Solutions to HW7

1. On $|z| = 1$,
\[
\left| \frac{8}{9}z^{10} + z^8 + 5z^5 + z \right| \leq 8 < 9.
\]
Hence by Rouche’s theorem, the constant function 9 and \( \frac{8}{9}z^{10} + z^8 + 5z^5 + z + 9 \) has the same number of roots in $|z| = 1$.

2. On the circle $|z| = 2$,
\[
|4z^2 - 1| \leq |z^6|.
\]
Therefore, $z^6 + 4z^2 - 1$ has 6 roots in $|z| < 2$.

On the circle $|z| = 1$,
\[
|z^6 - 1| \leq |4z^2|.
\]
Therefore $z^6 + 4z^2 - 1$ has 2 roots in $|z| < 1$. Consequently, it has 4 roots in the annulus $1 < |z| < 2$.

3. (1). By Nyquist criterion,
\[
N_0(P) = \frac{1}{2\pi} \left( 4\pi + 2\Delta_{\Gamma_+} (P(z)) \right).
\]
Here $\Gamma_+$ is the upper half imaginary axis. Let $z = yi, y \in (0, +\infty)$. Then
\[
P(yi) = y^4 - 2y^3i - 3y^2 + yi + 2.
\]
Let $P_R = y^4 - 3y^2 + 2, P_I = -2y^3 + y$. The real positive roots of $P_R$ are $1, \sqrt{2}$. The real positive root of $P_I$ is $\frac{\sqrt{2}}{2}$. As $y \to +\infty$,
\[
\frac{P_R}{P_I} \to -\infty.
\]
The sign of $(P_R, P_I)$ is:
- For $y \in (\sqrt{2}, +\infty)$, it is $(+, -)$.
- For $y \in (1, \sqrt{2})$, it is $(-, -)$.
- For $y \in \left(\frac{\sqrt{2}}{2}, 1\right)$, it is $(+, -)$.
- For $y \in \left(0, \frac{\sqrt{2}}{2}\right)$, it is $(+, +)$.

Hence $\Delta_{\Gamma_+} (\arg (P(z))) = 0$. The number of roots in the right half plane is 2.

(2). Let $\Gamma$ be the boundary of the right half disk $\{ z : |z| \leq R, \text{Re} z \geq 0 \}$. On $\Gamma$, for $R$ large,
\[
|e^{-z}| \leq |z - 3|.
\]
Then by Rouche’s theorem, the function \( z - 3 + e^{-z} \) has the same number of roots inside \( \Gamma \) as the function \( z - 3 \). We then deduce that it has 1 root in the right half plane.

4. 

\[
f(z) = \frac{z - 1}{(z + 1)(z - 2)} = \frac{2}{3} \frac{1}{z + 1} + \frac{1}{3} \frac{1}{z - 2}.
\]

In \( |z| < 1 \),

\[
f(z) = \frac{2}{3} \sum_{j=0}^{+\infty} (-z)^j - \frac{1}{6} \sum_{j=0}^{+\infty} \left(\frac{z}{2}\right)^j.
\]

In \( 1 < |z| < 2 \),

\[
f(z) = \frac{2}{3z} \sum_{j=0}^{+\infty} \left(-\frac{1}{z}\right)^j - \frac{1}{6} \sum_{j=0}^{+\infty} \left(\frac{z}{2}\right)^j.
\]

In \( |z| > 2 \),

\[
f(z) = \frac{2}{3z} \sum_{j=0}^{+\infty} \left(-\frac{1}{z}\right)^j + \frac{1}{3z} \sum_{j=0}^{+\infty} \left(\frac{2}{z}\right)^j.
\]

On the other hand,

\[
f(z) = \frac{2}{3} \frac{1}{z - 1 + 2} + \frac{1}{3} \frac{1}{z - 1 - 1}.
\]

Then in \( |z - 1| < 1 \),

\[
f(z) = \frac{1}{3} \sum_{j=0}^{+\infty} \left(-\frac{z - 1}{2}\right)^j - \frac{1}{3} \sum_{j=0}^{+\infty} (z - 1)^j.
\]