MULTI-VALUED FUNCTIONS

There are many functions whose inverse function is multi-valued. For instance:

\[ z = e^w, \quad z = w^2, \quad z = \cos w, \quad z = \sin w. \]

For each of these functions, a given value of \( z \) corresponds to more than one value of \( w \).

\[ f^{-1}(z) \text{ is multi-valued} \]
\[ f(w) \text{ is single valued. Given a } w, \text{ there is a unique value of } z \]

Goals:
(i) Determine all possible values of the inverse function \( w \).
(ii) Construct an inverse function that is single valued in some region of the complex plane.

LOGARITHM FUNCTION

Define the inverse function for \( z = e^w \).

For a given \( z = re^{i\varphi} \) with \( \varphi = \arg(z) \) we write \( w = u + iv \) where \( u, v \) to be found.

\[ re^{i\varphi} = e^{u+iv} \]

Taking the modulus we get \[ |re^{i\varphi}| = |e^{u+iv}| \rightarrow r = e^u \text{ so } u = \log r. \]

Then \[ v = \varphi + 2K\pi \quad K = 0, \pm 1, \pm 2, \ldots \]

Hence \[ w = \log r + i[\arg(z) + 2K\pi] \quad K = 0, \pm 1, \pm 2, \ldots \]

This is \[ w = \log |z| + i[\arg(z) + 2K\pi] \quad -\pi < \arg(z) < \pi \quad K = 0, \pm 1, \pm 2, \ldots \]
We define the multi-valued $\log z$ by

$$ w = \log z = \ln |z| + i(\arg z + 2k\pi) \quad k = 0, \pm 1, \pm 2, \ldots $$

It gives all the solutions $w$ to $z = e^w$.

Equivalently, we can define it as

$$ w = \log z = \ln |z| + i(\arg z) $$

since $\arg z$ is multi-valued.

We define the principal value of $\log z$ by

$$ w = \log z = \ln |z| + i\arg(z). $$

Since $-\pi < \arg(z) < \pi$ we notice that $w$ is not continuous at any point on the negative real axis.

A: As $z = x + iy$ with $y \to 0^+$ with $x < 0$ then

$$ \log z \to \ln |x| + i\pi. $$

B: As $z = x + iy$ with $y \to 0^-$ with $x < 0$ then

$$ \log z \to \ln |x| - i\pi. $$

Thus, $\log z$ is discontinuous on negative real axis.

Remark

(i) $\log z$ is called a "branch" of the multi-valued function $\log z$.

(ii) $\log z$ is continuous in the cut plane $\mathbb{C} \setminus \{\infty, 0\}$.

(iii) The point $z = 0$ is called a "branch point" of $\log z$, since if we encircle $z = 0$ by
A closed contour then \( \log z \) changes by an amount proportional to \( 2\pi i \).

Change in \( \log z \) around path \( C \) is \( 2\pi i \).

In the cut plane \( \log z \) is analytic and \( \frac{d}{dz} \log z = \frac{1}{z} \).

**Proof**

\[
\tilde{f}(z) = \log z = \frac{1}{2} \ln (x^2 + y^2) + i \tan^{-1} \left( \frac{y}{x} \right)
\]

with \(-\pi < \tan^{-1}(y/x) < \pi\).

Then

\[
\begin{align*}
U & = \frac{1}{2} \ln (x^2 + y^2) \\
V & = \tan^{-1} \left( \frac{y}{x} \right)
\end{align*}
\]

\[
\begin{align*}
U_x & = \frac{X}{X^2 + y^2} \\
V_x & = -\frac{y}{X^2} \\
U_y & = \frac{Y}{X^2 + Y^2} \\
V_y & = \frac{X}{X^2 + Y^2}
\end{align*}
\]

so \( U_x = V_y, \ U_y = -V_x \) provided \((x, y) \neq (0, 0)\).

And

\[
\tilde{f}'(z) = U_x + i V_x = \frac{X}{X^2 + Y^2} - \frac{i Y}{(X^2 + Y^2)} = \frac{\overline{z}}{|z|^2} = \frac{\overline{z}}{z \overline{z}} = \frac{1}{z}
\]

so \( \tilde{f}'(z) = 1/z \).

**Example** Calculate the following:

(i) \( \log (2i) \)  
(ii) \( \log \left( 1 + i\sqrt{3} \right) \)  
(iii) \( \log (-i) \)

**Solution**

(i) \( \log (2i) \): let \( z = 2i \). Then \( \arg(z) = \pi/2 \).

so \( \log (2i) = \ln 2 + i \left[ \frac{\pi}{2} + 2k\pi \right] \quad k = 0, \pm 1, \pm 2, \ldots \)
(ii) \[ 1 + i \sqrt{3} = 2 e^{\pi i/3}. \quad \text{Arg}(1 + i \sqrt{3}) = \pi/3 \]

so \[ \log(1 + i \sqrt{3}) = \ln 2 + i \pi/3 \]

(iii) \[ \log(-i). \text{ Let } z = -i. \quad |z| = 1, \quad \text{Arg} z = -\pi/2. \]

so \[ \log(-i) = \ln 1 + i (-\pi/2). \quad \text{so } \log(-i) = -\pi i/2. \]

One must be careful with identities involving \( \log z, \log z \).

The following results, as shown in HW, hold:

(i) \( \log(z_1 z_2) \neq \log(z_1) + \log(z_2) \)

(ii) \( \log(z_1 z_2) = \log z_1 + \log z_2 \)

(iii) \( \log(e^z) \neq z \)

(iv) \( \log(e^z) = z \) \text{ if and only if } -\pi < \text{Im}(z) \leq \pi.

(v) \( \log z = -\log(1/z) \)

(vi) \( z = e^{\log z} \)

(vii) \( \log(z^{1/p}) = \frac{1}{p} \log z \quad p: \text{positive integer} \)

(viii) \( \log(z^n) \neq n \log z \quad \text{in general } (n: \text{positive integer}) \)

We now give a proof for a few of these. You are asked to prove the others in the homework.
\[ \log z = \ln |z| + i [\arg z + 2k\pi]. \]

So
\[ e^{\log z} = e^{\ln |z| + i [\arg z + 2k\pi]} = |z|e^{i\arg(z)} = z. \]

**Proof (VII)** We will show that the sets of values of 
\[ \log (z^D) \quad \text{and} \quad n \log z \quad \text{do not coincide} \]

Let 
\[ z = \rho e^{i\varphi} \quad \text{with} \quad \varphi = \arg z. \]
We get 
\[ z^D = \rho^D e^{iD\varphi}. \]

Then
\[ \log (z^D) = \frac{\ln (\rho^D) + i [n \varphi + 2k\pi]}{n = 0, \pm 1, \pm 2,...} \]

\[ \log (z^n) = n \ln \rho + i [n \varphi + 2k\pi]. \]

But
\[ n \log z = n \left[ \ln \rho + i (\varphi + 2m\pi) \right] = n \ln \rho + i [n \varphi + 2m\pi]. \]

Comparing these two sets is equivalent to comparing
\[ \{ 2k\pi \} \quad \text{and} \quad \{ 2m\pi \} \]
\[ n = 0, \pm 1, \pm 2,... \quad \text{and} \quad m = 0, \pm 1, \pm 2,... \quad (n > 0 \text{ integer fixed}) \]

These are not in general the same. In particular if \( D = 2 \) then
\[ \{ 2k\pi \} = \{ 0, \pm 2\pi, \pm 4\pi, \ldots \} \]
\[ \{ 2m\pi \} = \{ 4\pi, 8\pi, \ldots \} \]

**Proof (VIII)** Show \( \log (e^z) \neq z \). Notice left-hand-side is multivalued, but right-hand-side is single valued.

Put 
\[ z = x + iy. \]

\[ e^z = e^x e^{iy} \quad \text{arg} \quad e^z = \text{arg} \quad e^{iy} = y, \quad \text{if} \quad -\pi < y \leq \pi \]

so
\[ \log (e^z) = \ln |e^z| + i [\arg + 2k\pi i] = \ln e^x + i (y + 2k\pi). \]

Thus
\[ \log (e^z) = z + i(2k\pi + y), \quad \text{if} \quad -\pi < y \leq \pi. \]
Proof (vii) show that the sets \( \log \left( z^{1/n} \right) \) and \( \frac{1}{n} \log z \) are the same where \( n \) is a positive integer.

Write \( z = re^{\theta i} \) with \( \theta = \arg(z) \). Then \( z^{1/n} = r^{1/n} e^{\left( \frac{\theta + 2k\pi}{n} \right)} \) \( k = 0, \ldots, n-1 \).

Thus \( \log \left( z^{1/n} \right) = \frac{1}{n} \log r + i \left[ \frac{\theta + 2k\pi}{n} \right] \) \( k = 0, 1, \ldots, n-1 \); \( p = 0, 1, 2, \ldots \).

Now \( \frac{1}{n} \log z = \frac{1}{n} \log r + i \left[ \frac{\theta}{n} + \frac{2p\pi}{n} \right] \) \( q = 0, 1, 2, \ldots \).

The set of values of \( \log \left( z^{1/n} \right) \) and \( \frac{1}{n} \log z \) are the same if the two sets \( \frac{1}{n} \left( k + pn \right) \) \( k = 0, \ldots, n-1 \); \( p = 0, 1, 2 \) coincide with the set \( \frac{1}{n} q \) \( q = 0, 1, 2 \).

Thus it true that for any \( k \) and \( p \) we get a \( q \) dividing \( k \) by \( n \) we get an integer and a remainder \( k \) in \( \{0, \ldots, n-1\} \).

Example of mapping involving \( \log z \)

Ex: Find the image of \( S = \{ z \mid \, \text{im} z \geq 0 \} \) under the mapping

\( W = 2 \log z \)

To parametrize \( z \)-plane let \( z = re^{\theta i} \). Then

\( W = 2 \left[ \log r + i \theta \right] \) with \( 0 \leq \theta \leq \pi \). Write \( W = U + iV \).

Hence \( U = 2 \log r \) \( V = 2 \theta \).

- Fix a ray with \( \theta \) fixed (line \( L \) in \( z \)-plane above). Then since \( 0 \leq r < \infty \) we get \( U \) in \( [-\infty, 0) \) and \( V \) fixed.

The image line \( L' \) is shown in \( w \)-plane above.

- Since \( 0 \leq \theta \leq \pi \) we get \( U \) in \( [-\infty, 0) \) and \( V \) in \( (0, 2\pi) \).

Hence \( S' = \{ W \mid 0 \leq \text{im} W \leq 2\pi \} \).
If $a$ is a complex number and $z \neq 0$ then we define \( z^a = e^{a \log z} \) (multi-valued).

Thus, \( z^a = e^{a [ \ln |z| + i \arg(z) + 2k\pi i]} \) \( k = 0, \pm 1, \pm 2, \ldots \).

This yields that \( z^a = |z|^a e^{i [a \arg(z) + 2k\pi]} \) \( k = 0, \pm 1, \pm 2, \ldots \).

There will be a finite number of values of $z^a$ only if $a$ is the ratio of two integers (i.e., is rational). In such a case $a/k$ is integer for some $k$.

The principal value of $z^a$ is defined by
\[
z^a = e^{a \log(z)} = e^{a [ \ln |z| + i \arg(z)]}
\]

Since $\log(z)$ is analytic in the slit domain $C \setminus (-\infty, 0)$ and $e^w$ is analytic, then $z^a$ is analytic in $C \setminus (-\infty, 0)$ and
\[
\frac{d}{dz} z^a = a z^{a-1} \quad \text{in} \quad C \setminus (-\infty, 0).
\]

Example: Find all solutions to $z^{1+i} = 4$. We write
\[
z^{1+i} = e^{(1+i)\log z} = e^{\ln 4}.
\]

Thus, \((1+i) \log z = \ln 4 + 2\pi ni\), \(n = 0, \pm 1, \pm 2\)

Thus \(2 \log z = (1-i) [\ln 4 + 2\pi ni] \Rightarrow \log z = (1-i) [\ln 2 + \pi n i] \).

Hence \(\log z = \ln 2 + \pi n + i(\ln 2 - \ln 2)\). Now exponentiating gives \(z = 2 e^{\pi n i} \left[ e^{i \ln 2} \right] = 2 e^{\pi n i} \left[ e^{i \frac{\pi}{2}} \right] = 2 e^{\pi n i} \left[ e^{i \frac{\pi}{2}} \right]
\]

Since \(e^{i \pi n} = (-1)^n\).
\( f(z) \) is a branch of the multi-valued function \( F(z) \) in a domain \( D \) if \( f(z) \) is single-valued and continuous in \( D \) and has the property that for each \( z \in D \) the value \( f(z) \) is one of the values of \( F(z) \).

To construct \( f(z) \) we introduce a curve emanating from a point (called the branch point) to ensure that \( f(z) \) is single-valued in the cut plane. A branch point is a point for which if we encircle it with an arbitrary sufficiently small curve the function \( F(z) \) changes discontinuously.

Although a deeper understanding of these issues requires more advanced topics (i.e. Riemann surface), we can still illustrate the idea with some examples.

**Example 1** Let \( F(z) = \log z \) (multi-valued).

The point \( z = 0 \) is a branch point since if we take a path \( C \) as shown below, then \( \log z \) does not return \( z \)-plane to its original value. The change \( [\log z]_C \)

\[
[\log z]_C = 2\pi i.
\]

Note: if we encircle any other point \( z \neq 0 \) with a small closed curve \( C \) (as shown)

\[
[\log z]_{C_1} = 0 \text{ thus } z \neq 0 \text{ is not a branch point.} \]
We must insert a curve, called the branch cut, to prevent complete circuits about the branch point, thus rendering the function single-valued. These cuts can be lines, curves, etc.

We then choose a range of argument to unambiguously define the function at each point in the cut plane.

Construct a branch of $f = \log z$ that is analytic except on the negative real axis and is real-valued on positive real axis.

This is

$$\tilde{f}(z) = \log z$$

for when $-\pi < \arg z \leq \pi$ and so for $z = x$ with $x > 0$ real,

$$\text{Im}[\tilde{f}(z)] = 0.$$

**Example** Consider the single-valued function

$$\tilde{f}(z) = \log (1 - z^2)$$

where is the function discontinuous?

**Solution** Since $\log (5)$ is analytic except on $\text{Im}(5) = 0$ and $\text{Re}(5) < 0$, we have that $\log (1 - z^2)$ is discontinuous only when $\text{Im}(1 - z^2) = 0$ and $\text{Re}(1 - z^2) < 0$. 
Let \( z = x + iy \),

set \( \text{IM}(1-z^2) = -2xy = 0 \)

\[ \text{RE}(1-z^2) = 1 - x^2 + y^2 < 0 \]

Hence either \( x = 0 \) or \( y = 0 \). But if \( x = 0 \) then \( 1 + y^2 < 0 \) is impossible. Hence \( y = 0 \) and \( 1 - x^2 + y^2 = 1 - x^2 < 0 \) implies \( |x| > 1 \).

Therefore the branch cuts are as shown.

\[ \text{in the cut plane as shown.} \]

**Example**

Let \( \Phi(z) = \log \left( \frac{z-1}{z-2} \right) \), where \( \log \) denotes the principal branch of multi-valued \( \log \) function.

Where is \( \Phi(z) \) analytic?

**Solution**

The only possible place where \( \Phi \) is not analytic is when

\[ \text{IM} \left( \frac{z-1}{z-2} \right) = 0 \ \text{and} \ \text{RE} \left( \frac{z-1}{z-2} \right) < 0. \]

Hence

\[ \frac{(z-1)}{(z-2)} = \frac{1}{(z-2)^2} \left[ z \bar{z} - z - 2z + 2 \right] = \frac{1}{|z-2|^2} \left[ x^2 + y^2 - (x+iy)(2-x-iy) \right] \]

so \( \text{IM} \left( \frac{z-1}{z-2} \right) < 0 \rightarrow y = 0 \)

\[ \text{RE} \left( \frac{z-1}{z-2} \right) < 0 \ \text{when} \ y = 0 \ \text{yield} \ x^2 - 3x + 2 = (x-2)(x-1) \leq 0. \]

Thus \( \Phi(z) \) is not analytic on branch cuts as shown.

\[ \text{in the cut plane as shown.} \]
MULTI-VALUED FUNCTIONS

Consider the function \( w = \frac{1}{z} \). (Multi-valued)

The point \( z = 0 \) is a branch point since if we encircle \( z = 0 \) by a simple closed curve \( C \), the change in \( z^{1/2} \), denoted by \( [z^{1/2}]_C \), is for \( C \) counterclockwise

\[
[z^{1/2}]_C = |z|e^{i\phi/2} \neq 0 \quad (\phi \to \phi + 2\pi)
\]

For any other point \( z_1 \), we have \( [z^{1/2}]_{C_1} = 0 \) where \( C_1 \) is a simple closed curve surrounding \( z_1 \) and not the origin.

Thus \( z_1 \neq 0 \) is not a branch point.

We must introduce a branch-cut emanating from \( z = 0 \) and extending to \( \infty \) to prevent encircling the origin, and hence rendering \( z^{1/2} \) analytic in the cut plane. Then, we choose a range of values for the argument of \( z \) to make it uniquely defined in cut plane.

Possible branch cuts for \( z^{1/2} \)
Example

Construct a branch of \( F(z) = z^{1/2} \) for which \( z^{1/2} \) is analytic in the cut plane \( C \setminus (-\infty, 0) \) and for which \( \text{Re}(\sqrt{z}) > 0 \).

Solution

We must have the cut as shown.

\[ z = |z| e^{i\varphi} \]

Hence, either \( -\pi < \varphi < 0 \)

or \( \pi < \varphi < 2\pi \).

Which range of angles works?

We calculate

\[ z^{1/2} = |z|^{1/2} e^{i\varphi/2} \]

Then \( \text{Re}(z^{1/2}) = |z|^{1/2} \cos(\varphi/2) \).

Hence, if \( -\pi < \varphi < 0 \), then \( \cos(\varphi/2) > 0 \) \( \Rightarrow \) \( \text{Re}(\sqrt{z}) > 0 \).

We then write (i)

\[ z^{1/2} = |z|^{1/2} e^{i\varphi/2} \quad \text{with} \quad -\pi < \varphi < 0. \]

Remark

(i) \( (i) \) is the principal branch of \( \sqrt{z} \).

It coincides precisely with the choice of branch

\[ z^{1/2} = e^{\frac{1}{2} \text{Log}(z)} = e^{\frac{1}{2} [\text{Log}|z| + i \text{Arg}z]} \quad -\pi < \text{Arg}z < \pi. \]

(ii) Calculate the principal value of \( (1 + i)^{1/2} \).

Solution

\[ \text{Arg}z = \pi/4 \quad |z| = \sqrt{2}. \quad \text{so} \]

\[ (1 + i)^{1/2} = (\sqrt{2})^{1/2} e^{i\pi/8}. \]

(iii) Construct a branch of \( z^{1/2} \) that is analytic in \( C \setminus (-\infty, 0) \) but has \( \text{Re}(z^{1/2}) < 0 \).

By repeating analysis above,

\[ z^{1/2} = |z|^{1/2} e^{i\varphi/2}, \quad -\pi < \varphi < 3\pi. \]
EXAMPLE SUPPOSE THAT $z^{1/2}$ DENOTE THE PRINCIPAL VALUE OF THE SQUARE ROOT. FIND ALL SOLUTIONS TO

$$(v) \quad z^{1/2} + z - i = 0. $$

**Solution** The principal value of $z^{1/2}$ is such that it is analytic in $C \setminus (-\infty, 0)$ and has $\operatorname{Re}(z^{1/2}) \geq 0$.

By taking $\operatorname{Re}(\ )$ of both sides in $(v)$ we obtain that $\operatorname{Re}(z^{1/2}) + 2 = 0 \implies \operatorname{Re}(z^{1/2}) = -2 < 0$ \implies contradiction.

Thus, with the principal value of $z^{1/2}$ $(v)$ has no solution.

**Remark** It is tempting but wrong to calculate as

$$z^{1/2} = i - 2$$

$$\implies (z^{1/2})^2 = (i - 2)^2 = 4 - 4i - 1 = 3 - 4i$$

so $z = 3 - 4i$.

EXAMPLE Construct a branch of $F(z) = (z^2 + 1)^{1/2}$ that is analytic in $|z| > 1$ and takes the value $F(2) = \sqrt{3}$.

**Solution** We use $z, z_1, z_2^{1/2} = (z, z_1)^{1/2}$ (multi-valued sets same) to write

$$F(z) = (z - 1)^{1/2} (z + 1)^{1/2}.$$

The only points in finite complex plane that are branch points are $z = -1$ and $z = 1$. We must have no branch cuts outside $|z| > 1$, so the easiest construction is to have
**Method I (Range of Angles)**

We write \( (z^2 - 1)^{1/2} = (z - 1)^{1/2} (z + 1)^{1/2} = (\Gamma_1 e^{i\Phi_1})^{1/2} (\Gamma_2 e^{i\Phi_2})^{1/2} \).

Hence \( f(z) = (z^2 - 1)^{1/2} = (\Gamma_1 \Gamma_2)^{1/2} e^{i(\Phi_1 + \Phi_2)/2} \) (X).

Specifying a branch is equivalent to choosing a range of angles.

Try \( -\pi < \Phi_1 < \pi, \quad -\pi < \Phi_2 < \pi \)

- Must check that discontinuity in \( f \) occurs between \(-1 < x < 1\).
- Must check that (X) gives \( f(2) = \sqrt{3} \).

<table>
<thead>
<tr>
<th>Point</th>
<th>( \Phi_1 )</th>
<th>( \Phi_2 )</th>
<th>( e^{i(\Phi_1 + \Phi_2)/2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
<td>( e^{i0} = 1 )</td>
</tr>
<tr>
<td>C'</td>
<td>0</td>
<td>0</td>
<td>( e^{i0} = 1 )</td>
</tr>
<tr>
<td>B</td>
<td>(-\pi)</td>
<td>0</td>
<td>( e^{-i\pi/2} = i )</td>
</tr>
<tr>
<td>B'</td>
<td>(-\pi)</td>
<td>0</td>
<td>( e^{-i\pi/2} = i )</td>
</tr>
<tr>
<td>D</td>
<td>(-\pi)</td>
<td>(-\pi)</td>
<td>( e^{-i\pi} = -1 )</td>
</tr>
<tr>
<td>D'</td>
<td>(-\pi)</td>
<td>(-\pi)</td>
<td>( e^{-i\pi} = -1 )</td>
</tr>
</tbody>
</table>

Thus the choice \( f(z) = (\Gamma_1 \Gamma_2)^{1/2} e^{i(\Phi_1 + \Phi_2)/2} \) with \(-\pi < \Phi_1 < \pi\)

And \(-\pi < \Phi_2 < \pi\) has a branch cut from \(-1 < x < 1\) as desired.

Now calculate \( f(2) \):

For \( z = 2 \) then \( \Gamma_1 = |z - 1| = 1, \Gamma_2 = |z + 1| = 3 \)

And \( \Phi_1 = \Phi_2 = 0 \) hence \( f(2) = \sqrt{1 \cdot 3} e^{i0} = \sqrt{3} \) as desired.
To calculate \( F(i) \) we draw

\[ \begin{align*}
&\quad \theta_1 = \theta_2 = \sqrt{2} , \\
&\phi_1 = 3\pi/4 , \quad \phi_2 = \pi/4 \quad \text{so} \quad F(i) = \left( \sqrt{2} \sqrt{2} \right)^{\frac{1}{2}} e^{i \left( 3\pi/4 + \pi/4 \right)/2} \\
&\quad \rightarrow F(i) = \sqrt{2} i.
\end{align*} \]

**Method 2 (Choosing a Branch of \( \log \))**

This method is less intuitive as it is not clear apriori which branch of \( \log \) to take.

For instance, consider several possible choices:

\begin{align*}
&\quad (A) \quad (z^2 - 1)^{\frac{1}{2}} = e^{\frac{1}{2} \log(z^2 - 1)} \quad \rightarrow \quad F(z) = e^{\frac{1}{2} \log |z - 1|} \\
&\quad (B) \quad (z^2 - 1)^{\frac{1}{2}} = \left[ -1 - z^2 \right]^{\frac{1}{2}} \quad \rightarrow \quad F(z) = \pm i e^{\frac{1}{2} \log |1 - z^2|} \\
&\quad (C) \quad (z^2 - 1)^{\frac{1}{2}} = \left[ z^2 (1 - 1/z^2) \right]^{\frac{1}{2}} \quad \rightarrow \quad F(z) = \pm z e^{\frac{1}{2} \log |1/z^2|} \\
&\quad (D) \quad (z^2 - 1)^{\frac{1}{2}} = \left[ -z^2 (-1 + 1/z^2) \right]^{\frac{1}{2}} \quad \rightarrow \quad F(z) = \pm i z e^{\frac{1}{2} \log |1 + 1/z^2|}
\end{align*}

Which one will give \( F(z) \) analytic in \(|z| > 1\) with \( F(z) = \sqrt{3} \)?

This is not clear without considerable extra effort.

Consider the obvious choice (A): \( F(z) = e^{\frac{1}{2} \log |z^2 - 1|} \).

To see if it works,

then \( F(z) \) is analytic except when

\[ \text{IM} (z^2 - 1) = 0 \quad \text{and} \quad \text{RE} (z^2 - 1) < 0. \]
Let $z = x + iy$

$\text{IM}(z^2 - 1) = 0 \rightarrow xy = 0 \rightarrow \text{either } x = 0 \text{ or } y = 0.$

$\text{RE}(z^2 - 1) = \text{RE} \left[ x^2 - y^2 + 2ixy - 1 \right] = x^2 - y^2 - 1 \leq 0.$

If $x = 0 \rightarrow \text{RE}(z^2 - 1) = -y^2 - 1 \leq 0 \text{ for all } y$

If $y = 0 \rightarrow \text{RE}(z^2 - 1) = x^2 - 1 \leq 0 \rightarrow |x| \leq 1.$

Thus, the choice has branch cut $\frac{1}{2} \log |1 - 1/z^2|.$

This is not what we want.

Hence choice A fails.

Consider choice (C) try $f(z) = \text{sgn } z \frac{1}{2} \log (|1 - 1/z^2|).$

Thus analytic except when $\text{IM} \left( 1 - \frac{1}{z^2} \right) = 0$

$\text{RE} \left( 1 - \frac{1}{z^2} \right) \leq 0.$

Thus

$\text{IM} \left( \frac{1}{z^2} \right) = \text{IM} \left( \frac{z}{|z|^4} \right) = \text{IM} \left( \frac{(x+iy)^2}{|z|^4} \right) = 0 \rightarrow xy = 0$

$\text{RE} \left( 1 - \frac{1}{z^2} \right) = 1 - \text{RE} \left( \frac{(x+iy)^2}{|z|^4} \right) \leq 0.$

Set $y = 0 \rightarrow \text{RE} \left( 1 - \frac{1}{z^2} \right) \leq 0 \rightarrow 1 - \frac{x^2}{x^4} \leq 0 \rightarrow |x| \leq 1.$

Set $x = 0 \rightarrow \text{RE} \left( 1 - \frac{1}{z^2} \right) \leq 0 \rightarrow \text{RE} \left( 1 - \frac{1}{(iy)^4} \right) = 1 + \frac{1}{y^4} \leq 0$

Impossible.

Thus $f(z) = \text{sgn } z \frac{1}{2} \log (|1 - 1/z^2|)$ has desired branch cut $\frac{1}{2} \log |1 - 1/z^2|.$
Now we must take a sign consistent with $F(2) = \sqrt{3}$.

Try a sign $\rightarrow F(2) = 2 e^{\frac{\pi}{2} \log (1 - \frac{3}{4})}$

\[ = 2 e^{\frac{1}{2} \left( \log \left( \frac{3}{4} \right) + i \arg \left( \frac{3}{4} \right) \right)} , \quad \arg \left( \frac{3}{4} \right) = 0 \]

\[ = 2 \left( e^{\frac{1}{2} \left( \log \left( \frac{3}{4} \right) \right)} \right) = 2 (\sqrt{\frac{3}{4}}) = \sqrt{3} \checkmark \]

Thus, the desired branch is

$$F(z) = z e^{\frac{\pi}{2} \log (1 - \frac{1}{z^2})}$$

This was not terribly clear in advance that this choice
would work. Notice $F(i) = i e^{\frac{\pi}{2} \log (2)} = i e^{\frac{\pi}{2} (\log 2)} = \sqrt{2} i$.

**Example**

Construct a branch of $F(z) = (z^2 - 1)^{\frac{1}{2}}$ that is analytic
in $|z| < 1$ and that takes the value $F(0) = i$.

**Solution**

**Method 1 (Range of Angle Method)**

We write $F(z) = (\Gamma_1 \Gamma_2)^{\frac{1}{2}} e^{i \left( \phi_1 + \phi_2 \right)/2}$

We try now the range $0 \leq \phi_1 < 2\pi$, $-\pi < \phi_2 \leq \pi$.

<table>
<thead>
<tr>
<th>POINT</th>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$e^{i(\phi_1 + \phi_2)/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>0</td>
<td>0</td>
<td>$e^{i0} = 1$</td>
</tr>
<tr>
<td>$C'$</td>
<td>$2\pi$</td>
<td>0</td>
<td>$e^{i2\pi} = 1$</td>
</tr>
<tr>
<td>$B$</td>
<td>$\pi$</td>
<td>0</td>
<td>$e^{i\pi} = -1$</td>
</tr>
<tr>
<td>$B'$</td>
<td>$\pi$</td>
<td>0</td>
<td>$e^{i\pi} = -1$</td>
</tr>
<tr>
<td>$D$</td>
<td>$\theta$</td>
<td>$\theta$</td>
<td>$e^{i\theta} = -1$</td>
</tr>
<tr>
<td>$D'$</td>
<td>$\pi$</td>
<td>$\theta$</td>
<td>$e^{i\pi} = -1$</td>
</tr>
</tbody>
</table>

\[ \rightarrow \text{This yields the branch cut as shown (what we want}) \]

\[ z\text{-plane} \quad \begin{array}{c|c|c}
-1 & -1 & \\
\end{array} \]
Hence, \( f(z) \) is analytic in \( |z| < 1 \).

Now at \( z = 0 \) we calculate \( \Phi = \Phi_1 + i \Phi_2 \), \( \Phi_1, \Phi_2 \in \mathbb{R} \), so that

\[
\Phi_1 = (z^2 - 1)^{1/2} = (1, 1)^{1/2} e^{i \pi/2} = (1, 1)^{1/2} e^{i \pi/2} = i,
\]

as required.

**Method 2**

We write

\[
(z^2 - 1)^{1/2} = [-(1 - z^2)]^{1/2} = \pm i (1 - z^2)^{1/2}.
\]

We now choose the principal value \((1 - z^2)^{1/2} = e^{1/2 \log (1 - z^2)}\).

Then,

\[
(z^2 - 1)^{1/2} = (\pm i) e^{1/2 \log (1 - z^2)}\]

This is choice B on page (M14.5).

Choose \( +i \) since at \( z = 0 \) \( \log (1) = 0 + 0 = 0 \).

Then \( F(0) = i \) as required.

Then \( F(0) = i \) as required.

We obtain

\[
(z^2 - 1)^{1/2} = i e^{1/2 \log (1 - z^2)}.
\]

By the example at bottom of page (M9) \( \log (1 - z^2) \) has branch cut as shown.

Hence \( (z^2 - 1)^{1/2} = i e^{1/2 \log (1 - z^2)} \) satisfies the requirement.

In particular if \( F(z) = (z^2 - 1)^{1/2} = i e^{1/2 \log (1 - z^2)} \).

Then \( F(i) = i e^{1/2 \log (1 - i^2)} = i e^{1/2 \log (2)} = i e^{1/2 \ln 2} = i e^{\ln \sqrt{2}} \).

Recall \( \log (1 - z^2) \) is analytic except for point \( z \) where

\[
\text{Im}(1 - z^2) = 0 \rightarrow xy = 0 \rightarrow \text{either } x = 0 \text{ or } y = 0.
\]

\[
\text{Re}(1 - z^2) < 0 \rightarrow 1 - (x^2 - y^2) < 0 \rightarrow \text{if } x = 0 \rightarrow 1 - y^2 < 0 \rightarrow \text{impossible}
\]

\[
\text{Re}(1 - z^2) < 0 \rightarrow 1 - (x^2 - y^2) < 0 \rightarrow \text{if } y = 0 \rightarrow 1 - x^2 < 0 \rightarrow |x| > 1.
\]
EXAMPLE

Construct a branch of \( f(z) = (z^2 + 1)^{\frac{1}{2}} \) that is analytic in \(|z| > 1\) and for which \( f(2i) = \sqrt{3}i\).

**Solution**

**Method 1**

\[ f(z) = (z + i)^{\frac{1}{2}}(z - i)^{\frac{1}{2}} = (\Gamma, \Gamma_1) \cdot e^{i(\Phi_1 + \Phi_2)/2} \]

We want a branch cut between \(-i\) and \(i\) as shown.

If we choose \(-\frac{\pi}{2} < \Phi_1 < \frac{3\pi}{2}, -\frac{\pi}{2} < \Phi_2 < \frac{3\pi}{2}\) then we get the desired branch cut. For then we have continuity at \(B'B\) and \(D'D'\).

For \(z = 2i\) we get \(\Phi_1 = \Phi_2 = \frac{3\pi}{2}\), \(\Gamma_1 = 1\), \(\Gamma_2 = 3\). Hence

\[ f(2i) = (1.3)^{\frac{1}{2}} e^{i(\frac{3\pi}{2} + \frac{3\pi}{2})/2} = i\sqrt{3} \]

**Method 2**

The choice \( (z^2 + 1)^{\frac{1}{2}} = e^{\frac{1}{2} \log(z^2 + 1)} \) clearly does not work since \( \log(z^2 + 1) \) is not analytic on \(z = iy\) with \(|y| > 1\).

Instead write \( (z^2 + 1)^{\frac{1}{2}} = (z^2 \left[ 1 + \frac{1}{z^2} \right])^{\frac{1}{2}} = z^{\frac{1}{2}} \log(1 + \frac{1}{z^2}) \).

Then choose \( (z^2 + 1)^{\frac{1}{2}} = z e^{\frac{1}{2} \log(1 + \frac{1}{z^2})} \).

Now \( z e^{\frac{1}{2} \log(1 + \frac{1}{z^2})} \) is analytic except on the segment for which

\[ \text{IM} \left( 1 + \frac{1}{z^2} \right) = 0 \text{ and RE} \left( 1 + \frac{1}{z^2} \right) < 0. \]

If we put \(z = x + iy\) then \( \text{IM} \left( 1 + \frac{1}{z^2} \right) = \text{IM} \left( \frac{z^2}{|z|^4} \right) = \frac{1}{|z|^4} (-2xy) = 0 \)

Hence either \(x = 0\) or \(y = 0\). But \( \text{RE} \left( 1 + \frac{1}{z^2} \right) = \text{RE} \left( \frac{z^2}{|z|^4} \right) + 1 = \frac{x^2 - y^2}{(x^2 + y^2)^2} + 1 < 0 \).

Clearly \(y = 0\) impossible. So \(x = 0\) yield \( -y^2 + 1 < 0 \to |y| < 1 \).

We conclude that \( (z^2 + 1)^{\frac{1}{2}} = z e^{\frac{1}{2} \log(1 + \frac{1}{z^2})} \) is analytic in \(|z| > 1\).

We calculate \( f(2i) = 2i e^{\frac{1}{2} \log(1 + \frac{1}{z^2})} = 2i e^{\frac{1}{2} \log(i^2)} = 2i \sqrt{\frac{1}{2}} = i\sqrt{3} \).
EXAMPLE

Construct a branch of \( f(z) = (z^3 + z^2 - 2z)^{1/2} \) that has a branch cut from \((0,1)\) and from \((-\infty, -2)\) along the real axis and for which \( f(z) = \sqrt{8} \).

Solution

\[
f(z) = \sqrt{z(z+2)(z-1)} = (\Gamma_1, \Gamma_2, \Gamma_3)^{1/2} e^{i(\Phi_1 + \Phi_2 + \Phi_3)/2}
\]

If we then choose

\[-\pi < \Phi_1 < \pi\]
\[-\pi < \Phi_2 < \pi\]
\[-\pi < \Phi_3 < \pi,\]

we will obtain the branch cut structure

\[\text{z-plane}\]

\[
\begin{array}{c}
-2 \\
\hline
\end{array}
\]

When \( z = 2 \) then \( \Phi_1 : \Phi_2 : \Phi_3 = 0 \), \( \Gamma_1 = 1 \), \( \Gamma_2 = 2 \), \( \Gamma_3 = 4 \).

Hence

\[
f(2) = (4 - 2i) e^{i0} = \sqrt{8}.
\]

Finally, we make a few additional miscellaneous comments.

Remark

Not everything with \( \sqrt{z} \) has a branch point at \( z = 0 \).

For which of the following is \( z = 0 \) a branch point?

(i) \( f(z) = \sin(\sqrt{z}) \)  (ii) \( f(z) = \sqrt{z} \sin(\sqrt{z}) \)

(iii) \( f(z) = (\arctan(\sqrt{z}) \).

In (ii) we use same choice of branch of \( \sqrt{z} \).
SOLUTION

Only \( \sin(\sqrt{z}) \) has a branch point at \( z = 0 \).

Let's check, in each case we encircle \( z = 0 \) by a simple closed counterclockwise curve and we calculate \( \left[ \frac{dF(z)}{dz} \right]_c \) (the change in \( F \) around the curve).

\[
\begin{align*}
(i) \quad \left[ \sin(\sqrt{z}) \right]_c &= \left[ \sin(\sqrt{r} e^{i\phi/2}) \right]_c \\
&= \sin(\sqrt{r} e^{i\pi/2}) - \sin(\sqrt{r} e^{i0}) \\
&= \sin(-\sqrt{r}) - \sin(\sqrt{r}) = -2 \sin(\sqrt{r}) \neq 0.
\end{align*}
\]

\[
\begin{align*}
(ii) \quad \left[ \sqrt{z} \sin(\sqrt{z}) \right]_c &= \left[ \sqrt{r} e^{i\phi/2} \sin(\sqrt{r} e^{i\phi/2}) \right]_c \\
&= -\sqrt{r} e^{i\pi/2} \sin(-\sqrt{r}) - \sqrt{r} e^{i0} \sin(\sqrt{r} e^{i3}) \\
&= -\sqrt{r} \sin(-\sqrt{r}) - \sqrt{r} \sin(\sqrt{r}) = 0.
\end{align*}
\]

\[
\begin{align*}
(iii) \quad \left[ \cos(\sqrt{z}) \right]_c &= \left[ \cos(\sqrt{r} e^{i\phi/2}) \right]_c \\
&= \cos(\sqrt{r} e^{i\pi/2}) - \cos(\sqrt{r} e^{i0}) = \cos(-\sqrt{r}) - \cos(\sqrt{r}) \\
&= 0 \quad \text{since} \quad \cos(0) = \cos(-0).
\end{align*}
\]

Remark 2: To classify whether \( z = 0 \) is a branch point of \( F(z) \) we must take a very large circle \( |z| = R \quad R \gg 1 \) and see if \( F(z) \) returns to its original value as we traverse the circle.

Equivalently, \( z = 0 \) is a branch point of \( F(z) \) iff \( z = 0 \) is a branch point of \( F(1/z) \) (i.e., let \( z = 1/z \)).
EXAMPLE 11 Z = \infty A BRANCH POINT FOR

(i) \( F(z) = \sqrt{(z+1)(z+2)(z-3)} \)

(ii) \( F(z) = \log \left( \frac{z+1}{z-1} \right) \)

(iii) \( F(z) = (z^3 - z)^{1/3} \)

SOLUTION

(i) \( \text{LET } s = \frac{1}{z} \text{ TO } F\left(\frac{1}{s}\right) = \sqrt{(1 + \frac{1}{s})(2 + \frac{1}{s})(-3 + \frac{1}{s})} = \frac{-3}{s}\sqrt{(1 + s)(1 + 2s)(1 - 3s)} \)

for \( |s| < 1 \), \( F\left(\frac{1}{s}\right) = s^{-3/2} \) SO THAT \( \left[ F\left(\frac{1}{s}\right) \right]_C \neq 0 \) WHERE C IS THE SMALL CIRCLE \( |s| = \delta \) \( \delta < 1 \).

SO \( Z = \infty \) IS A BP FOR \( F(z) \)

(ii) \( F(z) = \frac{1}{z} \log(1 + z) - \log(z - 1) \)

LET \( z = \frac{1}{s} \), THEN \( F\left(\frac{1}{s}\right) = \log(1 + \frac{1}{s}) - \log\left(\frac{1}{s} - 1\right) \)

\( = \log\left(\frac{1 + s}{s}\right) - \log\left(\frac{1 - s}{s}\right) \)

\( = \log(1 + s) - \log(1 - s) \).

LET \( C: |s| = \delta \) WITH \( \delta < 1 \). THEN \( \left[ F\left(\frac{1}{s}\right) \right]_C = 0 \).

\( s = 0 \) IS NOT A BP OF \( F\left(\frac{1}{s}\right) \rightarrow Z = \infty \) IS NOT A BP OF \( F(z) \).

(iii) \( \text{LET } z = \frac{1}{s} \), \( F\left(\frac{1}{s}\right) = \left(\frac{1}{s} - \frac{1}{s}\right)^{1/3} = \left(\frac{1 - s^2}{s^3}\right)^{1/3} = \left(\frac{1 - s^2}{s}\right)^{1/3} \)

NOW LET \( C: |s| = \delta \) WITH \( \delta < 1 \),

THEN \( \left[ F\left(\frac{1}{s}\right) \right]_C = 0 \). HENCE \( s = 0 \) IS NOT A BP OF \( F\left(\frac{1}{s}\right) \rightarrow Z = \infty \) IS NOT A BP OF \( F(z) \).
EXAMPLE

FIND ALL POSSIBLE VALUES OF

(i) \( \cos W = 2i \),
(ii) \( \sin W = i \)

SOLUTION

(i) Let \( z = \cos W \) = \( \frac{e^{iW} + e^{-iW}}{2} \) so \( e^{iW} + e^{-iW} = 2z \).

Hence \( e^{iW} + e^{-iW} = 4i \) \( \Rightarrow e^{iW} - 4i e^{-iW} + 1 = 0 \).

Let \( \Lambda = e^{iW} \) \( \Rightarrow \Lambda^2 - 4i\Lambda + 1 = 0 \).

Thus \( \Lambda = \frac{4i \pm \sqrt{-16 - 4}}{2} = 2i \pm i\sqrt{5} \).

Now \( e^{iW} = (2 \pm \sqrt{5})i \).

+ sign \( e^{iW} = (2 + \sqrt{5})i \) \( \Rightarrow iW = \log((2 + \sqrt{5})i) = \log(2 + \sqrt{5}) + i\left(\frac{\pi}{2} + 2\pi K\right) \quad K = 0, \pm 1, \pm 2, \ldots \).

Thus \( W = -i \log(2 + \sqrt{5}) + \frac{\pi}{2} + 2\pi K \quad K = 0, \pm 1, \pm 2, \ldots \).

- sign \( e^{iW} = (2 - \sqrt{5})i \) \( \Rightarrow iW = \log((2 - \sqrt{5})i) = \log(\sqrt{5} - 2) + i\left(\frac{3\pi}{2} + 2\pi K\right) \).

Thus \( W = -i \log(\sqrt{5} - 2) - \frac{3\pi}{2} + 2\pi K \quad K = 0, \pm 1, \pm 2, \ldots \).

Since \( \log(\sqrt{5} - 2) = \log((\sqrt{5} - 2)(\sqrt{5} + 2)/(\sqrt{5} + 2)) = \log(\sqrt{5} + 2) \).

Then we can write + sign and - sign together as

(i) \( W = \pm i \log(\sqrt{5} + 2) + \frac{\pi}{2} + 2\pi K \quad K = 0, \pm 1, \ldots \).

The symmetry in (i) follows from identity that

\( \cos W = \cos(-W) \).

(0) \( W_0 = (0) (-W_0) \).
(ii) For the \( \sin w \cdot i \) we put

\[
\frac{e^{iw} - e^{-iw}}{2i} = i \rightarrow e^{iw} - e^{-iw} = 2i.
\]

Thus \( e^{2iw} + 2e^{iw} - 1 = 0 \rightarrow \Lambda^2 + 2\Lambda - 1 = 0 \) with \( \Lambda = e^{iw} \).

So \( \Lambda = \frac{-2 \pm \sqrt{4 + 4}}{2} = -1 \pm \sqrt{2} \).

\( \sqrt{2} \) sign \( e^{iw} = -1 + \sqrt{2} \) \( \rightarrow \): \( iw = \log(\sqrt{2} - 1) = i\log(\sqrt{2} - 1) + 2k\pi i \). \( \therefore \) \( w = -i \log(\sqrt{2} - 1) + 2k\pi \), \( k = 0, \pm 1, \pm 2, \ldots \).

- \( -\sqrt{2} \) sign \( e^{iw} = -1 - \sqrt{2} \) \( \rightarrow \): \( iw = \log(-1 - \sqrt{2}) = i\log(\sqrt{2} + 1) + i(-\pi + 2k\pi) \).

So \( w = -i \log(\sqrt{2} + 1) + (-\pi + 2k\pi) \). \( k = 0, \pm 1, \pm 2, \ldots \).

Since \( \log(\sqrt{2} + 1) = -\log(\sqrt{2} - 1) \) it is clearly that the symmetry in the two results follow from the fact that if \( w = w_0 \) is a root of \( \sin(w) = z \) then

\( w = \pi - w_0 \).