Suppose that \( f(z) \) is defined on a domain \( S \) (open connected set) in the complex plane. If \( z_0 \) is a point in \( S \), then \( f(z) \) is continuous at \( z_0 \) if

\[
\lim_{{z \to z_0}} f(z) = f(z_0)
\]

That is, \( f \) is continuous at \( z_0 \) if the values of \( f(z) \) get arbitrarily close to \( f(z_0) \), so long as \( z \in S \) and \( z \) is sufficiently close to \( z_0 \). The technical definition is for any \( \varepsilon > 0 \), \( \exists \delta > 0 \) such that \( |f(z) - f(z_0)| < \varepsilon \) whenever \( 0 < |z - z_0| < \delta \).

The key point is:

\[
(\forall) \quad \text{For a function to be continuous at } z_0 \text{ we require that } f(z) \to f(z_0) \text{ as } z \to z_0 \text{ in any direction in the complex plane.}
\]

\( S \) \hspace{1cm} \text{let } C = \text{complex plane}

**Ex 1** \( f(z) = |z|^2 \) is continuous at every point \( z \in C \)

**Ex 2** \( f(z) = \frac{1}{4 - z} \) is continuous for \( z \in C \) except \( z = 4 \).

**Ex 3** \( f(z) = \frac{(z^4 - 1)}{(z - 1)} \) is continuous for \( z \in C \) provided that we define \( f(i) = -4i \).

**Ex 4** For \( f(z) = \frac{z}{z} \) then \( f(z) \) is not continuous at \( z = 0 \).

- Let \( z = x \) with \( x \to 0^+ \). Then \( \lim_{{z \to 0^+}} f(z) = 1 \), path 1
- Let \( z = iy \) with \( y \to 0^+ \). Then \( \lim_{{z \to 0^+}} f(z) = \lim_{{y \to 0^+}} \frac{iy}{y} = -1 \), path 2
Since the value of \( \lim_{z \to 0} f(z) \) is different on path 1 than on path 2, \( f(z) \) is not continuous at \( z = 0 \).

**Example (HW)** Identify any points of discontinuity of
\[
f(z) = \begin{cases} 
  z, & \text{if } |z| < 1 \\
  |z|^2, & \text{if } |z| > 1.
\end{cases}
\]

**Definition** A function \( f(z) \) for \( z \) in a domain \( S \) is differentiable at a point \( z_0 \) in \( S \) if
\[
(x) \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.
\]
Exits. If this limit exists we label it by \( f'(z_0) \).

**Key Point 1**: For \( f(z) \) to be differentiable at \( z = z_0 \) we require that the limit in \((x)\) give the same value for any path for which \( z \to z_0 \).

**Example 1** Show that \( f(z) = \bar{z} \) is not differentiable at any point \( z_0 \).

**Proof** We write \( h = \Delta z \) complex, and calculate
\[
L = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z) - z_0}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = 1.
\]

- **Path 1** Let \( \Delta z = \Delta x \) with \( \Delta x \to 0 \). Then
  \[
  L = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0} = 1.
  \]

- **Path 2** Let \( \Delta z = i \Delta y \) with \( \Delta y \to 0 \). Then
  \[
  L = \lim_{\Delta y \to 0} \frac{i \Delta y}{i \Delta y} = \lim_{\Delta y \to 0} = i \Delta y = -i.
  \]
Thus $f(z) = \overline{z}$ is not differentiable at any point $z_0$.

**Example 2**

Let $f(z) = iz^2$. $f(z)$ is continuous for all $z_0$.

However, we now show that $f(z)$ is not differentiable at any point $z_0 \neq 0$, but is differentiable at $z_0 = 0$.

**Proof**

Let $z_0$ be given. We calculate

$$L = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{[|z_0 + \Delta z|^2] - [z_0 \overline{z_0}]}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{[z_0 \overline{z_0} + \overline{z_0} \Delta z + z_0 \Delta z + |\Delta z|^2] - [z_0 \overline{z_0}]}{\Delta z}$$

So (7) $L = \overline{z_0} + z_0 \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z}$

From (7) we observe that if $z_0 = 0 \rightarrow L = 0$, i.e., differentiable.

But, if $z_0 \neq 0$, then since $\lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z}$ depends on path, for which $\Delta z \to 0$ as in Example 1 it follows that $L$ is not independent of the path for which $\Delta z \to 0$.

Thus $f(z) = iz^2$ is not differentiable for any $z \neq 0$.

**Definition**

A function $f(z)$ is **analytic** at a point $z_0$ if its derivative exists not only at $z_0$ but at any $z$ in a small neighborhood of $z_0$.

**Definition**

A function $f(z)$ is **analytic in a domain** $D$ if it has a derivative at every point in $D$. 
REMARKS

(i) \( F(z) = |z|^2 \) is differentiable at \( z = 0 \) but is not analytic at \( z = 0 \). Why? Because, we can find no small neighborhood about \( z = 0 \) for which \( F \) has a derivative at each point in the neighborhood. (Recall \( F(z) = |z|^2 \) is not differentiable for any \( z \neq 0 \).)

DEFINITION

\( F(z) \) is an entire function if it is analytic at each point in the complex plane.

EXAMPLES

(i) Polynomial \( P(z) = q_N z^N + \ldots + q_0 \) are analytic for all \( z \), i.e., entire function.

(ii) \( F(z) = \frac{z}{z^2 + 1} \) is analytic for all \( z \) except at \( z = \pm i \). Such points are "singularities".

(iii) \( F(z) = 8z + i \) is not differentiable at any point \( z \). Hence, nowhere analytic.

NOTE

Analytic at \( z_0 \) \( \longrightarrow \) Differentiable at \( z_0 \) \( \longrightarrow \) Continuity at \( z_0 \).

(Mean differentiable at \( z_0 \) and in any small neighborhood of \( z_0 \))

NOTE

If \( F(z) \) is differentiable at a point then "usual" rules of calculus still hold and can be proved from definition.
REMARKS

If \( f(z) \) and \( g(z) \) are differentiable at \( z \) then

"usual" formulae still hold:

\[
(fg)'(z) = f'(z)g(z) + f(z)g'(z)
\]

Product rule

\[
(fg)'(z) = f'(z)g(z)
\]

Chain rule

If \( f(z) \) and \( g(z) \) are differentiable for all \( z \), then

\[
\frac{d}{dz} f(g(z)) = f'(g(z))g'(z)
\]

CAUCHY RIEMANN EQUATIONS (SECTION 2.4)

We write \( f(z) = u(x,y) + iv(x,y) \) \( u = \text{Re}(f), v = \text{Im}(f) \).

THEOREM I

Suppose that \( f(z) \) is differentiable at a point \( z_0 \). Then the Cauchy-Riemann equations

\[
\begin{cases}
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\end{cases}
\]

are satisfied at \( z_0 = x_0 + iy_0 \).

PROOF

Since \( f(z) = u(x,y) + iv(x,y) \) is differentiable at \( z_0 = x_0 + iy_0 \)

then the limit

\[
\frac{f'(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}
\]

can be calculated by taking any path for which \( \Delta z \to 0 \).

PATH 1

Let \( \Delta z = \Delta x \to 0 \). Then

\[
\begin{align*}
\frac{f'(z_0)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + iv \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}
\end{align*}
\]
\[ f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + \frac{\partial v}{\partial x}(x_0, y_0), \quad (1) \]

**Path 2**

Let \( \Delta z = i \Delta y \) with \( \Delta y \to 0 \).

\[ f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i \Delta y} \]

\[ = -i \lim_{\Delta y \to 0} \left( \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} \right) + \lim_{\Delta y \to 0} \left( \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} \right) \]

Thus,

\[ f'(z_0) = -i \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0). \quad (2) \]

Since (1) and (2) must be the same, then

\[ \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ at } (x_0, y_0) \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ at } (x_0, y_0) \end{cases} \]

(x) are called Cauchy-Riemann equations.

**Remark**

(i) If the CR equation do not hold at \( (x_0, y_0) \) then \( f(z) \) is not differentiable at \( z_0 \).

(ii) Key point if \( f(z) \) is analytic on some domain \( D \), then CR equation must hold at every point in \( D \).

(iii) \( v \) is called the harmonic conjugate of \( u \) (explained later).
However, what is more useful is to determine for a given \( U(x,y) \) and \( V(x,y) \) whether \( F(z) = U(x,y) + iV(x,y) \) is an analytic function. For this we need:

**Theorem 2** Let \( F(z) = U(x,y) + iV(x,y) \) be defined in a domain \( S \) and let \( z_0 \) be a point in \( S \) (i.e. \( z_0 \in S \)). Then, if

1. \( U_x, U_y, V_x, V_y \) exist in a neighborhood of \( z_0 \) and are continuous at \( z_0 \)

and

2. If CR equations are satisfied at \( z_0 = x_0 + i y_0 \)

i.e. \( U_x = V_y, V_x = -U_y \) at \( (x_0, y_0) \)

then \( F \) is differentiable at \( z_0 \).

Thus if \( U_x, U_y, V_x, V_y \) exist and are continuous in \( S \) and CR hold in \( S \), then \( F(z) \) is analytic in \( S \). [\( \square \)]

**Remark** (i) The proof is technical (see p.75 of [55]).

(ii) Note that the continuity assumption in (i) is needed.

We can summarize Theorem 1 and 2 as the following:

**Theorem 1** If \( F(z) \) is differentiable at any \( z \in S \) (i.e. analytic in \( S \))

\[ \Rightarrow \text{CR equations are satisfied at each } z \text{ in } S. \]

Thus if CR not satisfied at some \( z_0 \in S \)

\[ \Rightarrow F(z) \text{ is not differentiable at } z_0.\]
(TH 2) IF CR ARE SATISFIED AT ANY \( z \in S \), 
AND \( u_x, u_y, v_x, v_y \) CONTINUOUS AT ANY \( z \in S \) 
\[ \Rightarrow f(z) \text{ IS ANALYTIC IN } S. \]

**Example 1**

**Let** \( f(z) = |z|^2 \), THEN WITH \( z = x + iy \)

\[ f(z) = x^2 + y^2 + i0 \]

\[ u = x^2 + y^2, \ v = 0 \Rightarrow u_x = v_y \Rightarrow 2x = 0 \]

\[ u_y = -v_x \Rightarrow 2y = 0. \]

**Thus CR satisfied only** AT \( x = y = 0 \). ALSO \( u_x, u_y, v_x, v_y \) 
ARE CONTINUOUS ALWAYS. THUS YIELD, \( f(z) \) IS DIFFERENTIABLE 
ONLY AT \( z = 0 \). IT IS NOT ANALYTIC AT \( z = 0 \) SINCE \( f(z) \)
IS NOT DIFFERENTIABLE AT ANY POINT IN A SMALL NEIGHBORHOOD 
OF \( z = 0 \).

**Example 2**

**Let** \( f(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y) \). 
SHOW THAT \( f(z) \) IS DIFFERENTIABLE ON COORDINATE AXES 
BUT IS NOWHERE ANALYTIC.

**Proof**

\[ u = x^3 + 3xy^2 - 3x \quad v = y^3 + 3x^2y - 3y \]

\[ u_x = 3x^2 + 3y^2 - 3 \quad v_y = 3y^2 + 3x^2 - 3. \]

\[ u_y = 6xy \quad v_x = 6xy \]

**Note:** \( u_x = v_y \) FOR ANY \( x, y \) BUT \( u_y = -v_x \Rightarrow 12xy = 0. \)

**Thus we need either** \( x = 0 \) OR \( y = 0 \) FOR CR TO BE 
SATISFIED.
So, \( sr \) are satisfied only on \( x = 0 \) or \( y = 0 \) and \( u_x, u_y, v_x, v_y \) are continuous \( \rightarrow \) \( F(z) \) is differentiable on \( x = 0 \) and on \( y = 0 \). Note: \( F(z) \) is nowhere analytic since we can never insert a small neighborhood about a point on coordinate axis for which \( F \) is differentiable at each point in the neighborhood.

\[ \text{Not differentiable in here, only on } x = 0. \]

Example 3: The function \( F(z) = \begin{cases} (\bar{z})^2/z & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \) is not differentiable at \( z = 0 \), but the CR equations are satisfied at \( z = 0 \). (See HW #2.)

Is this contradicting Theorem 2? No, one can show that if we write \( F(z) = u + iv \) then \( u_x, u_y, v_x, v_y \) are not all continuous at \( x = y = 0 \).

Example 4: Let \( u(x, y) \) be given, find the function \( v(x, y) \) (the harmonic conjugate) so that \( F = u + iv \) is analytic.

(i) Let \( u = x - 3xy^2 + y \). We will find \( v \) by CR equation.
\[ u_x = 3x^2 - 3y^2; \quad v_y. \]

Thus
\[ v = 3x^2 y - y^3 + \frac{h'(x)}{x}, \]
\[ v_x = 6xy + \frac{h'(x)}{x} - [u_x] = -[6xy + 1] \]

Thus, \( h'(x) = -1 \) or \( h(x) = -x \). (Ignore constant wlog.)

This gives
\[ v = 3x^2 y - y^3 - x. \]

So
\[ f(z) = x^3 - 3xy^2 + y + i[3x^2 y - y^3 - x]. \]

(ii) Let \( u = x^2 - y^2 \). Find harmonic conjugate \( v \).

Now
\[ u_x = v_y \quad \Rightarrow \quad 2x = v_y \quad \text{so} \quad v = 2xy + h(x) \]
\[ u_y = -v_x \quad \Rightarrow \quad -2y = -[2y + h'(x)] \quad \rightarrow \quad h'(x) = 0. \]

Take \( h(x) = 0 \) wlog.

So \( v = 2xy \) and \( f(z) = x^2 - y^2 + 2ixy \) is analytic.

Notice also \( f(z) = z^2 \).

Remarks

(i) Cauchy-Riemann in "polar" coordinate

\[ r^2 = x^2 + y^2 \]

Let \( f(z) = u(x, y) + iv(x, y) \)

\[ \tan \phi = y/x \]

\[ u(\Gamma, \phi) = u(\Gamma \cos \phi, \Gamma \sin \phi), \quad v(\Gamma, \phi) = v(\Gamma \cos \phi, \Gamma \sin \phi) \]

Now calculate \( \Gamma_x = x/\Gamma = \cos \phi, \quad \Gamma_y = \sin \phi, \quad \phi_x = -y/(x^2 + y^2) = -\sin \phi/\Gamma \)

\[ u_x = u_{\Gamma} \Gamma_x + u_{\phi} \phi_x \]

\[ \Rightarrow u_x = u_{\Gamma} \cos \phi + u_{\phi} \left(-\frac{\sin \phi}{\Gamma}\right) \]

\[ u_y = u_{\Gamma} \Gamma_y + u_{\phi} \phi_y = u_{\Gamma} \sin \phi + u_{\phi} \cos \phi/\Gamma. \]

Similarly
\[ v_x = v_{\Gamma} \cos \phi - \frac{1}{\Gamma} \sin \phi v_{\phi}, \quad v_y = v_{\Gamma} \sin \phi + \frac{1}{\Gamma} \cos \phi v_{\phi}. \]
Now set \( U_x = V_y \rightarrow (V_x - \frac{1}{r} V_q) \cos \phi - (V_y - \frac{1}{r} V_q) \sin \phi = 0 \)
\( U_y = -V_x \rightarrow (\frac{1}{r} V_q + V_x) \cos \phi + (V_y - \frac{1}{r} V_q) \sin \phi = 0 \).

This has the form
\[
\begin{pmatrix}
a & -b \\
b & a
\end{pmatrix}
\begin{pmatrix}
\cos \phi \\
\sin \phi
\end{pmatrix} = 0
\]
\( a = \frac{1}{r} V_q \quad b = \frac{1}{r} V_q \).

Thus taking the determinant and setting \( = 0 \) to ensure a nontrivial solution, we have:
\( a^2 + b^2 = 0 \)
\( \Rightarrow a = 0 \quad \text{and} \quad b = 0 \).

Thus
\[
\begin{align*}
\frac{1}{r} V_q &= U_x \\
\frac{1}{r} V_q &= V_x
\end{align*}
\]

are CR in polar form.

**Example**
(i) Show that \( U = \frac{\partial}{\partial \phi}, \ V = \frac{\partial}{\partial \phi} \sin \phi \), \( n > 0 \) an integer satisfy CR, and since they are smooth functions, it follows that \( f = U + i V \) is analytic.

(ii) Analytic functions must be in terms of \( z \).

Let \( f(x, y) = U(x, y) + i V(x, y) \). (1)

Suppose \( U, V \) are smooth functions and let
\[
\begin{align*}
(x) \quad & x = (z + \bar{z})/2 \\
y & = (z - \bar{z})/2i
\end{align*}
\]

Suppose that CR are satisfied. Show that if we substitute \((x)\) into \( (1) \) then there is no \( z \)-dependence.

**Derivation**
Let \( \tilde{f}(z, \bar{z}) = f \left( \frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right) \)

We calculate:
\[ \frac{d \hat{F}}{d \hat{Z}} = \frac{d \hat{F}}{d x} \frac{d x}{d \hat{Z}} + \frac{d \hat{F}}{d y} \frac{d y}{d \hat{Z}} = \frac{d \hat{F}}{d x} \frac{1}{2} - \frac{d \hat{F}}{d y} \frac{1}{2 i} \]

so \[ \frac{d \hat{F}}{d \hat{Z}} = \frac{1}{2} \left( \frac{d \hat{F}}{d x} + i \frac{d \hat{F}}{d y} \right) = \frac{1}{2} \left( (U_x + i V_x) + i (U_y + i V_y) \right) \]

thus \[ \frac{d \hat{F}}{d \hat{Z}} = \frac{1}{2} \left( (U_x - V_y) + i (V_x + U_y) \right) = 0 \text{ since } \]

\[ U_x = V_y \text{ by CR } \]
\[ U_y = -V_x \]

Hence \[ \frac{d \hat{F}}{d \hat{Z}} = 0 \Rightarrow \hat{F} = \Phi(z) \]

If (i) is analytic in a domain S, then there will be no \( z \) dependence if we substitute (x, y) into (i).

(iii) Theorem 3: If \( F(z) \) is analytic in a domain S and if \( F(z) = 0 \) everywhere in S then \( F(z) \) is a constant in S. (Recall: Domain is open and connected)

Proof: We will give the idea in class (see p. 76 and section 1.6 p. 40 of Saff-Snider.

Example: Suppose that \( \text{Re}[F(z)] \) is constant inside a domain S and \( F(z) \) is analytic in S. Prove that \( F(z) \) is constant in S.

Proof: \( \text{Re}[F(z)]. \) Since \( F(z) \) is constant, then \( U_x = U_y = 0. \) But by CR, we get \( V_x = V_y = 0. \)

Recall \( F'(z) = U_x + i V_x. \) Hence \( F'(z) = 0. \)

By Theorem 3, \( F(z) = \text{constant in S}. \)
(iv) **JACOBIAN**

Suppose that \( F(z) = u + iv \) with \( u(x,y) = U, v(x,y) = V \), is analytic in \( S \). Suppose we think of changing coordinate \( (x,y) \rightarrow (U,V) \) via

\[
U = U(x,y) \quad V = V(x,y).
\]

What is the Jacobián of the transformation?

\[
\begin{align*}
\Delta U &= U_x \Delta x + U_y \Delta y + \ldots \\
\Delta V &= V_x \Delta x + V_y \Delta y + \ldots
\end{align*}
\]

(from Taylor series in 2 variables)

Thus as a matrix

\[
\begin{pmatrix}
U_x & U_y \\
V_x & V_y
\end{pmatrix}
\begin{pmatrix}
\Delta x \\
\Delta y
\end{pmatrix} =
\begin{pmatrix}
\Delta U \\
\Delta V
\end{pmatrix}
\]

\( J = \begin{pmatrix}
U_x & U_y \\
V_x & V_y
\end{pmatrix} \) is Jacobián.

By Cauchy-Riemann equations,

\[
\det J = U_x V_y - V_x U_y = U_x^2 + V_y^2 = |F'(z)|^2.
\]

Thus

\[
\det J = |F'(z)|^2
\]

(v) **LEVEL CURVES**

If \( F(z) \) is analytic and we write

\[
F(z) = u(x,y) + iv(x,y)
\]

then we claim that the level curves

\[
u(x,y) = \text{constant} \quad \text{and} \quad v(x,y) = \text{constant}
\]

are orthogonal at every point where \( F'(z) = 0 \).

**Example:**

\[F(z) = \frac{z^2}{2}, \quad (x^2 - y^2) \]

Solid: level line for \( u = x^2 - y^2 = \text{constant} \)

Dotted: level line for \( v = 2xy = \text{constant} \)
Proof: The level lines are orthogonal if

\[ \nabla u \cdot \nabla v = 0 \]  
(recall \( \nabla u \perp \nabla v \) for constant).

\[ (u_x, u_y) \cdot (v_x, v_y) = u_x v_x + u_y v_y = -u_x u_y + u_y u_x = 0 \]

By Cauchy-Riemann equation.

Thus \( \nabla u \cdot \nabla v = 0 \).

Consequently, it is easy to find level curves that are orthogonal. Simply take the real and imaginary part of a complex function.

\[ \text{Ex.} \quad f(z) = z^2 = x^2 - y^2 + 2ixy \]

\[ f(z) = e^z = e^x \cos y + ie^x \sin y \]

\( (vi) \)  

Harmonic functions

A harmonic function \( H(x,y) \) is one for which \( H \) satisfies Laplace's equation

\[ H_{xx} + H_{yy} = 0. \]

Thus, \( H \) can be interpreted as a steady-state temperature distribution. Typically some boundary condition for \( H \) must be given.

We now show an important result. If \( f(z) = u(x,y) + iv(x,y) \) is analytic in a domain \( S \) then

\[ \begin{cases} \quad U_{xx} + U_{yy} = 0 \quad \text{in } S \\ V_{xx} + V_{yy} = 0 \quad \text{in } S \end{cases} \]

i.e. both \( U \) and \( V \) satisfy Laplace's equation.
REMARK (i) One might think that extra condition to ensure that $u_{xx}, u_{yy}$ etc. exist need to be imposed. We do not worry about this here. In fact we show later in course that if $f(z)$ is analytic then all higher derivatives $f', f'', f'''$, etc. exist!

The little proof of (x) is easy.

We have by analyticity that $\partial u / \partial x$ are satisfied

$$u_x = v_y$$

$$u_y = -v_x$$

This if $u, v$ smooth enough (not an extra condition by Remark 1)

Then

$$(u_x)_x = (v_y)_x = (v_x)_y = -u_y)_y$$.

Hence

$$u_{xx} + u_{yy} = 0$$

Similarly

$$v_{xx} + v_{yy} = 0$$.

We will give examples of solving Laplace's equation through elementary mappings in next section.