2.1 Scalars vs Vectors

**Scalar:** Any number in \( \mathbb{R} \) is referred to as a scalar, where \( \mathbb{R} \) is the set of all real numbers. A scalar is supposed to have a defined magnitude.

Note that: the concept can be extended to refer to any number in \( \mathbb{C} \) as a *complex scalar*, where \( \mathbb{C} \) is the set of all complex numbers.

**Vector:** Any quantity determined by two or more scalars arranged in predetermined order. A vector is supposed to have both a defined magnitude and direction.

**Eg 2.1.1:** A point in \( \mathbb{R}^2 \) (2D space) is a 2D vector.

![2D Vector](image)

**Eg 2.1.2:** A point in \( \mathbb{R}^3 \) (3D space) is a 3D vector.
Eg 2.1.3: Midterm performance of a student in Math 152 can be defined as a 19D vector.

\[(hw_1, \cdots, hw_{11}, lb_1, \cdots, lb_6, md_1, md_2) \in \mathbb{R}^{19}\]

where \(hw_1, \cdots, hw_{11}\) are the 11 homework assignment marks, \(lb_1, \cdots, lb_6\) the 6 labwork marks, and \(md_1, md_2\) the 2 midterm exam marks.

Remarks:

(i) Each scalar in a vector is called a component (or entry), e.g. the scalar 2 in vector \((3, 1, 2)\) is the 3rd component of the vector.

(ii) The number of components in a vector = dimension of the vector.

(iii) The order/position of each component in a vector must be consistent among all vectors, e.g. if \((2, -1)\) is a vector describing a point in a 2D space, 2 (but not \(-1\)) must be the \(x\) coordinate of the point.
In physics and geometry:

A vector is referred to as a quantity with both a magnitude and a direction.

**Question:** How to determine the magnitude and direction of a vector?

**Ans:** Given any vector \( \vec{x} = (x, y) \neq (0, 0) \in \mathbb{R}^2 \), draw an arrowed line connecting \((0, 0)\) to \((x, y)\) (see figure below), length of the line represents the magnitude while the arrow direction represents the direction of the vector.
Eg 2.1.4: \( \vec{a} = (1, 1) \) (or \([1, 1]\)). Its direction is 45° (see figure) and its magnitude, represented by double vertical bars

\[
||\vec{a}|| = \sqrt{1^2 + 1^2} = \sqrt{2},
\]

where \( ||\vec{a}|| \) defines the magnitude/length of the vector \( \vec{a} \). (Note that it differs from the notation \( |a| \) which denotes the magnitude or absolute value of a scalar \( a \).)

• Usually, \((1, 1)\) can be regarded as the location of the arrow head relative to \((0, 0)\) when the vector is drawn as an arrow connecting \((0, 0)\) and \((1, 1)\).

• But all arrows with identical direction and magnitude represent the same vector irrespective of the location of the arrow tail (see figure!)

• **Question**: what is the meaning of \((1, 1)\) when the vector is represented by the red arrow in the figure?  
  **Ans**: it represents the location of the arrow head relative to the location of the arrow tail!
Summary:

To graphically determine the direction and magnitude of any vector $\vec{a} = (a_1, a_2) \in \mathbb{R}^2$, one picks any starting point $T$ as the “tail” location, move $a_1$ units horizontally and $a_2$ units vertically to locate the ending point $H$ as the “head”, draw an arrowed line from $T$ to $H$. The arrow would point to the direction of $\vec{a}$ and the length of the line would represent its magnitude.
2.2 Vector addition and scalar multiplication

Let \( \vec{a} = (a_1, a_2) \), \( \vec{b} = (b_1, b_2) \) be vectors in \( \mathbb{R}^2 \), and \( c, d \) be scalars. Then,

(i) \( \vec{a} \pm \vec{b} = (a_1 \pm b_1, a_2 \pm b_2) \);
   i.e. adding/subtracting the corresponding components;

(ii) \( c\vec{a} = (ca_1, ca_2) \);
   i.e. multiplying each component by the scalar;

(iii) \( -\vec{a} = (-a_1, -a_2) \);

(iv) \( \vec{a} - \vec{a} = (0, 0) = \vec{0} \),
   \( \vec{0} \) is a vector with zero-length and no direction!
Properties of addition/scalar multiplication

Let $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{0}$ be vectors in $\mathbb{R}^n$ ($n \geq 2$), and $d$, $e$ be scalars. Then,

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutative);

2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$ (associative);

3. $\vec{a} + \vec{0} = \vec{a}$ (existence of zero);

4. $\vec{a} + (-\vec{a}) = \vec{a} - \vec{a} = \vec{0}$ (existence of negative);

5. $d(\vec{a} + \vec{b}) = d\vec{a} + d\vec{b}$ (distributive with scalar product);

6. $(d + e)\vec{a} = d\vec{a} + e\vec{c}$ (distributive with scalar product);

7. $1\vec{a} = \vec{a}$. 
Geometric meaning of vector addition/subtraction

Eg 2.2.1: Given that $\vec{a} = (2, 1)$, $\vec{b} = (1, 2)$. Demonstrate the geometric meaning of: (1) $\vec{c} = \vec{a} + \vec{b}$; (2) $\vec{c} = \vec{a} - \vec{b}$; (3) $\vec{c} = 2\vec{a}$, $\frac{1}{2}\vec{a}$; (4) $\vec{c} = -\vec{a}$.

Ans: (1) $\vec{c} = \vec{a} + \vec{b} = (2, 1) + (1, 2) = (3, 3)$.

- Plot $\vec{a}$ and $\vec{b}$ at identical starting point, they form two sides of a parallelogram (pgram). $\vec{c} = \vec{a} + \vec{b}$ is the diagonal vector of the pgram starting from the same point.
- Connecting $\vec{a}$ and $\vec{b}$ one-by-one, tail-to-head, irrespectively of order, the vector that connects the tail of the 1st and the head of the last represents the sum.
- When adding more than two vectors, one needs to connect them one-by-one, tail-to-head, irrespectively of order, the vector that connects the tail of the 1st and the head of the last vector represents the sum of all.
\( (2) \ \vec{c} = \vec{a} - \vec{b} = (2, 1) - (1, 2) = (1, -1). \)

- Plot \( \vec{a} \) and \( \vec{b} \) at identical starting point (i.e. join the tails of the two), they form two sides of a triangle. \( \vec{c} = \vec{a} - \vec{b} \) is the 3rd side of the triangle connecting the heads of the two, starting from the head of \( \vec{b} \) to that of \( \vec{a} \).

- Plot \( \vec{a} \) and \( \vec{b} \) at identical ending point (i.e. join the heads of the two), they form two sides of a triangle. \( \vec{c} = \vec{a} - \vec{b} \) is the 3rd side of the triangle connecting the tails of the two, starting from the tail of \( \vec{a} \) to that of \( \vec{b} \).
(c) \( \vec{c} = 2\vec{a} = (4, 2) \), \( \vec{c} = \frac{1}{2}\vec{a} = (1, \frac{1}{2}) \).

\[ \begin{array}{c}
\downarrow \\
\ \vec{a} \\
\uparrow \\
\end{array} \]

- \( \vec{c} = 2\vec{a} \) is a vector in the same direction as \( \vec{a} \) but with doubled length.

- \( \vec{c} = \frac{1}{2}\vec{a} \) is a vector in the same direction as \( \vec{a} \) but with half length.

- \( \vec{c} = \gamma\vec{a} \) (\( \gamma \neq 0 \) is scalar), a vector in the same direction as \( \vec{a} \) but with a length scaled by a factor \( \gamma \).

- Scalar multiplication of a vector does not change its direction but only its length.
(d) \( \vec{c} = -\vec{a} = (-2, -1). \)

- \( \vec{c} = -\vec{a} \) has the same length as \( \vec{a} \) but with direction reversed.
- The negative of a vector is a vector of identical length but opposite in direction.
Eg 2.2.2: Equation of a circle of radius 1 centred at $\vec{c} = (1, 2)$.

![Diagram of a circle with center at (1, 2) and radius 1]

**Ans:** Pick any point on the circle $\vec{x} = (x, y)$, $\vec{x} - \vec{c}$ defines the radial vector (red) starting from the centre of the circle pointing to that point. Thus, its length must be equal to the radius, i.e. 1. Therefore,

$$||\vec{x} - \vec{c}|| = 1$$

describes the set of all points $\vec{x}$ whose distance from the centre is equal to 1. This is exactly the circle itself.

- Note that $||\vec{x} - \vec{c}|| = \sqrt{(x - 1)^2 + (y - 2)^2} = 1$ becomes the equation of circle $(x - 1)^2 + (y - 2)^2 = 1^2$ that we know previously.

- However, $||\vec{x} - \vec{c}|| = 1$ can also describe geometrically similar objects in higher dimensions. If $\vec{x}, \vec{c} \in \mathbb{R}^3$, then it describes a sphere of radius 1 centred at $\vec{c}$. If $\vec{x}, \vec{c} \in \mathbb{R}^4$, then it describes a hypersphere of radius 1 centred at $\vec{c}$. If $\vec{x}, \vec{c} \in \mathbb{R}^n$, then it describes a n-sphere of radius 1 centred at $\vec{c}$. If $\vec{x}, \vec{c} \in \mathbb{R}^{\infty}$, then it describes a supesphere of radius 1 centred at $\vec{c}$. If $\vec{x}, \vec{c} \in \mathbb{R}^\infty$, then it describes a supersphere of radius 1 centred at $\vec{c}$.
\( \mathbb{R}^n (n > 3) \), then it describes a hypersphere of radius 1 centred at \( \vec{c} \) in \( n \)-dimensional space.
2.3 Vectors in orthogonal coordinates

In orthogonal coordinate systems, each vector can be expressed as a *linear combination* of unit vectors representing the directions of the orthogonal axes, also referred to as the *basis vectors*.

In $\mathbb{R}^2$, the basis vectors are

$$\hat{i} = \vec{e}_1 = (1, 0), \quad \hat{j} = \vec{e}_2 = (0, 1).$$

Thus, any vector in $\mathbb{R}^2$ can be expressed as

$$\vec{a} = (a_1, a_2) = a_1(1, 0) + a_2(0, 1) = a_1\hat{i} + a_2\hat{j},$$

where $a_1, a_2$ are the coordinates of $\vec{a}$ in orthogonal basis $\hat{i}, \hat{j}$.
In $\mathbb{R}^3$, the basis vectors are

\[ \hat{i} = \vec{e}_1 = (1, 0, 0), \quad \hat{j} = \vec{e}_2 = (0, 1, 0), \quad \hat{k} = \vec{e}_3 = (0, 0, 1). \]

Thus, any vector in $\mathbb{R}^3$ can be expressed as

\[ \vec{b} = (b_1, b_2, b_3) = b_1(1, 0, 0) + b_2(0, 1, 0) + b_3(0, 0, 1) \]
\[ = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}. \]

where $b_1$, $b_2$, $b_3$ are the coordinates of $\vec{b}$ in orthogonal basis $\hat{i}$, $\hat{j}$, $\hat{k}$. 
2.4 Inner/dot product

Def: Let \( \vec{a} = (a_1, a_2, \cdots, a_n) \), \( \vec{b} = (b_1, b_2, \cdots, b_n) \), \( \vec{c} \) be vectors in \( \mathbb{R}^n \) and that \( d \) be a scalar. The inner/dot product is defined as

\[
\vec{a} \cdot \vec{b} \equiv a_1b_1 + a_2b_2 + \cdots + a_nb_n,
\]

which is a scalar.

Important properties:

(i) \( \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \) (commutative);
(ii) \( \vec{a} \cdot \vec{a} = ||\vec{a}||^2 = a_1^2 + a_2^2 + \cdots + a_n^2 \);
(iii) \( \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \) (distributive);
(iv) \( (d\vec{a}) \cdot \vec{b} = \vec{a} \cdot (d\vec{b}) = d(\vec{a} \cdot \vec{b}) \) (commutative with scalar);
(v) \( \vec{0} \cdot \vec{a} = \vec{a} \cdot \vec{0} = \vec{0} \) (existence of zero);

*vi) \( \vec{a} \cdot \vec{b} = ||\vec{a}|| ||\vec{b}|| \cos \theta, \quad \theta = \text{smallest angle between } \vec{a}, \vec{b} \);
(vii) \( \vec{a} \cdot \vec{b} = 0 \text{ iff } \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0} \text{ or } \vec{a} \perp \vec{b} \).

All results above can be proved using the definition. (vi) is equivalent to the Law of Cosines. Let’s prove this result which can also be regarded as a proof of the Law of Cosines.
**Proof:** $\vec{a} \cdot \vec{b} = ||\vec{a}|| ||\vec{b}|| \cos \theta$.

**Law of Cosines:**

$$c^2 = a^2 + b^2 - 2ab \cos \theta.$$  

Based on the definition and other properties of inner product

$$||\vec{c}||^2 = \vec{c} \cdot \vec{c} = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = ||\vec{a}||^2 + ||\vec{b}||^2 - 2\vec{a} \cdot \vec{b}.$$  

Based on Pythagorean Theorem (see figure)

$$||\vec{c}||^2 = (||\vec{b}|| \sin \theta)^2 + (||\vec{a}|| - ||\vec{b}|| \cos \theta)^2 = ||\vec{a}||^2 + ||\vec{b}||^2 - 2||\vec{a}|| ||\vec{b}|| \cos \theta.$$
The last expression above is actually the Law of Cosines. By comparing the two expressions for $||\vec{c}||^2$, we conclude

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| ||\vec{b}|| \cos \theta.$$ 

**Eg 2.4.1:** Given $\vec{a} = (2, 3), \vec{b} = (1, -3) \in \mathbb{R}^2$. Calculate (a) $2\vec{a} + 4\vec{b}$;
(b) $\vec{a} \cdot \vec{b}$;
(c) $||\vec{a}||, ||\vec{b}||$;
(d) $\cos \theta, \theta$.

**Ans:**
(a) $2\vec{a} + 4\vec{b} = 2(2, 3) + 4(1, -3) = (8, -6)$;
(b) $\vec{a} \cdot \vec{b} = (2, 3) \cdot (1, -3) = 2 - 9 = -7$;
(c) $||\vec{a}|| = \sqrt{2^2 + 3^2} = \sqrt{13}, \ ||\vec{b}|| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$;
(d) $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}|| ||\vec{b}||} = \frac{-7}{\sqrt{130}}, \ \theta \approx 127.875^\circ \approx 2.23184 \text{ rad.}$
**Eg 2.4.2:** Find the angle between vectors connecting the centre of a cube to two neighbouring vertices.

**Ans:** Consider a cube of length 2 centred at $\vec{0}$.

Pick two vertices located at:

$$\vec{a} = (1, 1, 1), \quad \vec{b} = (1, -1, 1).$$

Applying the inner product law,

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}|| ||\vec{b}||} = \frac{1}{3} \Rightarrow \theta = \cos^{-1} \frac{1}{3} \approx 1.231 \text{ rad } \approx 70.5^\circ.$$ 

Using the Law of Cosines, we get the same result

$$2^2 = (\sqrt{3})^2 + (\sqrt{3})^2 - 2(\sqrt{3})^2 \cos \theta \Rightarrow \cos \theta = \frac{2}{6} = \frac{1}{3}.$$
2.5 Dot product and projection

**Def:** Projection of \( \vec{a} \) on \( \vec{b} \), denoted by \( \text{Proj}_b \vec{a} \), is defined as a vector in direction of \( \vec{b} \) with a length that is equal to the “shadow” of \( \vec{a} \) on \( \vec{b} \) (see figure).

Thus,

\[
\text{Proj}_b \vec{a} = \text{("shadow" length of } \vec{a} \text{ on } \vec{b}) \text{ (direction of } \vec{b})
\]

\[
= (||\vec{a}|| \cos \theta) \frac{\vec{b}}{||\vec{b}||} = (\vec{a} \cdot \vec{b}) \frac{\vec{b}}{||\vec{b}||^2}.
\]

This give the universal projection formula,

\[
\text{Proj}_b \vec{a} = (\vec{a} \cdot \vec{b}) \frac{\vec{b}}{||\vec{b}||^2}.
\]
If $\vec{b} = \vec{u}$ is a unit vector (i.e. $||\vec{u}|| = 1$), then

$$\text{Proj}_{\vec{u}} \vec{a} = (\vec{a} \cdot \vec{u}) \vec{u}. $$

**Eg 2.5.1:** Given $\vec{a} = (2, 3)$, $\vec{b} = (1, -3) \in \mathbb{R}^2$, find $\text{Proj}_{\vec{a}} \vec{b}$.

**Ans:** Using the projection formula

$$\text{Proj}_{\vec{a}} \vec{b} = (\vec{a} \cdot \vec{b}) \frac{\vec{a}}{||\vec{a}||^2} = \frac{-7}{13} \vec{a} = \left(\frac{-14}{13}, \frac{-21}{13}\right).$$

**Eg 2.5.2:** Force experienced by a pendulum parallel and perpendicular to the direction of motion.

**Ans:**

$$\vec{F}_\parallel = \text{Proj}_{\vec{u}_\parallel} \vec{F} = (\vec{F} \cdot \vec{u}_\parallel) \vec{u}_\parallel$$
$$= -mg \sin \theta \vec{u}_\parallel;$$
\[ \vec{F}_\perp = \text{Proj}_{\vec{u}_\perp} \vec{F} = (\vec{F} \cdot \vec{u}_\perp)\vec{u}_\perp = -mg \cos \theta \vec{u}_\perp. \]
2.6 Dot product, area of parallelograms (pgrams), and matrix determinants

Let $A=$ area of the pgram with sides $\vec{a}$, $\vec{b}$.

\[
A = (\text{base})(\text{height}) = ||\vec{a}|| ||\vec{b}|| \sin \theta = ||\vec{a}|| ||\vec{b}|| \cos(\frac{\pi}{2} - \theta).
\]

Let $\vec{a}_\perp$ be a vector perpendicular to $\vec{a}$ with identical length, i.e. $\vec{a}_\perp \cdot \vec{a} = 0$ and $||\vec{a}_\perp|| = ||\vec{a}||$. Based on the figure,

\[
A = ||\vec{a}|| ||\vec{b}|| \cos(\frac{\pi}{2} - \theta) = \vec{a}_\perp \cdot \vec{b},
\]

note that $\frac{\pi}{2} - \theta$ is the angle between between $\vec{a}_\perp$ and $\vec{b}$.

If $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$, then $\vec{a}_\perp = (-a_2, a_1)$. Thus,

\[
A = \vec{a}_\perp \cdot \vec{b} = a_1 b_2 - a_2 b_1.
\]
**Def:** A matrix is an array of scalars typically enclosed by square or round brackets.

**Eg 2.6.1:**

\[
\begin{bmatrix}
2 & 1 \\
-3 & 0 \\
1 & 4
\end{bmatrix}
\text{ or }
\begin{pmatrix}
2 & 1 \\
-3 & 0 \\
1 & 4
\end{pmatrix}
\text{ is a } 3 \times 2 \text{ matrix with 3 rows and 2 columns.}
\]

**Def:** A vector is a column or a row of scalars, or a special matrix with only one column or one row.

**Def:** A scalar is a \(1 \times 1\) matrix with one row and one column.

**Def:** For a \(2 \times 2\) matrix, the determinant is defined as

\[
\text{det} \begin{bmatrix}
\text{a}_1 & \text{a}_2 \\
\text{b}_1 & \text{b}_2
\end{bmatrix} = \left| \begin{array}{cc}
\text{a}_1 & \text{a}_2 \\
\text{b}_1 & \text{b}_2
\end{array} \right| = \text{a}_1\text{b}_2 - \text{a}_2\text{b}_1.
\]

Therefore, the area of a pgram with sides \(\vec{a}, \vec{b}\) is

\[
A = \vec{a}_\perp \cdot \vec{b} = \text{det} \begin{bmatrix}
\text{a}_1 & \text{a}_2 \\
\text{b}_1 & \text{b}_2
\end{bmatrix} = \text{abs} \left( \begin{array}{cc}
\text{a}_1 & \text{a}_2 \\
\text{b}_1 & \text{b}_2
\end{array} \right).
\]
**Eg:** Calculate the area of the pgram sided with $\vec{a} = (-1, 2)$ and $\vec{b} = (1, 3)$.

**Ans:**

$$A = \text{abs} \left( \begin{array}{cc} -1 & 2 \\ 1 & 3 \end{array} \right) = \text{abs}(-3 - 2) = 5.$$  

**Similarly**, one can calculate the volume of a parallelepiped (ppiped) with edges $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, and $\vec{c} = (c_1, c_2, c_3)$ in 3D space.

$$V = \text{det} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{abs} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$  

(Proof will come later!)
**Def:** For a $3 \times 3$ matrix, the determinant is defined as:

(1) in row expansion formula

\[
\text{det} \begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix}
\equiv
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix}
\equiv
a_1 \begin{vmatrix}
  b_2 & b_3 \\
  c_2 & c_3 \\
\end{vmatrix}
- a_2 \begin{vmatrix}
  a_1 & b_3 \\
  c_1 & c_3 \\
\end{vmatrix}
+ a_3 \begin{vmatrix}
  a_1 & b_2 \\
  c_1 & c_2 \\
\end{vmatrix}
\]

(2) in diagonal product formula

\[
\text{det} \begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix}
\equiv
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix}
\equiv
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix}
\equiv
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix}
\]

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Eg 2.6.2: Calculate the volume of the ppiped with edges

(1) \( \vec{a} = (3, 0, 0), \vec{b} = (2, 2, 0), \vec{c} = (1, -1, -1); \)

(2) \( \vec{a} = (1, 2, 3), \vec{b} = (2, 3, 4), \vec{c} = (1, 1, 1); \)

(3) \( \vec{a} = (1, 2, 3), \vec{b} = (2, 1, 2), \vec{c} = (1, 1, 1). \)

Ans:

(1)

\[
V_1 = \text{abs} \begin{vmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -1 & -1 \end{vmatrix} = \text{abs} \begin{vmatrix} 3 & 2 & 0 \\ -1 & -1 \end{vmatrix} = |-6| = 6.
\]

(2)

\[
V_2 = \text{abs} \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{vmatrix} = |3 + 6 + 8 - 9 - 4 - 4| = 0.
\]

This is because \( \vec{c} = \vec{b} - \vec{a} \) which means that the 3 vectors are in the same plane. \textit{Note that the determinant is zero if two rows or columns are identical or one is the linear combination of the other two.}

(3)

\[
V_3 = \text{abs} \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = |1 + 4 + 6 - 3 - 2 - 4| = 2.
\]
2.7 Cross product (between 3D vectors)

Def: Let \( \vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3) \) be vectors in \( \mathbb{R}^3 \). The cross product

\[
\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}
\]

\[
= \hat{i}(a_2b_2 - a_3b_2) - \hat{j}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_2 - a_2b_1)
\]

is a vector perpendicular to the plane span by \( \vec{a} \) and \( \vec{b} \) with a direction determined by the right-hand-rule (RHR).
Important properties: Let \( \vec{a}, \vec{b}, \vec{c} \) be vectors in \( \mathbb{R}^3 \), \( d \) be a scalar. Although we do not present proof of each of these properties, we can regard them as results rigorously proven and can be used in the proof of other theorems.

(1) \( \vec{a}, \vec{b}, \vec{a} \times \vec{b} \) are all vectors in \( \mathbb{R}^3 \);

*(2) \( \vec{a} \times \vec{b} \) is \( \perp \) to \( \vec{a}, \vec{b} \) following the RHR, i.e. \( \vec{a} \cdot (\vec{a} \times \vec{b}) = 0 = \vec{b} \cdot (\vec{a} \times \vec{b}) \);

*(3) \( ||\vec{a} \times \vec{b}|| = ||\vec{a}|| ||\vec{b}|| \sin \theta \) Area of pgram with sides \( \vec{a}, \vec{b} \), \( \theta \) = smallest angle between \( \vec{a}, \vec{b} \), (proof, p.28 course text);

(4) In a RH coordinate system:
\[
\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}.
\]

(5) \( \vec{a} \times \vec{b} = ||\vec{a}|| ||\vec{b}|| \sin \theta \vec{u}_\perp \), where
\( ||\vec{u}_\perp|| = 1 \) and \( \vec{u}_\perp \) is \( \perp \) to \( \vec{a}, \vec{b} \) following the RHR;

(6) \( \vec{a} \times \vec{b} = \vec{0} \) iff \( \vec{a} = \vec{0} \) or \( \vec{b} = \vec{0} \) or \( \vec{a} \parallel \vec{b} \);

(7) \( \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \);

(8) \( (d\vec{a}) \times \vec{b} = \vec{a} \times (d\vec{b}) = d(\vec{a} \times \vec{b}) \);

(9) \( \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \);

*(10) \( \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{b} \cdot (\vec{c} \times \vec{a}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \).
Proof of properties (10) and (2):

For (10), \( \vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot \left| \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right| \)

\[
= (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot \left[ \begin{array}{ccc} \hat{i} & b_2 & b_3 \\ \hat{j} & b_1 & b_3 \\ \hat{k} & b_1 & b_2 \end{array} \right] \\
= a_1 \left| \begin{array}{cc} b_2 & b_3 \\ c_2 & c_3 \end{array} \right| - a_2 \left| \begin{array}{cc} b_1 & b_3 \\ c_1 & c_3 \end{array} \right| + a_3 \left| \begin{array}{cc} b_1 & b_2 \\ c_1 & c_2 \end{array} \right| \\
= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.
\]

Similarly, one can show that

\[
(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{b} \cdot (\vec{c} \times \vec{a}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.
\]

For (2),

\[
\vec{a} \cdot (\vec{a} \times \vec{b}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0 \quad \text{(two identical rows!)}. 
\]

Similarly,

\[
\vec{b} \cdot (\vec{a} \times \vec{b}) = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0 \quad \text{(two identical rows!)}. 
\]

Thus, \( \vec{a} \) and \( \vec{b} \) are both \( \perp \) to \( \vec{a} \times \vec{b} \)!
**Theorem:** The volume of a ppiped with edges $\vec{a}$, $\vec{b}$, $\vec{c}$ is

$$V = \left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right| = \left| \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \right| = \text{abs} \left( \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \right).$$

**Proof:**

$$V = (\text{base area})(\text{height}) = ||\vec{b} \times \vec{c}|| ||\vec{a}|| \cos \theta = \vec{a} \cdot (\vec{b} \times \vec{c}).$$
**Eg 2.7.1:** Let \( \vec{a} = (1, 3, -2), \vec{b} = (-1, 2, 3), \vec{c} = (1, 1, 1). \) Find

1. area of the pgram with sides \( \vec{a} \) and \( \vec{b} \);
2. angle between \( \vec{a} \) and \( \vec{b} \);
3. volume of the ppipe with edges \( \vec{a}, \vec{b}, \vec{c} \).

**Ans:**

1. \( \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 2 & 3 \end{vmatrix} = 13\hat{i} - (1)\hat{j} + 5\hat{k} = (13, -1, 5). \)
   
   \[ A = ||\vec{a} \times \vec{b}|| = \sqrt{13^2 + (-1)^2 + 5^2} = \sqrt{195} \approx 13.96. \]

2. \( \cos \theta = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}|| ||\vec{b}||} = \frac{-1 + 6 - 6}{\sqrt{14}\sqrt{14}} = \frac{-1}{14}. \)
   
   \[ \theta = \cos^{-1} \left( \frac{-1}{14} \right) \approx 1.64 \text{ rad} \approx 94.10^\circ. \]

**Question:** why not use the formula \( ||\vec{a} \times \vec{b}|| = ||\vec{a}|| ||\vec{b}|| \sin \theta? \)

3. \( V = |\vec{a} \cdot (\vec{b} \times \vec{c})| = \abs{\begin{vmatrix} 1 & 3 & -2 \\ -1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix}} = 2 + 9 + 2 + 4 - 3 + 3 = 17. \)
2.8 Lines, curves, planes expressed in vector forms

2.8.1 Lines in 2D space

(1) Point-direction formula: Find the equation of line $L$ if one point on it and its direction are given. Let $\vec{p} = (p_1, p_2)$ be the point and $\vec{l} = (l_1, l_2)$ be the direction.

Based on the graph, any point $\vec{x} = (x, y)$ on $L$ should satisfy

\[ \vec{x} - \vec{p} = t\vec{l}, \quad \text{parametric form, } t \in \mathbb{R} \text{ is a parameter.} \]

**To put it in words:** the difference between any point $\vec{x}$ on $L$ and the point $\vec{p}$ lies in the same direction as $\vec{l}$.

Thus,

\[ \vec{x} = t\vec{l} + \vec{p}. \]
Remarks:

- When $t = 0$, $\vec{x} = \vec{p}$.
- When $t > 0$ and as $t$ increases, $\vec{x}$ moves away from $\vec{p}$ following the $\vec{l}$ direction.
- When $t < 0$ and as $t$ decreases, $\vec{x}$ moves away from $\vec{p}$ following the $-\vec{l}$ direction.
- If $\vec{p} = \vec{0}$, then line $L$ passes through the origin.
(2) **Point-normal formula:** Find the equation of line $L$ if one point on it and its normal direction are given. Let $\vec{p} = (p_1, p_2)$ be the point and $\vec{n} = (n_1, n_2)$ be the normal direction.

Based on the graph, any point $\vec{x} = (x, y)$ on $L$ should satisfy

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0.$$ 

To put it in words: *the difference between any point $\vec{x}$ on $L$ and the point $\vec{p}$ must be perpendicular to $\vec{n}$.*

**Remarks:**

- The point-normal formula does not involve a parameter.
- Given $\vec{l} = (l_1, l_2)$, one know that $\vec{n} = (l_2, -l_1)$ or $(-l_2, l_1)$.
- Given $\vec{n} = (n_1, n_2)$, one know that $\vec{l} = (n_2, -n_1)$ or $(-n_2, n_1)$.
**Eg 2.8.1:** Express the equation of line $L$ defined by $y = 3x + 7$ in two vector forms.

**Ans:** One needs a point on $L$: pick $\vec{p} = (0, 7)$ for simplicity.

One needs the direction of $L$ and its normal, given the slope $m = 3$,

$$\vec{l} = (1, 3), \quad \Rightarrow \quad \vec{n} = (3, -1).$$

Thus, in point-direction formula

$$\vec{x} = t\vec{l} + \vec{p} = t(1, 3) + (0, 7), \quad t \in \mathbb{R}.$$ 

Or

$$\begin{cases} 
    x = t, \\
    y = 3t + 7, \quad t \in \mathbb{R}.
\end{cases}$$

In point-normal formula,

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0, \quad \Rightarrow \quad (3, -1) \cdot (\vec{x} - (0, 7)) = 0.$$

Note that,

$$(3, -1) \cdot (x, y - 7) = 3x - y + 7 = 0, \quad \Rightarrow \quad y = 3x + 7.$$ 

Therefore, $y = 3x + 7$ is the actually simplified point-normal formula of the equation of $L$!
Eg 2.8.2: Consider line $L$:

$$\vec{x} = t\vec{l} + \vec{p} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} t + 3 \\ 2t - 2 \end{bmatrix}.$$  

Determine which of the following lines is parallel, perpendicular, or neither to $L$.

(a) $\vec{x} = \begin{bmatrix} 2t - 1 \\ -t + 3 \end{bmatrix}$; (b) $\vec{x} = \begin{bmatrix} -3t + 6 \\ -6t - 5 \end{bmatrix}$; (c) $\vec{x} = \begin{bmatrix} 2 + t \\ 1 - t \end{bmatrix}$.

Ans: We know that $\vec{l}_L = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(a) $\vec{l}_a = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, since $\vec{l}_a \cdot \vec{l}_L = 2 - 2 = 0$. $L_a$ is $\perp$ to $L$.

(b) $\vec{l}_b = \begin{bmatrix} -3 \\ -6 \end{bmatrix} = (-3) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -3\vec{l}_L$. $L_b$ is $\parallel$ to $L$.

(c) $\vec{l}_c = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{l}_c \cdot \vec{l}_L = 1 - 2 = -1$.

$$\cos \theta = \frac{\vec{l}_c \cdot \vec{l}_L}{||\vec{l}_c|| ||\vec{l}_L||} = \frac{-1}{\sqrt{10}} \Rightarrow \theta \approx 108.435^\circ.$$  

Thus, $L_c$ is at an angle $\theta \approx 108.435^\circ$ to line $L$.  

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2.8.2 Curves in 2D/ Curved surfaces in higher dimensions

(1) Circle of radius $R$ centred at $\vec{c}$.

$$||\vec{x} - \vec{c}|| = R, \quad (R > 0, \ R \in \mathbb{R}).$$

- In $\mathbb{R}^3$, it is a sphere of radius $R$ centred at $\vec{c}$.
- In $\mathbb{R}^n \ (n > 3)$, it is a “hypersphere” of radius $R$ centred at $\vec{c}$.

(2) Parabola with focus located at $\vec{f} = (0, p)$ and a directrix at $y = -p, \ (p > 0)$.

$$||\vec{x} - \vec{f}|| = y + p, \quad (p > 0, \ p \in \mathbb{R}).$$

In $\mathbb{R}^2: \quad ||\vec{x} - \vec{f}||^2 = x^2 + (y-p)^2 = (y+p)^2, \quad \Rightarrow \quad y = \frac{1}{4p}x^2.$

**Qusetion:** What does $||\vec{x} - \vec{f}|| = z + p$ represent in $\mathbb{R}^3$?
(3) Ellipse with foci located at $\vec{f}_1 = (-f, 0)$ and $\vec{f}_2 = (f, 0)$, $f^2 = a^2 - b^2$. 

$$||\vec{x} - \vec{f}_1|| + ||\vec{x} - \vec{f}_2|| = 2a, \quad (f, a, b > 0.)$$

In $\mathbb{R}^2$ it turns into: 

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ 

**Question:** How to derive the equation above? 

[ Hint: 

$$\sqrt{(x + f)^2 + y^2} + \sqrt{(x - f)^2 + y^2} = 2a.$$ 

Square both sides: 

$$X + \sqrt{X^2 - 4x^2f^2} = 2a^2,$$ 

where $X = x^2 + y^2 + f^2$. 

$$a^2X = a^4 - x^2f^2, \quad \Rightarrow \quad x^2(1 + \frac{f^2}{a^2}) + y^2 = a^2 - f^2$$ 

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ ]
2.8.3 Planes in 3D and intersections between them

(1) The point-normal formula of a plane in \( \mathbb{R}^3 \):

Let \( S \) be a plane in \( \mathbb{R}^3 \). Let \( \vec{p} \) be a point on \( S \) and \( \vec{n} \) be its normal.

Based on the graph, any point \( \vec{x} \) on \( S \) must satisfy

\[
\vec{n} \cdot (\vec{x} - \vec{p}) = 0 \quad \text{or} \quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}
\]

which is the point-normal formula of a plane in \( \mathbb{R}^3 \). In particular, if \( \vec{p} = \vec{0} \) (i.e. if the plane goes through the origin), the equation reduces to

\[
\vec{n} \cdot \vec{x} = 0.
\]
Eg 2.8.3: Find the equation of a plane with normal \( \vec{n} = (1, -1, 1) \) and a point \( \vec{p} = (0, 0, 6) \) on it.

Ans:

\[
\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p} = 6, \quad \iff \quad x - y + z = 6.
\]

Eg 2.8.4: For the plane \( x + 2y - 3z = 3 \), find its normal and a point on it.

Ans:

\[
x + 2y - 3z = (1, 2, -3) \cdot \vec{x} = \vec{n} \cdot \vec{x} \quad \Rightarrow \quad \vec{n} = (1, 2, -3).
\]

To find a point, we pick \( y = z = 0 \) for simplicity. Thus, \( x + 2y - 3z = 3 \Rightarrow x = 3 \) when \( y = z = 0 \). Therefore, \( \vec{p} = (3, 0, 0) \) is one point on the plane.

Eg 2.8.5: A plane \( S \), \( x - y + z = 6 \), and a point \( \vec{p}_0 = (2, 0, 1) \) are in \( \mathbb{R}^3 \).

(a) Show that \( \vec{p}_0 \) is not on \( S \).

(b) Find the (shortest) distance between \( \vec{p}_0 \) and \( S \).
Ans:

(a) Plug the point \( \vec{p}_0 = (2, 0, 1) \) into the equation,

\[
\text{lhs} = 2 - 0 + 1 = 3 \neq 6 = \text{rhs}.
\]
Thus, \( \vec{p}_0 \) is not on \( S \).

(b) See the graph, one needs to draw a normal vector that passes through \( \vec{p}_0 \) which intersects \( S \) at \( \vec{x}_0 \). The distance between \( p_0 \) and \( \vec{x}_0 \) is the distance we look for.

Normal of the plane: \( \vec{n} = (1, -1, 1) \). The point-direction formula gives the equation of the normal line that passes through \( \vec{p}_0 \):

\[
\vec{x} = t\vec{n} + \vec{p}_0 \quad \Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t + 2 \\ -t \\ t + 1 \end{bmatrix}.
\]

To find the point of intersection between the two, one needs to plug this into the equation of the plane

\[
x - y + z = (t + 2) - (-t) + (t + 1) = 3t + 3 = 6 \quad \Rightarrow \quad t^* = 1.
\]
Thus, \( \vec{x}_0 = \vec{x}(t^* = 1) = (3, -1, 2) \). This distance between the two are:

\[
d = ||\vec{x}_0 - \vec{p}_0|| = \sqrt{(3 - 2)^2 + (-1)^2 + (2 - 1)^2} = \sqrt{3}.
\]
(2) The parametric formula of a plane in $\mathbb{R}^3$: 

Let $\vec{l}_1, \vec{l}_2$ be two linearly independent (LI, i.e. the two are not parallel to each other) and $\vec{p}$ be a point on plane $S$, then the parametric equation of $S$ is given by

$$\vec{x} - \vec{p} = s\vec{l}_1 + t\vec{l}_2, \quad (s, t \in \mathbb{R} \text{ are parameters.})$$

Alternatively,

$$\vec{x} = \vec{p} + s\vec{l}_1 + t\vec{l}_2, \quad (s, t \in \mathbb{R} \text{ are parameters.})$$
Eg 2.8.6: A plane \( S \) is given in parametric formula as follows. Find its point-normal formula.

\[
\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.
\]

Ans: The normal is

\[
\vec{n} = \vec{l}_1 \times \vec{l}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = (-1)\hat{i} - (-1)\hat{j} + \hat{k} = (-1, 1, 1).
\]

When \( s = t = 0 \), \( \vec{x} = \vec{p} = (1, 2, 1) \) is a point on \( S \). Thus,

\[
\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p} \Rightarrow -x + y + z = 2.
\]

Or

\[
x - y - z = -2.
\]
**Eg 2.8.7:** A plane \( S \) is given by \( x + 2y + 3z = 6 \). Find one parametric formula of \( S \).

**Ans:** There are infinitely many apparently different answers! *An easier way of doing this will be shown later!* For now, One needs to find 3 points on \( S \) that are not located on a single line. For simplicity,

\[
\vec{p}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{p}_1 = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.
\]

Now, let

\[
\vec{l}_1 = \vec{p}_1 - \vec{p}_0 = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}, \quad \vec{l}_2 = \vec{p}_2 - \vec{p}_0 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.
\]

One parametric formula is given by

\[
\vec{x} = \vec{p}_1 + s\vec{l}_1 + t\vec{l}_2 = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.
\]
(3) The point-direction (parametric) formula of a line in $\mathbb{R}^3$:

Given a point $\vec{p}$ and a the direction $\vec{l}$ of a line $L$,

\[ \vec{x} - \vec{p} = t\vec{l} \quad \text{or} \quad \vec{x} = t\vec{l} + \vec{p}. \]
(4) The point-normal formula of a line in $\mathbb{R}^3$:

Given a point $\vec{p}$ and the normals $\vec{n}_1$, $\vec{n}_2$ of two planes that intersect at the line,

\[
\begin{aligned}
\vec{n}_1 \cdot (\vec{x} - \vec{p}) &= 0, \\
\vec{n}_2 \cdot (\vec{x} - \vec{p}) &= 0.
\end{aligned}
\]

or

\[
\begin{aligned}
\vec{n}_1 \cdot \vec{x} &= \vec{n}_1 \cdot \vec{p}, \\
\vec{n}_2 \cdot \vec{x} &= \vec{n}_2 \cdot \vec{p}.
\end{aligned}
\]

- Notice that $\vec{l} = \vec{n}_1 \times \vec{n}_2$.
- In parametric formula in any space dimension, the number of parameters is the dimension of the geometric object: 1 parameter for a line, 2 for a plane.
- In non-parametric form in $ND$ space, $N$ minus the number of equations is the dimension of the object: e.g., in 3-D space, 1 equation describes a 3-1=2 D plane, while 2 equations describe a 3-2=1 D line.

Number of equations =$\textbf{codimension}$ of the object.
Eg 2.8.8: A line \( L \) in \( \mathbb{R}^3 \) is defined by \[
\begin{align*}
x + y + z &= 3, \\
x - y + 2z &= -7.
\end{align*}
\]
(a) Find the normals of the two planes.
(b) Find the direction of \( L \).
(c) Find a point on \( L \).
(d) Write the equation of \( L \) in parametric form.

Ans:

(a) \( \vec{n}_1 = (1, 1, 1), \quad \vec{n}_2 = (1, -1, 2) \).

(b) Since the direction of \( L \), \( \vec{l} \), is \( \perp \) to \( \vec{n}_1 \) and \( \vec{n}_2 \),
\[
\vec{l} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 1 & 1 \\
1 & -1 & 2
\end{vmatrix} = (3)\hat{i} - (1)\hat{j} + (-2)\hat{k} = (3, -1, -2).
\]

(c) For simplicity, let \( z = 0 \). The equations become
\[
\begin{align*}
x + y &= 3, \\
x - y &= -7.
\end{align*}
\]
\[
\begin{align*}
(1)+(2) & \rightarrow 2x = -4 \quad \rightarrow \quad \begin{align*}
x &= -2, \\
y &= 5.
\end{align*}
\end{align*}
\]
Thus, \( \vec{p} = (-2, 5, 0) \) is one point on \( L \).

(d) \( \vec{x} = t\vec{l} + \vec{p} = t \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} + \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 3t - 2 \\ -t + 5 \\ -2t \end{bmatrix} \).
**Eg 2.8.9:** A line \( L \) in \( \mathbb{R}^3 \) is defined parametrically by
\[
\vec{x} = t(1, 2, 1) + (1, 1, 2).
\]
Find equations of two planes that intersect at \( L \).

**Ans:** There are infinitely many solutions. Starting from
\[
\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t + 1 \\ 2t + 1 \\ t + 2 \end{bmatrix},
\]
one simply needs to find distinct combinations of \( x, y, z \) that would cancel all terms that contain \( t \). For example,
\[
\begin{align*}
x - y + z &= (t + 1) - (2t + 1) + (t + 2) = 2. \\
x + y - 3z &= (t + 1) + (2t + 1) - 3(t + 2) = -4. \\
3x - y - z &= 3(t + 1) - (2t + 1) - (t + 2) = 0. \\
\end{align*}
\]
The list can continue to infinity. One can pick any two of them, e.g.
\[
\begin{align*}
3x - y - z &= 0, \\
x - y + z &= 2,
\end{align*}
\]
are equations of two planes that intersect at line \( L \).
2.9 Systems of equations & linear independence

Consider a system of 3 linear, algebraic equations

\[
\begin{align*}
    a_{11}x + a_{12}y + a_{13}z &= b_1, \\
    a_{21}x + a_{22}y + a_{23}z &= b_2, \\
    a_{31}x + a_{32}y + a_{33}z &= b_3,
\end{align*}
\]  

(2.9.1)

where \(a_{ij}, b_i, (i, j = 1, 2, 3)\) are scalars in \(\mathbb{R}\).

Each eqn defines a plane in \(\mathbb{R}^3\) with its normal given by

\[
\vec{n}_i = (a_{i1}, a_{i2}, a_{i3}), \quad (i = 1, 2, 3).
\]

There exist 3 possible situations:

- Intersect at one point
  - 3 normals are LI
    (not on the same plane)

- Intersect in a line
  - 3 normals are not LI
    (on the same plane)

- No common intersection
  - 3 normals are not LI
    (2 normals in same direction)
Linear dependence vs linear independence

Note that if \( \vec{n}_1, \vec{n}_2 \) are parallel to each other, then

\[
\vec{n}_1 = s\vec{n}_2 \quad \text{or} \quad \vec{n}_2 = s\vec{n}_1, \quad (s \neq 0 \in \mathbb{R}).
\]

Thus, one is a scalar multiple of the other. Two vectors are *linearly dependent* (LD) if they are parallel to each other or if one is a scalar multiple of the other.

On the contrary, if two vectors are not parallel (i.e. have different directions) or cannot be expressed as a scalar multiple of each other, then they are *linearly independent* (LI).

**Geometrically**, if the pgram with sides \( \vec{n}_1, \vec{n}_2 \) have non-zero area, then the two are LI. Otherwise, they are LD.

**Def:** Linear combination (LC). If \( \vec{a}_i, \ (i = 1, \cdots, m) \) are vectors in \( \mathbb{R}^n \), if

\[
\vec{b} = s_1\vec{a}_1 + s_2\vec{a}_2 + \cdots + s_m\vec{a}_m, \quad (\vec{b} \neq \vec{0})
\]

where the scalars, \( s_i \ (i = 1, \cdots, m) \in \mathbb{R} \), are not simultaneously zero, then \( \vec{b} \) is called a *linear combination* (LC) of vectors \( \vec{a}_i, \ (i = 1, \cdots, m) \).
**Def:** Linear dependence (LD) and linear independence (LI). Nonzero vectors $\vec{a}_i$ ($i = 1, \cdots, m$, $\vec{a}_i \neq \vec{0}$) are LI iff

$$s_1\vec{a}_1 + s_2\vec{a}_2 + \cdots + s_m\vec{a}_m = \vec{0}$$

is satisfied only when the scalars $s_i = (i = 1, \cdots, m)$ are simultaneously zero, i.e. $s_i = 0$ for all $i$. Otherwise, they are LD. If they are LD, then at least one of them can be expressed as a LC of the others.

**Put it in words:** A set of vectors are LI if none of them can be written as a linear combination of others.

**Eg:** If $s_1\vec{a}_1 + s_2\vec{a}_2 + \cdots + s_m\vec{a}_m = \vec{0}$ and $s_1 \neq 0$, then

$$\vec{a}_1 = \left(-\frac{s_2}{s_1}\right)\vec{a}_2 + \left(-\frac{s_3}{s_1}\right)\vec{a}_3 + \cdots + \left(-\frac{s_m}{s_1}\right)\vec{a}_m,$$

therefore, $\vec{a}_1$ must be a LC of the other vectors in the set.

**Basis of $\mathbb{R}^n$:** There are at most $n$ LI vectors in any set of vectors in $\mathbb{R}^n$. Any set of $n$ LI vectors can form a basis whose linear combinations can generate all vectors in $\mathbb{R}^n$. In practice, we often use orthonormal basis, i.e. $n$ unit vectors that are mutually orthogonal to each other.
Eg 2.9.1: Determine if vectors in the following sets are LI.

(a) \[ \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}; \]

(b) \[ \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}; \]

(c) \[ \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\}. \]

Ans:

(a) Not, not LI because \[ \begin{bmatrix} -3 \\ 6 \end{bmatrix} = (-3) \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \]

Or because \[ A_{pgram} = \begin{vmatrix} 1 & -2 \\ -3 & 6 \end{vmatrix} = 6 - 6 = 0. \]

(b) Yes, LI because \[ A_{pgram} = \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 2 + 1 = 3 \neq 0. \text{ Or, they are not scalar multiple of each other.} \]

(c) No, they can’t be LI because there could be at most 2 LI vectors in any set of vectors in \( \mathbb{R}^2 \).
Eg 2.9.2: Determine if vectors in the following sets are LI or LD.

(a) \{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \};

(b) \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \};

(c) \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}.

Ans:

(a) LD, because all 3 vectors are in the 2D \(yx\)-plane.

Or \[ \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} = 5 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}. \]

(b) LI, because they are not scalar multiple of each other.
(c) LI, because $V_{ppiped} = \text{abs} \left( \begin{vmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} \right)$

$$= | -1 + 1 + 0 - (-1) - 2 - 0 | = |1 - 2| = 1 \neq 0.$$ 

**Summary:**

- In $\mathbb{R}^2$, 2 vectors are LI if they point to different directions.
- In $\mathbb{R}^3$, 3 vectors pointing to 3 different directions can not guarantee that they are LI. They are LI only when they are NOT in the same plane.
- In $\mathbb{R}^n$, $n$ vectors are LI iff none of them can be expressed as a nontrivial LC of the others. Or when the volume of the “hyper-parallelepiped” with sides defined by these vectors is nonzero.