Closed book examination. No notes, texts, or calculators allowed. Time: 50 minutes

Marks

[10] 1. Define \( f(z) = (z^2 + 1)^{1/3} \).
   (i) Determine all of the branch points of \( f(z) \) in the extended complex plane.
   (ii) Construct a branch of \( f(z) \) that is analytic in the unit disk and that satisfies \( f(1) = 1 \). For this branch, calculate \( f(1 + i) \).

[10] 2. Let \( C \) be the curve \( z = e^{i\theta} \) with \( \pi/4 \leq \theta \leq 7\pi/4 \) oriented counterclockwise. Calculate the following integrals:
   (i) \( \int_{C} \frac{1}{z} \, dz \); (ii) \( \int_{C} z \cos(z^2) \, dz \)

[20] 3. Let \( C \) be the simple closed curve \( |z| = 2 \), oriented counterclockwise. Calculate the following integrals, providing justification for your results:
   (i) \( \int_{C} \frac{1}{(z-1)(z-2)} \, dz \)
   (ii) \( \int_{C} z^{-2} e^{2z} \, dz \)
   (iii) \( \int_{C} \frac{f(z)}{(z-1)(z-2)} \, dz \) where \( f(z) \) is analytic in \( |z| \geq 2 \) and bounded by \( |f(z)| < M \) in the region \( |z| \geq 2 \), where \( M \) is some positive constant.
   (iv) \( \int_{C_1} \frac{1}{4z + \sqrt{z}} \, dz \), where \( \sqrt{z} \) denotes the branch of the square root with a branch cut on the positive real axis in the \( z \)-plane for which \( \sqrt{-1} = i \). Here \( C_1 \) is the simple closed curve \( |z + 16| = 1 \) oriented counterclockwise.

[10] 4. Suppose that \( f(z) \) is analytic inside and on a simple closed curve \( C \). Assume also that \( |f(z) - 1| < 1 \) for \( z \) on \( C \). Prove that there is no point \( z_0 \) with \( \bigcap_{\int \text{inside } C} \) for which \( f(z_0) = 0 \).
   (i) Use this result to prove that there are no solutions to \( z = -3e^{-z^2} \) in the disk \( |z| < 1 \). (Hint: Determine a convenient choice for \( f(z) \) to use in the result above).

[50] Total Marks

The End
Problem 1

Let \( f(z) = (z^2 + 1)^{1/3} \).

(i) We factor \( f(z) = (z + i)^{1/3}(z - i)^{1/3} = (\Gamma_1 \Gamma_2)^{1/3} e^{i(\Phi_1 + \Phi_2)/3} \).

Clearly \( \Phi_1, \Phi_2 \) are branch points since if we increase \( \Phi_j \) by \( 2\pi \), i.e., \( \Phi_j \rightarrow \Phi_j + 2\pi \), \( f \) does not return to the same value.

Now check \( z = \infty \). Let \( z = \frac{1}{\delta} \), so \( f(\frac{1}{\delta}) = (\frac{1}{\delta} + i)^{1/3}(\frac{1}{\delta} - i)^{1/3} \) so that \( f(\frac{1}{\delta}) = \delta^{-1/3} (1 + i)^{1/3}(1 - i)^{1/3} \approx \delta^{-1/3} \) for \( |\delta| \to 0 \). Thus \( \delta \to 0 \) is a branch point of \( f(\frac{1}{\delta}) \). (\( \infty \) is also a branch point.

(ii) We want \( f(z) \) analytic in the unit disk, so try branch cuts.

As shown, \( \Gamma_1 \Gamma_2 \)

Thus \( f(z) = (\Gamma_1 \Gamma_2)^{1/3} e^{i(\Phi_1 + \Phi_2)/3} \).

\( -3\pi/2 < \Phi_1 < 3\pi/2 \)

\( -\pi/2 < \Phi_2 < 3\pi/2 \).

Now calculate \( f(0) \): For \( z = 0 \), \( \Phi_1 = -\pi/2 \), \( \Phi_2 = \pi/2 \), \( \Gamma_1 = \Gamma_2 = 1 \).

Thus \( f(0) = 1 \) as desired.

Now let \( z = 1 + i \). Then \( i \frac{1}{2} \Gamma_1 \Gamma_2 \) \( i \frac{1}{2} \Gamma_1 \Gamma_2 \)

\( \Gamma_1 = 1 \), \( \Phi_1 = 0 \)

\( \Gamma_2 = \sqrt{5} \), \( \Phi_2 = \tan^{-1}(2) \).

Thus \( f(1+i) = (\sqrt{5})^{1/3} e^{i\tan^{-1}(2)/3} = 5^{1/6} e^{i\tan^{-1}(2)/3} \).

Remark: Also could have chosen branch by writing

\( f(z) = \exp \left( \frac{1}{3} \log (z^2 + 1) \right) \log \text{principal branch} \).
Problem 2

Let $C$ be as shown.

(i) Define $\log z$ to be branch of $\log z$ with cut as shown on positive real axis. Then since $\log z$ is analytic in region containing $C$,

$$\int_C \frac{1}{z} \, dz = \log z_F - \log z_i = \left( \ln |z_F| + i \pi/4 \right) - \left( \ln |z_i| + i \pi/4 \right)$$

$$\int_C \frac{1}{z} \, dz = 3\pi i/2 \quad \text{since} \quad |z_F| = |z_i| = 1.$$

The other method is direct integration. Let $z = e^{it}$

so

$$\frac{dz}{z} = ie^{it} \, dt = -i \, dt.$$

$$\int_C \frac{dz}{z} = \int_{\pi/4}^{3\pi/4} i \, dt = 3\pi i/2.$$

(ii) \( \int_C z \cos(z^2) \, dz \). The antiderivative is

$$\frac{1}{2} \sin(z^2).$$

$$\int_C z \cos(z^2) \, dz = \frac{1}{2} \sin(z_F^2) - \frac{1}{2} \sin(z_i^2) = \frac{1}{2} \sin(e^{7\pi i/2}) - \frac{1}{2} \sin(e^{i\pi/2})$$

$$\int_C z \cos(z^2) \, dz = \frac{1}{2} \left[ \sin(-i) - \sin(i) \right] = -\sin(i),$$

but $\sin(i) = i \sinh(1)$, $\sin(-i) = -i \sinh(1)$.

So

$$\int_C z \cos(z^2) \, dz = -i \sinh(1).$$

Problem 3

[i] \( \int_C \frac{dz}{(z-1)(z-3)} \). Only $z = 1$ is inside contour.

$$\frac{1}{z-1} + \frac{A}{z-3} \quad \rightarrow \quad A(z-3) + B(z-1) = 1,$$

let $z = 3 \rightarrow B = 1/2$,

$z = 1 \rightarrow A = -1/2$. 
\[ \int_{C} \frac{dz}{z(z-1)(z-2)} = -\frac{1}{2} \int_{C_{1}} \frac{dz}{z-1} + \frac{1}{2} \int_{C_{2}} \frac{dz}{z-2} = -\frac{1}{2} (2\pi i) + 0 \quad \text{(since \( z \neq 3 \) not inside \( C \))}. \]

Thus \[ \int_{C_{0}} (z-1)(z-2) = -\pi i. \]

(ii) \[ I = \int_{C} \frac{e^{2z}}{z^2} \, dz : C = \text{iz1:2 counter-clockwise}. \]

Recall \[ f'(z_0) = \frac{1}{2\pi i} \int_{C} \frac{f(z)}{(z-z_0)^2} \]
when \( z_0 \) inside \( C \) and \( f \) analytic inside and on \( C \).

Thus set \( z_0 = 0 \) and \( f(z) = e^{2z} \).

Then \[ I = 2\pi i; f'(0) = 2\pi i; [2e^{2z}]_{z=0} = 4\pi i \rightarrow \int_{C} \frac{e^{2z}}{z^2} \, dz = 4\pi i. \]

(iii) Define \( g(z) = \frac{f(z)}{(z-1)(z-1-i)} \). Repeat that \( g(z) \) is analytic in \( |z| > 2 \)

Since \( f \) is analytic in \( |z| > 2 \) and \( z_1, z_2 \), \( z_2 = 1+i \) satisfy \( |z_1| < 2, |z_2| = \sqrt{2} < 2 \).

Thus, deform as shown

\[ I = \int_{C} g \, dz = J(R) = \int_{C_{R}} g(z) \, dz \]

But now let \( R \to \infty \) and estimate

\[ |J(R)| \leq \max_{C} \left| \frac{f(z)}{(z-1)(z-1-i)} \right| (2\pi R) \leq \frac{M}{(R-1)(R-\sqrt{2})} \to 0 \quad \text{as} \quad R \to \infty. \]

Thus \[ J(R) = I = 0 \quad \to \quad \int_{C} \frac{f(z)}{(z-1)(z-1-i)} \, dz = 0. \]

(iv) \[ I = \int_{C_{1}} \frac{dz}{z+i\sqrt{z}}. \] The contour \( C_1 \) and branch cut \( C_1 \)

are as shown

Thus the only possible singularity is if \( 4i + \sqrt{z} = 0 \) for \( z \) inside \( C_1 \),
but this is impossible since for branch cut with \( 0 \leq \theta < 2\pi \)
we have \( \sqrt{z} = r^{1/2} e^{i\theta/2} \rightarrow \text{IM} \left( \sqrt{z} \right) = r^{1/2} \sin \left( \frac{\theta}{2} \right) \geq 0 \) on \( 0 \leq \theta < 2\pi \).

Notice \( \sqrt{16} = 4i \). Thus \( \sqrt{-16} = 4i \) NOT \( \sqrt{-16} = -4i \). \( \implies I = 0. \)
Problem 4

Let \( g(z) = f(z) - 1 \). Then by assumption

\[ |g(z)| < 1 \text{ for } z \text{ on } C \]

\( g(z) \) analytic inside and on \( C \).

By Max-Modulus Principle, it follows that

\[ (\star) \quad |g(z)| < 1 \text{ for } z \text{ inside and on } C. \]

Suppose that there was a \( z_0 \) for which \( f(z_0) = 0 \) with \( z_0 \) inside \( C \). Then \( |g(z_0)| = 1 \), thus (contradict \( \star \)).

(i) Write \( z = -3e^{-z^2} \) and \( ze^{z^2} = -3 \). \( \Rightarrow \frac{1}{3} ze^{z^2} + 1 = 0 \).

So define \( f(z) = \frac{1}{3} ze^{z^2} + 1 \).

Then \( f(z) \) is analytic and \( f(z) - 1 = \frac{1}{3} ze^{z^2} \).

Now on \( |z| = 1 \), \( |f(z)| = \frac{1}{3} |ze^{z^2}| \leq \frac{1}{3} e^{1z^2} \leq \frac{e^1}{3} = \frac{2.71}{3} < 1 \).

(Here we used \( |e^w| \leq e^{|w|} \)).

Thus \( |f(z) - 1| < 1 \) for \( |z| = 1 \).

\( \Rightarrow \) \( f(z) \) has no zeros inside \( |z| < 1 \).

(i.e. \( z \neq -3e^{-z^2} \) for all \( z \) in \( |z| < 1 \))