Argument Principle

Mar. 1, 2019

References: Lecture notes on argument principle and Nyquist criterion. Section 6.7 of textbook.


Argument Principle:
Let $P$ be analytic inside the simple closed positively oriented curve $\Gamma$, then the number of zeroes (counted with multiplicity) of $P$ inside $\Gamma$ is equal to

\[
\frac{1}{2\pi i} \int_{\Gamma} P'(z) \frac{P(z)}{P(z)} \, dz.
\]

This integral is equal to

\[
\frac{\Delta_{\Gamma} (\arg P(z))}{2\pi}.
\]

Here $\Delta_{\Gamma} (\arg P(z))$ is the net excursion of $\arg (P(z))$ along the curve $P(z)$.

Formally, let $w = P(z)$, then the above integral will be transformed to

\[
\frac{1}{2\pi i} \int_{P(\Gamma)} \frac{1}{w} \, dw.
\]

Note that the image $P(\Gamma)$ may not be a simple closed curve (it can has self-intersection and may be repeated several times).

Ex1. $f(z) = z^3$.
For $z = e^{it}$, $f(z) = e^{3it}$. Hence it will traverse three times around the origin. This corresponds to the fact that $f$ has three zeroes in the unit disk.

Ex2. Find the number of zeroes of the polynomial $8z^4 + z^3 + \frac{8}{5}z^2 + z + 1$ in the unit disk.
For $z = e^{it}$, $P(z) = 8e^{4it} + e^{3it} + \frac{8}{5}e^{2it} + e^{it} + 1$.

\[
\left[ 8\cos(4t) + \cos(3t) + \frac{8}{5}\cos(2t) + \cos(t) + 1, 8\sin(4t) + \sin(3t) + \frac{8}{5}\sin(2t) + \sin(t) \right]
\]

The plot of $P(z)$ for $t \in [0, 2\pi]$ generated by computer:
The plot for $P(z)$ for $t \in [0, \pi]$

The plot of $P(z)$ for $t \in [\pi, 2\pi]$:
Hence $\Delta_r (\arg (P(z))) = 8\pi$. It follows that $P$ will has four zeroes inside the unit disk.