Ex1. Find a branch for the function $\log (z^2 + 1)$, such that it is analytic at 0, and equals $2\pi i$ at 0.

Consider the function $\text{Log} (z^2 + 1)$. The branch cut of this single valued function consists of those points $z$ satisfying

$$z^2 + 1 = a \in (-\infty, 0].$$

That is,

$$z = \pm i\sqrt{1-a}, \ a \in (-\infty, 0].$$

Therefore the branch cut of $\text{Log} (z^2 + 1)$ consists of two rays:

$$i\sqrt{1-a}, \ a \in (-\infty, 0];$$

$$-i\sqrt{1-a}, \ a \in (-\infty, 0].$$

In particular, it is analytic at 0. Moreover, we compute

$$\text{Log} (1) = 0.$$ 

We then choose the following branch for $\log (z^2 + 1)$:

$$\text{Log} (z^2 + 1) + 2\pi i.$$

This is a solution for our problem. Note that one may also choose other branches, the solutions are not unique.

Ex2. Find a branch for the function $\log (z^2 - 1)$, such that it is analytic at 0, and equals $-\pi i$ at 0.

Let us write

$$\log (z^2 - 1) = \log (z + 1) + \log (z - 1).$$

For $\log (z + 1)$, we choose the branch

$$\text{Log} (z + 1) - 2\pi i.$$

This branch is analytic at 0 and takes value $-2\pi i$ there.
For log \((z - 1)\), we choose the branch

\[ \ln|z - 1| + i \arg_{0} (z - 1). \]

This branch is also analytic at 0 and equals \(\pi i\) at 0.

The sum of the previous two branches will be a solution of our problem.

Note: For each fixed \(\tau \in \mathbb{R}\), the single valued function \(\arg_{\tau} (z)\) is a branch of \(\arg (z)\), and takes values in the interval \((\tau, \tau + 2\pi]\). Its branch cut is the ray making angle \(\tau\) with the positive \(x\) axis.