Lecture 6.3 Applications of Eigen-analysis II

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6.3.4 Systems of linear DEs with constant coefficients

Differential Equation (DE): An equation that relates one derivative of an unknown function $y(t)$ to other quantities that often include y itself and/or its other derivatives.

$$y^{(n)} = F(t,\ y,\ y',\ \dots\ ,\ y^{(n-1)}), \tag{1}$$

where t is the independent/free variable (many textbooks prefer to use x), $y^{(n)} \equiv \frac{d^n y}{dt^n},\ (n \geq 1)$ is the nth derivative of the unknown function.

One of the challenges of sciences and engineering:

Solve DE(s) for the unknown function(s).

Example 6.3.4.1:

$$x'(t) = ax(t), \qquad \text{(a is a constant)}, \tag{2}$$

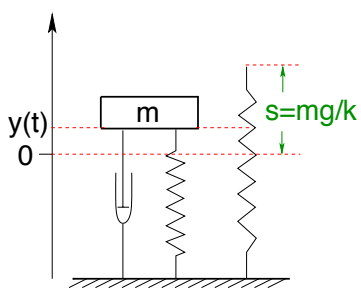
is a DE for an exponential process. Its solution is

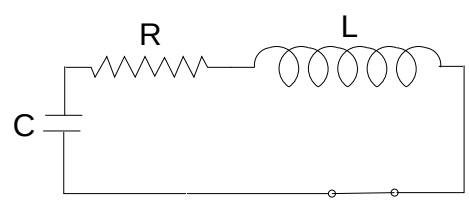
$$x(t) = e^{at}x(0).$$

Example 6.3.4.2:

$$my''(t) + cy'(t) + ky(t) = 0, \qquad \text{(m, c, k are constants)}, \tag{3}$$

models the displacement $y(t)$ of a mass suspended by a spring and a damper, often referred to as a mass-spring-damper system.





Example 6.3.4.3:

$$LI''(t) + RI'(t) + C^{-1}I(t) = 0, \qquad \text{(R, L, C are constants)}, \tag{4}$$

models the current $I(t)$ in a closed RLC circuit in the absence of voltage source. R, L, and C are, respectively, the resistance, inductance, and capacitance.

The DEs in **Examples 6.3.4.2** and **6.3.4.3** are 2nd-order that can often be reduced into a system of two 1st-order DEs.

Example 6.3.4.4: $LI''(t) + RI'(t) + C^{-1}I(t) = 0.$

By introducing a new variable $Z(t) = I'(t)$, it is reduced to

$$LZ'(t) + RZ(t) + C^{-1}I(t) = 0,$$

which is 1st-order. But now, we have a system of two, 1st-order DEs:

$$\begin{cases} I'(t) = Z(t), \\ Z'(t) = -\frac{1}{LC}I(t) - \frac{R}{L}Z(t), \end{cases}$$

which involves two unknown functions $I(t)$ and $Z(t)$.

Generally, a system of two linear DEs can be expressed as

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t), \\ x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t), \end{cases} \tag{5}$$

where $x_1(t),\ x_2(t)$ are unknown functions, a_{ij} $(i, j = 1, 2)$ are known scalars.

Def: vector function and its derivative.

A 2D vector function and its derivative are defined as

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \frac{d\vec{x}(t)}{dt} = \vec{x}'(t) = \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}.$$

This definition can be easily extended to n-dimensional vector functions and their derivatives. Such a definition allows one to express a system of DEs in the following matrix form,

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11}x_1(t) + a_{12}x_2(t) \\ a_{21}x_1(t) + a_{22}x_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

which further simplifies to

$$\vec{x}'(t) = Ax(t). \tag{6}$$

In eq.(6), the vector function $\vec{x}(t)$ can be n-dimensional and A can be $n \times n$ for any integer $n > 1$. But we shall focus on the simpler case of $n = 2$ in the rest of this lecture.

Striking similarities between a scalar and a matrix DEs:

$x'(t) = ax(t)$	$\vec{x}'(t) = A\vec{x}(t)$
a - a constant scalar	$A = [a_{ij}]$ - a constant matrix
IC: $x(0) = x(t=0)$	IC: $\vec{x}(0) = \vec{x}(t=0)$
$x(t) = e^{at}x(0)$	$\vec{x}(t) = e^{At}\vec{x}(0)$

Question: What is e^{At}?

Answer: $e^{At} = I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{n!}A^nt^n + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}A^nt^n.$

So what?

It is still not practical to evaluate a series of matrices involving $A,\ A^2,\ \cdots,\ A^n$!

We need an alternative method to solve $\vec{x}'(t) = A\vec{x}(t)$!

Principle of Superposition: For any linear, homogeneous system

$$L\vec{x} = \vec{0} \qquad \text{(L is a linear operator)}$$

if $\vec{x}_1$ and $\vec{x}_2$ are both solutions, so is a linear combination (LC) of the two

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2,$$

where $c_1,\ c_2$ are arbitrary scalars.

Proof:

$$\vec{x}_1 \text{ is a solution} \quad \Rightarrow \quad L\vec{x}_1 = \vec{0};$$

$$\vec{x}_2 \text{ is a solution} \quad \Rightarrow \quad L\vec{x}_2 = \vec{0}.$$

Then,

$$L(c_1\vec{x}_1 + c_2\vec{x}_2) \overset{\text{L is linear}}{=} c_1L\vec{x}_1 + c_2L\vec{x}_2 = c_1\vec{0} + c_2\vec{0} = \vec{0}$$

$$\Rightarrow \quad \vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 \quad \text{is also a solution!}$$

Remarks:

(1) The result generalizes to a LC of n solutions!

(2) For the homogeneous system $A\vec{x} = \vec{0}$, $L = A$.

For the homogeneous system $\frac{d\vec{x}}{dt} = A\vec{x}$, $\left(\frac{d}{dt} - A\right)\vec{x} = \vec{0}$,

$L = \frac{d}{dt} - A$ is a differential operator.

Def: Linear independence of vector functions.

If $\vec{x}_1(t),\ \vec{x}_2(t),\ \cdots,\ \vec{x}_n(t) \in \mathbb{R}^n$ are a list of n vector functions, then if

$$\det[\vec{x}_1(t)\ \ \vec{x}_2(t)\ \ \cdots\ \ \vec{x}_n(t)] \neq 0,$$

for all t in an open interval in $\mathbb{R}$, then they are linearly independent (LI).

Example 6.3.4.5: The vector functions

$$\vec{x}_1(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, \qquad \vec{x}_2(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

are LI because

$$\det[\vec{x}_1(t)\ \ \vec{x}_2(t)] = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} = \cos^2 t + \sin^2 t = 1 \neq 0, \quad \text{for all t.}$$

Theorem: General solution of a linear system of DEs.

For $\vec{x}' = A\vec{x}$, where A is $n \times n$, it is always possible to find n LI solutions of the form

$$\vec{x}_j = e^{\lambda_j t}\vec{s}_j, \qquad (j = 1, 2, \ldots, n)$$

such that all possible solutions of the system can be expressed in the form

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t),$$

where c_j $(j = 1, 2, \ldots, n)$ are arbitrary constants, λ_j $(j = 1, 2, \ldots, n)$ are eigenvalues of A. Such a solution is referred to as the **general solution**.

If A has n distinct eigenvalues or if $AM = GM$ for all the repeated eigenvalues, then $\vec{s}_j$ $(j = 1, 2, \ldots, n)$ are the corresponding eigenvectors and are constant (i.e., independent of t). *This will be the case considered in this course.*

If for a repeated eigenvalue λ_k, $AM(\lambda_k) > GM(\lambda_k)$, then the corresponding $\vec{s}_k$ may be explicitly dependent on t. *This case will NOT be considered in this course.*

Proof: Beyond the level of this course, find reference on the internet or in

other textbooks.

Theorem: 2×2 **systems with two LI eigenvectors.**

Consider the linear system $\vec{x}' = A\vec{x}$, where A is 2×2.

If the two eigenvalues are distinct, i.e. $\lambda_1 \neq \lambda_2$, or if $\lambda = \lambda_1 = \lambda_2$ but $GM(\lambda) = AM(\lambda) = 2$, then there exits two LI eigenvectors $\vec{v}_1$ and $\vec{v}_2$. Then, the general solution is

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2,$$

where $c_1,\ c_2$ are arbitrary scalars.

Furthermore, if $\vec{x}(0)$ is given, then

$$\vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2 = [\vec{v}_1 \quad \vec{v}_2] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = T\vec{c}, \quad \Rightarrow \quad \vec{x}(0) = T\vec{c}.$$

It can be solved by using row reduction, or $\vec{c} = T^{-1}\vec{x}(0)$, or Cramer's rule:

$$c_1 = \frac{\det[\vec{x}(0) \quad \vec{v}_2]}{\det T}, \qquad c_2 = \frac{\det[\vec{v}_1 \quad \vec{x}(0)]}{\det T},$$

where $T = [\vec{v}_1 \quad \vec{v}_2]$ is the matrix formed by eigenvectors of A in its columns.

In this case, the solution contains no arbitrary constants and is unique.

Proof: First, we show that if $\{\lambda_j,\ \vec{v}_j\}$ is an eigen-pair, i.e. $A\vec{v}_j = \lambda_j \vec{v}_j$, then $\vec{x}_j = e^{\lambda_j t}\vec{v}_j$ is a solution to $\vec{x}' = A\vec{x}$.

Substitute $\vec{x}_j$ into both sides of $\vec{x}' = A\vec{x}$:

$$LHS = (\vec{x}_j)' = \left(e^{\lambda_j t}\vec{v}_j\right)' = \left(e^{\lambda_j t}\right)' \vec{v}_j = \lambda_j e^{\lambda_j t}\vec{v}_j.$$

$$RHS = A\vec{x}_j = Ae^{\lambda_j t}\vec{v}_j = e^{\lambda_j t}A\vec{v}_j = e^{\lambda_j t}\lambda_j\vec{v}_j = \lambda_j e^{\lambda_j t}\vec{v}_j.$$

$$LHS = RHS \qquad \Rightarrow \qquad \vec{x}_j = e^{\lambda_j t}\vec{v}_j \ \text{ is a solution.}$$

We then show that $\vec{x}_1$ and $\vec{x}_2$ are LI.

$$\det[\vec{x}_1 \ \ \vec{x}_2] = \det[e^{\lambda_1 t}\vec{v}_1 \ \ e^{\lambda_2 t}\vec{v}_2] = e^{\lambda_1 t}e^{\lambda_2 t}\det[\vec{v}_1 \ \ \vec{v}_2] = e^{(\lambda_1+\lambda_2)t}\det T.$$

Therefore, if $\vec{v}_1,\ \vec{v}_2$ are LI, $\det[\vec{x}_1 \ \ \vec{x}_2] \neq 0$ for all t.

Example 6.3.4.6: For $\begin{cases} x_1'(t) = -3x_1(t) + x_2(t), \\ x_2'(t) = x_1(t) - 3x_2(t). \end{cases}$

(a) Express the system in matrix form.

(b) Find the general solution.

(c) Given $x_1(0) = 2,\ x_2(0) = -1$, find the unique solution.

(d) What happens to the general solution as $t \to \infty$?

Answer:

(a) Let

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}, \qquad \vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Then, in matrix form, the system becomes

$$\vec{x}'(t) = A\vec{x}(t), \qquad \vec{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

(b) $Tr = -6,\ Det = 8$. Thus,

$$\lambda^2 + 6\lambda + 8 = 0 \quad \Rightarrow \quad (\lambda + 2)(\lambda + 4) = 0 \quad \Rightarrow \quad \lambda_1 = -2,\ \lambda_2 = -4.$$

$$\vec{v}_1 = \ker(A - \lambda_1 I) = \ker(A + 2I) = \ker \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}}_{\vec{c}_1 + \vec{c}_2 = \vec{0}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\vec{v}_2 = \ker(A - \lambda_2 I) = \ker(A + 4I) = \ker \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{\vec{c}_1 - \vec{c}_2 = \vec{0}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Thus, the general solution is

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 = c_1 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$\text{Or} \qquad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-2t} + c_2 e^{-4t} \\ c_1 e^{-2t} - c_2 e^{-4t} \end{bmatrix}.$$

(c) Using Cramer's rule,

$$c_1 = \frac{\det[\vec{x}(0) \quad \vec{v}_2]}{\det[\vec{v}_1 \quad \vec{v}_2]} = \frac{\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{-2+1}{-1-1} = \frac{1}{2}.$$

$$c_2 = \frac{\det[\vec{v}_2 \quad \vec{x}(0)]}{\det[\vec{v}_1 \quad \vec{v}_2]} = \frac{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{-1-2}{-1-1} = \frac{3}{2}.$$

Therefore,

$$\vec{x}(t) = \frac{1}{2}e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3}{2}e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-2t} + 3e^{-4t} \\ e^{-2t} - 3e^{-4t} \end{bmatrix}.$$

(d)

$$\vec{x}(\infty) = \lim_{t\to\infty}\left(c_1 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{0}.$$

Example 6.3.4.7: For $\vec{x}' = A\vec{x}$, $A = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}$.

(a) Find the general solution.

(b) Given $\vec{x}(0) = [1\ 3]^T$, find the unique solution.

(c) What happens to the general solution as $t \to \infty$?

Answer:

(a) $Tr = -2,\ Det = 1 - (-9) = 10$. Thus,

$$\lambda^2 + 2\lambda + 10 = 0 \quad \Rightarrow \quad (\lambda+1)^2 + 9 = 0 \quad \Rightarrow \quad \lambda_{1,2} = -1 \pm 3i.$$

$$\vec{v}_1 = \ker(A - \lambda_1 I) = \ker(A - (-1+3i)I) = \ker \underbrace{\begin{bmatrix} -3i & -3 \\ 3 & -31 \end{bmatrix}}_{\vec{c}_1 + i\vec{c}_2 = \vec{0}} = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

$$\vec{v}_2 = \overline{\vec{v}_1} = \overline{\begin{bmatrix} 1 \\ i \end{bmatrix}} = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

(b) The general solution is

$$\vec{x}(t) = c_1 e^{(-1+3i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{(-1-3i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

(c) Using Cramer's rule,

$$c_1 = \frac{\det[\vec{x}(0) \quad \vec{v}_2]}{\det[\vec{v}_1 \quad \vec{v}_2]} = \frac{\begin{vmatrix} 1 & 1 \\ 3 & -i \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ i & -i \end{vmatrix}} = \frac{-3-i}{-2i} = \frac{1-3i}{2}.$$

$$c_2 = \frac{\det[\vec{v}_1 \quad \vec{x}(0)]}{\det[\vec{v}_1 \quad \vec{v}_2]} = \frac{\begin{vmatrix} 1 & 1 \\ i & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ i & -i \end{vmatrix}} = \frac{3-i}{-2i} = \frac{1+3i}{2}.$$

The unique solution is

$$\vec{x}(t) = \frac{1-3i}{2} e^{(-1+3i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1+3i}{2} e^{(-1-3i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Notice that the solution contains two terms that are complex conjugates of each other. Thus, it should be a real-valued vector. However, a lot of computations are required to get to that form.

(d)

$$\vec{x}(\infty) = \lim_{t\to\infty} \left(e^{-t}e^{3it}c_1\vec{v}_1 + e^{-t}e^{-3it}c_2\vec{v}_2\right) = \vec{0}.$$

Theorem: Solutions in real form when eigenvalues are complex.

Consider the linear system $\vec{x}' = A\vec{x}$, where A is 2×2. If

$$\lambda_{1,2} = \alpha \pm \beta i, \qquad (\alpha,\ \beta \in \mathbb{R})$$

with corresponding eigenvector $\vec{v}_1,\ \overline{\vec{v}_1}$, then

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2\overline{e^{\lambda_1 t}\vec{v}_1},$$

is the general solution in complex form in which, $c_1,\ \lambda_1,\ \vec{v}_1$ are typically complex-valued. The *general solution in real form* is

$$\vec{x}(t) = c_1 Re\{e^{\lambda_1 t}\vec{v}_1\} + c_2 Im\{e^{\lambda_1 t}\vec{v}_1\}.$$

Proof: Since both

$$e^{\lambda_1 t}\vec{v}_1 \qquad \text{and} \qquad \overline{e^{\lambda_1 t}\vec{v}_1}$$

are solutions to the system, based on Principle of Superposition, so are the following linear combinations of them

$$\vec{x}_R = Re\{e^{\lambda_1 t}\vec{v}_1\} = \frac{1}{2}\left(e^{\lambda_1 t}\vec{v}_1 + \overline{e^{\lambda_1 t}\vec{v}_1}\right);$$

$$\vec{x}_I = Im\{e^{\lambda_1 t}\vec{v}_1\} = \frac{1}{2i}\left(e^{\lambda_1 t}\vec{v}_1 - \overline{e^{\lambda_1 t}\vec{v}_1}\right).$$

It is possible to show that $\vec{x}_R$ and $\vec{x}_I$ are LI (not shown here, try it as an exercise!). Therefore, the general solution in real form is

$$\vec{x}(t) = c_1\vec{x}_R + c_2\vec{x}_I = c_1 Re\{e^{\lambda_1 t}\vec{v}_1\} + c_2 Im\{e^{\lambda_1 t}\vec{v}_1\}.$$

Example 6.3.4.8: For $\vec{x}' = A\vec{x}$, $A = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}$. Find the general solution in real form and the unique solution for $\vec{x}(0) = [1 \;\; 3]^T$.

Answer: The general solution in complex form was found In **Example 6.3.4.7**.

$$\vec{x}_1 = e^{(-1+3i)t}\begin{bmatrix} 1 \\ i \end{bmatrix}.$$

The solutions in real form are its real and imaginary parts, respectively. Thus,

$$\vec{x}_1 = e^{-t}(\cos 3t + i\sin 3t)\begin{bmatrix} 1 \\ i \end{bmatrix} = e^{-t}\begin{bmatrix} \cos 3t + i\sin 3t \\ i\cos 3t - \sin 3t \end{bmatrix}$$

$$= e^{-t}\begin{bmatrix} \cos 3t \\ -\sin 3t \end{bmatrix} + ie^{-t}\begin{bmatrix} \sin 3t \\ \cos 3t \end{bmatrix}.$$

Therefore,

$$\vec{x}_R = Re\{e^{\lambda_1 t}\vec{v}_1\} = e^{-t}\begin{bmatrix} \cos 3t \\ -\sin 3t \end{bmatrix}, \quad \vec{x}_I = Im\{e^{\lambda_1 t}\vec{v}_1\} = e^{-t}\begin{bmatrix} \sin 3t \\ \cos 3t \end{bmatrix}.$$

The general solution in real form is

$$\vec{x}(t) = c_1\vec{x}_R + c_2\vec{x}_I = c_1 e^{-t}\begin{bmatrix} \cos 3t \\ -\sin 3t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} \sin 3t \\ \cos 3t \end{bmatrix}.$$

Substitute the IC $\vec{x}(0) = [1 \;\; 3]^T$ into the solution, one gets

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \vec{x}(0) = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which yields $c_1 = 1,\ c_2 = 3$.

Or, using Cramer's rule

$$c_1 = \frac{\det[\vec{x}(0) \quad \vec{x}_I(0)]}{\det[\vec{x}_R(0) \quad \vec{x}_I(0)]} = \frac{\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = 1.$$

$$c_2 = \frac{\det[\vec{x}_R(0) \quad \vec{x}(0)]}{\det[\vec{x}_R(0) \quad \vec{x}_I(0)]} = \frac{\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = 3.$$

Thus, the unique solution is

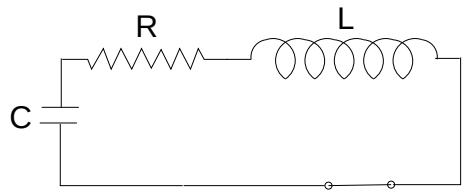
$$\vec{x}(t) = \vec{x}_R + 3\vec{x}_I = e^{-t} \begin{bmatrix} \cos 3t \\ -\sin 3t \end{bmatrix} + 3e^{-t} \begin{bmatrix} \sin 3t \\ \cos 3t \end{bmatrix} = e^{-t} \begin{bmatrix} \cos 3t + 3\sin 3t \\ 3\cos 3t - \sin 3t \end{bmatrix}.$$

Remarks:

(a) In most cases, we prefer the general solution in real form. If not specifically mentioned, we always assume that we are looking for the solution in real form.

(b) The unique solution found in **Example 6.3.4.8** is identical to that found in **Example 6.3.4.7(c)**. (For your own exercise, show it!)

Example: Solving a realistic RLC circuit: $LI''(t) + RI'(t) + \frac{1}{C}I(t) = 0$.



In matrix form, the system is

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \vec{x} = A\vec{x}, \qquad \vec{x}(t) = \begin{bmatrix} I(t) \\ I'(t) \end{bmatrix}.$$

Let $\frac{1}{LC} = 1$, $\frac{R}{L} = 2\gamma, (\gamma > 0)$. Then, $A = \begin{bmatrix} 0 & 1 \\ -1 & -2\gamma \end{bmatrix}$.

(a) Find values of γ for which the "under-damped" or "oscillatory" solutions occur in this system.

(b) Find the general solution in real form when γ is in the range found in (a).

(c) Find the solution for $\gamma = 0.6$ and for $\vec{x}(0) = [2 \quad -1]^T$.

(d) What happens to the circuit as $t \to \infty$?

Answer:

(a) Oscillations occur when the eigenvalues are complex because complex eigenvalues give rise to sine and cosine functions.

Notice that $Tr = -2\gamma$ and $Det = 1$,

$$\lambda^2 + 2\gamma\lambda + 1 = \lambda^2 + 2\gamma\lambda + \gamma^2 + 1 - \gamma^2 = (\lambda + \gamma)^2 + 1 - \gamma^2 = 0$$

Now, it is obvious, complex eigenvalues occur when $1 - \gamma^2 > 0 \Rightarrow \gamma < 1$. Under this condition,

$$\lambda_{1,2} = -\gamma \pm i\sqrt{1 - \gamma^2} = -\gamma \pm i\omega,$$

where $\omega = \sqrt{1 - \gamma^2}$ or $\omega^2 = 1 - \gamma^2$.

(b) For $\lambda_1 = -\gamma + \omega i$:

$$\vec{v}_1 = \ker(A - (-\gamma + \omega i)I) = \ker \begin{bmatrix} \gamma - \omega i & 1 \\ -1 & -\gamma - \omega i \end{bmatrix} \xrightarrow[\gamma^2+\omega^2=\gamma^2+1-\gamma^2=1]{R_1' = (\gamma+\omega i)R_1}$$

$$\ker \begin{bmatrix} 1 & \gamma + \omega i \\ -1 & -\gamma - \omega i \end{bmatrix} \xrightarrow{R_2' = R_2 + R_1} \ker \underbrace{\begin{bmatrix} 1 & \gamma + \omega i \\ 0 & 0 \end{bmatrix}}_{\vec{c}_1 - (\gamma - \omega i)\vec{c}_2 = \vec{0}} = \begin{bmatrix} 1 \\ \omega i - \gamma \end{bmatrix}.$$

Thus, complex-valued solution is

$$\vec{x}_1 = e^{\lambda_1 t}\vec{v}_1 = e^{(-\gamma+\omega i)t} \begin{bmatrix} 1 \\ \omega i - \gamma \end{bmatrix} = e^{-\gamma t}(\cos \omega t + i \sin \omega t) \begin{bmatrix} 1 \\ \omega i - \gamma \end{bmatrix}$$

$$= e^{-\gamma t}\begin{bmatrix} \cos\omega t + i\sin\omega t \\ -\gamma\cos\omega t - \omega\sin\omega t + i(\omega\cos\omega t - \gamma\sin\omega t) \end{bmatrix}$$

$$= e^{-\gamma t}\begin{bmatrix} \cos\omega t \\ -\gamma\cos\omega t - \omega\sin\omega t \end{bmatrix} + ie^{-\gamma t}\begin{bmatrix} \sin\omega t \\ \omega\cos\omega t - \gamma\sin\omega t \end{bmatrix}.$$

Therefore, the general solution in real form is

$$\vec{x}(t) = c_1 Re\{e^{\lambda_1}\vec{v}_1\} + c_2 Im\{e^{\lambda_1}\vec{v}_1\}$$

$$= c_1 e^{-\gamma t}\begin{bmatrix} \cos\omega t \\ -\gamma\cos\omega t - \omega\sin\omega t \end{bmatrix} + c_2 e^{-\gamma t}\begin{bmatrix} \sin\omega t \\ \omega\cos\omega t - \gamma\sin\omega t \end{bmatrix}.$$

(c) For $\gamma = 0.6$, $\omega = \sqrt{1-\gamma^2} = \sqrt{0.64} = 0.8$. The solution now reads

$$\vec{x}(t) = c_1 e^{-0.6t}\begin{bmatrix} -\cos 0.8t \\ 0.6\cos 0.8t + 0.8\sin 0.8t \end{bmatrix} + c_2 e^{-0.6t}\begin{bmatrix} \sin 0.8t \\ 0.8\cos 0.8t - 0.6\sin 0.8t \end{bmatrix}$$

$$= c_1\vec{r}_1(t) + c_2\vec{r}_2(t).$$

Using the initial condition $\vec{x}(0) = [2 \quad -1]^T$ and Cramer's rule,

$$c_1 = \frac{\det[\vec{x}(0) \quad \vec{r}_2(0)]}{\det[\vec{r}_1(0) \quad \vec{r}_2(0)]} = \frac{\begin{vmatrix} 2 & 0 \\ -1 & 0.8 \end{vmatrix}}{\begin{vmatrix} -1 & 0 \\ 0.6 & 0.8 \end{vmatrix}} = \frac{1.6}{-0.0} = -2.$$

$$c_2 = \frac{\det[\vec{r}_1(0) \quad \vec{x}(0)]}{\det[\vec{r}_1(0) \quad \vec{r}_2(0)} = \frac{\begin{vmatrix} -1 & 2 \\ .6 & -1 \end{vmatrix}}{\begin{vmatrix} -1 & 0 \\ 0.6 & 0.8 \end{vmatrix}} = \frac{1 - 1.2}{-0.8} = \frac{1}{4}.$$

Therefore, the unique solution is

$$\vec{x}(t) = -2e^{-0.6t} \begin{bmatrix} -\cos 0.8t \\ 0.6\cos 0.8t + 0.8\sin 0.8t \end{bmatrix} + \frac{e^{-0.6t}}{4} \begin{bmatrix} \sin 0.8t \\ 0.8\cos 0.8t - 0.6\sin 0.8t \end{bmatrix}.$$

(d) As $t \to \infty$, $e^{-0.6t} \to 0$, thus

$$\vec{x}(\infty) = \lim_{t\to\infty} \vec{x}(t) = \vec{0}.$$

Therefore, the oscillation eventually damps out after a very long time.