Cauchy integral formula and its applications

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References: Lecture Notes on Cauchy formula and its consequences; Sec. 4.5 of the textbook.

Keep in mind: Cauchy integral formula is

\[ f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta. \]

Here \( \Gamma \) is a simple closed curve with positive orientation, \( z \) is inside \( \Gamma \) and \( f \) is analytic inside (and on) \( \Gamma \).

Ex2. Compute the integral:

\[ \int_\Gamma \frac{z^4+2}{(z^2+1)(z-1)} \, dz, \]

where \( \Gamma : |z| = 5 \), with counter clockwise direction.

Step 1. Locate the singularities (inside the contour). 1, \( i \), \( -i \). Let \( C_1, C_2, C_3 \) be small circles (with radius \( < \frac{1}{2} \)) around these singularities, also with counter clockwise direction.

Step 2. We have, by Cauchy theorem for multiply connected domain:

\[ \int_\Gamma \frac{z^4+2}{(z^2+1)(z-1)} \, dz = \int_{C_1} + \int_{C_2} + \int_{C_3} \frac{z^4+2}{(z^2+1)(z-1)} \, dz. \]

Step 3. For each \( C_j \), we apply Cauchy integral formula (with different \( f \) for different \( C_j \)):

\[ \int_{C_1} \frac{z^4+2}{(z^2+1)(z-1)} \, dz = \int_{C_1} \frac{z^4+2}{z^2+1} \, dz \]

\[ = \int_{C_1} \frac{z^4+2}{z+1} \, dz \]

\[ = 2\pi i \frac{1^4+2}{1^2+1} = 2\pi i \frac{3}{2}. \]
\[ \int_{C_1} \frac{z^4 + 2}{(z^2 + 1)(z - 1)} \, dz = \int_{C_2} \frac{\frac{z^4 + 2}{z + i}}{z - i} \, dz = 2\pi i \frac{i^4 + 2}{(i + i)(i - 1)} = 2\pi i \frac{3}{2i(i - 1)}. \]

\[ \int_{C_3} \frac{z^4 + 2}{(z^2 + 1)(z - 1)} \, dz = \int_{C_3} \frac{\frac{z^4 + 2}{z + i}}{z - i} \, dz = 2\pi i \frac{i^4 + 2}{(-i - i)(-i - 1)} = 2\pi i \frac{3}{(-2i)(-i - 1)}. \]

Hence

\[ \int_{\Gamma} \frac{z^4 + 2}{(z^2 + 1)(z - 1)} \, dz = 2\pi i \frac{3}{2} + 2\pi i \frac{3}{2i(i - 1)} + 2\pi i \frac{3}{(-2i)(-i - 1)} = 0. \]

**Ex2.** \[ \int_0^{2\pi} e^{2\cos \theta} \cos (2 \sin \theta) \, d\theta. \]

The main idea to compute this real integral is transforming it into a complex integral and using Cauchy integral formula.

First, we compute, by Cauchy integral formula,

\[ \int_{|z|=1} \frac{e^{2z}}{z} \, dz = 2\pi i, \text{ (positive orientation).} \quad (1) \]

Let us compute this integral using parametrization technique. Let \( z = e^{i\theta}. \) Then

\[ \int_{|z|=1} \frac{e^{2z}}{z} \, dz = i \int_0^{2\pi} e^{2(\cos \theta + i \sin \theta)} \, d\theta = i \int_0^{2\pi} e^{2\cos \theta} (\cos (2 \sin \theta) + i \sin (2 \sin \theta)) \, d\theta. \]

Comparing this identity with (1), we obtain

\[ \int_{0}^{2\pi} e^{2\cos \theta} (\cos (2 \sin \theta) + i \sin (2 \sin \theta)) \, d\theta = 2\pi. \]
That is,

\[\int_{0}^{2\pi} e^{2\cos \theta} \cos (2 \sin \theta) \, d\theta = 2\pi,\]
\[\int_{0}^{2\pi} e^{2\cos \theta} \sin (2 \sin \theta) \, d\theta = 0.\]

As a matter of fact, the same argument as above tells us, for any real number \(a\),

\[\int_{0}^{2\pi} e^{a\cos \theta} \cos (a \sin \theta) \, d\theta = 2\pi.\]

Note that for \(a = 0\), this is trivial.