MATH 305: MIDTERM 1: October 14th, 2011 (M. WARD)

Closed Book and Notes. 50 minutes. Total 50 points

**PROBLEM 1:** (12 Points) Find all solutions in the complex plane to the following:

(i) \( z^4 = 8iz \)

(ii) \( \sin z = \cosh 2 \)

(iii) \( e^{1/z} = e^{10}(1 + i) \)

(Hint: you will need the identity \( \sin(x + iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y) \))

**PROBLEM 2:** (8 Points)

Let \( f(z) = y^3 + 3x^2y - 3y + i(x^3 + 3xy^2 - 3x) \) where \( z = x + iy \). Where is \( f(z) \) differentiable in the complex plane? Where is \( f(z) \) analytic? Explain your reasoning carefully.

**PROBLEM 3:** (18 Points) Establish the validity of each of the following statements. If it is true, then provide a proof. If it is false, carefully explain why.

i) \( \text{Arg}(z^2) = 2 \text{Arg}(z) \) for all \( z \neq 0 \).

ii) \( \log(e^z) = z \) for all \( z \).

iii) \( |e^z| \leq |z|^2 \) for all \( z \).

iv) \( \text{Re}(i/\bar{z}) = -\text{Im}(z)/|z|^2 \) for all \( z \neq 0 \).

v) If \( f(z) = u(x, y) + iv(x, y) \) is an entire function of \( z = x + iy \), then \( e^u \cos v \) is a harmonic function.

vi) \( \text{Log}(z^3) \) is an analytic function everywhere in the complex \( z \)-plane except on the negative real axis.

**PROBLEM 4:** (12 Points) Find the image of the set \( S \) under the map \( w = f(z) \) for each of the following:

i) \( S = \{ z \mid |z - i| \leq 2 \} \) and \( f(z) = 2i(z + 1) \)

ii) \( S = \{ z \mid 1 \leq \text{Re}(z) \leq \frac{\pi}{2} + 1 \text{ with } \text{Im}(z) \geq 0 \} \) and \( f(z) = e^{2i(z-1)} \).
(i) \( z^4 = 8iz \)

One root if \( z = 0 \) so that

\[ z^3 = 8i = 8e^{i\pi/2} \]

Now put \( z = r e^{i\phi} \) so that

\[ r^3 e^{3i\phi} = 8 e^{i\pi/2} \]

Hence taking modules \( r = 2 \)

and \( 3\phi = \pi/2 + 2k\pi \quad k = 0, 1, 2 \).

In summary root are

\[ z = 0 \text{ and } z_k = 2e^{i(\pi/6 + 2k\pi/3)} \quad k = 0, 1, 2 \]

(ii) \( \sin z = \cosh 2 \)

We write

\[ \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y \]

\[ \cosh 2 = \cos x \sinh y \]

Thus \( \cosh 2 = \sinh \cosh y \)

\[ O = \cos x \sinh y \]

We must have \( y \neq 0 \) so

\[ x_n = (2n+1)\pi/2 \quad n = 0, \pm 1, \pm 2, \ldots \]

But we need \( \sin(x_n) = 1 > 0 \).

Hence \( x_n = (2n+1)\pi/2 \quad n = 0, \pm 2, \pm 4, \ldots \)

and \( \cosh y = \cosh 2 \rightarrow y = \pm 2 \).

Hence

\[ z = (2n+1)\pi/2 + 2i \]

\[ n = 0, \pm 2, \pm 4, \ldots \]

(iii) \( e^{1/z} = e^{10(1+i)} = \sqrt{2} e^{i\pi/4} \)

Let \( w = 1/z \). Then

\[ e^w = \sqrt{2} e^{10} e^{i\pi/4} \]

\[ w = \log[\sqrt{2} e^{10} e^{i\pi/4}] \]

so \( w_k = ln(\sqrt{2} e^{10}) + i(\frac{\pi}{4} + 2k\pi) \)

\[ k = 0, \pm 1, \pm 2, \ldots \]

Thus root are

\[ z_k = \frac{1}{w_k} = \frac{1}{ln(\sqrt{2} e^{10}) + i(\frac{\pi}{4} + 2k\pi)} \]

Notice that \( |z_k| \rightarrow 0 \) at \( |k| \rightarrow \infty \).
PROBLEM 2

\[ F = y^3 + 3x^2y - 3y + (x^3 + 3xy^2 - 3x). \]

\[ U = y^3 + 3x^2y - 3y \quad \quad V = x^3 + 3xy^2 - 3x \]

\[ U_x = 6xy \quad \quad V_y = 6xy \]

\[ U_y = 3y^3 + 3x^2 - 3 \quad \quad V_x = 3x^2 + 3y^2 - 3 \]

Now \[ U_x = V_y \quad \Rightarrow 6xy = 6xy \quad \text{always true} \]

\[ U_y = -V_x \quad \Rightarrow 6x^2 + 6y^2 = 6 \quad \Rightarrow x^2 + y^2 = 1 \]

Thus the equation holds on circle \( x^2 + y^2 = 1 \).

- \( F \) is differentiable at each point on \( |z| = 1 \)
- BUT \( F \) is nowhere analytic since we cannot have any small disk centered at a point on \( |z| = 1 \) for which \( F \) is differentiable everywhere inside the disk.

**Problem 3**

(i) \( \text{ARG}(z^2) = 2 \text{ARG}(z) \) is FALSE.

Let \( z = e^{\frac{3\pi i}{4}} \). Then \( \text{ARG}(z^2) = \text{ARG}(e^{\frac{3\pi i}{2}}) = -\frac{\pi}{2} \)

\[ 2 \text{ARG}(z) = 2 \text{ARG}(e^{\frac{3\pi i}{4}}) = 2 \left( \frac{3\pi}{4} \right) = \frac{3\pi}{2}. \]

(ii) \( \log(e^z) = z \) is FALSE IN GENERAL.

Notice LHS is MULTI-VALUED, while RHS is SINGLE-VALUED.
In fact if \( z = x + iy \) then
\[
\log(e^z) = \log(e^{x+iy}) = \log(e^x e^{iy}) = \log(e^x) + i(y + 2\pi k) \\
\text{Hence} \quad \log(e^z) = z + 2\pi k
\]

(iii) \( |e^z| = e^{|z|^2} \) is true.

Let \( z = x + iy \), then \( |e^z| = |e^{x^2 - y^2 + 2ixy}| = e^{x^2 - y^2} \leq e^{x^2 + y^2} \).

Hence \( |e^z| = e^{x^2 + y^2} \leq e^{|z|^2} \).

(iv) \( \text{RE} \left( \frac{i}{z} \right) = \frac{-1 \text{IM}(z)}{|z|^2} \) for all \( z \neq 0 \) is true.

We write \( \text{RE} \left( \frac{i}{z} \right) = \text{RE} \left( \frac{iZ}{|z|^2} \right) = \frac{-1}{|z|^2} \text{RE} \left( i(x + iy) \right) = \frac{-y}{|z|^2} \).

Thus \( \text{RE} \left( \frac{i}{z} \right) = \frac{-y}{|z|^2} = \frac{-1 \text{IM}(z)}{|z|^2} \).

(vi) \( \log(z^5) \) is analytic except on paths for which \( \text{RE}(z^5) < 0 \) and \( \text{IM}(z^5) = 0 \).

If we let \( \text{IM}(z^5) = 0 \rightarrow \sin(5\varphi) = 0 \rightarrow \varphi = \frac{n\pi}{5}, \, n = 0, \ldots, 9 \).

Choose the path with \( n = 3 \). Then \( \varphi = 3\pi/5 \) as shown.

\[ \text{On this path,} \]
\[ \text{RE}(z^5) = |z|^5 \cos \left( \frac{3\pi}{5} \right) = -|z|^5 < 0. \]

This is a path other than \( z < 0, z \) real for which \( \log(z^5) \) is not analytic.

(vii) True if \( f(z) \) is analytic \( \rightarrow g(z) = e^{f(z)} = e^u \) is analytic.

\[ \text{RE} \left[ g(z) \right] = e^u \text{ is harmonic (since real part of analytic function)} \]
Problem 4

(i) Let \( F(z) = 2i(z+1) \)
\[ S' = \{ z \mid |z - i| \leq 2 \} \]

Define \( w = 2i(z+1) \) so \( z = -1 + \frac{w}{2i} \), \( \rightarrow |z - i| \leq 2 \) yields \( |\frac{w}{2i} + i| \leq 2 \).

Hence \( S' = \{ w \mid |w/2i - 1 - i| \leq 2 \} \).

Now \( \left| \frac{w + 2i(-1-i)}{2i} \right| = \frac{1}{2} |w - (2i-2)| \leq 2 \) \( \rightarrow |w - (2i-2)| \leq 4 \).

Hence \( S' = \{ w \mid |w - (2i-2)| \leq 4 \} \).

Alternatively we can proceed by picture:

(ii) \( S = \{ z \mid |z - i| \leq 2 \} \) with \( \text{IM} z > 0 \)

Now let \( w = e^{\frac{z}{2i}} \) so \( u + iv = e^{\frac{x}{2i}} \cos y + i e^{\frac{x}{2i}} \sin y \)

This gives \( u = e^{\frac{x}{2i}} \cos y \) \( \quad 0 < y < \pi \)
\( v = e^{\frac{x}{2i}} \sin y \) \( \quad -\pi < x < 0 \)
\( \Rightarrow u^2 + v^2 = \left( e^{\frac{x}{2i}} \right)^2, \quad v > 0. \)

Now \( e^{\frac{x}{2i}} \) ranges from \((0, 1)\)
\( \Rightarrow -\pi < x \leq 0 \)
\( \Rightarrow S' = \{ w \mid |w| \leq 1 \text{ with } \text{IM} w > 0 \} \).