Main purpose:

To introduce some basic knowledge of complex numbers to students so that they are prepared to handle complex-valued roots when solving the characteristic polynomials for eigenvalues of a matrix.

Eg: In high school, students learned that the roots of a quadratic equation \( ax^2 + bx + c = 0 \) \( (a \neq 0) \) are given by

\[
x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

where the sign of the discriminant \( \Delta = b^2 - 4ac \) determines the following three outcomes

\[
\begin{align*}
\Delta > 0, & \quad 2 \text{ real roots;} \\
\Delta = 0, & \quad 1 \text{ (repeated) real root;} \\
\Delta < 0, & \quad \text{no real root.}
\end{align*}
\]

When complex-valued roots are allowed as in the case when solving eigenvalues, however, a polynomial of degree \( n \) always has \( n \) roots (Gauss’ Fundamental Theorem of Algebra), of which some or all of them can be identical (repeated). Thus, a quadratic equation always has 2 roots irrespective of the sign of \( \Delta \).
5.1 Definitions and basic concepts

The imaginary number $i$:

\[ i \equiv \sqrt{-1} \iff i^2 = -1. \quad (1) \]

Every *imaginary number* is expressed as a real-valued multiple of $i$:

\[ \sqrt{-9} = \sqrt{9} \sqrt{-1} = \sqrt{9i} = 3i. \]

A complex number:

\[ z = a + bi, \quad (2) \]

where $a$, $b$ are real, is the sum of a real and an imaginary number.

The real part of $z$: $Re\{z\} = a$ is a real number.

The imaginary part of $z$: $Im\{z\} = b$ is a also a real number.
A complex number represents a point \((a, b)\) in a 2D space, called the complex plane. Thus, it can be regarded as a 2D vector expressed in form of a number/scalar. Therefore, there exists a one-to-one correspondence between a 2D vectors and a complex numbers.

![Figure 1: A complex number \(z\) and its conjugate \(\bar{z}\) in complex space. Horizontal axis contains all real numbers, vertical axis contains all imaginary numbers.](image)

The complex conjugate:

\[ \bar{z} = a - bi, \]

which is obtained by reversing the sign of \(\text{Im}\{z\}\).

Notice that:

\[ \text{Re}\{z\} = \frac{z + \bar{z}}{2} = a, \quad \text{Im}\{z\} = \frac{z - \bar{z}}{2i} = b. \]

Therefore, both \(\text{Re}\{z\}\) and \(\text{Im}\{z\}\) are linear combinations of \(z\) and \(\bar{z}\).
5.2 Basic computations between complex numbers

Addition/subtraction:

If \( z_1 = a_1 + b_1i \), \( z_2 = a_2 + b_2i \) (\( a_1, a_2, b_1, b_2 \in \mathbb{R} \)), then

\[
z_1 \pm z_2 = (a_1 + b_1i) \pm (a_2 + b_2i) = (a_1 \pm a_2) + (b_1 \pm b_2)i.
\]

Or, real parts plus/minus real parts, imaginary parts plus/minus imaginary parts.

Multiplication by a real scalar \( \alpha \):

\[
\alpha z_1 = \alpha a_1 + \alpha b_1i.
\]

Multiplication between complex numbers:

\[
z_1z_2 = (a_1+b_1i)(a_2+b_2i) = a_1a_2+a_1b_2i+a_2b_1i+b_1b_2i^2 = (a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)i.
\]

All rules are identical to those of multiplication between real numbers, just remember that \( i^2 = -1 \).
**Length/magnitude** of a complex number $z = a + bi$

$$|z| = \sqrt{zz^\ast} = \sqrt{(a + bi)(a - bi)} = \sqrt{a^2 + b^2},$$

which is identical to the length of a 2D vector $(a, b)$.

**Division between complex numbers:**

$$\frac{z_1}{z_2} = \frac{z_1z_2^\ast}{|z_2|^2} = \frac{(a_1 + b_1i)(a_2 - b_2i)}{|z_2|^2} = \frac{(a_1a_2 + b_1b_2) + (a_2b_1 - a_1b_2)i}{a_2^2 + b_2^2}.$$

**Eg 5.2.1** Given that $z_1 = 3 + 4i$, $z_2 = 1 - 2i$, calculate

1. $z_1 - z_2$;
2. $\frac{z_1}{2}$;
3. $|z_1|$;
4. $\frac{z_2}{z_1}$.

**Ans:**

1. $z_1 - z_2 = (3 - 1) + (4 - (-2))i = 2 + 6i$;
2. $\frac{z_1}{2} = \frac{3}{2} + \frac{4}{2}i = 1.5 + 2i$;
3. $|z_1| = \sqrt{z_1z_1^\ast} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$;
4. $\frac{z_2}{z_1} = \frac{z_2z_1^\ast}{z_1z_1^\ast} = \frac{(1-2i)(3-4i)}{5^2} = \frac{-5-10i}{25} = -\frac{1}{5} - \frac{2}{5}i$. 

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5.3 Complex-valued exponential and Euler’s formula

Euler’s formula:
\[ e^{it} = \cos t + i \sin t. \]  \hspace{1cm} (3)

Based on this formula and that \( e^{-it} = \cos(-t) + i \sin(-t) = \cos t - i \sin t \):

\[ \cos t = \frac{e^{it} + e^{-it}}{2}, \quad \sin t = \frac{e^{it} - e^{-it}}{2i}. \]  \hspace{1cm} (4)

Why? Here is a way to gain insight into this formula.

Recall the Taylor series of \( e^t \):
\[ e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}. \]

Suppose that this series holds when the exponent is imaginary.

\[ e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n \text{ even}} \frac{(it)^n}{n!} + \sum_{n \text{ odd}} \frac{(it)^n}{n!} = \sum_{m=0}^{\infty} \frac{(it)^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(it)^{2m+1}}{(2m+1)!} \]

\[ = \sum_{m=0}^{\infty} \frac{i^{2m}t^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{i^{2m+1}t^{2m+1}}{(2m+1)!} = \sum_{m=0}^{\infty} \frac{(i^2)^mt^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{i(i^2)^mt^{2m+1}}{(2m+1)!} \]
\[= \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m+1}}{(2m + 1)!} = \cos t + i \sin t.\]

**Remarks:**

- Sine and cosine functions are actually linear combinations of exponential functions with imaginary exponents.
- Similarly, hyperbolic sine and cosine functions are linear combinations of exponential functions with real exponents.

\[\sinh(t) = \frac{e^t - e^{-t}}{2}, \quad \cosh(t) = \frac{e^t + e^{-t}}{2}.\]
5.4 Polar representation of complex numbers

For any complex number $z = x + iy \ (\neq 0)$, its length and angle w.r.t. the horizontal axis are both uniquely defined.

![Diagram](image)

Figure 2: A complex number $z = x + iy$ can be expressed in the polar form $z = \rho e^{i\theta}$, where $\rho = \sqrt{x^2 + y^2}$ is its length and $\theta$ the angle between the vector and the horizontal axis. The fact $x = \rho \cos \theta$, $y = \rho \sin \theta$ are consistent with Euler’s formula $e^{i\theta} = \cos \theta + i \sin \theta$.

One can convert a complex number from one form to the other by using the Euler’s formula:

$$z = x + iy \iff z = \rho e^{i\theta}, \quad \text{where}$$

$$x = \rho \cos \theta, \quad y = \rho \sin \theta; \quad \rho = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \frac{y}{x},$$

where we often restrict $0 \leq \theta \leq 2\pi$ or $-\pi \leq \theta \leq \pi$. Otherwise, the conversion from Cartesian to polar coordinates is not unique, $\theta$ can differ by an integer multiple of $2\pi$. 

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Eg 5.4.1 Convert the following complex numbers from one form to the other.

1. \( z = 3i \);
2. \( z = 1 \);
3. \( z = 1 + i\sqrt{3} \);
4. \( z = -2 - 2i \);
5. \( z = e^{-i\frac{\pi}{6}} \);
6. \( z = 5e^{i\frac{\pi}{4}} \);
7. \( z = -5e^{-i\frac{\pi}{3}} \).

Ans:

1. \( z = 3i = 3e^{i\frac{\pi}{2}} \);
2. \( z = 1 = e^{i0} = 1 \), (for a positive real number, there is no change!);
3. \( z = 1 + i\sqrt{3} = \sqrt{1^2 + (\sqrt{3})^2}e^{i\tan^{-1}\frac{\sqrt{3}}{1}} = 2e^{i\frac{\pi}{3}} \);
4. \( z = -2 - 2i = \sqrt{(-2)^2 + (-2)^2}e^{i\tan^{-1}\left(-\frac{2}{2}\right)} = 2\sqrt{2}e^{i\frac{5\pi}{4}} = 2\sqrt{2}e^{-i\frac{3\pi}{4}} \);
5. \( z = e^{-i\frac{\pi}{6}} = \cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - i\frac{1}{2} \);
6. \( z = 6e^{i\frac{\pi}{4}} = 6\cos\frac{\pi}{4} + i6\sin\frac{\pi}{4} = 3\sqrt{2} + 3\sqrt{2}i = 3\sqrt{2}(1 + i) \);
7. \( z = -4e^{-i\frac{\pi}{3}} = (-4)\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] = (-4)\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] = -2 + 2\sqrt{2}i \).
**Remark:** It is important to know that the collection of all complex numbers of the form $z = e^{i\theta}$ form a circle of radius one (unit circle) in the complex plane centered at the origin. In other words, the equation for a unit circle centered at the origin in complex plane is $z = e^{i\theta}$ (see figure).

![Figure 3: The collection of all complex numbers of the form $z = e^{i\theta}$ form a unit circle centered at the origin in the complex plane.](image)

**Remark:** Rotation of a vector represented by a complex number $z = \rho e^{i\theta}$ counter-clockwise by angle $\phi$ is achieved by multiplying $e^{i\phi}$ to it:

$$e^{i\phi} z = e^{i\phi} \rho e^{i\theta} = \rho e^{i(\theta + \phi)}.$$  

**Remark:** The product between $z_1 = \rho_1 e^{i\theta_1}$ and $z_2 = \rho_2 e^{i\theta_2}$ yields

$$z_1 z_2 = \rho_1 e^{i\theta_1} \rho_2 e^{i\theta_2} = (\rho_1 \rho_2) e^{i(\theta_1 + \theta_2)}$$

which is a vector of length $\rho_1 \rho_2$ and an angle $\theta_1 + \theta_2$. 
Eg 5.4.2 Find all roots of \(3\sqrt{1}\), complex and real.

Ans:

\[ 1 = e^0 = e^{i(0)} = e^{(2\pi n)i}, \quad \text{for any integer } n. \]

Therefore,

\[ 3\sqrt{1} = 1^{\frac{1}{3}} = \left(e^{(2\pi n)i}\right)^{\frac{1}{3}} = e^{\frac{2\pi n}{3}i}, \quad \text{for any integer } n. \]

Often, the angle \(\theta\) for a complex number expressed in form of \(e^{\theta i}\) is restricted in the range \(0 \leq \theta < 2\pi\).

If so, \(3\sqrt{1}\) has only three roots in this range: \(3\sqrt{1} = 1, \quad e^{\frac{2\pi}{3}i}, \quad e^{\frac{4\pi}{3}i}.\)

Figure 4: The cubic roots of number 1 in complex plane.
5.5 Polynomials of degree \( n \) must have \( n \) roots!

**Eg 5.5.1** Find all roots of \( z^2 + 2z + 10 = 0 \).

**Ans:** Notice that

\[
z^2 + 2z + 10 = z^2 + 2z + 1 + 9 = (z + 1)^2 + 9 = 0.
\]

There is no real root! But there are two complex-valued roots forming a pair of complex conjugates.

\[
(z + 1)^2 + 9 = 0 \Rightarrow (z + 1)^2 = -9 \Rightarrow z + 1 = \pm \sqrt{-9} \Rightarrow z = -1 \pm 3i.
\]

**Final remarks:**

(a) Any polynomial of degree \( n \) can always be factored into the product of \( n \) terms in which \( z_i \), \( (i = 1, \ldots, n) \) are the \( n \) roots.

\[
P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = a_n (z-z_n)(z-z_{n-1}) \cdots (z-z_1).
\]

(b) Complex roots of a polynomial always occur as a pair of complex conjugates: \( z_\pm = a \pm bi \).