Residue Theorem

Mar. 11

References: Lecture notes on Residue calculus. Section 6.1 of the textbook.

Suppose $z_0$ is an isolated singularity, and around $z_0$

$$w(z) = \sum_{j=-\infty}^{+\infty} a_j (z - z_0)^j .$$

Define the residue of $w$ at $z_0$ to be

$$\text{Res}(w, z_0) := a_{-1} .$$

The following residue theorem is a direct consequence of the Cauchy theorem for multiply connected domains.

**Theorem 1 (Residue theorem):** Suppose $w$ has finitely many singularities $z_1, ..., z_n$ inside $\Gamma$. Then

$$\int_\Gamma w(z) \, dz = 2\pi i \sum_{j=1}^{n} \text{Res}(w, z_j) .$$

Residue calculus:
Suppose $z_0$ is a pole of $w$ of order $m$. Then

$$\text{Res}(w, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (w(z) (z - z_0)^m) . \tag{1}$$

Order of the pole:
In the special case that $w(z) = \frac{f(z)}{g(z)}$, where $f, g$ are analytic. Assume $z_0$ is a zero of $f$ of order $n$, and a zero of $g$ of order $p$. Then $z_0$ is a pole of $w$ of order $p - n$. 

1
In the special case that \( p = 1, n = 0 \). Then by (1),

\[
\text{Res} (w, z_0) = \frac{f(z_0)}{g'(z_0)}.
\] (2)

Ex1. Compute \( \text{Res} \left( (z + 1) e^{\frac{1}{z}}, 0 \right) \).
Observe that

\[
e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2} \frac{1}{z^2} + \frac{1}{3!} \frac{1}{z^3} + ...\]

We get

\[
(z + 1) e^{\frac{1}{z}} = (z + 1) \left( 1 + \frac{1}{z} + \frac{1}{2} \frac{1}{z^2} + \frac{1}{3!} \frac{1}{z^3} + ... \right).
\]

Expand it, we see that the coefficient before the term \( 1/z \) will be

\[
\frac{1}{2} + 1 = \frac{3}{2}.
\]

Hence \( \text{Res} \left( (z + 1) e^{\frac{1}{z}}, 0 \right) = \frac{3}{2} \).

Ex2. \( \text{Res} \left( \frac{e^z}{\sin z}, 0 \right) \)
Using formula (2), we get

\[
\text{Res} \left( \frac{e^z}{\sin z}, 0 \right) = \frac{e^z}{\cos z} \bigg|_{z=0} = 1.
\]