Question 1: [3 points each] The following questions require little or no computation. **Put your answers in the boxes.**

(a) What is the average rate of change of $f(x) = x^3$ on the interval $-1 \leq x \leq 1$ (i.e., $[-1, 1]$)?

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(b) What is the absolute maximum value of the function $f(x) = 3x - x^3$ on the interval $-3 \leq x \leq 3$ (i.e., $[-3, 3]$)?

$f(-3) = 18$

(c) What is the derivative of the function $f(x) = a^x$? (Assume that $a > 0$)

$(lna)a^x$

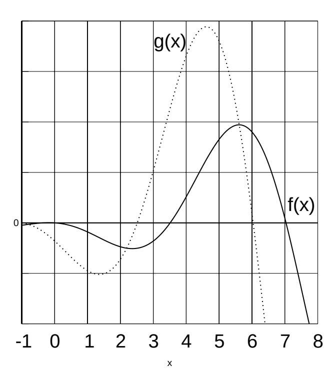
(d) If $\frac{dy}{dt} = y$ and $y(0) = 10$, then what is the value of $y(2)$?

$10e^2 \approx 73.89$

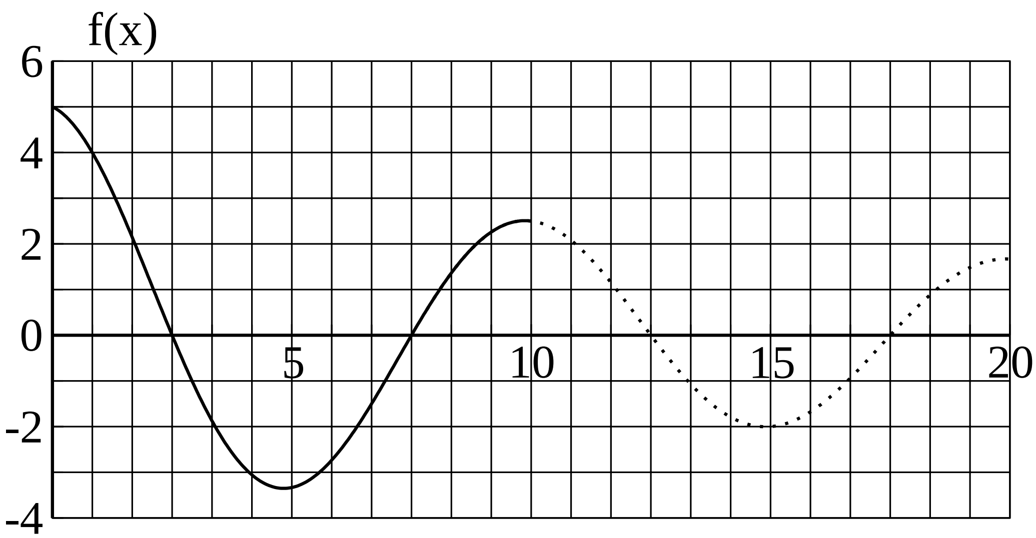
(e) Find the slope and the y-intercept of the tangent line of the circle $x^2 + y^2 = 4$ at $(-1, \sqrt{3})$.

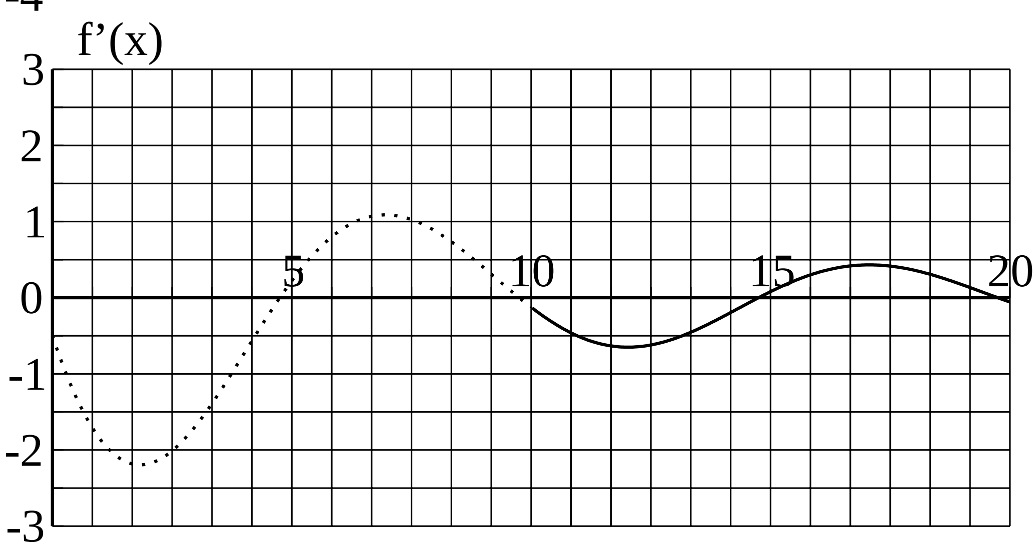
$m = \frac{1}{\sqrt{3}} \approx .57735$, $b = \frac{4}{\sqrt{3}} \approx 2.3094$

(f) Given the graph of $f(x)$, sketch the graph of $g(x) = 2f(x+1)$ on the same plot.



Question 2: [10 points] The graphs of $f(x)$ (top) and $f'(x)$ (bottom) are plotted, respectively, on the intervals [0, 10] and [10, 20]. Sketch the missing portion in each of the graphs on the interval [0, 20].





Question 3: Blood Alcohol Level (BAL), the amount of alcohol in your blood stream (here represented by $B(t)$), is measured in milligrams of alcohol per 100 milliliters of blood. At the end of a party (time $t = 0$), a drinker is found to have $B(0) = 0.08$ (the legal level for driving impairment), and after that time $B(t)$ satisfies the differential equation

$$\frac{dB}{dt} = -kB$$

where k is a positive constant that represents the rate of removal of alcohol from the blood-steam by the liver.

(a) If the drinker had waited for three hours before driving, (until $t = 3$) his BAL would have dropped to 0.04. Determine the value of the rate constant k (specifying appropriate units) for this drinker.

Answer: The blood level is an exponential function: $B(t) = B(0)e^{-kt} = 0.08e^{-kt}$. At $t = 3$, $B(3) = 0.04 = 0.08e^{-3k}$. Thus, $e^{3k} = 2 \Rightarrow 3k = ln2 \Rightarrow k = ln2/3 = 0.231$.

(b) According to this model, how much longer would it take for the BAL to drop to 0.01?

Answer: We look for t^* such that $B(t^*) = 0.01 = 0.08e^{-kt^*} \Rightarrow e^{kt^*} = 8$ $kt^* = ln8 = 3ln2 \Rightarrow t^* = 3ln2/(ln2/3) = 9$ hrs.

Therefore, after the drinker had waited for three hours and began to drive, it would take 6 more hours for the BAL to drop to 0.01.

Question 4: [10 points] Consider the curve whose equation is

$$x^6 - 3xy + y^6 = 1$$

(a) Find the slope of the tangent line at the point (1, 0) on this curve.

Answer: Differentiate both sides of $x^6-3xy+y^6 = 1$ with respect to x: $6x^5-3y-3xy'+6y^5y' = 0$ $\Rightarrow y'(6y^5 - 3x) = 3y - 6x^5 \Rightarrow y' = (3y - 6x^5)/(6y^5 - 3x)$.

$$y' = \frac{y - 2x^5}{2y^5 - x} \quad \Rightarrow \quad y'|_{(1,0)} = \frac{0-2}{0-1} = 2$$

Thus, the slope is $m = 2$.

(b) Determine whether the curve is concave up or concave down at the point (1, 0).

Answer: Differentiate again,

$$y'' = \frac{(y' - 10x^4)(2y^5 - x) - (y - 2x^5)(10y^4y' - 1)}{(2y^5 - x)^2}$$

Thus, $y''|_{(1,0)} = \frac{(2-10)(-1)-(-2)(-1)}{(-1)^2} = \frac{8-2}{1} = 6 > 0$, the curve is concave up.

Question 5: [10 points] In Fish River, the number of salmon (in thousands), x, in a given year, is linked to the the number of salmon (in thousands), y, in the following year by the function

$$y = Axe^{-bx}$$

where $A,\ b$ are positive constants.

(a) For what number of salmon is there no change in the number from one year to the next?

Answer: $y = x \Rightarrow x = Axe^{-bx} \Rightarrow x(1 - Ae^{-bx}) = 0$. Thus, either $x = 0$ or $1 - Ae^{-bx} = 0 \Rightarrow e^{bx} = A \Rightarrow x = lnA/b$.

(b) Find the number of salmon that would yield the largest number of salmon in the following year.

Answer: $y' = Ae^{-bx} - Abxe^{-bx} = Ae^{-bx}(1 - bx) = 0 \Rightarrow x = 1/b$. Since $y'' = Ae^{-bx}[-b(1 - bx) - b] = -Abe^{-bx}[2 - bx]$, thus $y''|_{x=1/b} = -Abe^{-1} < 0$. Therefore, $x = 1/b$ is the unique maximum.

Question 6: [10 points] In Big River, the number of salmon (in thousands), x, in a given year is linked to the number (in thousands), y, in the following year by the function

$$y = 3x^2e^{-x}$$

(a) For what number of salmon is there no change in the number from one year to the next? Use Newton's method with an initial guess $x_0 = 0.5$, show how you set up the method, and give values of the intermediate steps. The answer should be correct to 4 decimal places.

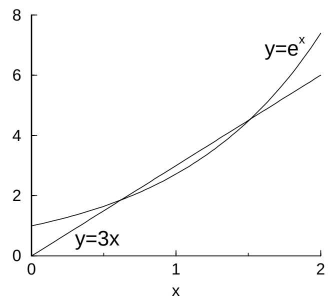
Answer: $y = x \Rightarrow x = 3x^2e^{-x} \Rightarrow xe^x = 3x^2 \Rightarrow x(3x - e^x) = 0$. Thus, either $x = 0$ or $3x - e^x = 0$. Let $f(x) = 3x - e^x = 0$. Newton's method gives the following iteration:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - e^{x_n}}{3 - e^{x_n}}$$

$x_0 = 0.5$
$x_1 = 0.61005965$
$x_2 = 0.61899677$
$x_3 = 0.61906128$
$x_4 = 0.61906129$

(b) Do you expect other possible solutions to this problem? Indicate how many and why?

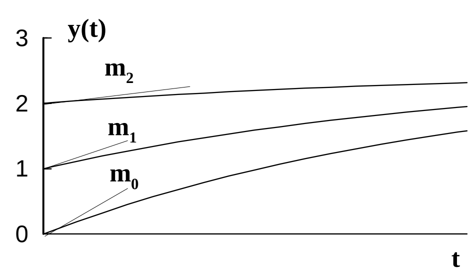
Answer: In addition to solutions $x = 0$ and $x = 0.61906129$, there is another solution on the interval $(1,\ 2)$ near $x = 1.5$. This is because the graphs of $y = 3x$ and $y = e^x$ intersects at two different points (see figure) at which $3x = e^x$. These interacetion points are zeros of $f(x) = 3x - e^x$.



Question 7: [12 points] Solutions to the differential equation

$$\frac{dy}{dt} = a - by$$

starting at three different initial values are shown in the graph. Also shown are the tangent lines to these curves at $t = 0$. You are given the following information about the slopes of these tangent lines:
(i) The slope m_0 is five times the slope m_2.
(ii) The slope m_1 is 3.
Use this information to answer the following questions:



(a) Determine the values of the constants a and b. (Justify your answer.)

Answer: The slope of the solution curves are determined by $m(y) = \frac{dy}{dt} = a - by$. Thus,
$m(0) = m_0 = a = 5m_2$,
$m(1) = m_1 = a - b = 3$,
$m(2) = m_2 = a - 2b$.

Therefore, $a = 5m_2 = 5a - 10b \Rightarrow 4a = 10b \Rightarrow a = 5b/2$. We solve for $a = 5$ and $b = 2$.

(b) Determine the value that $y(t)$ will approach after a long time on any of these curves.

Answer: After a long time, the value $y(t)$ will approach the stable steady-state $y = 5/2 = 2.5$. This state is stable since the slope of $m(y) = 5 - 2y$ is $m'(y) = -2 < 0$.

An alternative way to answer this question is to solve the eqn $\frac{dy}{dt} = 5 - 2y = -2(y - 2.5)$. Let $\delta = y - 2.5$, thus $\delta' = -2\delta$ which implies $\delta(t) = \delta(0)e^{-2t} = (y(0) - 2.5)e^{-2t}$. Therefore, $y(t) = 2.5 + \delta = 2.5 + (y(0) - 2.5)e^{-2t}$. We can calculate the limit $y(\infty) = 2.5$.

(c) Find the value of $y(t)$ at time $t = 0.2$ given that $y(0) = 1$. Round off your answer to 2 decimal places and justify how this answer was obtained.

Answer: METHOD I. Solve the initial value problem $y' = 5 - 2y$, $y(0) = 1$. We obtain (see above), $y(t) = 2.5 - 1.5e^{-2t}$. $y(0.2) = 2.5 - 1.5e^{-2(0.2)} \approx 1.49$.

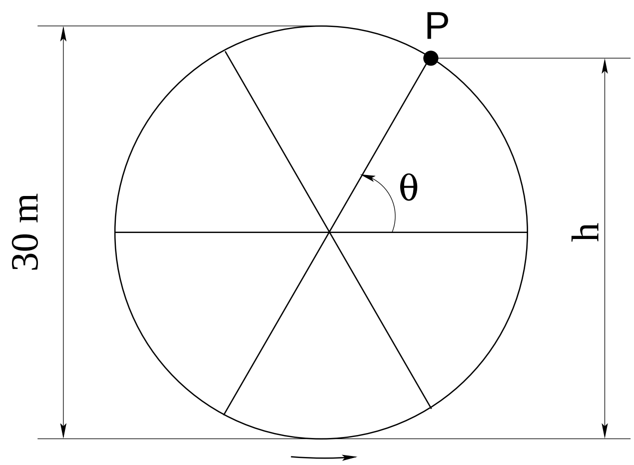
METHOD II. Euler's method with $h = 0.1$:
$y_1 = y_0 + hf(y_0) = 1 + 0.1(5 - 2) = 1.3$
$y_2 = y_1 + hf(y_1) = 1.3 + 0.1(5 - 2 * 1.3) = 1.3 + 0.24 = 1.54$.

Question 8: [10 points] A ferris wheel 30 meters high turns counterclockwise at a uniform rate of 3 revolutions per minute. Find the rate of change in the altitude (h) of a passenger at the instant when he is located at point P. Use the fact that $\theta = \pi/3$ radians at that instant. Specify the units of your result.

Answer: Step I. Identify the variables (h, θ) and the constants ($d = 30$ m, $r = 15$ m, $\theta' = d\theta/dt = 3\ rev/min = 3(2\pi)\ Rad/min = 6\pi\ Rad/min$.

Step II. Find the relatioship between the ariables. $h = rsin\theta + r = r(sin\theta + 1) = 15(sin\theta + 1)$.

Step III. Differentiate to determine the relatioship between the rates. $dh/dt = 15cos\theta d\theta/dt = 15cos(\pi/3)6\pi = 45\pi\ m/min = 141.37\ m/min = 0.75\pi\ m/s = 2.36\ m/s$.



Question 9: [10 points] A wedge-shaped region is cut out of a circle of fixed radius r and the cut edges are joined to produce a cone, as shown in the figure. The remaining part of the circle's perimeter whose length is $(2\pi - \theta)r$ forms the bottom edge of the cone. For what angle θ is the volume of the resulting cone greatest?

Recall: the volume of a cone with base area A and height h is $V = \frac{1}{3}Ah$.

Answer: The radius of the circular bottom edge of the cone is $R = (2\pi - \theta)r/(2\pi) = pr$, where $p = (2\pi - \theta)/(2\pi) = 1 - \frac{\theta}{2\pi}$. Thus the area of the base is $A = \pi R^2 = \pi p^2 r^2$.

Using trigonometry, $r^2 = R^2 + h^2 \Rightarrow h = \sqrt{r^2 - R^2} = r[1 - p^2]^{1/2}$.

Thus,

$$V = \frac{1}{3}Ah = \frac{1}{3}(\pi p^2 r^2)(r[1 - p^2]^{1/2}) = \frac{\pi r^3}{3} p^2 [1 - p^2]^{1/2}$$

Therefore,

$$\frac{dV}{d\theta} = \frac{dV}{dp}\frac{dp}{d\theta} = \frac{\pi r^3}{3}\{2p[1-p^2]^{1/2} - p^3[1-p^2]^{-1/2}\}(-\frac{1}{2\pi})$$
$$= -\frac{r^3}{6}p[1-p^2]^{-1/2}[2(1-p^2)-p^2] = -\frac{r^3}{6}p[1-p^2]^{-1/2}(2-3p^2)$$

Thus, $dV/d\theta = 0 \Rightarrow 2-3p^2 = 0 \Rightarrow p = \sqrt{\frac{2}{3}} \Rightarrow \theta^* = 2\pi(1-\sqrt{\frac{2}{3}}) = 1.153\ rad.$

The complexity of the derivation can be greatly reduced if V is expressed in terms of the area $A = \pi R^2$: $V = \frac{1}{3}Ah = \frac{1}{3}A\sqrt{r^2 - A/\pi}$.

$$\frac{dV}{d\theta} = \frac{dV}{dA}\frac{dA}{d\theta} = \frac{1}{3}(r^2 - A/\pi)^{-1/2}[r^2 - A/\pi - \frac{A}{2\pi}]A' = \frac{1}{3}(r^2 - A/\pi)^{-1/2}[r^2 - \frac{3A}{2\pi}]A'$$

Thus, $dV/d\theta = 0 \Rightarrow r^2 = \frac{3A}{2\pi} \Rightarrow 1 = \frac{3p^2}{2} \Rightarrow p = \sqrt{\frac{2}{3}} \Rightarrow \theta^* = 2\pi(1-\sqrt{\frac{2}{3}}) = 1.153\ rad.$

In order to check if this is a maximum, we check the sign of $dV/d\theta$ for $\theta < \theta^*$ and $\theta > \theta^*$. For $\theta < \theta^*$, $p > \sqrt{\frac{2}{3}}$, thus $dV/d\theta > 0$. For $\theta > \theta^*$, $p < \sqrt{\frac{2}{3}}$, thus $dV/d\theta < 0$. This means $d^2V/d^2\theta < 0$ at $\theta = \theta^*$ thus it is a maximum.