Solutions to HW6.

1. Proof. Since $f^{(5)}$ is bounded and entire, by Liouville’s theorem, $f^{(5)}$ is a constant. That is

$$f^{(5)}(z) = c_1.$$ 

Then

$$f^{(4)}(z) = c_1 z, \ldots, f(z) = \frac{c_1}{120} z^5 + \ldots + c_5 z + c_6.$$ 

2. Proof. Let $c = a + bi$, where $a, b$ are real constants to be determined later on. We compute

$$cf = (a + bi)(u + vi) = au - bv + (bu + av)i.$$ 

Moreover,

$$|e^{cf}| = e^{au - bv}.$$ 

Now we choose $a = -1, b = 1$, using the assumption that $u + v > 0$, we get

$$|e^{cf}| = e^{-u - v} < 1.$$ 

Hence by Liouville’s theorem $e^{cf}$ is a constant:

$$e^{cf} = A.$$ 

Taking derivative, we get

$$f' e^{cf} = 0.$$ 

Hence $f' = 0$ and $f$ is a constant.

3. Since the function $z^2 + 4z - 1$ is analytic, the maximum of its modulus is achieved on the boundary $|z| = 1$. For $z = e^{it}, t \in [0, 2\pi]$, we have

$$|z^2 + 4z - 1|^2 = (e^{2it} + 4e^{it} - 1) (e^{-2it} + 4e^{-it} - 1)$$

$$= 1 + 4e^{it} - e^{2it} + 4e^{-it} + 16 - 4e^{it} - e^{-2it} - 4e^{-it} + 1$$

$$= 18 - 2 \cos (2t).$$

Hence $|z^2 + 4z - 1| \leq \sqrt{20}$.

4. $u := \text{Re} \left( e^{xz} \right) = e^{2x} \cos (2y)$. It is a harmonic function. By the maximum principle, its maximum is achieved on the boundary of $[0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$.

On the left boundary, $u = \cos (2y)$. Here max $u = 1$

On the right boundary, $u = e^x \cos (2y)$. Here max $u = e^x$

On the bottom boundary, $u = e^{2x}$. Here max $u = e^x$

On the top boundary, $u = -e^{2x}$. Here max $u = -1$. 

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Hence max \( u = e^\pi \).

5. Let \( z = e^{it}, t \in [0, 2\pi] \). Then

\[
P(e^{it}) = 6e^{4it} + e^{3it} - 2e^{2it} + e^{it} - 1
\]

\[
= 6 \cos (4t) + \cos (3t) - 2 \cos (2t) + \cos (t) - 1
\]

\[
+ (6 \sin (4t) + \sin (3t) - 2 \sin (2t) + \sin t) i
\]

We plot the curve \( P(\Gamma) \), as \( t \) increases from 0 to \( 2\pi \):

\[
[6 \cos (4t) + \cos (3t) - 2 \cos (2t) + \cos (t) - 1, (6 \sin (4t) + \sin (3t) - 2 \sin (2t) + \sin t)]
\]

Hence \( \arg_{\Gamma} (P(z)) = 8\pi \). By the argument principle, the number of zeros of \( P \) in the unit disk is

\[
\frac{1}{2\pi} \arg_{\Gamma} (P(z)) = 4.
\]