Some simple analytic functions (Exponential, Trigonometric, Hyperbolic)

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Ex1. Find the image of \( \{ z : |\text{Re} z| < \frac{\pi}{2}, \text{Im} z > 0 \} \) under the map \( f (z) = \sin z \).

We write, using trigonometric identity,

\[
\sin z = \sin x \cosh y + i \cos x \sinh y.
\]

Denote \( u = \sin x \cosh y, v = \cos x \sinh y \). Let us analyze the image of the boundary curve of the original domain.

If \( x = -\frac{\pi}{2} \), then

\[
u = - \cosh y, v = 0.
\]

Note the \( y \in (0, +\infty) \). This corresponds to a half straight line on the real axis.

If \( x = \frac{\pi}{2} \), then

\[
u = \cosh y, v = 0.
\]

If \( y = 0 \), then \( u = \sin x, v = 0 \).

For general fixed \( x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) with \( x \neq 0 \), we have the following relation between \( u \) and \( v \):

\[
\left( \frac{u}{\sin x} \right)^2 - \left( \frac{v}{\cos x} \right)^2 = 1.
\]

This is part of a hyperbola. Note that since \( y > 0 \), always we have \( v > 0 \).

In conclusion, the image is the upper half plane.

Ex2. Solve \( \sin z = 2 \).

This equation is equivariant to

\[
\sin x \cosh y + i \cos x \sinh y = 2
\]

We therefore get

\[
\begin{align*}
\cos x \sinh y &= 0, \\
\sin x \cosh y &= 2.
\end{align*}
\]
From the first equation, we get $y = 0$, or $x = \frac{\pi}{2} + 2k\pi$, or $x = -\frac{\pi}{2} + 2k\pi$.

Case 1. $y = 0$.
In this case, the second equation reads as $\sin x = 2$, which has no solution. (Remember that $x$ is real)

Case 2. $x = -\frac{\pi}{2} + 2k\pi$.
In this case, the second equation is $-\cosh y = 2$. This equation has no real solution.

Case 3. $x = \frac{\pi}{2} + 2k\pi$.
In this case, the second equation is $\cosh y = 2$. That is

$$\frac{e^y + e^{-y}}{2} = 2.$$ 

Solving this equation, we find $e^y = 2 \pm \sqrt{3}$. Hence $y = \ln (2 \pm \sqrt{3})$.

In summary, the solutions of the equation $\sin z = 0$ are given by

$$z = \frac{\pi}{2} + 2k\pi + i \ln \left(2 \pm \sqrt{3}\right), \; k \in \mathbb{Z}.$$