Some simple analytic functions (Exponential, Trigonometric, Hyperbolic)

Jan. 18, 2019

Suggested problems for this week: Exercises 2.5: 3, 7, 9, 17, 19, Exercises 3.4: 4. Exercises 3.2: 5, 9, 13, 17.

Ex1. Find the image of

\[ \{ z : |\text{Re} z| < \frac{\pi}{2}, \, \text{Im} z > 0 \} \]

under the map \( f(z) = \sin z \).

We write, using trigonometric identity,

\[ \sin z = \sin x \cosh y + i \cos x \sinh y. \]

Denote \( u = \sin x \cosh y, v = \cos x \sinh y \). Let us analyze the image of the boundary curve of the original domain.

If \( x = -\frac{\pi}{2} \), then

\[ u = -\cosh y, \, v = 0. \]

Note the \( y \in (0, +\infty) \). This corresponds to a half straight line on the real axis.

If \( x = \frac{\pi}{2} \), then

\[ u = \cosh y, \, v = 0. \]

If \( y = 0 \), then \( u = \sin x, \, v = 0. \)

For general fixed \( x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) with \( x \neq 0 \), we have the following relation between \( u \) and \( v \):

\[ \left( \frac{u}{\sin x} \right)^2 - \left( \frac{v}{\cos x} \right)^2 = 1. \]

This is part of a hyperbola. Note that since \( y > 0 \), always we have \( v > 0 \).

In conclusion, the image is the upper half plane.

Ex2. Solve \( \sin z = 2 \).

This equation is equivariant to

\[ \sin x \cosh y + i \cos x \sinh y = 2 \]

We therefore get

\[ \begin{cases} 
\cos x \sinh y = 0, \\
\sin x \cosh y = 2. 
\end{cases} \]
From the first equation, we get $y = 0$, or $x = \frac{\pi}{2} + 2k\pi$, or $x = -\frac{\pi}{2} + 2k\pi$.

Case 1. $y = 0$.

In this case, the second equation reads as $\sin x = 2$, which has no solution. (Remember that $x$ is real)

Case 2. $x = -\frac{\pi}{2} + 2k\pi$.

In this case, the second equation is $-\cosh y = 2$. This equation has no real solution.

Case 3. $x = \frac{\pi}{2} + 2k\pi$.

In this case, the second equation is $\cosh y = 2$. That is

$$\frac{e^y + e^{-y}}{2} = 2.$$

Solving this equation, we find $e^y = 2 \pm \sqrt{3}$. Hence $y = \ln \left(2 \pm \sqrt{3}\right)$.

In summary, the solutions of the equation $\sin z = 0$ are given by

$$z = \frac{\pi}{2} + 2k\pi + i \ln \left(2 \pm \sqrt{3}\right), \ k \in \mathbb{Z}. $$