Solutions to HW1.

1. \(-\frac{253}{4225} + \frac{33596}{4225}i\).

2. We compute
   
   \[
   6 - (3 + i) = 3 - i, \\
   4 + 4i - (3 + i) = 1 + 3i.
   \]
   
   \[\text{Arg}\left(\frac{1 + 3i}{3 - i}\right) = \text{Arg}(i) = \frac{\pi}{2}.\]
   
   Hence it is a right triangle.

3. The distance from \(z\) to 2 is equal to the distance from \(z\) to the line \(\text{Re}\, z = -1\). Hence it is a parabola.

4. (1) \(\text{Arg}\left(\frac{-5}{3 - i}\right) = -\frac{3}{4}\pi.\) (2) \(\arg\left((\sqrt{3} - i)^2\right) = \{-\frac{\pi}{3} + 2k\pi, k \in \mathbb{Z}\}.

5. 
   
   \[
   \int_0^{2\pi} (\cos \theta)^6 \, d\theta = \int_0^{2\pi} \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^6 \, d\theta \\
   = \frac{1}{2^6} \int_0^{2\pi} \left(e^{6i\theta} + 6e^{4i\theta} + 15e^{2i\theta} + 20 + 15e^{-2i\theta} + 6e^{-4i\theta} + e^{-6i\theta}\right) \, d\theta \\
   = \frac{5\pi}{8}.
   \]

6. The equation is
   
   \[
   \left(1 + \frac{1}{z}\right)^6 = 1.
   \]
   
   Hence we get
   
   \[1 + \frac{1}{z} = 1^\frac{1}{6} = e^{i\theta}, \quad \theta = \frac{k\pi}{3}, k = 0, 1, ..., 5.\]
   
   We then find the solutions
   
   \[
   z = \frac{1}{1 - e^{i\theta}} = \frac{1 - \cos \theta + i \sin \theta}{(1 - \cos \theta)^2 + (\sin \theta)^2} \\
   = \frac{2 \sin^2 \frac{\theta}{2} + 2i \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{4 \sin^2 \frac{\theta}{2}} \\
   = \frac{\sin \frac{\theta}{2} + i \cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} \\
   = \frac{1}{2} + \frac{i}{2} \cot \frac{\theta}{2}.
   \]

7. (1) It is a translation in negative \(y\) direction. The image is \(\text{Im}\, z < 0\).
   (2) The image in the \(w\) plane is \(\frac{1}{2} < |w| < +\infty\).

8. (1) \(u = \sqrt{|xy|}, v = 0\). In the region \(D: \{z: xy \neq 0\}\), \(u\) and \(v\) are differentiable. But the Cauchy-Riemann equation does not hold there. Hence
$f$ is not differentiable in $D$. This also implies that $f$ is not analytic anywhere in the complex plane. Next let us analyze the differentiability at the points in the axes. Fix $z_0 = y_0i$. We have

$$\lim_{z \to 0} \frac{f(z_0 + z) - f(z_0)}{z} = \lim_{x+yi \to 0} \frac{\sqrt{|x(y + y_0)|}}{x + yi}.$$ 

This has no unique limit. Hence $f$ is also not differentiable on the axes.

(2) $f(z) = |z|^2 + 4\bar{z} = x^2 + y^2 + 4x - 4yi$.

$$\partial_x u = 2x + 4, \partial_y u = 2y$$

$$\partial_x v = 0, \partial_y v = -4.$$ 

The Cauchy-Riemann equation holds at $z = -4$. Hence $f$ is differentiable at $-4$ but nowhere analytic.

(3) $f(z) = z^2$. It is analytic everywhere.