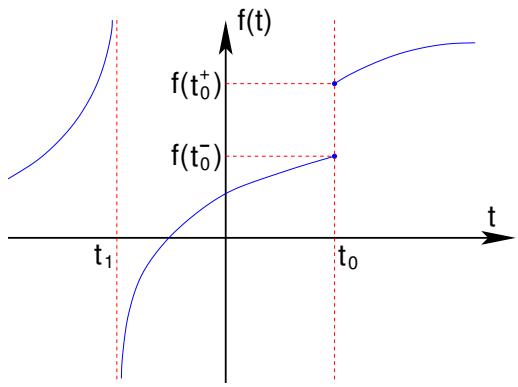



Figure 26: $f(t)$ has a jump discontinuity at $t_0$ but not at $t_1$.

**Definition of piecewise continuity:** $f(t)$ is *piecewise continuous*

(i) in $[0,\ T]$, if $f(0),\ f(T)$ are both finite, and that $f(t)$ has a finite number of jump discontinuities in this interval;

(ii) in $[0,\ \infty)$, if it is piecewise continuous in $[0,\ T]$ for all $T > 0$.

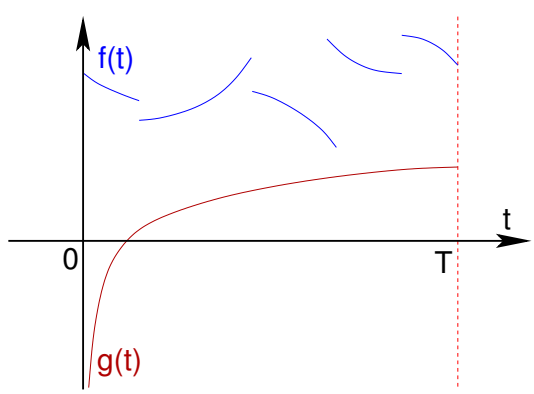



Figure 27: In $[0,\ T]$, $f(t)$ is piecewise continuous although it has several jump discontinuities. $g(t)$ is not although it has no discontinuities at all. But its value at $t = 0$ is not finite.

**Definition of exponential order:** $f(t)$ is of *exponential order* $s_0$ as $t \to \infty$, if

$$|f(t)| \leq Ke^{s_0 t}, \qquad \text{for } t > M,$$

where $K,\ M > 0,\ s_0,\ K,\ M$ are real numbers.

**Remark:** $f(t)$ is of exponential order $s_0$ simply means that $f(t)$ cannot approach infinity faster than the exponential function $e^{s_0 t}$ as $t \to \infty$.

**Theroem 4.4.3 Existence of LT:** If $f(t)$ is piecewise continuous in $[0,\ \infty)$ and of exponential order $s_0$, then $F(s) = \mathcal{L}[f(t)]$ exists for $s > s_0$.

**Proof:** Read Boyce and DiPrima textbook.

**Example 4.5.1:** Find the LT of the following piecewise continuous function

$$f(t) = \begin{cases} 1, & \text{if } 0 \leq t < 4; \\ t, & \text{if } 4 \leq t. \end{cases}$$

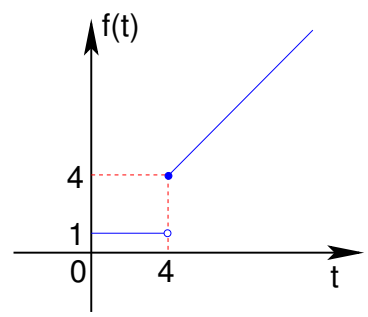



Figure 28:

**Answer:**

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt = \int_0^4 e^{-st}(1)dt + \int_4^\infty e^{-st} t dt = I_1 \ + \ I_2.$$

$$I_1 = \int_0^4 e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_0^4 = \frac{1 - e^{-4s}}{s}, \qquad (s \neq 0).$$

$$I_2 = \int_4^\infty e^{-st} t dt \overset{t=u+4}{=} \int_0^\infty e^{-s(u+4)}(u+4)d(u+4) = e^{-4s} \int_0^\infty e^{-su}(u+4)du$$

$$= e^{-4s}\left[\int_0^\infty e^{-su} u du + 4\int_0^\infty e^{-su} du\right] \overset{u=t}{=} e^{-4s}\left[\int_0^\infty e^{-st} t dt + 4\int_0^\infty e^{-st} dt\right]$$

$$= e^{-4s}\left(\mathcal{L}[t] + 4\mathcal{L}[1]\right) = e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right), \qquad (s > 0).$$

Therefore,

$$\mathcal{L}[f(t)] = I_1 \ + \ I_2 = \frac{1-e^{-4s}}{s} + e^{-4s}\left(\frac{1}{s^2}+\frac{4}{s}\right) = \frac{1}{s} + \frac{e^{-4s}}{s^2} + 3\frac{e^{-4s}}{s}, \qquad (s>0).$$

**Remark:** We shall demonstrate later that, using unit step functions, we can make calculating the LT of piecewise continuous functions much easier.

## 6.6 Laplace transform of derivatives

**Theroem 4.5.1:** Suppose $f(t)$ and $f'(t)$ are continuous in $[0,\ \infty)$ and are of exponential order $s_0$, and that $f''(t)$ is piecewise continuous in $[0,\ \infty)$. Then, $f,\ f',\ f''$ have Laplace transform for $s > s_0$,

$$\mathcal{L}[f'] = s\mathcal{L}[f] - f(0);$$
$$\mathcal{L}[f''] = s^2\mathcal{L}[f] - f'(0) - sf(0).$$

Notice that the initial conditions $f(0),\ f'(0)$ naturally show up in the LTs of derivatives.

**Proof:**

$$\mathcal{L}[f'] \overset{\text{by definition}}{=} \int_0^\infty e^{-st} f' dt \overset{\frac{df}{dt}=f' \text{ or } df=f'dt}{=} \int_0^\infty e^{-st} df \overset{\int udv = uv - \int vdu}{=} e^{-st} f(t)\Big|_0^\infty - \int_0^\infty f(t) d(e^{-st})$$

$$= -f(0) - \int_0^\infty f(t)(-s)e^{-st}dt = s\int_0^\infty f(t)e^{-st}dt - f(0) = s\mathcal{L}[f] - f(0).$$

Now, let $g(t) = f'(t)$.

$$\mathcal{L}[f''] = \mathcal{L}[g'] \overset{\text{use prev. formula}}{=} s\mathcal{L}[g] - g(0) \overset{\text{g(t)=f'(t)}}{=} s\mathcal{L}[f'] - f'(0)$$

$$\overset{\text{use prev. formula again}}{=} s\left(s\mathcal{L}[f] - f(0)\right) - f'(0) = s^2\mathcal{L}[f] - sf(0) - f'(0).$$

## 6.7 Use Laplace transform to solve IVPs

**Example 4.6.1:** Use Laplace transform to solve the following IVP:

$$\begin{cases} y' - ay = 0, \\ y(0) = y_0. \end{cases}$$

**Answer:** Apply LT to both sides of the ODE

$$\mathcal{L}[y'] - a\mathcal{L}[y] = 0 \quad \Rightarrow \quad s\mathcal{L}[y] - y(0) - a\mathcal{L}[y] = 0 \quad \Rightarrow \quad sY(s) - y_0 - aY(s) = 0 \quad \Rightarrow \quad Y(s) = \frac{y_0}{s-a}.$$

Now,

$$y(t) = \mathcal{L}^{-1}[Y(s)] = y_0\mathcal{L}^{-1}[\frac{1}{s-a}] \overset{e^{at} \leftrightarrow \frac{1}{s-a}}{=} y_0 e^{at}.$$

**Example 4.6.2:** Use Laplace transform to solve the following nonhomogeneous IVP. (Note this is identical to forced vibration in the absence of damping!)

$$\begin{cases} y'' + \omega_0^2 y = F\cos(\omega t), & (\omega \neq \omega_0) \\ y(0) = 0, \quad y'(0) = 0. \end{cases}$$

**Answer:** Apply LT to both sides of the ODE

$$\mathcal{L}[y''] + \omega_0^2\mathcal{L}[y] = F\mathcal{L}[\cos(\omega t)] \quad \Rightarrow \quad s^2\mathcal{L}[y] - sy(0) - y'(0) + \omega_0^2\mathcal{L}[y] = \frac{Fs}{s^2+\omega^2}$$

$$\Rightarrow \quad (s^2+\omega_0^2)Y(s) = \frac{Fs}{s^2+\omega^2} \quad \Rightarrow \quad Y(s) = \frac{Fs}{(s^2+\omega^2)(s^2+\omega_0^2)}$$

$$\overset{\text{partial fractions}}{\Longrightarrow} \quad Y(s) = \frac{F}{\omega_0^2-\omega^2}\left[\frac{s}{s^2+\omega^2} - \frac{s}{s^2+\omega_0^2}\right].$$

Now,

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{F}{\omega_0^2-\omega^2}\left[\mathcal{L}^{-1}[\frac{s}{s^2+\omega^2}] - \mathcal{L}^{-1}[\frac{s}{s^2+\omega_0^2}]\right] = \frac{F}{\omega_0^2-\omega^2}\left[\cos\omega t - \cos\omega_0 t\right],$$

where the fact $\mathcal{L}^{-1}$ is a linear operator was used. Note that this IVP consisting of a nonhomoeneous ODE and two ICs is solved directly using Laplace transform method. No need to find $y_h(t)$ and $y_p(t)$ sparately, no need to determine the integration constants.

**Definition of inverse Laplace transform $\mathcal{L}^{-1}$:**

$$f(t) \equiv \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)ds,$$

where the integration is done along the vertical line $Re(s) = \gamma$ in the complex plane such that $\gamma$ is greater than the real part of all singularities of F(s). Since $\mathcal{L}^{-1}$ is also an integral operator, it is also linear as can be proofed in a similar way as in Theorem 4.4.1. Therefore,

$$\mathcal{L}^{-1}[c_1F_1(s) + c_2F_2(s)] = c_1\mathcal{L}^{-1}[F_1] + c_2\mathcal{L}^{-1}[F_2].$$

**Remark:** We almost never use this definition to calculate $\mathcal{L}^{-1}[F(s)]$. Such a path integral in complex plane is beyond most students in second year. Since each time we calculate one $\mathcal{L}[f(t)] = F(s)$, it establishes a pairwise relationship between $f(t)$ and $F(s)$. We then can use the result for finding $f(t)$ given $F(s)$ as well as finding $F(s)$ given $f(t)$.

### 6.7.1 Solving IVPs involving extensive partial fractions

**Example 4.7.1:** Use Laplace transform solve

$$\begin{cases} y'' - 6y' + 5y = 3e^{2t}, \\ y(0) = 2, \quad y'(0) = 3. \end{cases}$$

**Answer:** Laplace transform both sides of the ODE and let $Y(s) = \mathcal{L}[y]$:

$$\overbrace{s^2Y - sy_0 - y_0'}^{\mathcal{L}[y'']} - 6\overbrace{(sY - y_0)}^{\mathcal{L}[y']} + 5Y = \overbrace{\frac{3}{s-2}}^{\mathcal{L}[3e^{2t}]} \qquad \Longrightarrow$$

$$s^2Y - 2s - 3 - 6sY + 12 + 5Y = \frac{3}{s-2} \qquad \Longrightarrow \qquad \overbrace{(s^2 - 6s + 5)Y}^{y''-6y'+5y} = \overbrace{2s - 9}^{\text{ICs}} + \overbrace{\frac{3}{s-2}}^{\text{g(t)}}.$$

Notice that $s^2 - 6s + 5 = (s-1)(s-5)$, we obtain

$$(s-1)(s-5)Y = 2s - 9 + \frac{3}{s-2} \qquad \Longrightarrow \qquad Y(s) = \frac{2s-9}{(s-1)(s-5)} + \frac{3}{(s-1)(s-5)(s-2)}.$$

A little bit of algebra allows us to combine the two terms into the following one rational function

$$Y(s) = \frac{(s-3)(2s-7)}{(s-1)(s-2)(s-5)}.$$

It was relatively easy for us to solve for $Y(s)$ but the solution we are looking for is

$$y(t) = \mathcal{L}^{-1}[Y(s)].$$

Unless we can find a way to break $Y(s)$ into partial fractions, we do not know $\mathcal{L}^{-1}[Y(s)] =?$

Our goal is to achieve the following:

$$Y(s) = \frac{(s-3)(2s-7)}{(s-1)(s-2)(s-5)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-5},$$

where the constants $A,\ B,\ C$ can be determined by **Heaviside's method:**

$$A = (s-1)Y(s)|_{s=1} = \left.\frac{(s-3)(2s-7)}{(s-2)(s-5)}\right|_{s=1} = \frac{(-2)(-5)}{(-1)(-4)} = \frac{10}{4} = 2.5.$$

$$B = (s-2)Y(s)|_{s=2} = \left.\frac{(s-3)(2s-7)}{(s-1)(s-5)}\right|_{s=2} = \frac{(-1)(-3)}{(1)(-3)} = -\frac{3}{3} = -1.$$

$$C = (s-5)Y(s)|_{s=5} = \left.\frac{(s-3)(2s-7)}{(s-1)(s-2)}\right|_{s=5} = \frac{(2)(3)}{(4)(3)} = \frac{6}{12} = 0.5.$$

Thus,

$$Y(s) = \frac{2.5}{s-1} + \frac{(-1)}{s-2} + \frac{0.5}{s-5}$$

Therefore,

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{2.5}{s-1} + \frac{(-1)}{s-2} + \frac{0.5}{s-5}\right] = 2.5\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + 0.5\mathcal{L}^{-1}\left[\frac{1}{s-5}\right]$$

$$= 2.5e^{t} - e^{2t} + 0.5e^{5t}.$$

**Theorem 4.7.1:** Suppose that

$$Y(s) = \frac{P(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)},$$

where the degree of $P(s)$ is smaller than $n$ and $s_1,\ s_2,\ \cdots, s_n$ are distinct numbers, then

$$Y(s) = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_n}{s-s_n},$$

where

$$A_j = (s-s_j)Y(s)|_{s=s_j}, \qquad (\text{for all } j = 1,\ 2, \cdots, n).$$

**Proof:**

$$(s-s_j)Y(s) = (s-s_j)\frac{A_1}{s-s_1} + \cdots + (s-s_j)\frac{A_{j-1}}{s-s_{j-1}} + A_j + (s-s_j)\frac{A_{j+1}}{s-s_{j+1}} + \cdots + (s-s_j)\frac{A_n}{s-s_n}.$$

Since all $s_j$ are distinct, only in the $j-th$ term, this multiplicative factor cancels the denominator leaving $A_j$ by itself. Use sigma sum notation

$$(s-s_j)Y(s) = A_j + (s-s_j)\sum_{k=1,\ k\neq j}^{n} \frac{A_k}{s-s_k}.$$

When $s = s_j$, the second term of the $rhs$ vanishes, thus

$$A_j = (s-s_j)Y(s)\big|_{s=s_j}.$$

### 6.7.2 Solving general IVPs involving a linear ODE with constant coefficients

**Theorem 4.7.2:** Consider

$$\begin{cases} ay'' + by' + cy = g(t), \\ y(0) = y_0, \quad y'(0) = y_0'. \end{cases}$$

Laplace transform both sides of the ODE and let $Y(s) = \mathcal{L}[y]$, $G(s) = \mathcal{L}[g]$:

$$a\overbrace{(s^2Y - sy_0 - y_0')}^{\mathcal{L}[y'']} + b\overbrace{(sY - y_0)}^{\mathcal{L}[y']} + cY = G \qquad \Longrightarrow$$

$$as^2Y - ay_0s - ay_0' + bsY - by_0 + cY = G \qquad \Longrightarrow \qquad \overbrace{(as^2+bs+c)Y}^{ay''+by'+cy} = \overbrace{(as+b)y_0 + ay_0'}^{\text{ICs}} + \overbrace{G}^{\text{g(t)}}.$$

Therefore,

$$Y(s) = \frac{(as+b)y_0 + ay_0' + G(s)}{as^2+bs+c} = \overbrace{\frac{(as+b)y_0 + ay_0'}{as^2+bs+c}}^{y_h \text{ with ICs}} + \overbrace{\frac{G(s)}{as^2+bs+c}}^{y_p}.$$

**Example 4.7.2:** Solve the following IVP

$$\begin{cases} y'' + 3y' + 2y = e^{-3t}, \\ y(0) = 0, \quad y'(0) = 1. \end{cases}$$

**Answer:** Notice that $as^2 + bs + c = s^2 + 3s + 2 = (s+1)(s+2)$ and that

$$G(s) = \mathcal{L}[g] = \mathcal{L}[e^{-3t}] = \frac{1}{s+3}.$$

Based on Theorem 4.7.2:

$$Y(s) = \frac{1}{(s+1)(s+2)} + \frac{1}{(s+1)(s+2)(s+3)}.$$

Using Heaviside's method

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+3} = \frac{3}{2}\frac{1}{s+1} - 2\frac{1}{s+2} + \frac{1}{2}\frac{1}{s+3}.$$

Therefore,

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{3}{2}\mathcal{L}^{-1}[\frac{1}{s+1}] - 2\mathcal{L}^{-1}[\frac{1}{s+2}] + \frac{1}{2}\mathcal{L}^{-1}[\frac{1}{s+3}] = \frac{3}{2}e^{-t} - 2e^{-2t} + \frac{1}{2}e^{-3t}.$$

**Remark:** Heaviside's method applies because in this example the ch. eq. $as^2 + bs + c = 0$ gives two distinct real roots, and that $g(t)$ is not identical to either of the homogeneous solutions.

**Example 4.7.3:** Solve the following IVP

$$\begin{cases} y'' + 2y' + y = t, \\ y(0) = 0, \quad y'(0) = 1. \end{cases}$$

**Answer:** Notice that $as^2 + bs + c = s^2 + 2s + 1 = (s+1)^2$ and that

$$G(s) = \mathcal{L}[g] = \mathcal{L}[t] = \frac{1}{s^2}.$$

Based on Theorem 4.7.2:

$$Y(s) = \frac{1}{(s+1)^2} + \frac{1}{s^2(s+1)^2}.$$

Now, the inverse of first term can be readily solved but Heaviside's method does not apply to the second term. Notice that

$$\frac{1}{s^2(s+1)^2} = \frac{1}{s^2} - \frac{1}{(s+1)^2} - \frac{2s}{s^2(s+1)^2} = \frac{1}{s^2} - \frac{1}{(s+1)^2} - 2\left[\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right]$$

Thus

$$Y(s) = \frac{1}{s^2} - 2\frac{1}{s} + 2\frac{1}{s+1} + 2\frac{1}{(s+1)^2}.$$

Therefore,

$$y(t) = \mathcal{L}^{-1}[Y(s)] = t - 2 + 2e^{-t} + 2te^{-t}.$$

**Remarks:**

- Heaviside's method does not apply because in this example the ch. eq. $as^2 + bs + c = 0$ gives two repeated real roots.

- We shall demonstrate later that using the convolution theorem, we can calculate the inverse LT of this kind without having to solve the partial fractions.

**Example 4.7.4:** Solve the following IVP

$$\begin{cases} y'' + 2y' + 2y = 2, \\ y(0) = 0, \quad y'(0) = 1. \end{cases}$$

**Answer:** Notice that $as^2 + bs + c = s^2 + 2s + 2 = (s+1)^2 + 1$ and that

$$G(s) = \mathcal{L}[g] = \mathcal{L}[1] = \frac{2}{s}.$$

Based on Theorem 4.7.2:

$$Y(s) = \frac{1}{(s+1)^2 + 1} + \frac{2}{s[(s+1)^2 + 1]}.$$

Now, the inverse of first term can be readily solved but Heaviside's method does not apply to the second term. Notice that

$$\frac{2}{s[(s+1)^2 + 1]} = \frac{A}{s} - \frac{B(s+1) + C}{(s+1)^2 + 1},$$

where $A = B = C = 1$.

Thus,

$$Y(s) = \frac{1}{s} - \frac{s+1}{(s+1)^2 + 1}.$$

Therefore,

$$y(t) = \mathcal{L}^{-1}[Y(s)] = 1 - e^{-t}\cos t.$$

**Remarks:**

- Heaviside's method does not apply because in this example the ch. eq. $as^2 + bs + c = 0$ gives two complex roots, $-\alpha \pm i\beta$. In this case, $as^2 + bs + c = (s+\alpha)^2 + \beta^2$.
- We shall demonstrate later that using the convolution theorem, we can calculate the inverse LT of this kind without having to solve the partial fractions.

### 6.7.3 Some inverse LTs where Heaviside's method fails to apply

**Example 4.7.5:** Find $y(t) = \mathcal{L}^{-1}[Y(s)]$ for

$$Y(s) = \frac{8-(s+2)(4s+10)}{(s+1)(s+2)^2}.$$

**Answer:** In this case

$$Y(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2},$$

where

$$A = (s+1)Y(s)\big|_{s=-1} = \frac{8-(s+2)(4s+10)}{(s+2)^2}\bigg|_{s=-1} = 8-(1)(6) = 2.$$

$$C = (s+2)^2Y(s)\big|_{s=-2} = \frac{8-(s+2)(4s+10)}{s+1}\bigg|_{s=-2} = \frac{8-(0)(2)}{(-1)} = -8.$$

$B$ has to be calculated separately using the fact

$$Y(s) = \frac{2}{s+1}+\frac{B}{s+2}-\frac{8}{(s+2)^2} = \frac{2(s+2)^2+B(s+1)(s+2)-8(s+1)}{(s+1)(s+2)^2} = \frac{8-(s+2)(4s+10)}{(s+1)(s+2)^2} \quad \Longrightarrow$$

$$2(s+2)^2+B(s+1)(s+2)-8(s+1) = 8-(s+2)(4s+10) \quad \overset{s=0}{\Longrightarrow} \quad 8+2B-8 = 8-(2)(10) \quad \Longrightarrow$$

$$2B = -12 \quad \Longrightarrow \quad B = -6.$$

Therefore

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[\frac{2}{s+1}] - \mathcal{L}^{-1}[\frac{6}{s+2}] - \mathcal{L}^{-1}[\frac{8}{(s+2)^2}]$$
$$= 2e^{-t} - 6e^{-2t} - 8te^{-2t}.$$

**Example 4.7.6:** Find $y(t) = \mathcal{L}^{-1}[Y(s)]$ for

$$Y(s) = \frac{s^2-5s+7}{(s+2)^3}.$$

**Answer:** In this case

$$Y(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3},$$

where

$$C = (s+2)^3Y(s)\big|_{s=-2} = s^2-5s+7\big|_{s=-2} = 4+10+7 = 21.$$

$A,\ B$ cannot be determined this way.

An alternative way is to express the numerator as a polynomial of $(s+2)$:

$$s^2 - 5s + 7 = [(s+2)-2]^2 + 5[(s+2)-2] + 7 = (s+2)^2 - 9(s+2) + 21.$$

Now substitute back into $Y(s)$:

$$Y(s) = \frac{s^2 - 5s + 7}{(s+2)^3} = \frac{(s+2)^2 - 9(s+2) + 21}{(s+2)^3} = \frac{1}{s+2} - \frac{9}{(s+2)^2} + \frac{21}{(s+2)^3}.$$

Therefore, $A = 1$, $B = -9$, and $C = 21$.

Therefore

$$\begin{aligned} y(t) = \mathcal{L}^{-1}[Y(s)] &= \mathcal{L}^{-1}[\frac{1}{s+2}] - \mathcal{L}^{-1}[\frac{9}{(s+2)^2}] + \mathcal{L}^{-1}[\frac{21}{(s+2)^3}] \\ &= e^{-2t} - 9te^{-2t} + \frac{21}{2}t^2e^{-2t}. \end{aligned}$$

**Example 4.7.1:** Find $y(t) = \mathcal{L}^{-1}[Y(s)]$ for

$$Y(s) = \frac{1 - s(5+3s)}{s[(s+1)^2+1]}.$$

**Answer:** Our goal is

$$Y(s) = \frac{1 - s(5+3s)}{s[(s+1)^2+1]} = \frac{A}{s} + \frac{B(s+1)+C}{(s+1)^2+1} = \frac{A[(s+1)^2+1] + Bs(s+1) + Cs}{s[(s+1)^2+1]}.$$

Equating the numerators on both sides, we obtain

$$1 - s(5+3s) = A[(s+1)^2+1] + Bs(s+1) + Cs.$$

Now,

$$s = 0, \qquad \Longrightarrow \qquad 2A = 1 \qquad \Longrightarrow \qquad A = \frac{1}{2}.$$

$$s = -1, \qquad \Longrightarrow \qquad A - C = 3 \qquad \Longrightarrow \qquad C = -\frac{5}{2}.$$

$$s = 1, \qquad \Longrightarrow \qquad 5A + 2B + C = -7 \qquad \Longrightarrow \qquad B = -\frac{7}{2}.$$

Thus,

$$Y(s) = \frac{\frac{1}{2}}{s} + \frac{-\frac{7}{2}(s+1) - \frac{5}{2}}{(s+1)^2+1} = \frac{1}{2}\frac{1}{s} - \frac{7}{2}\frac{s+1}{(s+1)^2+1} - \frac{5}{2}\frac{1}{(s+1)^2+1}.$$

Therefore

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{2}\mathcal{L}^{-1}[\frac{1}{s}] - \frac{7}{2}\mathcal{L}^{-1}[\frac{s+1}{(s+1)^2+1}] - \frac{5}{2}\mathcal{L}^{-1}[\frac{1}{(s+1)^2+1}]$$

$$= \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t.$$

**Example 4.7.8:** Find $y(t) = \mathcal{L}^{-1}[Y(s)]$ for

$$Y(s) = \frac{6+3s}{(s^2+1)(s^2+4)}.$$

**Answer:** Our goal is

$$Y(s) = (6+3s)\left[\frac{A}{s^2+1} - \frac{B}{s^2+4}\right].$$

In this example it is easy to find that $A = B = \frac{1}{3}$. Thus,

$$Y(s) = \frac{1}{3}(6+3s)\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right] = (2+s)\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right].$$

Therefore

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{2}{s^2+1} - \frac{2}{s^2+4}\right] + \mathcal{L}^{-1}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right]$$
$$= 2\sin t - \sin 2t + \cos t - \cos 2t.$$

## 6.8 Unit step/Heaviside function

**Unit step (Heaviside) function** is a function whose value steps up by a unit at $t = 0$ from 0 to 1. It is defined as

$$u(t) = \begin{cases} 0, & t < 0; \\ 1, & t \geq 0. \end{cases} \tag{77}$$

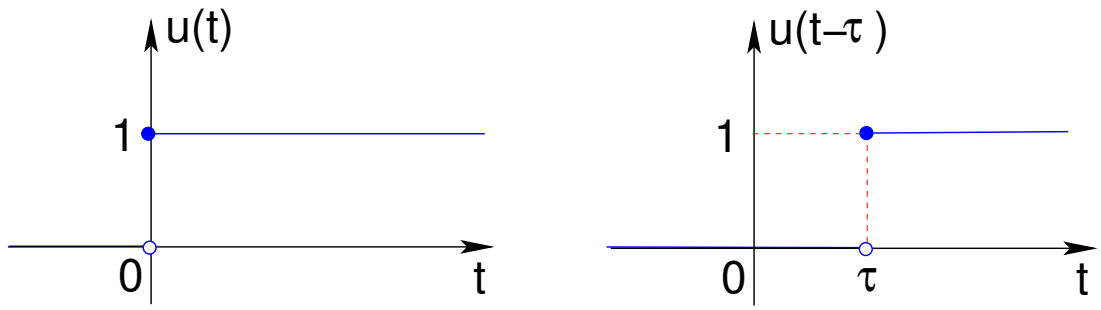



Figure 29: Heaviside function $u(t)$ and shifted Heaviside function $u(t - \tau)$.

**A shifted unit step function** is a function whose value steps up by a unit at $t = \tau$ from 0 to 1 ($\tau > 0$).

$$u(t - \tau) = \begin{cases} 0, & t < \tau; \\ 1, & t \geq \tau. \end{cases} \tag{78}$$

### 6.8.1 Laplace transform of $u(t)$ and $u(t - \tau)$

Based on definition,

$$\mathcal{L}[u(t)] = \int_0^\infty e^{-st} u(t)\, dt = \int_0^\infty e^{-st}\, dt = \mathcal{L}[1] = \frac{1}{s}, \qquad (s > 0).$$

$$\mathcal{L}[u(t-\tau)] = \int_0^\infty e^{-st} u(t-\tau)\, dt = \int_\tau^\infty e^{-st}\, dt = \int_0^\infty e^{-s(t+\tau)}\, d(t+\tau) = e^{-s\tau} \int_0^\infty e^{-st} dt = \frac{e^{-s\tau}}{s}, \quad (s > 0).$$

**Second Shifting Theorem:** Suppose that $\tau > 0$ and $F(s) = \mathcal{L}[f(t)]$ exists for $s > s_0$. Then,

$$\mathcal{L}[u(t - \tau) f(t - \tau)] = e^{-s\tau} F(s), \qquad (s > s_0).$$

**Proof:** Based on definition

$$\mathcal{L}[u(t - \tau) f(t - \tau)] = \int_0^\infty e^{-st} u(t - \tau) f(t - \tau) dt = \int_\tau^\infty e^{-st} f(t - \tau) dt$$

$$= \int_0^\infty e^{-s(t+\tau)} f(t + \tau - \tau) d(t + \tau) = e^{-s\tau} \int_0^\infty e^{-st} f(t) dt = e^{-s\tau} F(s).$$

### 6.8.2 Expressing piecewise continuous function in terms of $u(t-\tau)$

Consider the following piecewise continuous function

$$f(t) = \begin{cases} f_0(t), & \text{if } 0 \le t < \tau; \\ f_1(t), & \text{if } \tau \le t. \end{cases}$$

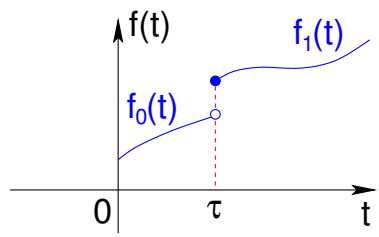



Figure 30: Piecewise continuous function differentially defined on two intervals $[0,\ \tau)$ and $[\tau,\ \infty)$.

Using unit step function $u(t-\tau)$, we can express this function in the following form

$$f(t) = f_0(t) + u(t-\tau)(f_1(t) - f_0(t)) = \begin{cases} f_0(t), & \text{if } 0 \le t < \tau; \\ f_0(t) + f_1(t) - f_0(t) = f_1(t), & \text{if } \tau \le t. \end{cases}$$

**Example 4.8.1:** Express the following function in terms of unit step function, then calculate its LT.

$$f(t) = \begin{cases} 2t+1, & \text{if } 0 \le t < 2; \\ 3t, & \text{if } 2 \le t. \end{cases}$$

**Answer:**

$$\begin{aligned} f(t) &= f_0(t) + u(t-2)(f_1(t) - f_0(t)) = 2t + 1 + u(t-2)(3t - (2t+1)) \\ &= 2t + 1 + u(t-2)(t - 2 + 1) = 2t + 1 + u(t-2)(t-2) + u(t-2). \end{aligned}$$

The LT is

$$\mathcal{L}[f(t)] = \mathcal{L}[2t+1] + \mathcal{L}[u(t-2)(t-2)] + \mathcal{L}[u(t-2)(1)] = \frac{2}{s^2} + \frac{1}{s} + e^{-2s}\frac{1}{s^2} + e^{-2s}\frac{1}{s}.$$

**Back to Example 4.5.1:** Find the LT of the following piecewise continuous function

$$f(t) = \begin{cases} 1, & \text{if } 0 \le t < 4; \\ t, & \text{if } 4 \ge t. \end{cases}$$

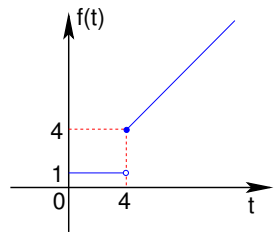



Figure 31: A piecewise continuous function.

**Answer:**

$$f(t) = f_0(t) + u(t-4)(f_1(t) - f_0(t)) = 1 + u(t-4)(t-1) = 1 + u(t-4)(t-4) + u(t-4)(3).$$

The LT is

$$\mathcal{L}[f(t)] = \mathcal{L}[1] + \mathcal{L}[u(t-4)(t-4)] + 3\mathcal{L}[u(t-4)] = \frac{1}{s} + \frac{e^{-4s}}{s^2} + 3\frac{e^{-4s}}{s}.$$

**Example 4.8.2:** Express the following function in terms of unit step function, then calculate its LT.

$$f(t) = \begin{cases} \sin t, & \text{if } 0 \le t < \pi; \\ 0, & \text{if } \pi \le t. \end{cases}$$

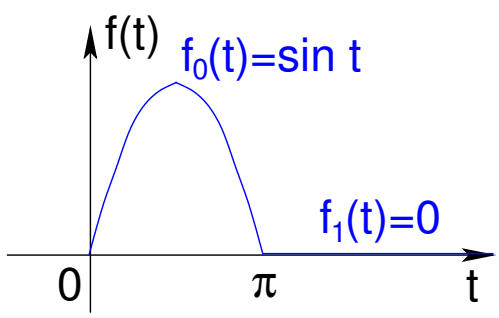



Figure 32: A sine function with only half a cycle.

**Answer:**

$$f(t) = f_0(t) + u(t-\pi)(f_1(t) - f_0(t)) = \sin t + u(t-\pi)(0 - \sin t) = \sin t - u(t-\pi)\sin t.$$

Notice that

$$\sin t = \sin(t-\pi+\pi) = \sin(t-\pi)\cos\pi + \cos(t-\pi)\sin\pi = -\sin(t-\pi).$$

We now have

$$f(t) = \sin t - u(t-\pi)\sin t = \sin t + u(t-\pi)\sin(t-\pi).$$

The LT is

$$\mathcal{L}[f(t)] = \mathcal{L}[\sin t] + \mathcal{L}[u(t-\pi)\sin(t-\pi)] = \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1} = (1+e^{-\pi s})\frac{1}{s^2+1}.$$

### 6.8.3 Solving nonhomogeneous ODEs with piecewise continuous forcing

**Example 4.8.3:** Use Laplace transform to solve the following IVP

$$\begin{cases} y'' + 4y = g(t), \\ y(0) = 0, \quad y'(0) = 1, \end{cases}$$

where $g(t)$ is defined as

$$g(t) = \begin{cases} \sin t, & \text{if } 0 \le t < \pi; \\ 0, & \text{if } \pi \le t, \end{cases}$$

which is identical to that defined **Example 4.8.2.**

**Answer:** Based on the results obtained in **Example 4.8.2**,

$$G(s) = \mathcal{L}[g(t)] = (1+e^{-\pi s})\frac{1}{s^2+1}.$$

Based on Theorem 4.7.2:

$$Y(s) = \frac{(as+b)y_0 + ay_0' + G(s)}{as^2+bs+c} = \overbrace{\frac{(as+b)y_0 + ay_0'}{as^2+bs+c}}^{y_h \text{ with ICs}} + \overbrace{\frac{G(s)}{as^2+bs+c}}^{y_p}.$$

Substitute the ICs and the expression for $G(s)$ into this equation, we obtain

$$Y(s) = \frac{1}{s^2+4} + \frac{G(s)}{s^2+4} = \frac{1}{s^2+4} + \frac{1+e^{-\pi s}}{(s^2+1)(s^2+4)}.$$

Notice that

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right].$$

Plug it back,

$$\begin{aligned}
Y(s) &= \frac{1}{s^2+4} + \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right] + \frac{e^{-\pi s}}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right] \\
&= \frac{1}{3}\frac{2}{s^2+4} + \frac{1}{3}\frac{1}{s^2+1} + \frac{e^{-\pi s}}{6}\left[\frac{2}{s^2+1} - \frac{2}{s^2+4}\right]
\end{aligned}$$

Do the inverse transform term by term, we obtain

$$y(t) = \mathcal{L}[Y(s)] = \frac{1}{3}\sin(2t) + \frac{1}{3}\sin t + \frac{1}{3}u(t-\pi)\sin(t-\pi) - \frac{1}{6}u(t-\pi)\sin(2(t-\pi)).$$

## 6.9 Laplace transform of integrals and inverse transform of derivatives

**6.9.1 Transform of integrals.** Let $\mathcal{L}[f(t)] = F(s)$. Thus,

$$\boxed{\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}.}$$

**Proof.** Let $A(t)$ be the anti-derivative of $f(t)$ such that $A'(t) = f(t)$ and that $A(0) = 0$. Thus,

$$\int_0^t f(\tau)d\tau = A(\tau)|_0^t = A(t) - A(0) = A(t) \qquad \Rightarrow$$

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \mathcal{L}\left[A(t)\right] = \int_0^\infty e^{-st}A(t)dt = \frac{-1}{s}\int_0^\infty A(t)d\left(e^{-st}\right)$$

$$= \frac{-1}{s}\left[A(t)e^{-st}|_0^\infty - \int_0^\infty e^{-st}dA(t)\right] = \frac{1}{s}\int_0^\infty e^{-st}A'(t)dt = \frac{1}{s}\int_0^\infty e^{-st}f(t)dt = \frac{F(s)}{s}.$$

**E.g.** Find $\mathcal{L}^{-1}\left[\frac{1}{s(s^2+4)}\right]$. We can use the Laplace transform pair in the opposite direction.

**Ans:**

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+4)}\right] = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s}\frac{2}{s^2+2^2}\right] = \frac{1}{2}\int_0^t \sin(2\tau)d\tau = \frac{-1}{4}\cos(2\tau)|_0^t = \frac{1}{4}[1-\cos(2t)].$$

**6.9.2 Inverse transform of derivatives.**

Let $\mathcal{L}[f(t)] = F(s)$. Thus,

$$\boxed{\mathcal{L}\left[(-t)f(t)\right] = F'(s) \quad \text{or} \quad \mathcal{L}^{-1}\left[F'(s)\right] = (-t)f(t).}$$

**E.g.** Calculate $\mathcal{L}^{-1}\left[\frac{2s}{(s^2+1)^2}\right]$.

**Ans:**

$$\mathcal{L}^{-1}\left[\frac{2s}{(s^2+1)^2}\right] = \mathcal{L}^{-1}\left[-\left(\frac{1}{s^2+1}\right)'\right] = -\mathcal{L}^{-1}\left[\left(\frac{1}{s^2+1}\right)'\right] = -\left[(-t)\sin t\right] = t\sin t.$$

More generally, we have

$$\boxed{\mathcal{L}\left[(-t)^n f(t)\right] = F^{(n)}(s) \quad \text{or} \quad \mathcal{L}^{-1}\left[F^{(n)}(s)\right] = (-t)^n f(t).}$$

## 6.10 Impulse/Dirac delta function

Let's define a square-wave function centred at $t = 0$ with unit area under it.

$$I_a(t) = \begin{cases} \frac{1}{2a}, & \text{if } |t| < a, \\ 0, & \text{if } |t| \geq a, \end{cases}$$

where $a > 0$.

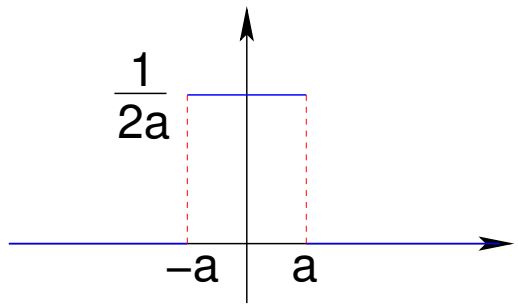



Figure 33: An impulse of width $2a$ and height $\frac{1}{2a}$ for $0 < a << 1$.

Notice that

$$\text{The area under the curve} = \int_{-\infty}^{\infty} I_a(t)dt = \int_{-a}^{a} \frac{1}{2a}dt = 1.$$

**A Dirac delta/impulse function** is defined by making such a function infinitely high and infinitely narrow such that the area under the curve remains one.

$$\delta(t) \equiv \lim_{a\to 0} I_a(t) = \begin{cases} \infty, & \text{if } t = 0, \\ 0, & \text{elsewhere.} \end{cases} \tag{79}$$

Important properties of an impulse function:

(1) Area under the curve is equal to 1.

$$\int_{-\infty}^{\infty} \delta(t)dt = \int_{-\infty}^{\infty} \lim_{a\to 0} I_a(t)dt = \lim_{a\to 0} \int_{-a}^{a} I_a(t)dt = \lim_{a\to 0} \int_{-a}^{a} \frac{1}{2a}dt = \lim_{a\to 0} 1 = 1.$$

(2) Can be shifted to give $\delta(t - \tau)$

$$\delta(t-\tau) \equiv \lim_{a\to 0} I_a(t-\tau) = \begin{cases} \infty, & \text{if } t = \tau, \\ 0, & \text{elsewhere,} \end{cases}$$

which also has an area of 1 under it.

$$\int_{-\infty}^{\infty} \delta(t-\tau)dt = \int_{\tau-a}^{\tau+a} \lim_{a\to 0} \frac{1}{2a}dt = \lim_{a\to 0} \int_{\tau-a}^{\tau+a} \frac{1}{2a}dt = 1.$$

(3)

$$\int_{-\infty}^{\infty} \delta(t-\tau) f(t) dt = f(\tau), \qquad (f(t) \text{ is continuous}).$$

**Proof:**

$$\int_{-\infty}^{\infty} \delta(t-\tau) f(t) dt = \int_{\tau-a}^{\tau+a} \lim_{a\to 0} \frac{1}{2a} f(t) dt = \lim_{a\to 0} \frac{1}{2a} \int_{\tau-a}^{\tau+a} f(t) dt$$

$$\overbrace{=}^{F'(t)=f(t) \text{ or } F(t)=\int f(t)dt} \lim_{a\to 0} \frac{1}{2a} \left[F(\tau+a) - F(\tau-a)\right]$$

$$= \lim_{a\to 0} \frac{1}{2a} \left[F(\tau+a) - F(\tau) + F(\tau) - F(\tau-a)\right]$$

$$= \frac{1}{2} \lim_{a\to 0} \left[\frac{F(\tau+a) - F(\tau)}{a} + \frac{F(\tau) - F(\tau-a)}{a}\right] = \frac{1}{2}[F'(\tau) + F'(\tau)] = f(\tau).$$

(4)

$$u'(t-\tau) = \delta(t-\tau).$$

**Proof:** It is easy to show that $u'(t-\tau)$ satisfies all the properties of $\delta(t-\tau)$ listed above.

(5) Any function centred at $t = 0$ with a unit area under it approaches a Dirac delta function when its height approaches infinity and width approaches zero. For example, when a Gaussian function becomes infinitely high and infinitely narrow, it becomes a Dirac delta function.

$$\delta(t-\tau) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\tau)^2}{2\sigma^2}}.$$

**Laplace transform of $\delta(t-\tau)$:**

$$\mathcal{L}[\delta(t-\tau)] = \int_0^{\infty} e^{-st} \delta(t-\tau) dt = e^{-s\tau}, \qquad \mathcal{L}[\delta(t)] = e^0 = 1.$$

**Transfer function:** Consider the following IVP

$$\begin{cases} ay'' + by' + cy = \delta(t), \\ y(0) = 0, \quad y'(0) = 0. \end{cases}$$

Apply Laplace transform to both sides,

$$(as^2 + bs + c)Y(s) = 1, \qquad \Longrightarrow \qquad Y(s) = \frac{1}{as^2 + bs + c} \equiv H(s),$$

where $H(s)$ is referred to as the *transfer function*. $h(t) = \mathcal{L}^{-1}[H(s)]$ is the solution to the IVP above.

**Theorem 4.7.2:** Can now be restated as follows. For

$$\begin{cases} ay'' + by' + cy = g(t), \\ y(0) = y_0, \quad y'(0) = y_0', \end{cases}$$

after applying LT to both sides, we obtain

$$\overbrace{(as^2 + bs + c)Y}^{ay''+by'+cy} = \overbrace{(as+b)y_0 + ay_0'}^{\text{ICs}} + \overbrace{G(s)}^{\text{g(t)}}$$

which leads to

$$Y(s) = \frac{1}{as^2 + bs + c}\left[\overbrace{(as+b)y_0 + ay_0'}^{\text{ICs}} + \overbrace{G(s)}^{\text{g(t)}}\right] = H(s)\left[\overbrace{(as+b)y_0 + ay_0'}^{\text{ICs}} + \overbrace{G(s)}^{\text{g(t)}}\right]$$

$$= H(s)[(as+b)y_0 + ay_0'] + H(s)G(s),$$

where the first terms gives the solution to the homogenenous ODEs with the given ICs, the second gives one particular solution to the nonhomogeneous ODE with zero/trivial ICs.

## 6.11 Convolution product

**Definition of convolution:** Suppose that $f(t)$, $g(t)$, $h(t)$ are continuous function and $c$ is a constant, convolution product is defined as

$$f(t) * g(t) \equiv \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau.$$

Convolution obeys the following rules:

(i) $f * g = g * f$, (commutative);

(ii) $f * (g + h) = f * g + f * h$, (distributive);

(iii) $f * (g * h) = (f * g) * h$, (associative);

(iv) $c(f * g) = (cf) * g = f * (cg)$;

(v) $f * 0 = 0 * f = 0$.

**Theorem 4.10.1 Laplace transform of convolution product:**

If $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$ both exist for $s > a \geq 0$, then for

$$h(t) = f(t) * g(t), \qquad H(s) = \mathcal{L}[h(t)] = \mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)] = F(s)G(s), \quad (s > a).$$

Therefore, $f(t) * g(t) \leftrightarrow F(s)G(s)$ form a Laplace transform pair.

**Proof:** See textbooks.

**Remark:** The LT transform of the convolution product $f * g$ is the product of the respective transforms $F(s)G(s)$, the inverse transform of the product $F(s)G(s)$ is the convolution product of the respective inverses $f * g$. Besides many other uses of this theorem, it definitely makes many tough inverse LTs easier to calculate.

**Example 4.10.1:** Given that $H(s) = \frac{1}{(s^2+1)^2}$ (the partial fraction $\frac{1}{(s^2+1)^2} = \frac{1}{2}[\frac{1}{s^2+1} + \left(\frac{s}{s^2+1}\right)']$ is not easy to figure out), find $\mathcal{L}^{-1}[H(s)]$ using the convolution product.

**Answer:**

$$\mathcal{L}^{-1}[H(s)] = \mathcal{L}^{-1}[\frac{1}{s^2+1}\frac{1}{s^2+1}] = \mathcal{L}^{-1}[\frac{1}{s^2+1}] * \mathcal{L}^{-1}[\frac{1}{s^2+1}] = \sin t * \sin t = \int_0^t \sin\tau \sin(t-\tau) d\tau$$

Notice that

$$\sin(t-\tau) = \sin t \cos\tau - \cos t \sin\tau.$$

Now the above convolution becomes

$$\sin t * \sin t = \sin t \int_0^t \sin\tau \cos\tau d\tau - \cos t \int_0^t \sin^2\tau d\tau = \frac{1}{2}\sin^3 t - \frac{1}{2}t\cos t + \frac{1}{4}\cos t \sin 2t.$$

Further simplification might be possible using trig identities (exercise for you). This is the solution of the following IVP:

$$\begin{cases} y'' + y = \sin t, \\ y(0) = 0, \quad y'(0) = 0, \end{cases}$$

where the forcing term resonates with the intrinsic harmonic oscillations.

**Remark:** Many inverse Laplace transforms that were difficult to solve using partial fractions can be solved with convolution product. The trade off is to calculate the convolution integral which could be challenging for some problems.

**Example 4.10.2:** Use convolution product to solve for $\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2(s-1)}\right]$.

**Answer:** The partial fraction for this function is

$$Y(s) = \frac{1}{(s+1)^2(s-1)} = \frac{1}{4}\left[\frac{1}{s-1} - \frac{s+3}{(s+1)^2}\right] = \frac{1}{4}[\frac{1}{s-1} - \frac{1}{s+1} - \frac{2}{(s+1)^2}]$$

which is not really easy for some students. Now, using convoltion

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[\frac{1}{(s+1)^2}] * \mathcal{L}^{-1}[\frac{1}{s-1}] = te^{-t} * e^{t} = \int_0^t \tau e^{-\tau} e^{t-\tau} d\tau = e^t \int_0^t \tau e^{-2\tau} d\tau$$

$$\overbrace{=}^{\text{integration by parts}} -\frac{1}{2}e^t\left[te^{-2t} + \frac{1}{2}e^{-2t} - \frac{1}{2}\right] = \frac{1}{4}\left[e^t - e^{-t} - 2te^{-t}\right].$$

**Remark:** In this question, the convolution integral is not difficult at all. Whether you choose partial fractions or convolution is your own choice based on your preferences.

**Example 4.10.3:** Solve the following IVP for any function $g(t)$ whose LT $G(s) = \mathcal{L}[g(t)]$ exists.

$$\begin{cases} y'' + y = g(t) \\ y(0) = 3, \quad y'(0) = -1. \end{cases}$$

**Answer:** Based on Theorem 4.7.2:

$$Y(s) = H(s)[(as+b)y_0 + ay_0'] + H(s)G(s) = \frac{3s-1}{s^2+1} + \frac{1}{s^2+1}G(s).$$

Therefore, the solution is

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \overbrace{3\cos t - \sin t}^{\text{IC-related}} + g(t) * \sin t.$$

**Remark:** Now, changing the forcing term does not change the parts of the solution that is related to the ICs. For any given $g(t)$, we just need to calculate the corresponding convolution integral to get the answer.