DEFINITION LET \( t \) BE REAL-VALUED AND
\[
W(t) = C(t) + iD(t)
\]
WHERE \( C(t), D(t) \) ARE REAL-VALUED.

THEN WE DEFINE
\[
\int_{a}^{b} W(t) \, dt = \int_{a}^{b} C(t) \, dt + i \int_{a}^{b} D(t) \, dt
\]

EX.
\[
\int_{0}^{\pi/2} (e^t + 2i \sin t) \, dt = \int_{0}^{\pi/2} e^t \, dt + 2i \int_{0}^{\pi/2} \sin t \, dt = e^{\pi/2} - 1 + 2i.
\]

KEY PROPERTIES IF \( Z(t) = a(t) + i b(t), W(t) = C(t) + i D(t) \)
AND \( \chi \) IS A COMPLEX NUMBER, THEN

(i) \( \frac{d}{dt} \{ Z(t) + W(t) \} = \dot{Z}(t) + \dot{W}(t) \)

(ii) \( \frac{d}{dt} \{ Z(t) W(t) \} = W \dot{Z} + \dot{W} Z \)

(iii) \( \int_{a}^{b} \{ W(t) + \chi Z(t) \} \, dt = \int_{a}^{b} W(t) \, dt + \chi \int_{a}^{b} Z(t) \, dt \)

(iv) \( \int_{a}^{b} \dot{W}(t) \, dt = W(b) - W(a) \)

(v) \( \left| \int_{a}^{b} W(t) \, dt \right| \leq \int_{a}^{b} |W(t)| \, dt \leq \text{MAX} \{ |W(t)| \} \, (b - a) \)
\( \alpha \leq t \leq b \)

WHEN \( W(t) \) IS CONTINUOUS ON \( \alpha \leq t \leq b \).

THE PROOFS OF THESE ARE STRAIGHTFORWARD AND ARE OMITTED
EXCEPT FOR THE IMPORTANT (V), WHICH WE NOW SHOW.

PROOF OF (V) SINCE \( W(t) \) IS COMPLEX, THEN FOR SOME \( \rho > 0 \)
AND \( \varphi \) WE HAVE
\[
\int_{a}^{b} W(t) \, dt = \rho e^{i\varphi}.
\]
WHERE \( \rho = \left| \int_{a}^{b} \frac{W(t)}{dt} \right| \) OF THIS COMPLEX NUMBER.
\[ \rho = \left| \int_{\alpha}^{\beta} w(t) \, dt \right| = e^{-i\eta} \int_{\alpha}^{\beta} w(t) \, dt = \int_{\alpha}^{\beta} e^{-i\eta} w(t) \, dt. \tag{X} \]

Now recall that for any complex number \( z \), then
\[ \text{RE}(z) \leq |\text{RE}(z)| \leq |z|. \]

Hence, since RE of (X) is real (it must be \( \rho \)) then,
\[ \rho = \int_{\alpha}^{\beta} \left| \text{RE} \left( e^{-i\eta} w(t) \right) \right| \, dt \leq \int_{\alpha}^{\beta} \left| \text{RE} \left( e^{-i\eta} w(t) \right) \right| \, dt \]
\[ \leq \int_{\alpha}^{\beta} \left| e^{-i\eta} w(t) \right| \, dt = \int_{\alpha}^{\beta} |w(t)| \, dt. \]

Thus,
\[ \left| \int_{\alpha}^{\beta} w(t) \, dt \right| = \rho \leq \int_{\alpha}^{\beta} |w(t)| \, dt. \tag{Q} \]

Next we define a path or contour in the complex plane.

**Definition.** A path or contour \( C \) in complex plane is a piecewise smooth function \( z(t) \) with \( a \leq t \leq b \) with \( z \) complex. A contour has an orientation or direction as \( t \uparrow \).

(i) \( z(t) = z_0 + \text{Re} \, e^{i\pi t} \), \( 0 \leq t \leq 1 \) is a semi-circle centered at \( z_0 \) oriented counterclockwise and of radius \( \Gamma \)

(ii) \( z(t) = z_0 (1-t) + z_1 \), \( 0 \leq t \leq 1 \) is a straight line between \( z_0 \) and \( z_1 \).
(iii) \[ z(t) = t^2 + it \text{ for } 0 \leq t \leq 1 \] parameterizes parabola \( x = y^2 \).

(iv) The path may be the union of straight line segments:

\[ \begin{array}{c}
Z_0 \\ C_0 \\ Z_1 \\ C_1 \\ Z_2 \\ C_2 \\ Z_3 \\ C_3
\end{array} \]

Here \( C = C_1 \cup C_2 \cup C_3 \).

**Definition.** Let \( C \) be a smooth contour on \( a \leq t \leq b \).

Then \[ \int_C f(z) \, dz = \int_a^b f(z(t)) z'(t) \, dt \]

when \( F \) is continuous on the contour.

If \( C \) is piecewise smooth and can be decomposed as

\[ C = C_1 \cup C_2 \cup \ldots \cup C_N \]

where \( z_j(t) \) parameterizes \( C_j \), then

\[ \int_C f(z) \, dz = \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} f(z_j(t)) z_j'(t) \, dt \]

**Picture:**

\[ \begin{array}{c}
Z_0 \\ t_0 \\
Z_1 \\ t_1 \\
Z_2 \\ t_2 \\
Z_3 \\ t_3
\end{array} \]

**Some simple properties are:**

(i) \[ \int_C (a_1 f_1 + a_2 f_2) \, dz = a_1 \int_C f_1 \, dz + a_2 \int_C f_2 \, dz \]

(ii) \[ \int_{-C} f \, dz = -\int_C f \, dz \quad \text{changing orientation of path introduces sign.} \]
(iii) The value of \( \int_C f(z) \, dz \) is independent of how one parametrize the path, provided that the re-parametrization is 1–1, and the orientation of the path.

(iv) Suppose \( |f(z)| \) is bounded on the contour. This occurs, for instance, when \( f(z) \) is analytic on \( C \).

Then \( \int_C |f(z)| \, dz \leq \max_C |f(z)| \cdot (\text{length of } C) \)

whenever \( \text{length of } C \) is finite.

**Proof:** Define \( I = \int_C f(z) \, dz \)

Then,

\[ I = \int_a^b f(z(t)) z'(t) \, dt. \]

Recall that if \( w(t) = f(z(t)) z'(t) \) then by (v) on page (x),

\[ |I| = \left| \int_a^b w(t) \, dt \right| \leq \int_a^b |w(t)| \, dt = \int_a^b |f(z(t))| |z'(t)| \, dt. \]

Now \( |f(z(t))| \leq \max_{a \leq t \leq b} |f(z(t))| = \max_C |f(z)| \).

So \( |I| = \left| \int_C f(z) \, dz \right| \leq \max_C |f(z)| \int_a^b |z'(t)| \, dt. \) (x)

Now write \( z(t) = x(t) + iy(t) \). Then \( |z'| = \sqrt{x'^2 + y'^2} \).

Recall \( \int_a^b \sqrt{x'^2 + y'^2} \, dt = \text{length}(C) \).

Thus, from (x) \( |I| \leq \max_C |f(z)| \cdot (\text{length of } C) \).
Proof of (iii) Let \( z(t) \) be a parametrization of \( C \) on \( a \leq t \leq b \).

Let \( t = \phi(s) \) with \( \phi(s) \) and \( s \) real such that
\[
\phi(a) = a, \quad \phi(b) = b.
\]
Define \( \tilde{z}(s) = z(\phi(s)) \).

\[
\int_a^b f(z(\phi(s))) z'(\phi(s)) \phi'(s) \, ds = \int_a^b f(z(s)) z'(s) \, ds.
\]

Now change variable: \( t = \phi(s) \) so that
\[
\int_a^b f(z(\phi(s))) z'(\phi(s)) \phi'(s) \, ds = \int_a^b f(z(t)) z'(t) \, dt.
\]

**Theorem (Fundamental Theorem of Calculus)** Suppose that \( f(z) \) is continuous in a domain \( D \) and has an antiderivative \( \Phi(z) \) throughout \( D \) (i.e., \( d\Phi/dz = f(z) \) for each \( z \) in \( D \)). Then for any contour \( C \) in \( D \) with initial point \( z_i \) and end point \( z_f \) we get
\[
\int_C f(z) \, dz = \Phi(z_f) - \Phi(z_i).
\]

**Fundamental Theorem Calculus**

**Remark (i)** This means that \( \Phi(z) \) is analytic and continuous in \( D \).

**Proof** Suppose that \( C \) is a contour in \( D \) joining \( z_i \) to \( z_f \).

Then if \( C \) is piecewise smooth,
\[
\int_C f(z) \, dz = \sum_{j=1}^{n} \int_{c_j} f(z) \, dz = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} f(z(t)) z'(t) \, dt \quad (\star)
\]

Now on each separate interval \( dz/dt \) exist and is continuous.

Therefore,
\[
\frac{d}{dt} \Phi(z(t)) = \Phi'(z(t)) z'(t) = f(z(t)) z'(t),
\]
from \((\star)\) we get
\[
\int_C f(z) \, dz = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{d}{dt} \Phi(z(t)) \, dt = \sum_{j=1}^{n} \left( \Phi(z(t_j)) - \Phi(z(t_{j-1})) \right)
\]
telescoping sum
\[
\rightarrow \int_C f(z) \, dz = \Phi(z_f) - \Phi(z_i).
\]
**Definition**  
C is a closed contour or a loop if its initial and terminal points coincide. A simple closed contour is a closed contour with no multiple points other than its initial-terminal point; in other words, if \( z(t) \) for \( a \leq t \leq b \) is a parametrization of the closed contour, then \( z(t) \) is 1-1 on \([a, b)\).

![Diagram of closed contours]

**Corollary**  
If \( f(z) \) is continuous in a domain \( D \) and has an anti-derivative \( F(z) \) throughout \( D \), then \( \int_C f(z) \, dz = 0 \) for all loops \( C \) lying in \( D \). Thus, in this case the integral is independent of the specific path.

The proof is immediate by the theorem.

Since the integral only depends on initial and end state, which are the same, then \( \int_C f(z) \, dz = F(z_f) - F(z_i) = F(z_i) - F(z_i) = 0 \).
EXAMPLE 1 \[ \text{CALCULATE } I_1 = \int_{C_1} z^2 \, dz, \quad I_2 = \int_{C_2} z^2 \, dz \]

WHERE
(i) \( C_1 \) is straight line from \( z = 0 \) to \( z = 2 + i \)

(ii) \( C_2 \) is path from \( z = 0 \) to \( z = 2 + i \) as shown

\[ \text{FOR } C_1: \text{LET } z = (2 + i)t \to \frac{dz}{dt} = 2 + i \quad \text{FOR } 0 \leq t \leq 1. \]

\[ \text{THEN} \quad I_1 = \int_0^1 (2 + i)^2 (2 + i) t^3 \, dt = \int_0^1 (z(t))^3 z'(t) \, dt \]

\[ \quad I_1 = (2 + i)^3 \int_0^1 t^3 \, dt = (2 + i)^3 \frac{t^4}{4} \bigg|_0^1 = (2 + i)^3 = \frac{1}{3} (2 + 11i). \]

\[ \text{FOR } C_2: \text{FIRST LET } z = 2t, \quad \frac{dz}{dt} = 2 \quad \text{dt} \]

\[ \text{THEN} \quad I_{21} = \int_{C_{21}} z^3 \, dz = \int_0^1 z_1^3 (1) \frac{dz}{dt} \, dt = \int_0^1 (4t^3) 2t \, dt = 8 \frac{t^4}{4} \bigg|_0^1 = 8. \]

Now \( I_{22} = \int_{C_{22}} z^2 \, dz. \) Let \( z = 2 + it, \quad 0 \leq t \leq 1. \)

\[ \frac{dz}{dt} = i. \]

\[ I_{22} = \int_0^1 (2 + it)^2 i \, dt = \int_0^1 [4 + 4it - t^2] i \, dt = 4i - 4it \bigg|_0^1 = i t^2 \bigg|_0^1 = i. \]

\[ I_{22} = 4i - 2 - i/3 = -2 + 11i/3. \]

Thus adding together, \( \int = \int_{C_{21}} + \int_{C_{22}} = \frac{1}{3} (8 - 6 + 11i) = \frac{1}{3} (2 + 11i). \)

Notice that \( \int_{C_1} = \int_{C_2} \rightarrow \text{INDEPENDENCE OF PATH.} \)

Now by the theorem on p. 15 (FTC), \( z^2 \) is continuous \( \forall z \)

and the antiderivative is \( \Phi(z) = \frac{z^3}{3}. \)

Then, \( \int_C z^2 \, dz = \Phi(2 + i) - \Phi(0) = \frac{(2 + i)^3}{3}. \)
**Example 2** Calculate \( I_1 = \int_{C_1} \bar{z} \, dz \) and \( I_2 = \int_{C_2} \bar{z} \, dz \)

**Where**

(i) \( C_1: \ z = e^{it}, \ 0 \leq t \leq \pi \)

(ii) \( C_2: \ z = e^{it}, \ 0 \leq t \leq -\pi \)

Notice that both paths have the same initial and final state.

For \( C_1: \ z = e^{it}, \ 0 \leq t \leq \pi \rightarrow \frac{dz}{dt} = ie^{it} \)

so \( \int_{C_1} \bar{z} \, dz = \int_0^\pi \bar{z}(t)z'(t) \, dt = \int_0^\pi e^{-it} ie^{it} \, dt = \pi i \).

For \( C_2: \ z = e^{it}, \ 0 \leq t \leq -\pi \rightarrow \frac{dz}{dt} = ie^{it} \)

so \( \int_{C_2} \bar{z} \, dz = \int_0^{-\pi} \bar{z}(t)z'(t) \, dt = \int_0^{-\pi} e^{-it} ie^{it} \, dt = -\pi i \).

Notice that \( I_1 \neq I_2 \) even though initial and end states are identical.

Also notice that \( \bar{z} \) does not have an anti-derivative. Hence the theorem (FTC) cannot be applied.

**Example 3** Calculate \( I = \int_C z^3 \, dz \) where \( C \) is the portion of ellipse \( x^2 + 4y^2 = 1 \) joining \( z = 1 \) to \( z = i/2 \)

**Solution** \( \Phi(z) = z^{4/4} \) is the anti-derivative of \( F(z) \). Then, FTC \( \int_C z^3 \, dz = \Phi(i/2) - \Phi(1) = (i/2)^{4/4} - 1^{4/4} = 1/64 - 1/4 = -15/64 \).

**Example** Calculate \( \int_C e^{z} \, dz \) where \( C \) is quarter-circle \( z = e^{it}, \ 0 \leq t \leq \pi/2 \)

**Solution** \( e^z \) has the anti-derivative \( \Phi(z) = e^z \) defined \( \forall z \).

Thus by FTC, \( \int_C F(z) \, dz = \int_C e^{z} \, dz = \Phi(i) - \Phi(1) = e^i - e^1 = \cos(1) - e^1 + i \sin(1) \).

Notice that \( \int_C e^{z} \, dz = \int_0^1 e^{i t} \, i e^{it} \, dt = \int_0^1 e^{i t} \, [\cos(t) + i \sin(t)] \, dt \).

Thus \( \text{RE} \left[ \int_0^1 e^{i t} \, [\cos(t) + i \sin(t)] \, dt \right] = \cos(1) - e^1 \).
Calculate \[ I = \int_{C} \frac{1}{z} \, dz \] over the \( \frac{1}{2} \) circle \( z = 2e^{it} \) from \( t = 0 \) to \( t = \frac{\pi}{2} \).

Now the anti-derivative \( \Phi(z) = \sin z \) exists for all \( z \).

Hence,
\[
\int_{C} \frac{1}{z} \, dz = \Phi(-2) - \Phi(2) = +\sin(-2) - \sin(2) = -\sin(2) - \sin(2) = -2\sin(2).
\]

An important integral is to calculate
\[
I = \int_{C} (z - z_0)^n \, dz \quad n: \text{integer}
\]
and \( C \) is a circle of radius \( R \) centered at \( z_0 \) oriented counterclockwise.

We will first calculate directly. Let \( z = z_0 + Re^{it} \) with \( 0 \leq t \leq 2\pi \).

Then
\[
I = \int_{C} (z - z_0)^n \, dz = \int_{0}^{2\pi} (z(t) - z_0)^n \, \frac{dz(t)}{dt} \, dt = \int_{0}^{2\pi} R^n e^{int} Re^{it} \, dt.
\]

\((\ast)\) \[ I = iR^{n+1} \int_{0}^{2\pi} e^{i(n+1)t} \, dt. \]

There are two cases:

- If \( n \neq -1 \) then \[ I = iR^{n+1} \int_{0}^{2\pi} e^{i(n+1)t} \, dt = 0 \]

since \[ \int_{0}^{2\pi} \cos(mt) \, dt = \int_{0}^{2\pi} \sin(mt) \, dt = 0 \] for \( m \neq 0 \) \( m: \text{integer} \)

Thus \( I = 0 \) if \( n \neq -1 \).

(Also note: \( \ast \) \)

- If \( n = -1 \) then \( \ast \) gives \[ I = i \int_{0}^{2\pi} 1 \, dt = 2\pi i. \]

Thus
\[
\int_{C} (z - z_0)^n \, dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}
\]

We now calculate indirectly. Suppose \( n \neq -1 \) then let \( D \) be any domain excluding the point \( z_0 \) which contains \( C \) (see the figure).
The domain D is the annulus between dotted lines.

In D, the anti-derivative for \(\Phi = -1\) is \(F(z) = (z - z_0)^{\Phi + 1}\). By FTC, since initial and final state are same, \(\int_C (z - z_0)^{\Phi} dz = 0\), \(\Phi = -1\).

Notice, if \(\Phi = -1\) then there is no anti-derivative defined in any annular domain around \(z_0\), which contains C, since \(\log(z - z_0)\) is not continuous across the branch cut \(\text{Im}(z - z_0) = 0, \text{Re}(z - z_0) < 0\).

i.e. \(\frac{d}{dz}\log(z - z_0) = \frac{1}{z - z_0}\) in \(C \setminus \{\text{Im}(z - z_0) = 0, \text{Re}(z - z_0) < 0\}\).

Thus for \(\Phi = -1\), FTC cannot be invoked.

Example calculate \(I = \int_C \frac{1}{z} dz\)

When C is the closed contour defined by the polar equation \(z = e^{i\Phi}\) with \(\Gamma = 2 - \sin^2(\frac{\Phi}{4})\) with \(0 < \Phi < 4\pi\).

We note that the path is as shown.

The path circle, the origin twice.

We calculate \(dz/d\Phi = r'e^{i\Phi} + i e^{i\Phi}\).

Thus \(I = \int_C \frac{1}{e^{i\Phi}} \frac{dz}{d\Phi} = \int_0^{4\pi} \frac{i}{r e^{i\Phi}} [r'e^{i\Phi} + i e^{i\Phi}] d\Phi\).

So \(I = \int_0^{4\pi} \frac{1}{r} (r' + i) d\Phi = \ln(r(\Phi))\int_0^{4\pi} + 4\pi i\) since \(r \neq 0\) for any \(\Phi\).

But \(r(4\pi) = r(0)\) so \(\ln(r(\Phi))\int_0^{4\pi} = 0\) and \(I = 4\pi i\).
Example show by a limiting procedure that
\[ \int_C \frac{1}{z} \, dz = 2\pi i \quad \text{where } C \text{ is a circle of radius } R \]
centered at \( z = 0 \) oriented counterclockwise.

As remarked earlier, we cannot use FTC. Instead we consider the picture.

We define \( \log z \) to be the principal branch of \( \log z \) with
\[ \log z = \ln |z| + i\varphi \quad -\pi < \varphi < \pi. \]
\[ \frac{d}{dz} \log z = \frac{1}{z} \quad \text{for } z \text{ in } C \setminus \{-\infty, 0\}. \]

We parameterize \( z = R e^{i\theta} \) with \(-\pi + \varepsilon < \theta < \pi - \varepsilon\) for \( \varepsilon > 0 \). Call this \( C_{\varepsilon} \).

Then we can use FTC for any \( \varepsilon > 0 \) to get
\[ \int_{C_{\varepsilon}} \frac{1}{z} \, dz = \log z_f - \log z_i \quad z_i = Re^{i(-\pi + \varepsilon)} \quad z_f = Re^{i(\pi - \varepsilon)} \]
\[ \int_{C_{\varepsilon}} \frac{1}{z} \, dz = (\ln R + i(\pi - \varepsilon)) - (\ln R + i(-\pi + \varepsilon)) \]
\[ = 2i\pi - 2 \varepsilon \]

Now let \( \varepsilon \to 0^+ \) \( \to \int_C \frac{1}{z} \, dz = 2i\pi. \)

Remark: This can clearly be extended to prove that
\[ \int_C \frac{1}{z} \, dz = 2\pi i \quad \text{for any simple closed contour containing origin}. \]

In addition
\[ \int_C \frac{1}{z^n} \, dz = 0 \quad \text{for } n \neq 1 \quad \text{where} \]
\( C \) is any closed simple contour containing \( z = 0 \).

(\text{It follows by existence of anti-derivative}).
Next we give a few examples to illustrate bounds on integrals. We recall that if \( f(z) \) is bounded on \( C \) then
\[
\left| \int_C f(z) \, dz \right| \leq \max_C |f(z)| \text{ length}(C).
\]

**Example 1** Estimate \( \left| \int_C \frac{1}{z^4} \, dz \right| \) where \( C \) is the line joining \( z = -1 \) to \( z = i \).

Now let \( z = -1 + t(i+1) \) when \( t = 0 \to z = -1 \)
\( t = 1 \to z = i \).

Now \[
\max_C \left| \frac{1}{z^4} \right| = \frac{1}{\min_C |z^4|} = \frac{1}{\min_C |z|^4} \leq \frac{1}{|\frac{1}{2}(-1+i)|^4} = \frac{1}{\left| \frac{\sqrt{2}}{2} \right|^4} = 4.
\]

Length \( C = \sqrt{2} \).

Thus,
\[
\left| \int_C \frac{1}{z^4} \, dz \right| \leq 4\sqrt{2}.
\]

**Example 2** Estimate \( \left| \int_C \frac{1}{z^2+1} \, dz \right| \) where \( C \) is the quarter circle \( z = 2e^{i\theta} \) with \( 0 \leq \theta \leq \pi/2 \).

We recall the \( \Lambda \)-inequality \( |z_1 + z_2| \geq |z_1| - |z_2| \).

Thus \[
\frac{1}{|z^2+1|} \leq \frac{1}{|z|^2 - 1} \quad \text{(note: } |z^2 - (1)| \geq |z^2| - 1 \text{).}
\]

But since \( |z|^2 = 4 \) on \( C \) we get
\[
\frac{1}{|z^2+1|} \leq \frac{1}{4 - 1} = \frac{1}{3}
\]

Thus \[
\max_C \frac{1}{|z^2+1|} \leq \frac{1}{3} \text{ length } C = \frac{\pi}{2} \text{ of } C = \frac{\pi}{2}.
\]

Thus
\[
\left| \int_C \frac{1}{z^2+1} \, dz \right| \leq \frac{\pi}{3}.
\]

**Example 3** Estimate \( \left| \int_C (e^z - \bar{z}) \, dz \right| \) where \( C \) is the circle \( |z| = 2 \).

We use \[
|e^z - \bar{z}| \leq |e^z| + |\bar{z}| = |e^{\text{Re}(z)} + i \text{Im}(z)| + |\bar{z}| = |e^{\text{Re}(z)}| + |\bar{z}|
\]
Continuing on \[
|e^z - \bar{z}| \leq e^{\text{Re}(z)} + |\bar{z}| \leq e^{\text{Re}(z)} + |z|^2 \leq e^{\text{Re}(z)} + 2 |z| \leq e^2 + 2.
\]

Now \text{ length } (C) = 2\pi \text{ of } C. \text{ Thus, } \left| \int_C (e^z - \bar{z}) \, dz \right| \leq (e^2 + 2) \pi.
Let $C_R$ denote the semi-circle $z = Re^{i\theta}$ with $0 \leq \theta \leq \pi$.

Show that

(i) \[ \left| \int_{C_R} \frac{z}{z^3 + 1} \, dz \right| \to 0 \quad \text{as} \quad R \to \infty \]

(ii) \[ \left| \int_{C_R} \frac{\log z}{z^2 + 1} \, dz \right| \to 0 \quad \text{as} \quad R \to \infty \quad \log z \text{ is the P.V. of } \int \log z \]

(iii) \[ \left| \int_{C_R} \frac{e^{iz}}{z^2 + 1} \, dz \right| \to 0 \quad \text{as} \quad R \to \infty \quad \text{for any } K > 0 \text{ real.} \]

Proof (i) \[ |z^3 + 1| \geq |z|^3 - 1 = R^3 - 1 \quad \text{for} \quad R > 1. \]

Thus \[ \left| \frac{z}{z^3 + 1} \right| \leq \frac{|z|}{R^3 - 1} = \frac{R}{R^3 - 1} \quad \text{on} \quad C_R. \]

Now \[ \text{length } (C_R) = \pi R. \]

Thus \[ \left| \int_{C_R} \frac{z}{z^3 + 1} \, dz \right| \leq \pi \left( \frac{R}{R^3 - 1} \right) R = \frac{\pi R^2}{R^3 - 1} \to 0 \quad \text{as} \quad R \to \infty. \]

(ii) \[ |\log z| = |\log |z| + i\theta| \leq |\log R + i\pi| = \sqrt{(\log R)^2 + \pi^2} \quad \text{on} \quad C_R. \]

Thus \[ \left| \int_{C_R} \frac{\log z}{z^2 + 1} \, dz \right| \leq \frac{\sqrt{(\log R)^2 + \pi^2}}{R^2 - 1} \pi R = O \left( \frac{R |\log R|}{R^2} \right) \quad \text{for} \quad R \to \infty. \]

(iii) Let \[ z = x + iy. \]

Thus \[ |e^{iz}| = |e^{ix - K\pi}| = e^{-KY} \geq 1 \quad \text{for} \quad K > 0 \quad \text{and} \quad Y > 0. \]

Thus \[ |e^{iz}| \leq 1 \quad \text{on} \quad C_R. \]

Thus \[ |z^2 + 1| \geq |z|^2 - 1 = R^2 - 1 \quad \text{on} \quad C_R. \]

Hence \[ \left| \int_{C_R} \frac{e^{iz}}{z^2 + 1} \, dz \right| \leq \frac{1}{R^2 - 1} \to 0 \quad \text{as} \quad R \to \infty. \]