Be sure that this examination has 2 pages.

The University of British Columbia
Final Examinations - December 2010

Mathematics 305

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Closed book examination. No notes, texts, or calculators allowed. Time: 2 1/2 hours

Special Instructions: No notes, book, or calculator allowed

Marks

[40] 1. Identify whether each of the following statements are true or false. You must give reasons for your answers.

(i) $\text{Arg}(z_1z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$.
(ii) $\text{Re}(i/\bar{z}) = -\text{Im}(z)/|z|^2$.
(iii) $\sin(n\theta) = \text{Im}\{(\cos \theta + i \sin \theta)^n\}$ where $n$ is a positive integer.
(iv) $f(z) = |z|^2$ is analytic at $z = 0$ but not at any other point.
(v) $u = r^n \cos(n\theta)$ is a harmonic function, where $n$ is a positive integer, $r^2 = x^2 + y^2$ and $\tan \theta = y/x$.
(vi) If $f(z) = u + iv$ is an entire function, then $u^2 - v^2$ is a harmonic function.
(vii) Let $M = \max(|e^{iz^2}|)$ over the disk $|z| \leq 2$. Then, $M = 1$.
(viii) $|\sin(z)|$ is bounded as $|z| \to \infty$.
(ix) the equation $\sqrt{z} + (1 - i) = 0$, where $\sqrt{z}$ is the principal branch of the square root function, has no solution.
(x) $|e^{z^2}| \leq e^{|z|^2}$ for all $z$.
(xi) $\log(e^z) = z$.
(xii) $\int_C z^{-1/2} \sin(\sqrt{z})dz = 0$ where $C$ is the simple closed curve $|z| = 1$ oriented counterclockwise, and $\sqrt{z}$ is the principal branch of the square root function.

[15] 2. Consider the function $f(z)$ defined by

$$f(z) = \frac{z}{z^2 - z - 2}.$$
(i) Determine the Laurent series of \( f(z) \) centered at \( z_0 = 0 \) that converges in the region \( |z| > 2 \).

(ii) By using the Laurent series in (i), and by integrating it term by term, evaluate \( \int_C f(z) \, dz \) where \( C \) is the simple closed curve \( |z| = 4 \) oriented counterclockwise. Confirm your result by using the residue theorem applied to the function \( f(z) \) on the region \( |z| \leq 4 \).

[15] 3. Consider the following function \( f(z) \) defined by

\[
f(z) = \frac{1}{z(1 - \cos(\sqrt{z}))(z - \pi^2)}.
\]

(i) Identify and then classify all of the singular points of \( f(z) \) in the complex plane.

(ii) Calculate \( \int_C f(z) \, dz \) where \( C \) is the circle \( |z| = 10 \) oriented in a counterclockwise sense.

[15] 4. Let \( a > 0 \) with \( a \) real. By using residue theory, calculate values for the following integrals in as compact a form as you can:

\[
\begin{align*}
\text{(i) } I &= \int_0^{2\pi} \frac{1}{a + \cos \theta} \, d\theta, \quad \text{with } a > 1; \\
\text{(ii) } I &= \int_0^{\infty} \frac{x \sin x}{x^2 + a^2} \, dx.
\end{align*}
\]

[15] 5. By using residue theory, calculate the following integrals:

\[
\begin{align*}
\text{(i) } I &= \int_0^{\infty} \frac{\sin x}{x(x^2 + 1)} \, dx; \\
\text{(ii) } I &= \int_0^{\infty} \frac{\sqrt{x}}{(x^2 + 1)} \, dx.
\end{align*}
\]

[100] Total Marks

The End
(i) \[ \text{ARG}(z_1 z_2) = \text{ARG}(z_1) + \text{ARG}(z_2) \]

**FALSE:** If \( z_1 = e^{\frac{3\pi i}{4}} \) and \( z_2 = e^{\frac{3\pi i}{4}} \),

then \( \text{ARG}(z_1 + z_2) = 3\pi/4 + 3\pi/4 = 3\pi/2 \)

\[ \text{ARG}(z_1 z_2) = \text{ARG}(e^{\frac{3\pi i}{2}}) = -\pi/2 \]

(ii) \[ \text{RE}(\frac{i}{z}) = -\frac{\text{IM}(z)}{|z|^3} \quad \text{true} \]

\[
\frac{1}{z} = \frac{i z}{|z|^2} = \frac{i x - y}{|z|^2} \quad \text{so} \quad \frac{\text{IM}(\frac{i}{z})}{|z|^2} = \frac{x}{|z|^2} = \frac{\text{RE}(\frac{z}{|z|^2})}{|z|^2}
\]

\[ \text{RE}(\frac{i}{z}) = -\frac{\text{IM}(z)}{|z|^2} \]

(iii) \[ \sin \left( n \varphi \right) = \text{IM} \left[ e^{i n \varphi} \right] \]

\[ n = 0, 1, 2, \ldots \]

**true** let \( z = e^{i \varphi} \)

then \( z^n = \left( e^{i \varphi} \right)^n = \sum_{k=0}^{n} \frac{n!}{k! (n-k)!} e^{i k \varphi} \)

\[ \text{let} \quad \Gamma = 1 \text{ and EQUATE IMAGINARY PARTS.} \]

\[ \sin \left( n \varphi \right) = \text{IM} \left[ \left( \cos \varphi + i \sin \varphi \right)^n \right] \]

(iv) \[ f(z) = |z|^2 = (x^2 + y^2) \] is **not analytic anywhere since CR EQUATION

\[ u_x = v_y \rightarrow x = 0 \text{ only hold at the isolated } \]

\[ u_y = -v_x \rightarrow y = 0 \text{ point (0, 0) and not in neighborhood of (0, 0).} \quad \text{FALSE} \]

(v) \[ u = \sum_{n=1}^{\infty} c_n (x^2 + y^2)^n \] \( u \) **HARMONIC** since \( f(z) = z^2 \)

\[ u \text{ ANALYTIC} \forall z \] and \( u = \text{RE} \left[ f(z) \right] \text{.} \quad \text{TRUE} \]

(vi) **true** let \( f = u + iv \) be entire. Then

\[ f^2 = u^2 - v^2 + 2iuv \] entire function.

\[ u^2 - v^2 = \text{RE} \left[ f^2 \right] \text{ is HARMONIC, since } f^2 \text{ is entire.} \]
(Vii) \text{FALSE}

The max occur on the boundary by max module principal.

Thus \[ M = \max |e^{i z^2}| = e^{\max |z^2|} = e^{4} \]

let \( i z = \theta \) \( 4 \rightarrow z = -4i = 4 e^{i \pi/2} \)

\[ z = 4 e^{3 \pi i/4} \]

\[ \rightarrow z = 2 e^{3 \pi i/4} \]

(Viii) \text{FALSE}

\( |\sin(z)| \) unbounded at \( |z| \to \infty \) when \( z = iy \)

with \( y \to \infty \).

\[ \sin(iy) = \frac{e^{i iy} - e^{-i iy}}{2i} = -\frac{e^{y} - e^{-y}}{2i} = i \left( \frac{e^{y} - e^{-y}}{2} \right) \]

\[ \sin(iy) = i \sinh(y) \]

so \( |\sin(iy)| = |\sinh(y)| \frac{e^{y/2}}{2} \)

so unbounded if we set \( x = 0 \) and let \( |y| \to \infty \).

(Vii) \[ \sqrt{z} = \Gamma \left( \frac{1}{2} \right) e^{i \phi/2} \]

\( \Gamma \left( \frac{1}{2} \right) e^{i \phi/2} \)

\( -\pi < \phi < \pi \)

\[ \text{true} \]

\[ \text{Re} \left( \sqrt{z} \right) = \Gamma \left( \frac{1}{2} \right) \left( \frac{\phi}{2} \right) \geq 0 \), \text{guaranteed by branch choice.} \]

\[ \text{so} \]

\[ \text{Re} \left( \sqrt{z} \right) + 1 = 0 \text{ is impossible, since } \text{Re} \left( \sqrt{z} \right) > 0. \]

(Vi) \[ |e^w| \leq e^{|w|} \text{ true. let } w = z^2 \text{ and use } |z^2| + |z|. \]

Proof: let \( w = u + iv \).

\[ |e^{u+iv}| = e^u \leq e \]

\[ |w|^2 \]

\[ \text{with } u \leq |w| \]
\( (x) \quad \text{FALSE} \quad \log(e^z) = \log(e^{x+i\gamma}) = \log(e^{x+i\gamma}) = \ln(e^x) + i + 2k\pi i. \)

so \( \log(e^z) = z + 2k\pi i. \)

\( (xi) \quad \int_c z^{-1/2} \sin(\sqrt{z}) \, dz = 0 \quad \text{true.} \)

\text{Note:}\quad \frac{\sin(\sqrt{z})}{\sqrt{z}} = \frac{\sqrt{z} - z^{3/2}/3 + z^{5/2}/5 + \ldots}{\sqrt{z}} = 1 - z^{1/2}/3 + z^{3/2}/5 - z^{5/2}/7 + \ldots \)

\text{Analogy}\quad \forall z.

By \( \quad \int_c z^{-1/2} \sin(\sqrt{z}) \, dz = 0. \)

\text{Problem 2}\quad \frac{F(z)}{(z-2)(z+1)} = A + \frac{B}{z-2} \rightarrow z: A(z+1) + B(z-2)

(i) \quad \text{let} \quad z = 2 \rightarrow A = \frac{2}{3}

\quad z = -1 \rightarrow B = \frac{1}{3}.

\( \quad F(z) = \frac{2}{3(z-2)} + \frac{1}{3(z+1)} = \frac{2}{3z(z-2)} + \frac{1}{3z(z+1)} \)

\( \quad F(z) = \sum_{j=0}^{\infty} \frac{z^j}{3z} \left( \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!} \right) \quad \text{converge, if } |z| > 1; \)

\( \quad F(z) = \frac{2}{3z} \left( 1 + \frac{2}{z} + \frac{4}{z^2} + \ldots \right) + \frac{1}{3z} \left( 1 - 1/z + \frac{1}{z^2} - \frac{1}{z^3} + \ldots \right) \)

(ii) \quad \text{now integrate term by term since } |z| = 4 \text{ is in zone of convergence. we recall } \int \frac{1}{z^p} \, dz = 0 \text{ for } p > 1, \text{ and } \int \frac{1}{z^2} \, dz = 2\pi i. \)
Thus, \( I = \int \frac{f(z)}{z} \, dz = \int \left( \frac{1}{3z} + \frac{1}{3z} \right) \, dz = 2\pi i \).

Now we restate the theorem.

\[
I = \int_{|z|=4} \frac{f(z)}{z} \, dz = 2\pi i \left( \text{Re} \left[ \int_{|z|=1} f(z) \, dz \right] + \text{Re} \left[ \int_{|z|=2} f(z) \, dz \right] \right)
\]

\[
= 2\pi i \left( \frac{z}{2z-1} \int_{|z|=1} + \frac{z}{2z-1} \int_{|z|=2} \right) = 2\pi i \left( \frac{1 + \frac{2}{3}}{3} \right) = 2\pi i.
\]

**Problem 3**

(i) \( z = \pi^2 \) is a simple pole.

\[
R = \max \left\{ \frac{1}{|z-\pi^2|} \right\} = \frac{1}{|\pi-\pi^2|} = \frac{1}{\pi-\pi^2}
\]

\[
\cos \left( \sqrt{z} \right) = 1 \quad \text{with} \quad z \neq 0 \quad \text{is a simple pole}
\]

\[
\sqrt{z} = 2\pi i, \quad z = 4\pi^2, \quad \text{for} \quad k = 1, 2, 3, \ldots, \text{simple pole}
\]

Near \( z = 0 \), we get

\[
\cos \left( \sqrt{z} \right) \sim 1 - \frac{z}{2} + \frac{z^2}{4} + \ldots
\]

\[
1 - \cos \left( \sqrt{z} \right) \sim \frac{z}{2} - \frac{z^2}{2} + \frac{z^3}{24} = \frac{z}{2} \left( 1 - \frac{z}{12} \right)
\]

So near \( z = 0 \),

\[
f(z) \sim \frac{1}{z} \left( z^{\frac{1}{2}} - z^{\frac{3}{2}} + \ldots \right) = \frac{1}{z^2}
\]

\( z = 0 \) is a pole of order 2.

(ii) Now inside \( |z| = 4 \) we have a simple pole at \( z = \pi^2 \) and a double pole at \( z = 0 \).

\[
\text{So,} \quad \int_{|z|=1} \frac{f(z)}{z} \, dz = 2\pi i \left( \text{Re} \left[ \int_{|z|=0} f(z) \, dz \right] + \text{Re} \left[ \int_{|z|=2} f(z) \, dz \right] \right)
\]

\[
= 2\pi i \left( \frac{1 + \frac{2}{3}}{3} \right) = 2\pi i.
\]
\[ f(z) \propto \left[ \frac{1}{\sqrt{\pi}} \left( 1 - \cos(\pi) \right) \right] \propto \frac{1}{2\pi^2 (z - \pi^{'})}. \]

\[ \text{Re} \left[ f(z) \right] = \frac{1}{2\pi^{'}}. \]

Now near \( z = 0 \),

\[ f(z) \propto \frac{1}{z \left( \frac{z^2 - z'^2}{2} \right) \left( z - \pi^{'}) \right)} \propto \frac{1}{z - \pi^{'}} \]

\[ \propto \frac{1}{\frac{z^2}{2} \left( 1 - \frac{z}{\pi^{'}} \right) \left( 1 - \frac{z}{\pi} \right) \left( \pi^{' - 1} + z \right) + \frac{2}{\pi^{'}} \left( \frac{z}{\pi} \right)} \]

\[ f(z) \propto -\frac{2}{\pi^{'}} \left( \frac{z}{\pi^{'}} \right) \left( 1 + z \left( \frac{1}{\pi^{'}} + \frac{1}{\pi} \right) \right) \]

\[ \text{Re} \left[ f(z) \right] \propto -\frac{2}{\pi^{'}} \left( \frac{1}{\pi^{'}} + \frac{1}{\pi} \right). \]

So

\[ \int_c f(z) \, dz = 2\pi i \left[ \frac{1}{2\pi^{'}} \left( \frac{1}{\pi^{'}} + \frac{1}{\pi} \right) \right]. \]

\[ C: |z| = 10 \quad = 2\pi i \left[ \frac{1}{2\pi^{'}} \left( \frac{1}{\pi^{'}} + \frac{1}{\pi} \right) \right]. \]
Problem 4

(i) \[ I = \int_0^{2\pi} \frac{1}{a + \cos \phi} \, d\phi, \quad a > 1. \]

\[ \cos \phi = \frac{z + \sqrt{z^2 - 4}}{2}, \quad \frac{dz}{i} \]

\[ I = \int_C \frac{1}{a + \left( \frac{z + \sqrt{z^2 - 4}}{2} \right)} \frac{dz}{i z} = -i \int_C \left( \frac{1}{(a z + \frac{z^2 + 1}{2})} \right) \frac{dz}{i} \]

\[ I = -2i \int_C \frac{dz}{z^2 + 2a z + 1} \quad C: |z| = 1 \text{ counterclockwise.} \]

Roots are simple poles at \[ z = -2a \pm \sqrt{4a^2 - 4} = -2a \pm \sqrt{a^2 - 1} \]

Only root inside \[ z^+ = \frac{2}{2} \sqrt{a^2 - 1} \text{ since } |z| > 1 \]

Then \[ I = -2i \left[ 2\pi i \text{ Re} \left( f, z^+ \right) \right] = 4\pi \frac{1}{2z^+ + 2a} \]

\[ I = 2\pi i \frac{1}{z^+ + 1} = 2\pi i \frac{1}{\sqrt{a^2 - 1}} \text{ valid for } a > 1. \]

(ii) \[ I = \frac{1}{2} \text{ Im}(J), \quad J = \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + a^2} \, dx. \]

Now by Jordan's Lemma, we integrate over semi-circle in upper 1/4 plane.

\[ J = \lim_{R \to \infty} \int_{C_R} e^{ix} dz = 2\pi i \text{ Re} \left( \frac{z e^{iz}}{z^2 + a^2} \right) \]

\[ = 2\pi i \left( \frac{i a e^{-a}}{2ja} \right) = \pi e^{-a}. \]

\[ I = \frac{1}{2} \text{ Im} \left( \pi i e^{-a} \right) = \pi e^{-a}/2. \]
Problem 5

(ii). We let $\sqrt{z}$ be the principal branch of $\sqrt{\cdot}$.

We integrate along the contour shown. Over top of branch cut,

\[
\lim_{R \to \infty} \left( I_L + I_R + C_R \right) = 2\pi i \text{ Re} \left( \frac{\sqrt{z}}{z^2+1} \right)
\]

\[
= 2\pi i \left( \frac{2i}{4} \right) = \pi i/2
\]

Now \[
\lim_{R \to \infty} \left| \frac{\sqrt{z}}{z^2+1} \right| dz \leq \left( \frac{R^{1/2}}{R^2} \right) R \to 0, R \to \infty.
\]

Now on $I_R$, $z = R e^{i \pi}$, $dz = e^{i \pi} i$ $dr$.

So \[
I_R = \int_0^\infty -\left( R^{1/2} e^{i \pi/2} \right) \frac{1}{R^2 + 1} \frac{1}{R} \, dR = i \int_0^\infty \frac{1}{R^{1/2} + 1} \, dR.
\]

Thus let \[
I = \int_0^\infty \frac{\sqrt{x}}{x^{1/2} + 1} \, dx.
\]

Then \[
(1 + i) I = \pi \left( 1 + i \right) \sqrt{2},
\]

\[
\Rightarrow I = \pi \sqrt{2}.
\]

(iii) \[
I = \text{Im} \left( \int_0^\infty \frac{e^{ix}}{x(x^2+1)} \, dx \right).
\]

Let \[
J = \int_0^\infty \frac{e^{ix}}{x(x^2+1)} \, dx
\]

We need indented contour as shown:

\[
\text{Re} \left( \frac{e^{iz}}{z(z^2+1)} \right)
\]

\[
J + \text{Im} \left( \int_0^\infty \frac{e^{iz}}{z(z^2+1)} \, dz \right) = 2\pi i \text{ Re} \left( \frac{e^{iz}}{z(z^2+1)} \right)
\]

\[
J \approx \frac{\pi}{2i} = 2\pi i \left[ \frac{e^{-1/i}}{i} \right] = \pi e^{-1} = i\pi e^{-1}.
\]
so \( \mathcal{J} = i \mathcal{R} (1 - e^{-1}) \).

\[ \tilde{I} = \frac{1}{2} \text{IM} (\mathcal{J}) \]

so \( \tilde{I} = \frac{\tilde{I}}{2} (1 - e^{-1}) \).
Marks

[30] 1. Identify whether each of the following statements are true or false. You must give
reasons for your answers to receive credit. (Hint: very little calculation is needed to
solve these).

(i) $\log(z^2) = 2 \log(z)$.

(ii) $|e^{-z^2}| \leq 1$ when $|\arg(z)| \leq \pi/2$.

(iii) $f(z) = |z|^2$ is differentiable at $z = 0$ but is not analytic at $z = 0$.

(iv) If $f(z) = u + iv$ is an entire function, then $uv$ is a harmonic function.

(v) Let $f(z) = z(z - i)$. Then $\max_{|z| \in D} |f(z)| = 2$, where $D$ is the region $|z| \leq 1$.

(vi) Suppose that $f(z)$ is an entire function that satisfies $|f(z)| > 1$ for all $z$. Then,
$f(z)$ must be the constant function.

[10] 2. Let $f(z) = (z^2 + 1)^{1/3}$. We seek to construct a branch of $f(z)$ that is analytic in $|z| < 1$,
with branch cuts on portions of the imaginary axis, and that satisfies $f(0) = 1$.

(i) Show how to construct this branch by specifying the range of angles $\arg(z - i)$
and $\arg(z + i)$ appropriately.

(ii) Next, define this branch of $f(z)$ in terms of the principal value of some logarithm
function.

(iii) For this branch of $f(z)$ calculate $f(2 + 2i)$.

Continued on page 2
3. Calculate each of the following integrals over the simple closed curve $C$:
   (i) $\int_C z^5/(z^6 + 2z) \, dz$ where $C$ is the counter-clockwise circle $|z| = 2$.
   (ii) $\int_C z^3 e^{1/z} \, dz$ where $C$ is the counter-clockwise circle $|z| = 1$.
   (iii) $\int_C z^{-4} \sin(3z) \, dz$ where $C$ is the counter-clockwise circle $|z| = 1$.
   (iv) $\int_C (z - 2i)^{-2} \log(z) \, dz$, where $\log(z)$ denotes the principal value of the logarithm function, and $C$ is the counter-clockwise circle $|z - 2i| = 1$.

4. Consider the function $f(z)$ defined by
   $$f(z) = \frac{\sin(iz/4)}{z^2(1 - e^z)}.$$  
   (i) Identify and then classify all of the singular points of $f(z)$ in the complex plane.
   (ii) Calculate the first two terms in the Laurent expansion of $f(z)$ in powers of $z$ which converges in $0 < |z| < r_1$. What is the radius $r_1$ of convergence of this series?
   (iii) Calculate $I = \int_C f(z) \, dz$ where $C$ is the counter-clockwise circle $|z| = 1$.

5. Calculate the following integrals in as explicit a form as you can:
   (i) $I = \int_0^\infty \frac{x^{1/3}}{x^2 - 4x + 8} \, dx$
   (ii) $I = \int_0^{2\pi} \frac{\cos(n\theta)}{1 + k \cos(\theta)} \, d\theta$.
   In (ii), $n$ is a non-negative integer and $k$ is real with $k^2 < 1$. (Hint: In (ii) it may be helpful to first write $\cos(n\theta) = \text{Re}(e^{in\theta})$.)

6. Suppose that $p(z) = a_0 + a_1 z + \cdots + a_N z^N$ is a polynomial of degree $N \geq 2$ with $a_N \neq 0$ and $a_0 \neq 0$. Suppose that $z_1, \ldots, z_N$ are distinct roots of $p(z) = 0$. By using residue theory applied to the integral
   $$\int_C \frac{p'(z)}{z^2 p(z)} \, dz,$$
   where the contour $C$ is to be chosen appropriately, derive an explicit formula for the sum
   $$S = \sum_{j=1}^N \frac{1}{z_j^2},$$
   in terms of some of the coefficients $a_0, \ldots, a_N$ of the polynomial. Does your formula still work if the roots of the polynomial $p(z)$ are not distinct?

Total Marks

The End
Problem 1

(i) \( \log(z^2) \neq 2 \log(z) \) \( \iff \) False

let \( z = e^{3\pi i/4} \)

\[
\log(z^2) = -i\pi/2
\]

\[
2 \log(z) = 3\pi i/2
\]

(ii) \( |e^{-z^3}| \leq 1 \) \( \iff \) \( \text{arg} \ z \leq \pi/2 \) \( \text{False} \)

let \( z = re^{i\phi} \)

\[
|e^{-z^3}| = |e^{-r^3 \cos(3\phi) - i r^3 \sin(3\phi)}| = e^{-r^3 \cos(3\phi)} < 1
\]

when \( \cos(3\phi) > 0 \) \( \rightarrow \) \( -\pi/6 < 3\phi < \pi/6 \) \( \rightarrow \) \( -\pi/6 < \phi < \pi/6 \)

(iii) \( \) True:

\[
f = x^2 + y^2 + i0 \quad \text{so} \quad u = x^2 + y^2, \quad v = 0
\]

\[
\begin{align*}
\frac{\partial u}{\partial x} &= 2x, & \frac{\partial u}{\partial y} &= -2y \\
\frac{\partial v}{\partial x} &= 0, & \frac{\partial v}{\partial y} &= 0
\end{align*}
\]

\( f \) is differentiable at \( z = 0 \)

\( f \) is not analytic at \( z = 0 \).

(iv) \( f(z) = z(z-i) \)

By Max Modulus Principle

\[
\max_{|z|\leq1} |f(z)| = \max_{|z|\leq1} |z(z-i)| = \max_{|z|\leq1} |z-i| = 2, \quad \text{occur when}
\]

\( z = i \)

(v) True. There is no point \( z_0 \) where \( f(z_0) = 0 \). Since \( |f(z_0)| > 1 \).

Thus \( g(z) = 1/f(z) \) is an analytic function that satisfies:

\[
|g(z)| \leq 1 \quad \forall \ z \quad \text{by Liouville's Theorem,} \quad g(z) \text{ and hence } f(z) \text{ is the constant function.} \]
Problem 2

(i) \[ f(z) = \left( z + i \right)^{\frac{1}{3}} \left( z - i \right)^{\frac{1}{3}} = \left( e^{i \theta_1}, e^{i \theta_2} \right)^{\frac{1}{3}} e^{i \left( \frac{\theta_1 + \theta_2}{3} \right)} \]

Choose \(-\frac{\pi}{2} \leq \theta_1 \leq \frac{3\pi}{2}\)

\(-\frac{\pi}{2} \leq \theta_2 \leq \frac{\pi}{2}\)

Then at \( z = 0 \):

\( \theta_1 = -\frac{\pi}{2}, \theta_2 = \frac{\pi}{2} \)

\[ f(0) = 1. \]

(ii) \[ (z^2 + 1)^{\frac{1}{3}} = e^{\frac{1}{3} \log(z^2 + 1)} \]

TAKE \( f(z) = e^{\frac{1}{3} \log(z^2 + 1)} \)

Now \( f(z) \) is analytic except where \( \text{Re} \left( z^2 \right) + 1 < 0 \) \( \uparrow |y| \geq 1 \).

\( |\text{Im} \left( z^2 \right) | = 0 \rightarrow z = iy \)

(iii) Now for \( z = 2 + 2i \).

\[ f(2 + 2i) = e^{\frac{1}{3} \log(1 + 3i)} = e^{\frac{1}{3} \log \left( \sqrt{5} \right) + \frac{1}{3} \tan^{-1} \left( 3 \right) \tan^{-1} \left( 3 \right)} \]

\[ f(2 + 2i) = \sqrt{5} e^{\frac{1}{2} \tan^{-1} \left( 3 \right)} \]
PROBLEM 3

(i) \( I = \int_{C} \frac{z^5}{z^6 + 2z} \, dz \quad C: |z| = 2. \)

Singularities at \( z = 0 \) and \( z^6 = -2 \) \( \Rightarrow |z| = 2^{1/6} < 2. \)

All singularities inside \( C. \)

So \( I \sim \int_{C} \frac{z^5}{z^6} \, dz = 2\pi i. \)

(ii) \( I = \int_{C} z^3 \left( 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \frac{1}{5z^5} \right) \, dz = \frac{2\pi i}{4!} \cdot \frac{\pi i}{12} \)

(iii) \( I = \int_{C} \frac{\sin 3z}{z^4} \, dz = \int_{C} \frac{1}{z^4} \left( 3z - \left( \frac{3z}{3!} \right)^3 \right) \, dz = \frac{-27}{6} \cdot \frac{2\pi i}{2\pi i} = -9\pi i \)

(iv) \( I = \int_{C} \frac{1}{(z - 2i)^2} \log(z) \, dz = 2\pi i \left[ \frac{d}{dz} \log(z) \right] \left|_{2i}^{2i} \right. = \frac{2\pi i}{2i} = \pi. \)

PROBLEM 4

(i) \( z = 0 \) is a double pole, \( z = 2m\pi i \) \( m: 0, 1, 2, \ldots \)

Now if \( m = \text{odd} \) then simple pole.

if \( m = \text{even} \) then removable singularity.

(ii) Radius of convergence is \( 2\pi. \)
A. \( z \to 0, \)

\[
\mathcal{F}(z) \sim \frac{\left(\frac{iz}{4}\right) - \left(\frac{iz}{4}\right)^3}{\left(\frac{i^3}{4}\right)^3} \frac{1}{z} = \frac{\left(\frac{iz}{4}\right)}{z^2} \left[ 1 + \frac{z^2}{q_0} \right] \frac{1}{z} = \frac{\left(\frac{iz}{4}\right)}{z^2} \left[ 1 - \frac{z^2}{q_0} \right] \left[ 1 - \frac{z}{z} \right]
\]

\[
\mathcal{F}(z) \sim -\frac{i}{4z^2} \left[ 1 + \frac{z^2}{q_0} \right] \left[ 1 - \frac{z}{z} \right] \sim -\frac{i}{4z^2} \left( 1 - \frac{z}{z} \right) \sim \frac{i}{4z^2} + \frac{i}{z}
\]

Thus \( q_1 = \frac{i}{4} \)

\[
\int_C \mathcal{F}(z) \, dz = 2\pi i \left( \frac{i}{4} \right) = -\pi i
\]

**Problem 5**

\[
\text{Nov} \quad \left| \int_C \mathcal{F}(z) \, dz \right|_{R \to 0} \to 0.
\]

\[
z^3 - 4z + 8 = 0 \quad \Rightarrow \quad z = \frac{1}{2} \sqrt[3]{16 - 32} = 2 \pm 2i.
\]

Notice that \( z_1 = 2 + 2i \) inside \( C. \)

\[
\text{Thus} \quad \int_{C} \mathcal{F}(z) \, dz = 2\pi i \left( \frac{z_{1/3}}{z_1 - 4} \right) = 2\pi i \frac{i^{1/3}}{2i} = 2\pi i \frac{i^{1/3}}{2i} = \frac{\pi}{2} i^{1/3}
\]

Now on \( I_1: \)

\[
dz = dr \Rightarrow \int_{I_1} = -\int_0^r \frac{1}{2} \, e^{i\pi/3} \, dr.
\]

Thus \( e^{i\pi/3} J + I = \frac{\pi}{2} i^{1/3} e^{i\pi/2} \)

J: \( \int_0^r \frac{1}{r^{1/3}} \, dr \)

I: \( \int_0^r \frac{1}{r^{1/3}} \, dr. \)
\[ I + e^{-i\pi/3} I = \frac{\pi}{2} \sqrt{\beta^{1/3}} e^{-i\pi/4} \]

Then
\[ IM\left( e^{-i\pi/3} I \right) = \frac{\pi}{2} \sqrt{\beta^{1/3}} IM\left( e^{-i\pi/4} \right) \]

For
\[ I \begin{align*}
\sin \left( \frac{\pi}{3} \right) &= \frac{\pi}{2} \sqrt{8^{1/3}} \sqrt{2} = \frac{\pi}{4} \\
I \frac{\sqrt{3}}{2} &= \frac{\pi}{2} \sqrt{8^{1/3}} \sqrt{2} = \frac{\pi}{4} \sqrt{2} \left[ 8^{1/6} \right]
\end{align*} \]

so
\[ I \frac{\sqrt{3}}{2} = \frac{\pi}{4} \left( 2^{1/2} \right)^{1/6} = \frac{\pi}{2} 2 = \pi. \]

so
\[ I = \frac{\pi}{\sqrt{3}}. \]

(ii)
\[ I = \int_0^{2\pi} \frac{e^{i\eta} \eta}{1 + \frac{4}{\kappa^2} \cos \theta} \, d\theta = \text{RE} \left( \int_0^{2\pi} \frac{e^{i\eta} \eta}{1 + \frac{4}{\kappa^2} \cos \theta} \, d\theta \right). \]

Let \( z = e^{i\theta} \) so
\[ I = \text{RE} \left( \int_C \frac{z^n}{1 + \frac{4}{\kappa^2} \frac{1}{z^{1/2}}} \, \frac{1}{iz} \, dz \right) \]

I = \text{RE} \left[ -i \int_C \frac{z^n}{1 + \frac{4}{\kappa^2} \frac{1}{z^{1/2}}} \, dz \right] = \frac{2\pi}{\kappa} \text{RE} \left[ -i \int_C \frac{z^n}{z^{2/2} + 2z + \frac{1}{\kappa}} \, dz \right]. \]

Now pole at
\[ z = -\frac{2}{\kappa} + \frac{\sqrt{4 + \frac{4}{\kappa^2}}}{\kappa}, \quad -\frac{1}{\kappa} \sqrt{-1} \quad \text{inside } C. \]

\[ I = \frac{2\pi}{\kappa} \text{RE} \left[ -i \frac{2\pi i \text{RE}}{z^n} \left( \frac{z^n}{z^{2/2} + 2z + \frac{1}{\kappa}} ; z^* \right) \right] = \frac{4\pi}{\kappa} \text{RE} \left( \frac{z^n}{2z^* + \frac{2}{\kappa}} \right). \]

Now
\[ 2z^* + \frac{2}{\kappa} = 2\sqrt{1/\kappa^2 - 1} \]

\[ I = \frac{2\pi}{\kappa} \left[ -\frac{1}{\kappa} + \sqrt{\frac{1}{\kappa^2} - 1} \right] \left( \frac{Z^*_+}{\sqrt{1/\kappa^2 - 1}} \right)^n \]
So \[ I = \frac{2\pi}{\eta^D} \frac{\left[ -1 + \sqrt{\eta^2 - 1} \right]^D}{\sqrt{1 - \eta^2}}. \]

**Problem 6**

Let \( C \) enclose \( z = 0 \) and all zeroes of \( p(z) \).

Then since \( \left| \frac{p'(z)}{p(z)} \right| < \frac{d}{R} \) for \( |z| = R > 1 \)

we deform \( \int_C \) to \( \int_{CR} \) by C.I.T. and let \( R \to \infty \)

\[ \int_C \frac{p'}{z^2 p} \, dz = \lim_{R \to \infty} \int_{CR} \frac{p'}{z^2 p} \, dz = 0 \quad \text{since} \quad \left| \int_{CR} \frac{p'}{z^2 p} \, dz \right| \leq \frac{d}{R} \frac{1}{R^2} \quad \text{as} \quad R \to \infty. \]

Then by residue theorem

\[ 0 = \int_C \frac{p'}{z^2 p} \, dz = 2\pi i \sum_{j=1}^{N} \text{Res} \left( \frac{p'}{z^2 p} ; z_j \right) + 2\pi i \text{Res} \left( \frac{p'}{z^2 p} ; 0 \right). \]

Thus \( (4) \), \( \sum_{j=1}^{N} \frac{1}{z_j^2} = - \text{Res} \left[ \frac{p'}{z^2 p} ; 0 \right] \)

\( z = 0 \) is a double pole.

The L-series

\[ \frac{p'}{z^2 p} = \frac{a_1 + 2a_2 z + \cdots}{z^2 (a_0 + a_1 z + \cdots)} = \frac{\left[ a_0 + 2a_1 z \right]}{a_0 z^2} \left[ 1 - \frac{a_1}{a_0} z \right]. \]

\[ \sum_{j=1}^{N} \frac{1}{z_j^2} = \frac{a_1^2}{a_0^2} - 2 \frac{a_2}{a_0} : \]

check \( p = (2z-1)(4z+1)(z-1) = 0. \)

\( z_1 = 1/2, z_2 = -1/4, z_3 = 1 \to \sum_{j=1}^{3} \frac{1}{z_j^2} = \text{work}. \)

Now \( p = (8z^2 - 2z - 1)(z-1) = 8z^3 - 2z^2 - z - 8z^2 + 2z + 1 = 8z^3 - 10z^2 + z + 1. \)

\( a_3 = 1, a_1 = 1, a_2 = -10 \to \text{work}. \)