Notes on Elementary Matrices

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1 Invertible matrices and row operations

Introduction:

Elementary row operations can transform each invertible matrix $A_{n \times n}$ into the identity matrix $I_n$. The latter is actually the “reduced row echelon form” of an invertible matrix.

$$\begin{bmatrix} A & I \end{bmatrix} \xrightarrow{\text{elementary row operations}} \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

**Elementary row operations** include:

1. Exchanging/swapping between two rows.
2. Multiplying a row by a scalar.
3. Adding a scalar multiple of one row to another.

Each elementary row operation is a linear transformation.

Each elementary row operation is invertible and one-to-one.

Each elementary matrix is obtained by “doctoring” the identity matrix $I$ in one way or the other (see text below for details).
Summary:

If $A$ is invertible, then

$$E_k E_{k-1} \cdots E_1 A = I \iff A^{-1} = E_k E_{k-1} \cdots E_1,$$

where $E_l$ ($l = 1, 2, \ldots, k$) are all elementary matrices.

Since $E_l$ ($l = 1, 2, \ldots, k$) are all invertible,

$$A = (A^{-1})^{-1} = (E_k E_{k-1} \cdots E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}.$$

Furthermore, we shall demonstrate that

$$\det A = \det E_1^{-1} \det E_2^{-1} \cdots \det E_k^{-1}.$$

This is the key step in the proof of the relation $\det(AB) = \det A \det B$ for all $n \times n$ matrices.

These formulas provide an alternative method for calculating $A^{-1}$ and $\det A$ if it is invertible.
2 Exchanging/swapping between two rows

To swap between the two rows of $A_{2 \times 2}$, multiply it by $I_{1\leftrightarrow 2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

$$I_{1\leftrightarrow 2}A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}.$$ 

Swapping between the $k^{th}$ and $l^{th}$ rows of $I_n$ gives rise to the elementary matrix that does the swapping for $n \times n$ matrices.

$$I_{k\leftrightarrow l} = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

Summary:

- $I_{k\leftrightarrow l}^{-1} = I_{k\leftrightarrow l}$ or $I_{k\leftrightarrow l}^2 = I$.
- $\det I_{k\leftrightarrow l} = -\det I = -1$.
- $\det(I_{k\leftrightarrow l}A) = -\det A = \det I_{k\leftrightarrow l} \det A$. 
3 Multiplying one row by a scalar

Replacing the 1 in the $k^{th}$ row of $I_n$ by a scalar $s \neq 0$ gives the elementary matrix that multiplies the $k^{th}$ row of a matrix by $s$.

$$I_{ssk} = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

Summary:

- $I_{ssk}^{-1} = I_{\frac{1}{s}sk}$.
- $\det I_{ssk} = s \det I = s$.
- $\det(I_{ssk}A) = s \det A = \det I_{ssk} \det A$. 

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4 Adding a scalar multiple of a row to another

Replacing the 0 in the \((k, l)^{th}\) location of \(I_n\) by a scalar \(s\) gives the elementary matrix adds \(s\) times the \(k^{th}\) row to the \(l^{th}\) row of an \(n \times n\) matrix.

\[
I_{s \cdot k \rightarrow l} = \begin{bmatrix}
1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \\
0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & s & \cdots & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \\
\end{bmatrix}
\]

Summary:

- \(I_{s \cdot k \rightarrow l} = I_{-s \cdot k \rightarrow l}^{-1}\).
- \(\det I_{s \cdot k \rightarrow l} = 1 = \det I_{s \cdot k \rightarrow l}^{-1}\).
- \(\det(I_{s \cdot k \rightarrow l}A) = \det A = \det I_{s \cdot k \rightarrow l} \det A\).
5 Implications of Row Operations

Based on the properties derived above,

1. Exchanging two rows in a matrix results in a change of sign in the determinant.
2. Multiplying a row by a scalar $s$ results in a change in its determinant by a factor $s$.
3. Adding a constant multiple of one row to another causes no change in the determinant.
4. All conclusions reached above apply equally to column operations. Just change the word “row” into “column” in all statements above. (Think now the matrix $A^T$. Applying all the above results to rows of $A^T$ is equivalent to applying them to columns of $A$!)

Conclusion:

∀ invertible matrix $A$, ∃ a series of elementary matrices such that

$$E_k E_{k-1} \cdots E_1 A = I \iff A^{-1} = E_k E_{k-1} \cdots E_1.$$  

Thus,

$$A = (A^{-1})^{-1} = (E_k E_{k-1} \cdots E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1},$$

and that

$$\det A = \det E_1^{-1} \det E_2^{-1} \cdots \det E_k^{-1}.$$