The logarithmic function

Ex1. Compute \( \log (-2i) \)

\[
\log (-2i) = \ln |-2i| + i \arg (-2i) \\
= \ln 2 + i \left( -\frac{\pi}{2} + 2k\pi \right), k \in \mathbb{Z}.
\]

On the other hand, \( \text{Log} (-2i) = \ln 2 - \frac{\pi}{2} i \).

Ex2. Find solutions of \( \sin z = \cos z \).

The equation is equivariant to

\[
\frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iz} - e^{-iz}}{2i},
\]

Hence \((1 + i) e^{iz} = (i - 1) e^{-iz} \), from which we get

\[
e^{2iz} = i.
\]

Therefore, by the definition of \( \log \), we have

\[
2iz = \log (i) = \left( \frac{\pi}{2} + 2k\pi \right) i. \\
z = \frac{\pi}{4} + k\pi, k \in \mathbb{Z}.
\]

Ex3. Simplify \( \text{Log} (e^z) \).

Writing \( z = x + yi \), we have

\[
\text{Log} (e^z) = \text{Log} (e^{x+yi}) \\
= \ln (e^x) + i \text{Arg} (e^{y+yi}) \\
= x + i (y - 2k^*\pi),
\]

where the integer \( k^* \) is chosen such that \( y - 2k^*\pi \in (-\pi, \pi] \). In particular,

\[
\text{Log} (e^z) = z, \text{ if and only if } y \in (-\pi, \pi].
\]
Ex4.

$$\log(z_1 z_2) = \log(z_1) + \log(z_2).$$

Indeed,

$$\log(z_1 z_2) = \ln |z_1 z_2| + i \arg(z_1 z_2)$$
$$= \ln |z_1| + \ln |z_2| + i ((\arg z_1) + \arg z_2)$$
$$= \ln |z_1| + i \arg(z_1) + \ln |z_2| + i \arg(z_2)$$
$$= \log(z_1) + \log(z_2).$$

Note that log is indeed multiple-valued function.