Application of Cauchy integral formula–Derivatives of analytic functions

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References: Lecture Notes on Cauchy formula and its consequences; Sec. 4.5 of the textbook.

Suggested exercises for this(and previous) week: Exercises 4.3: 1.a,c,f,i. Exercise 4.4: 10a,c, 13,15,16,17,20. Exercises 4.5: 3,4,6.

Let $f$ be analytic inside $\Gamma$, which is positively oriented. Then we have the following formula for the derivatives of $f$:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$  

This formula can be used to compute certain integrals with finite order singularities(of the form $\frac{f(\xi)}{(\xi - a)^n}$, with $n$ being a natural number).

Note that there are functions with singularities of infinite order(To be discussed later on).

Ex1. Compute $\int_{\Gamma} \frac{e^z}{z^5} dz$, where $\Gamma$ is $|z| = 1$, with positive orientation.

Note that the function $\frac{e^z}{z^5}$ has a singularity at 0. Applying the derivative formula, we get

$$\int_{\Gamma} \frac{e^z}{z^5} dz = \frac{2\pi i}{4!} (e^z)^{(4)}|_{z=0}$$

$$= \frac{\pi i}{12}.$$  

Ex2. $\int_{\Gamma} \frac{z+1}{z(z-1)^2} dz$, $\Gamma$ is the figure eight contour. Surrounding 0, 1, positively orientated around 1 and negatively oriented around 0.

Let $C_1$ be the part of contour around 1 and $C_2$ be that of 0. Note that $C_2$ is negatively oriented.
Then we have

\[
\int_{\Gamma} \frac{z+1}{z(z-1)^3} \, dz = \int_{C_1} + \int_{C_2} \frac{z+1}{z(z-1)^3} \, dz
\]

\[
= \frac{2\pi i}{2!} \left( \frac{z+1}{z} \right) ^{(2)} \bigg|_{z=1} - 2\pi i \left( \frac{z+1}{(z-1)^3} \right) \bigg|_{z=0}
\]

\[
= 2\pi i + 2\pi i
\]

\[
= 4\pi i.
\]